BENDING OF THIN SHELLS OF REVOLUTION OF VARIABLE THICKNESS

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Reprinted from :
INDIAN JOURNAL OF MECHANICS AND MATHEMATICS
Volume IV, No. 2, 1966.

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ABSTRACT : The Love-Meissner's equations for the bending of thin shells have been solved approximately for all shells of revolution in general, assuming that the wall thickness decreases exponentially from the edge. The method of obtaining the solutions of a specific problem with prescribed surface loads and boundary conditions has been discussed from the standpoint of superposition. Numerical results are given for a clamped spherical shell subjected to different types of axi-symmetrical loads.

Nomenclature :

Fig. 1 represents an element of the surface bounded by two adjacent parallel circles and meridional planes. The following symbols have been used.

θ	coordinate of longitude,
ϕ	angle between the principal normal in the meridional plane and the axis of revolution,
r_1, r_2	principal radii of curvature,
r_0	radius of the horizontal circle,
h	thickness of the shell,
D	flexural rigidity of the shell, $=Eh^3/12\,(1-\nu^2)$,
E	Young's modulus,
ν	Poisson's ratio,
N_θ	unit hoop force,
N	force per unit length of horizontal circle,
M_θ	bending moment per unit length of the meridian,
M_ϕ	bending moment per unit length of horizontal circle,
Q_ϕ	Shearing force per unit length of the horizontal circle,
S	displacement along increasing direction of r_0,
v	displacement along increasing direction of φ,
W	normal displacement, positive inwards,
V	rotation of the edge, $=r_2\,Q_\phi$,
Z	intensity of load along inward drawn normal.

1. Introduction :

The generalised Love-Meissner equations for the bending of thin shells have been solved by Spotts,[1], in the case of a symmetrically loaded spherical shell having the square root of its thickness varying linearly from the edge. Assuming the same variation in thickness, Rygol [2], has shown that Love-Meissner's approximate equations can be solved exactly for any surface of revolution. He has obtained the expressions for stresses, moments, displacement and rotation of the edge in simple forms and deduced the corresponding results for a spherical shell as a particular case. Flugge [3], has treated the same problem in his literature. Basuli [4], has solved the problem, assuming the thickness varying linearly as the arc-length from the top and deduced the corresponding results for a spherical shell. The object of this paper is to show that the the approximate Love-Meissner's equations can be solved approximately for all shells of revolution whose wall thickness decreases exponentially from the edge. At first, the problems of bending under edge loading and edge moments have been solved; then the method of obtaining the solutions of a specific problem with prescribed surface loads and boundary conditions has been discussed from the point of view superposition. Numerical results are given for a clamped spherical shell subjected to various types of axi symmetrical loads.

2. Equations of Equlibrium :

The problem of axi-symmetrical bending of thin shells having the form of a surface of revolution by moments and horizontal forces distributed over the edges of the shell involves the solution of the Love-Meissner's differential equations [5],

$$\frac{r_2}{r_1^2}\frac{d^2U}{d\varphi^2}+\frac{1}{r_1}\left[\frac{d}{d\varphi}\left(\frac{r_2}{r_1}\right)+\frac{r_2}{r_1}\cot\varphi-\frac{r_2}{r_1}\frac{1}{h}\frac{dh}{d\varphi}\right]\frac{dU}{d\varphi}$$
$$-\frac{1}{r}\left(\frac{r_1}{r_2}\cot^2\varphi-\nu-\frac{\nu}{h}\frac{dh}{d\varphi}\cot\varphi\right)U=E\,h\,V, \quad \ldots \quad (2.1)$$

$$\frac{r_2}{r_1^2}\frac{d^2V}{d\varphi^2}+\frac{1}{r_1}\left[\frac{d}{d\varphi}\left(\frac{r_2}{r_1}\right)+\frac{r_2}{r_1}\cot\varphi+\frac{3r_2}{r_1}\frac{1}{h}\frac{dh}{d\varphi}\right]\frac{dV}{d\varphi}$$
$$-\frac{1}{r_1}\left(\nu-3\nu\frac{\cot\varphi}{h}\frac{dh}{d\varphi}+\frac{r_1}{r_2}\cot^2\varphi\right)V=-\frac{U}{D}. \quad \ldots \quad (2.2)$$

These differential equations have been solved for a thin spherical shell of uniform thickness and it has been shown that Q_φ and V are of such magnitude that these functions and their first derivatives can be neglected in comparison with their second derivatives with this approximation the differential equations (2.1) and (2.2) can be written in the forms

$$\frac{r_2^2}{r_1^2}\frac{d^2Q_\varphi}{d\varphi^2}=EhV, \qquad \frac{1}{r_1^2}\frac{d^2V}{d\varphi^2}=-\frac{Q_\varphi}{D}. \qquad \dots \qquad (2.3)$$

Eliminating V from these equations we get

$$\frac{d^4Q_\varphi}{d\varphi^4}+12(1-\nu^2)\left(\frac{r_1^2}{r_2h}\right)^2Q_\varphi=0. \qquad \dots \qquad (2.4)$$

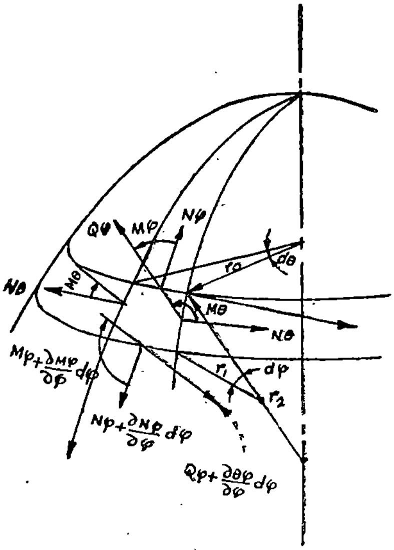

Fig.—1

With these approximations for Q_φ and its derivatives, N_θ, N_φ, V, M_φ and δ can be expressed in terms of Q_φ and its derivatives, and they are represented by,

$$N_\varphi=-Q_\varphi\cot\varphi, \qquad N_\theta=-\frac{r_2}{r_1}\frac{dQ_\varphi}{d\varphi}, \qquad V=\frac{1}{Eh}\frac{r_2^2}{r_1^2}\frac{d^2Q_\varphi}{d\varphi^2},$$

$$M_\varphi=-D\left(\frac{1}{r_1}\frac{dV}{d\varphi}+\nu\frac{V}{r_2}\cot\varphi\right)=-\frac{D}{Eh}\frac{r_2^2}{r_1^3}\frac{d^3Q_\varphi}{d\varphi^3},$$

$$\delta=\frac{r_0}{Eh}(N_\theta-\nu N_\varphi)=-\frac{r_0}{Eh}\frac{r_2}{r_1}\frac{dQ_\varphi}{d\varphi}. \qquad \dots \qquad (2.5)$$

3. Solution of the Differential Equation :

The coefficient of Q_ϕ in the differential equation (2.4) is a variable quantity. Let us assume that the thickness of the shell is given by

$$h=h_0+\lambda\, e^{ka\phi}, \qquad \dots \qquad (3.1)$$

where h_0, λ, a are arbitrary constants and K is a small parameter. Substituting this in equation (2.4) and dividing by $(h_0+\lambda)^2$ we find that,

$$\frac{(h_0+\lambda\, e^{ka\varphi})^2}{(h_0+\lambda)^2}\frac{d^4Q_\varphi}{d\varphi^4}+4\alpha^4Q_\varphi=0, \quad \ldots \quad (3.2)$$

in which

$$\alpha^4=\frac{3(1-\nu^2)}{(h_0+\lambda)^2}\left(\frac{r_1^2}{r_2}\right)^2. \quad \ldots \quad (3.3)$$

The following method taking the quantity K as a parameter proves to be most efficient in handling the present problem. Considering Q_ϕ as a function of ϕ and K we can express Q_φ (ϕ, K) in the form of the power series

$$Q_\varphi=\sum_{m=0}^{\infty}Q_{\varphi_m}K^m \quad \ldots \quad (3.4)$$

in which m is an integer and the coefficients Q_{φ_m} are merely functions of ϕ. Substituting the expression (3.4) in equation (3.2) and equating to zero the coefficients of successive powers of K, we obtain a sequence of differential equations :

$$\frac{d^4Q_{\varphi_0}}{d\varphi^4}+4\alpha^4Q_{\varphi_0}=0,$$

$$\frac{d^4Q_{\varphi_1}}{d\varphi}+4\alpha^4Q_{\varphi_1}=\left(\frac{a\lambda}{h_0+\lambda}\right)8\alpha^4\phi\, Q_{\varphi_0},$$

$$\frac{d^4Q_{\varphi_2}}{d\varphi^2}+4\alpha^4Q_{\varphi_2}=\left(\frac{a\lambda}{h_0+\lambda}\right)8\alpha^4\phi\, Q_{\varphi_1}$$

$$+\left\{\frac{\lambda(h_0-2\lambda)a^2}{2(h_0+\lambda)^2}\right\}8\alpha^4\phi^2\, Q_{\varphi_0}, \quad \ldots \quad (3.5)$$

... ...

The solution of the first equation of (3.5) can be written as $Q_{\phi_0}=A_0\, e^{\alpha\varphi}\cos(\alpha\varphi+\epsilon_0)+A_0'\, e^{-\alpha\varphi}\cos(\alpha\varphi+\epsilon_0')$. Considering the case in which there is an hole at the top and shell is bent by forces and moments uniformly distributed along the edge, we need consider only the first term of Q_{φ_0}, since that term decreases as the angle ϕ decreases. Similar argument holds good for the solutions of all other equations of (3.5). In writing down the solutions of (3.5) we can now omit the the terms contaning $e^{-\alpha\varphi}$. The solutions of the equations (3.5) are:

$$Q_{\varphi_0}=A_0\, e^{\alpha\varphi} \cos(\alpha\varphi+\epsilon_0)$$

$$Q_{\varphi_1}=A_1\, e^{\alpha\varphi} \cos(\alpha\varphi+\epsilon_1)C_1\, A_0\, e^{\alpha\varphi}\{ 3\alpha\varphi-\alpha^2\varphi^2) \cos(\alpha\varphi+\epsilon_0)$$
$$+\alpha^2\varphi^2 \sin(\alpha\varphi+\epsilon_0)\}$$

$$Q_{\varphi_2}=A_2\, e^{\alpha\varphi} \cos(\alpha\varphi+\epsilon_2)+C_1A_1\, e^{\alpha\varphi}\{ (3\alpha\varphi-\alpha^2\varphi^2) \cos(\alpha\varphi+\epsilon_1)$$
$$+\alpha^2\varphi^2 \sin(\alpha\varphi+\epsilon_1)\}-A_0\, e^{\alpha\varphi}[\, C_2\{(4\alpha^3\varphi^3-18\alpha^2\varphi^2+15\alpha\varphi)$$
$$\cos(\alpha\varphi+\epsilon_0)-(4\alpha^3\varphi^3-15\alpha\varphi) \sin(\alpha\varphi+\epsilon_0)\}+C_3\{(12\alpha^3\varphi^3$$
$$-30\alpha^2\varphi^2+15\alpha\varphi) \cos(\alpha\varphi+\epsilon_0)+(4\alpha^4\varphi^4-12\alpha^3\varphi^3+15\alpha\varphi)$$
$$\sin(\alpha\varphi+\epsilon_0)\}], \qquad \ldots \qquad (3.6)$$

where

$$C_1=\left(\frac{a\lambda}{h_0+\lambda}\right)\frac{1}{4\alpha},\quad C_2=\frac{\lambda(2h_0-\lambda)a^2}{96\alpha^2(h_0+\lambda)^2},\quad C_3=\frac{a^2\lambda^2}{64(h_0+\lambda)^2\alpha^2} \qquad \ldots \qquad (3.7)$$

The arbitrary constants A_0, ϵ_0, A_1, ϵ_1, A_2, ϵ_2, etc. are to be determined for each particular problem from its edge conditions. Moreover, from consideration of orders of magnitude the fourth and higher terms will be negligible in comparison to the sum of the first three terms in Q_φ. While calculating the derivatives of Q_φ, we obtain, N_φ, N_θ, V, M_φ and δ.

4. Edge Loading:

For a shell subjected to uniformly distributed horizontal load R along its edge $\phi=\zeta$ (Fig. 2), the boundary conditions are:

$$N_\varphi=R\cos\zeta,\ Q_\varphi=-R\sin\zeta,\ M_\varphi=0 \text{ for } \phi=\zeta. \qquad \ldots \qquad (4.1)$$

Imposing these boundary conditions in (3.4) and using the expressions for N_φ, Q_φ, M_φ, the arbitrary constants A_0, ϵ_0 etc. are given by the relations:

$$\alpha\zeta+\epsilon_0=-\pi/4,\quad A_0=-R\sqrt{(2)}\, e^{-\alpha\zeta}\sin\zeta,$$

$$\tan(\alpha\zeta+\epsilon_1)=\frac{3(\alpha\zeta+1)}{2\alpha^2\zeta^2-3\alpha\zeta},\qquad A_1=\frac{C_1A_0}{\sqrt{(2)}}(3\alpha\zeta+3)\operatorname{cosec}(\alpha\zeta+\epsilon_1),$$

$$\tan(\alpha\zeta+\epsilon_2)=\frac{\lambda^2(6\alpha^4\zeta^4-9\alpha^2\zeta^2-33\alpha\zeta-21)+4h_0\lambda(9\alpha^2\zeta^2+3\alpha\zeta-6)}{\lambda^2(6\alpha^4\zeta^4-44\alpha^3\zeta^3+9\alpha^2\zeta^2)+4h_0\lambda(4\alpha^3\zeta^3-9\alpha^2\zeta^2)},$$

$$A_2=\frac{C_1^2A_0}{6\sqrt{(2)}}\left\{(\lambda^2)(6\alpha^4\zeta^4-44\alpha^3\zeta^3+9\alpha^2\zeta^2)+4h_0\lambda(4\alpha^3\zeta^3-9\alpha^2\zeta^2)\right\}.$$
$$\times\sec(\alpha\zeta+\epsilon_2). \qquad \ldots \qquad (4.2)$$

Evaluating A_0, ϵ_0, A_1, ϵ_1, A_2, ϵ_2 from the relation (4.2) and substituting their values in (3.4) and (3.8), the expressions for stresses and moments can be obtained.

The results for spherical shells can be had from the following equations by putting $r_1 = r_2 = a$, the radius of the shell.

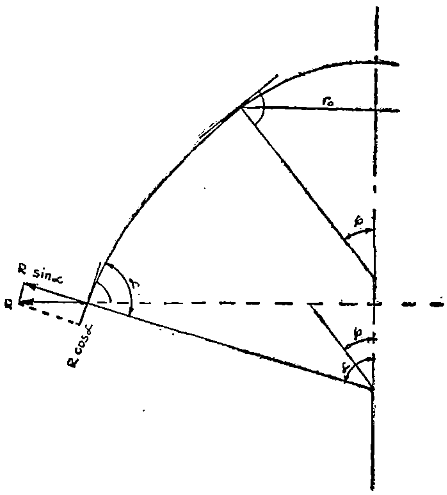

FIG. 2.

For the above spherical dome, we have

$$A_0 = .8112 \ \mathrm{R}e^{-\alpha\zeta}, \qquad \alpha\zeta + \epsilon_0 = -\pi/4,$$
$$A_1 = -33.47 \ \mathrm{R}e^{-\alpha\zeta} \qquad \alpha\zeta + \epsilon_1 = 24.8°,$$
$$A_2 = 907.8 \ \mathrm{R}e^{-\alpha\zeta} \qquad \alpha\zeta + \epsilon_2 = -88.6°.$$

and hence Q_φ, N_φ, N_θ, V, C_2, M_φ, δ_1 can be obtained.

5. Edge Moment :

If M be the uniformly distributed moment for unit length of the horizontal circle $\varphi = \zeta$, we have on the boundary,

$$Q_\varphi = N_\varphi = 0 \text{ and } M_\varphi = M. \qquad \ldots \qquad (5.1)$$

Using these boundary conditions, we get the arbitrary constants A_0, ϵ_0 etc. from the following relations :

$$\alpha\zeta + \epsilon_0 = \pi/2, \ A_0 = \frac{2E}{D\alpha^3} \frac{r_1^3}{r_2^2} M e^{-\alpha\zeta},$$

$$\tan(\alpha\zeta + \epsilon_1) = -\frac{\alpha^2\zeta^2 + 3(\alpha\zeta - 1)}{\alpha^2\zeta^2}, \ A_1 = -C_1 A_0 \alpha^2\zeta^2 \sec(\alpha\zeta + \epsilon_1),$$

$$\tan(\alpha\zeta+\epsilon_2)=-\frac{\lambda(28\alpha^3\zeta^3+18\alpha^2\zeta^2-165\alpha\zeta+75)+4h_0\lambda(4\alpha^3\zeta^3+18\alpha^2\zeta^2-21\alpha\zeta+3)}{\lambda(12\alpha^4\zeta^4+28\alpha^3\zeta^3-36\alpha^2\zeta^2-15\alpha\zeta)+4h_0\lambda(4\alpha^3\zeta^3-15\alpha\zeta)}$$

$$A_2=-\frac{C_1^2 A_0}{12}\left\{\lambda^2(12\alpha^4\zeta^4+28\alpha^3\zeta^3-36\alpha^2\zeta^2-15\alpha\zeta)+4h_0\lambda(4\alpha^3\zeta^3-15\alpha\zeta)\right\}$$
$$\times\sec(\alpha\zeta+\epsilon_2). \quad \ldots \quad (5.2)$$

Evaluating these constants and substituting them in equations (3.4), (3.8), we get the expressions for stresses and moments. For the spherical shell under consideration these constants are given by,

$$A_0 = .04872\ \mathrm{M}e^{-\alpha\zeta}, \qquad \alpha\zeta+\epsilon_0=\pi/2,$$
$$A_1 = -3.165\ \mathrm{M}e^{-\alpha\zeta}, \qquad \alpha\zeta+\epsilon_1=-55.5°,$$
$$A_2 = -115.1\ \mathrm{M}e^{-\alpha\zeta}, \qquad \alpha\zeta+\epsilon_2=-28.04°.$$

In this case the expressions for stresses and moments are Q_φ, N_φ, N_θ, V_2 M_φ, δ_2 can be obtanied.

6. Shells under surface loads and prescribed boundary conditions :

To solve any particular problem having specified boundary conditions and surface loads, we have to superpose the solutions obtained for edge loadings and edge moments on the corresponding membrane solutions loaded with the given surface loads. The appropriate R and M are to be determined from the boundary conditions. For the sake of completeness of the problem, the known membrane solutions for a surface of revolution are given below [5],

$$2\pi r_0 N_\varphi \sin\varphi+P=0, \quad \frac{N_\varphi}{r_1}+\frac{N_\theta}{r_2}=-Z, \quad \ldots \quad (6.1)$$

P being the resultant load above the parallel circle r_0,

$$V=\frac{\cot\varphi}{Eh\, r_1}\left[(r_1+\nu r_2)N_\varphi-(r_2+\nu r_1)N_\theta\right]-\frac{1}{r_1}\frac{d}{d\varphi}\left[\frac{r_2}{Eh}(N_\theta-\nu N_\varphi)\right], \quad \ldots \quad (6.2)$$

$$\delta=\frac{r_0}{Eh}\left(N_\theta-\nu N_\varphi\right). \quad \ldots \quad (6.3)$$

7. Clamped Spherical Shells under Loads :

Case I : *Uniform normal pressure intensity p.*

For a spherical shell subjected to uniform normal pressure intensity p, taking $r_1=r_2=a$ and $r_0=a\sin\phi$, we have from the equations (6.1) (6.3),

$$N_\theta=N_\phi=-\frac{pa}{2}, \quad V=V_3=-\frac{pa^2(1-\nu)K}{2Eh}\left(1-\frac{h_0}{h}\right),$$
$$\ldots \quad (7.1)$$
$$\delta=\delta_3=-\frac{pa^2(1-\nu)}{2\,Eh}\sin\phi, \quad M_\phi=M_\theta=0.$$

For complete solution, these quantities are to be added with the corresponding quantities for edge loadings and edge moments. For clamped edge at the edge $\phi=\zeta$, the rotation and displacement will be zero, which leads to the results,

$$V_1+V_2+V_3=0,\ \ \delta_1+\delta_2+\delta_3=0$$

enabling us to determine Rand M.

For spherical shell under consideration they are given by

$$R=9.694\ p,\ \ M=-39.98\ p.$$

Case II: *Applied surface loading of intensity q.*

For an applied surface loading of intensity q, the membrane results are

$$N_\varphi=-\frac{qa}{1+\cos\varphi},\ N_\theta=qa\left(\frac{1}{1+\cos\varphi}-\cos\varphi\right),$$

$$V=-\frac{qa(2+\nu)}{Eh}\sin\varphi+\frac{Ka^2q}{Eh}\left(\frac{1+\nu}{1+\cos\varphi}-\cos\varphi\right)\left(1-\frac{h_0}{h}\right),$$

$$\delta=\frac{qa^2\sin\varphi}{Eh}\left(\frac{1+\nu}{1+\cos\varphi}-\cos\varphi\right).$$

R and M will be given by,

$$R=5.498\ q,\qquad M=-30.04\ q,$$

Also $\frac{N_\theta}{q},\ \frac{M_\varphi}{q}$ can be plotted against ϕ

Case III: *Dead load of intensity q_0.*

For a shell under its own weight the membrane results are

$$N_\varphi=-\frac{q_0ah}{1+\cos\varphi}+\frac{q_0\,ah}{\sin\varphi}\left\{\left(1-\frac{h_0'}{h}\right)\frac{1}{\sin\varphi}-Ka\left(1-\frac{h_0}{h}\right)\right\},$$

$$N_\theta=q_0\,ah\left(\frac{1}{1+\cos\varphi}-\cos\varphi\right)-\frac{q_0\,ah}{\sin\varphi}\left\{\left(1-\frac{h_0'}{h}\right)\frac{1}{\sin\varphi}-Ka\left(1-\frac{h_0}{h}\right)\right\},$$

$$V=-\frac{q_0a\,(2+\nu)\sin\varphi}{E}+\frac{Kq_0(1+\nu)\,a^2}{E\sin^2\varphi}\left(1-\frac{h_0}{h}\right)$$
$$\times\left\{Ka\left(1-\frac{h_0}{h}\right)\sin\varphi+\frac{h_0'}{h}-\cos\varphi\right\},$$

$$\delta=\frac{q_0\,a^2\sin\varphi}{E}\left(\frac{1+\nu}{1+\cos\varphi}-\cos\varphi\right)+\frac{q_0\,a^2(1+\nu)}{E}$$
$$\times\left\{Ka\left(1-\frac{h_0}{h}\right)-\left(1-\frac{h_0'}{h}\right)\frac{1}{\sin\varphi}\right\},$$

where h_0' is the thickness of the shell at the top and q_0 is the weight per unit volume. R and M will be given by

$$R=-73.48\ q_0, \qquad M=299.9\ q_0,$$

I am thankful to Dr. S. Basuli for his kind help and guidance throughout the work.

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Paper received on 20. 9. 65.

Published By :

Jadavpur Mathematical Association, Jadavpur University, Calcutta-32

STRESSES ON SOME NON-CIRCULAR CYLINDRICAL SHELLS

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Reprinted from
JOURNAL OF SCIENCE AND ENGINEERING RESEARCH
VOL. XI, PART 2, 1967

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WEST BENGAL, INDIA.

ABSTRACT : The object of this paper is to investigate the stresses on cylindrical shells with the characteristic curves as cardioide, lemniscate or equiangular spiral. A comparative study of stresses has also been made in different shells with the same span.

1. NOMENCLATURE :

N_x, N_ϕ = unit normal forces.

T = unit central shear.

$P_x = P_x(\phi), P_y = P_y(\phi), P_z = P_z(\phi)$; surface load components.

P_E = [force/surface] ; weight of the shell per unit of middle-surface.

P_S = [force/surface] ; the snow load per unit of the plan of middle-surface.

P_W = [force/surface] ; the wind load per unit of middle-surface.

A vertical line with the upward direction as the positive direction is chosen as the initial line.

r = the radius vector.

θ = the vectorial angle.

ϕ = the angle made by the normal at any point on the curve with the initial line.

$\rho = f(\phi)$; the radius of curvature.

The clockwise direction is chosen to be the positive direction in measuring θ or ϕ. The directions of $x, y, z-$axes are shown in Fig. 1.

In this case a section normal to the generators would bisect the middle surface in a curve which is characteristic of the barrel and in the following we shall refer to it as the 'directrix'. From these considerations we find that ρ the radius of curvature is a free variable of ϕ.

2. INTRODUCTION :

Shells are used for roof structures over large columnless areas and for storage tanks. A great number of air-craft hangers, covered markets, factory and car sheds, planetoriums and rail road terminals have been erected with shell construction. Some machine parts or rocket bodies are designed from the idea of shell structures. A detailed study of different shells is necessary for the considerations of variety, economy and architectural show-manship in building construction.

The Membrane Theory of general cylindrical shells, e.g., circular, elliptic, parabolic, cycloidal vaults has been discussed in classical literature on shells. In semi-circular or semi-elliptical or cycloidal barrel vaults the hoop force N_ϕ vanishes along the longitudinal edges for dead or snow load, because of the fact that the directrix in any one of them has only a vertical tangent at the free edges. Since there are no forces acting in a vertical direction no supports are needed along the edges. This conclusion is of great practical importance for roofs since the lateral walls of the building can be kept light and large openings do not present a problem. This condition will also be satisfied if some other curve is taken provided $\varphi = \pm\pi/2$ at the longitudinal edges.

The present endeavour is to find out the stresses, in detail, for barrel vaults with the characteristic curves as cardioide, lemniscate equiangular spiral under different load conditions. To ensure the edge values of ϕ as discussed the suitable portions of the curves are to be chosen for each individual problem. A table is also given here representing the various stresses for these shells and a circular shell of same span for comparison and ready use by the designers.

3. EQUILIBRIUM OF THE SHELL ELEMENT :

By examining the equilibrium of the forces acting on the shell element in Fig 2, we can immediately conclude, from the condition of equilibrium of the moments around the shell normal, that the conjugated central shears must be equal as shown in Fig. 2. As for the equilibrium of forces in the x-direction, we obtain

$$\frac{\partial}{\partial x}(N_x\, \rho d\phi)\, dx + \frac{\partial}{\partial \varphi}(T\, dx)\, d\phi + P_x\, dx\, \rho d\phi = 0$$

and accordingly for the equilibrium in the y and z directions

$$\frac{\partial}{\partial \varphi}(N\varphi\, dx)\, d\varphi + \frac{\partial}{\partial x}(T\rho\, d\varphi) dx + P_y\, dx\, \rho d\varphi = 0,$$

$$N_\varphi\, dx\, d\varphi + P_z\, dx\, \rho d\varphi = 0.$$

Dividing all three equations by $dx\,d\phi$ and considering that ρ varies as a function of ϕ but is independent of x, we find

$$\frac{\partial N_x}{\partial x}\rho + \frac{\partial T}{\partial \varphi} + P_x\,\rho = 0 \qquad \ldots \qquad (3a)$$

$$\frac{\partial N_\phi}{\partial \varphi} + \frac{\partial T}{\partial x}\rho + P_y\,\rho = 0 \qquad \ldots \qquad (3b)$$

$$N_\phi + P_z\rho = 0. \qquad \ldots \qquad (3c)$$

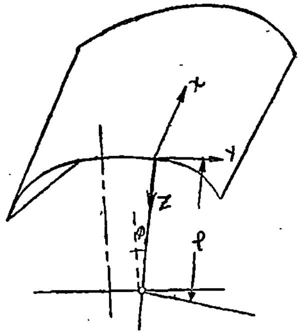

FIG.-1

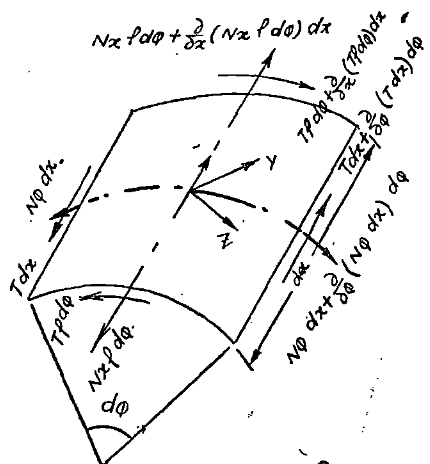

FIG.-2

4. LOAD COMPONENTS :

Dead Weight : We are concerned with the load components in the x, y, z-directions. The most important load is dead weight composed of the weight of the structure and of the roofing.

Following the Fig. 3, when the x-axis is along a generating line, y-axis is tangent to the directrix and z-axis is along the normal to the surface inward forming a right handed system, we obtain

$$P_x = 0 \qquad \ldots \qquad (4a_1)$$

$$P_y = P_E \sin\varphi \qquad \ldots \qquad (4b_1)$$

$$P_z = P_E \cos\varphi. \qquad \ldots \qquad (4c_1)$$

Snow Load :

To calculate the snow load it is practical to assume a load unifomly distributed over the plan of the shell. From Fig. 4, we get

$$P_x = 0 \qquad \ldots \qquad (4a_2)$$

$$P_y = P_s \sin\varphi \cos\varphi \qquad \ldots \qquad (4b_2)$$

$$P_z = P_s \cos^2\varphi. \qquad \ldots \qquad (4c_2)$$

Wind Load :

The wind load of the shell is composed of pressure on the wind side and suction on the leeward side. Only the load component acting perpendicularly to the middle surface P_z is of importance, since the components P_x and P_y are due to the friction forces and are almost equal to zero. In our case we shall consider here the wind load components as follows :

$$P_x = 0 \quad \ldots \quad (4a_3)$$

$$P_y = 0 \quad \ldots \quad (4b_3)$$

$$P_z = P_w \sin \phi. \quad \ldots \quad (4c_3)$$

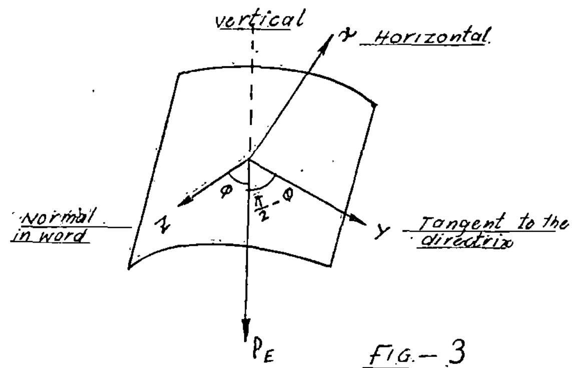

FIG.—3

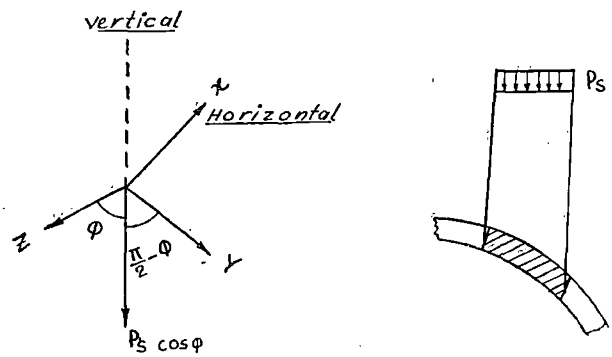

FIG.-4

5. SOLUTIONS OF DIFFENTIAL EQUATIONS :

In any one of the load conditions under our discussion we find that $P_x = 0$ and $P_y = P_y(\phi)$, $P_z = P_z(\phi)$, moreover $\rho = f(\phi)$.

From equation (3c) we immediately obtain the hoop force N_ϕ as

$$N_\phi = -P_z\,\rho \qquad \ldots \qquad (5.1)$$

Hence

$$\frac{\partial N_\phi}{\partial \varphi} = -(P_z'\,\rho + P_z\,\rho')$$

Substituting this expression for $\partial N_\phi/\partial\varphi$ in equation (3b), $\partial T/\partial x$ stands as

$$\frac{\partial T}{\partial x} = -\frac{1}{\rho}\left[P_y\,\rho - P_z'\,\rho - P_z\,\rho' \right].$$

On integration we have,

$$T = -\frac{x}{\rho}\left[(P_y - P_z')\,\rho - P_z\,\rho' \right] + F_1(\varphi) \qquad \ldots \qquad (5.2)$$

where the dash indicates differentiation with respect to ϕ and $F_1(\phi)$ is a function of ϕ only. To determine it the boundary conditions have to be used. Suppose the vault is identically supported at both ends, that is at $x=0$ and $x=l$. Then the entire stress distribution must be symmetrical with respect to the bisecting plane $x = l/2$. The unit central shear T, especially, must vanish for all values of ϕ, because any shearing stress not equal to zero in this locus would amount to asymmetry. Thus we obtain from equation (5.2)

$$-(l/2\rho)\,[\rho(P_y - P_z') - P_z\,\rho'] + F_1(\phi) = 0.$$

Hence

$$F_1(\phi) = (l/2\rho)\,[\rho(P_y - P_z') - P_z\,\rho'].$$

So, the expression for T becomes

$$T = \frac{(l-2x)}{2}\left[P_y - P_z' - P_z\,\frac{\rho'}{\rho} \right] \qquad \ldots \qquad (5.3)$$

The partial derivative of T with respect to ϕ is

$$\frac{\partial T}{\partial \varphi} = \frac{(l-2x)}{2\rho^2}\left[\rho^2(P'_y - P''_z) - \rho\,(\rho' P'_z + \rho'' P_z) + P_z \rho'^2 \right],$$

whence the equation (3a) leads us to

$$\frac{\partial N_x}{\partial x} = -\frac{(l-2x)}{2\rho^3}\left[\rho^2(P'_y - P''_z) - \rho(\rho' P'_z + \rho'' P_z) + P_z\,\rho'^2 \right]$$

which on integration gives

$$N_x = -\frac{x(l-x)}{2\rho^3}\left[\rho^2(P'_y - P''_z) - \rho\,(\rho' P'_z + \rho'' P_z) + P_z\,\rho'^2 \right] + F_2(\phi) \qquad \ldots \qquad (5.4)$$

To determine the function $F_2(\phi)$, which is again exclusively dependent on ϕ, we may require that at $x=0$ and $x=l$, the supports of the shell should not be able to resist forces of any magnitude in the direction of the generator. Thus we require $N_x = 0$ for all ϕ at these loci. These conditions are satisfied if we put $F_2(\phi) = 0$ in (5.4). Now the results can be written as,

$$N_\phi = -P_z \rho \qquad \ldots \qquad (5.5)$$

$$T = \frac{(l-2x)}{2}\left[P_y - P'_z - P_z \frac{\rho'}{\rho}\right] \qquad \ldots \qquad (5.6)$$

$$N_x = -\frac{x(l-x)}{2\rho^3}\left[\rho^2(P'_y - P''_z) - \rho(\rho' P'_z + \rho'' P_z) + P_z (\rho')^2\right]. \qquad \ldots \qquad (5.7)$$

6. A PORTION OF A CARDIOIDE AS DIRECTRIX :

As a first example of non-circular vault, let us examine the cardioidal vault. The equation of the directrix in this case is taken as $r = (4/3\sqrt{3})a (1 + \cos\theta), -60^\circ \leqslant \theta \leqslant 60^\circ$, which in our ρ, ϕ notations stands as $\rho = (16a/9\sqrt{3}) \cos \phi/3$. We shall consider the portion $-\pi/2 \leqslant \phi \leqslant \pi/2$ only. The horizontal

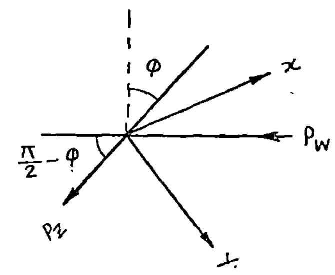

FIG.— 5

length covered by this part of the curve will be equal to $2a$. Now from equations (5.5), (5.6), (5.7) and those of § 4 we get,

for *Dead Weight :*

$$N_\phi = -\frac{16}{9\sqrt{3}}\, a\, P_E \cos\phi \cos\frac{\phi}{3}\,, \qquad \ldots \qquad (6.1)$$

$$T = \frac{(l-2x)}{6} P_E \left(6 \sin\phi + \cos\phi \tan\frac{\phi}{3}\right), \qquad \ldots \qquad (6.2)$$

$$N_x = -\frac{\sqrt{3}x(l-x)}{64a} P_E \left(2\cos\phi + 36\cos\phi\cos^3\frac{\phi}{3} - 3\sin\phi\sin\frac{2\phi}{3}\right)\sec^3\frac{\phi}{3} \qquad \ldots \qquad (6.3)$$

for *Wind Load*

$$N_\varphi = -\frac{16a}{9\sqrt{3}} P_W \sin\phi \cos\frac{\phi}{3}, \quad \ldots \quad (6.4)$$

$$T = -\frac{(l-2x)}{2} P_W\left(\cos\varphi - \tfrac{1}{3}\sin\phi\tan\frac{\phi}{3}\right), \quad \ldots \quad (6.5)$$

$$N_x = -\frac{\sqrt{3}\,x(l-x)}{64a} P_W\left(18\sin\phi\cos^2\frac{\phi}{3} + 3\cos\varphi\sin\frac{2\phi}{3} + 2\sin\phi\right)\sec^3\frac{\phi}{3}, \quad \ldots \quad (6.6)$$

for *Snow Load*

$$N_\phi = -\frac{16a}{9\sqrt{3}} P_S \cos^2\phi\cos\frac{\phi}{3}, \quad \ldots \quad (6.7)$$

$$T = \frac{(l-2x)}{12} P_S\left(9\sin 2\phi + 2\cos^2\phi\tan\frac{\phi}{3}\right), \quad \ldots \quad (6.8)$$

$$N_x = -\frac{\sqrt{3}x\,(l-x)}{64a} P_S\left(54\cos 2\phi\cos^2\frac{\phi}{3} + 2\cos^2\phi - 3\sin 2\phi\sin\frac{2\phi}{3}\right)\sec^3\frac{\phi}{3}. \quad \ldots \quad (6.9)$$

7. A PORTION OF A LEMNISCATE AS DIRECTRIX :

The equation of a lemniscate is chosen as $r^2 = 8a^2\cos 2\theta$, $-30° \leqslant \theta \leqslant 30°$. Therefore $\rho = (2\sqrt{2}a/3)\ \sqrt{(\sec 2\phi/3)}$ is the equation of the directrix where ϕ is given by $-\pi/2 \leqslant \phi \leqslant \pi/2$. The span will be of length $2a$ in this case also.

As before, considering equations (5.5), (5.6), (5.7) and from the equations of § 4 we obtain ;

in case of *Dead Weight*,

$$N_\phi = -\frac{2\sqrt{2}a}{3} P_E\cos\phi\sqrt{\left(\sec\frac{2\phi}{3}\right)}, \quad \ldots \quad (7.1)$$

$$T = \frac{(l-2x)P_E}{6}\left(6\sin\phi - \cos\phi\tan\frac{2\phi}{3}\right) \quad \ldots \quad (7.2)$$

$$N_x = -\frac{x(l-x)}{4\sqrt{2}a} P_E\left(6\cos\phi + \sin\phi\tan\frac{2\phi}{3} - \tfrac{2}{3}\cos\phi\sec^2\frac{2\phi}{3}\right)\sqrt{\left(\cos\frac{2\phi}{3}\right)} \quad \ldots \quad (7.3)$$

in case of *Wind Load*,

$$N_\phi = -\frac{2\sqrt{2a}}{3} P_W \sin\phi \sqrt{\left(\sec\frac{2\phi}{3}\right)} \qquad \ldots \quad (7.4)$$

$$T = -\frac{(l-2x)}{6} P_W \left(3\cos\phi + \sin\phi\tan\frac{2\phi}{3}\right) \qquad \ldots \quad (7.5)$$

$$N_x = \frac{x(l-x)P_W}{4\sqrt{2}\,a}\left(\cos\phi\tan\frac{2\phi}{3} - 3\sin\phi + \tfrac{2}{3}\sin\phi\sec^2\frac{2\phi}{3}\right)\sqrt{\left(\cos\frac{2\phi}{3}\right)}. \qquad \ldots \quad (7.6)$$

for *Snow Load*,

$$N_\phi = -\frac{2\sqrt{2a}}{3} P_S \cos^2\phi \sqrt{\left(\sec\frac{2\phi}{3}\right)} \qquad \ldots \quad (7.7)$$

$$T = \frac{(l-2x)P_S}{12}\left(9\sin 2\phi - 2\cos^2\phi\tan\frac{2\phi}{3}\right), \qquad \ldots \quad (7.8)$$

$$N_x = -\frac{x(l-x)P_S}{4\sqrt{2}\,a}\left(9\cos 2\phi + \sin 2\phi\tan\tfrac{2}{3}\phi - \tfrac{2}{3}\cos^2\phi\sec^3\frac{2\phi}{3}\right)\sqrt{\left(\cos\frac{2\phi}{3}\right)} \qquad \ldots \quad (7.9)$$

8. A PORTION OF EQUIANGULAR SPIRAL AS DIRECTRIX :

In polar co-ordinates the equation of an equiangular spiral is $r = a' e^{\theta \cot \alpha}$, α being the angle made by the radius with the tangent at (r, θ). Its equation in ρ, ϕ notations is

$$\rho = [a' \operatorname{cosec}\alpha\, e^{(\pi/2-\alpha)\cot\alpha}]\, e^{\phi\cot\alpha}. \qquad \ldots \quad (8.1)$$

By spiral (*i*), we shall mean the directrix given by the equation (8.1) for the portion $-\pi/2 \leqslant \phi \leqslant \pi/2$ and for spiral (*ii*), the values for ϕ will be between o and $\pi/2$ including the end points.

The stresses and shears under different load condition will be given by the following equations.

For *Dead Weight :*

$$N_\phi = -A\, P_E\, e^{\phi\cot\alpha}\cos\phi \qquad \ldots \quad (8.2)$$

$$T = \frac{(l-2x)P_E}{2}(2\sin\phi - \cot\alpha\cos\phi) \qquad \ldots \quad (8.3)$$

$$N_x = -\frac{x(l-x)P_E}{2A\, e^{\phi\cot\alpha}}(2\cos\phi + \cot\alpha\sin\phi). \qquad \ldots \quad (8.4)$$

For *Wind Load* :

$$N_\phi = - AP_W e^{\phi \cot \alpha} \sin \phi, \qquad \dots \quad (8.5)$$

$$T = - \frac{(l-2x)P_W}{2} (\cos \phi + \cot \alpha \sin \phi), \qquad \dots \quad (8.6)$$

$$N_x = - \frac{x(l-x)P_W}{2A\, e^{\phi \cot \alpha}} (\sin \phi - \cot \alpha \cos \phi). \qquad \dots \quad (8.7)$$

For Snow Load,

$$N_\phi = - AP_S e^{\phi \cot \alpha} \cos^2 \phi, \qquad \dots \quad (8.8)$$

$$T = - \frac{(l-2x)}{4} P_S (2 \cot \alpha \cos^2 \phi - 3 \sin 2\phi), \qquad \dots \quad (8.9)$$

$$N_x = - \frac{x(l-x)P_S}{2A\, e^{\phi \cot \alpha}} (3 \cos 2\phi + \cot \alpha \sin 2\phi). \qquad \dots \quad (8.10)$$

In case of Spiral (*i*), we shall replace

$$a' \quad \text{by} \quad \frac{2a\, e^{a \cot \alpha}}{\sin \alpha\, (1+e^{\pi \cot \alpha})}$$

and

$$A \quad \text{by} \quad \frac{2a\, e^{(\pi/2) \cot \alpha}}{\sin^2 \alpha (1+e^{\pi \cot \alpha})},$$

so that the horizontal portion covered by the directix will be equal to $2a$.

For Spiral (*ii*) we take

$$a' = \frac{2a\, e^{a \cot \alpha}}{(\sin \alpha\, e^{\pi \cot \alpha} - \cos \alpha\, e^{(\pi/2) \cot \alpha})}$$

$$A = \frac{2a}{\sin \alpha (\sin \alpha\, e^{(\pi/2) \cot \alpha} - \cos \alpha)}.$$

The idea behind this is that the span will be of length $2a$. For our calculations we have chosen $\alpha = \pi/4$.

9. SEMICIRCULAR BARREL OF RADIUS a :

In this case $\rho = a$ and ϕ lies between $-\pi/2$ to $\pi/2$. Following article numbers 2 and 4 we have

Dead Weight.

$$N_\phi = -aP_E \cos \phi \qquad \dots \quad (9.1)$$

$$T = (l-2x) P_E \sin \phi \qquad \dots \quad (9.2)$$

$$N_x = - \frac{x(l-x) P_E}{a} \cos \phi. \qquad \dots \quad (9.3)$$

Wind Load,

$$\mathbf{N}_\phi = - a P_W \sin \phi, \qquad \ldots \quad (9.4)$$

$$T = -\frac{(l-2x) P_W}{2} \cos \phi, \qquad \ldots \quad (9.5)$$

$$N_x = -\frac{x(l-x) P_W}{2a} \sin \phi. \qquad \ldots \quad (9.6)$$

Snow Load,

$$N_\phi = - a P_S \cos^2 \phi, \qquad \ldots \quad (9.7)$$

$$T = \frac{3(l-2x)}{4} P_S \sin 2\phi, \qquad \ldots \quad (9.8)$$

$$N_x = -\frac{3x(l-x)P_S}{2a} \cos 2\phi. \qquad \ldots \quad (9.9)$$

	Dead Weight $-\frac{N_\phi}{aP_E}$					Wind Load $-\frac{N_\phi}{aP_W}$				
	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
Circle	0	.7071	1	.7071	0	—1	—.7071	0	.7071	1
Cardioide	0	.7008	1.027	.7008	0	— .8890	—.6285	0	.6285	.8890
Lemniscate	0	.7161	.9427	.7161	0	—1.332	—.7161	0	.7161	1.332
Spiral (i)	0	.2568	.7969	1.237	0	— .1655	—.2568	0	1.237	3.836
Spiral (ii)	—	—	1.049	1.628	0	—	—	0	1.628	5.051

	Dead Weight $\frac{T}{(l-2x)P_E}$					Wind Load $\frac{T}{(l-2x)P_W}$				
	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
Circle	—1	—.7071	0	.7071	1	0	.7071	1	.7071	0
Cardioide	—1	—.7386	0	.7386	1	—.1925	.6440	1	.6440	—.1925
Lemniscate	—1	—.6389	0	.6389	1	.5774	.8433	1	.8433	.5774
Spiral (i)	—1	—1.061	—.5	.3536	1	— 1	0	1	1.414	1
Spiral (ii)	—	—	—.5	.3536	1	—	—	1	1.414	1

	Dead Weight $-\frac{N_x a}{x(l-x)P_E}$					Wind Load $-\frac{N_x a}{x(l-x)P_W}$				
	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
Circle	0	.7071	1	.7071	0	— .5	— .3535	0	.3535	.5
Cardioide	—.0108	.7241	1.028	.7241	—.0108	— .6043	— .3568	0	.3568	.6043
Lemniscate	.2164	.6616	.9428	.6616	.2164	— .0417	— .1785	0	.1785	.0417
Spiral (i)	—3.021	.9965	1.255	.6067	.1303	—3.021	—1.947	—.6275	0	.1303
Spiral (ii)	—	—	.9532	.4609	.0990	—	—	.4766	0	.0990

	$-\frac{N_\phi}{aP_S}$		Snow Load			$-\frac{2T}{(l-2x)P_S}$				
	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
Circle	0	.5	1	.5	0	0	1.5	0	−1.5	0
Cardioide	0	.4973	1.026	.4973	0	0	1.545	0	−1.545	0
Lemniscate	0	.5065	.9427	.5065	0	0	1.403	0	−1.403	0
Spiral (i)	0	.1816	.7969	.8745	0	0	2	1	−1	0
Spiral (ii)	—	—	1.049	1.151	0	—	—	1	−1	0

Snow Load $-\frac{aN_x}{x(l-x)P_S}$

	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
Circle	−1.5	0	1.5	0	−1.5
Cardioide	−1.689	— .0150	1.516	—.0150	−1.689
Lemniscate	−1.125	.0219	1.473	.0219	−1.125
Spiral (i)	−9.061	−1.377	1.882	.2860	— .3910
Spiral (ii)	—	—	1.430	.2074	— .2970

I am indebted to Professor S. Basuli for his constant help and encouragement during the preparation of this paper.

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Published by

Indian Institute of Technology, Kharagpur (S. E. Rly.), India.