STUDY OF INTERLOCKED MULTIPLET PROBLEMS IN ANISOTROPICALLY SCATTERING MEDIA

USING DISCRETE-ORDINATE METHOD

A Thesis
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Summary

This thesis is intended to solve the radiative transfer equations of the interlocked multiplet lines for anisotropically scattering medium. This thesis contains five chapters and three appendices and is organized as follows.

The chapter-1 is "General Introduction". At the beginning of this chapter,the focus is on "the interlocking problem in Radiative Transfer" and the formation of the general equation of the radiative transfer equations for m number of interlocked multiplet lines, following the radiative transfer equation for interlocked triplets by Woolley and Stibbs. Chandrasekhar's discrete ordinates method is discussed with the simplest form of a radiative transfer equation . Attempt has been made to list the different works on Radiative Transfer by using discrete ordinate methods, developed by Chandrasekhar and modified by different workers from time to time. Effort has also been made to enlist the works done by the different authors in connection with interlocked multiplet lines.

In Chapter-2, mentioning the azimuth free planetary phase function, the works of a few workers on Radiative Transfer where this phase function is used are focused. The chapter actually opens with an attempt to solve radiative transfer equation of the r^{th} interlocked line involving the planetary phase function and the linear Planck function. The same equation with an exponential form of Planck function is also solved in this chapter.

In Chapter-3, the author tries to solve the radiative transfer equation of r^{th} interlocked line from the set of m equations for interlocked multiplets of order m with two different phase functions. One is Rayleigh phase function and the other is Pomraning phase function. Some works of different astrophysicists by using these two phase functions are also mentioned.

The chapter-4 is concerned with the derivation of the diffusely reflected intensity and the emergent intensity from the radiative transfer equation of r^{th} interlocked line with three term scattering indicatrix and the linear form of Planck function. A few works on Radiative Transfer by using the three terms scattering indicatrix are also highlighted in the chapter.

In chapter-5, the values of H-functions for doublet lines with or without interlocking are derived by approximating the H-functions following the technique of Abu-Shumays. The results are compared with those obtained by Busbridge and Stibbs as well as Karanjai and Deb by preparing tables and drawing graphs. The residual intensities for three different cases those chosen by Eddington as well as Busbridge and Stibbs are obtained and results are compared with those obtained by Eddington as well as Busbridge and Stibbs.

In appendix-I, a few words on H-functions for interlocked multiplets has been said.

In appendix-II, some identities are derived. These identities are used in different chapters of this thesis.

In appendix-III, an important identity is derived. This is used to derive an expression of the emergent intensity for the isotropic case.

Contents

Chapter 1

General Introduction

Scattering is said to be ***coherent*** when radiation is emitted in the same frequency in which it was absorbed. On the other hand, when frequency of the emitted radiation differs from that of the absorbed radiation, it is said to be ***non-coherent*** scattering. The term ***'completely non-coherent scattering'*** is sometimes used to mean that the scattering involves not only a change in the frequency but also a complete redistribution in frequency i.e. scattering in which the frequency of re-emission is uncorrelated with the frequency absorbed. It was pointed out by Thomas,[202] and earlier by Henyey,[84] Unno[207,208] and Edmonds[70] that from practical point of view, purely coherent scattering (in frequency) is not possible in strict sense in stellar atmospheres. We designate the scattering as coherent or non-coherent according to our theoretical consideration of the problem. In the word of Jefferies,[98] "The generalization from coherent to non-coherent scattering made necessary the consideration of the simultaneous transfer of photons at all frequencies in the line along with the parallel recognition that the photons belong to the whole line and no longer to a particular frequency. ...."(vide Athay[13])

When an atom absorbs energy of certain frequency, ν, the probability that the energy will be re-emitted in the same frequency will be maximum if

(i). the atom is at rest

(ii). the atom is in the lowest quantum state

(iii). the atom is in a weak radiation field.

Departure from any of the above three conditions will cause non coherent scattering.

In one of his classical papers entitled "The Formation of Absorption Lines", Eddington[68] quoted," The crucial question is whether light absorbed in one part of a line is re-emitted in precisely the same part of the line. If so, the blackening in this frequency is independent of what is happening in neighbouring frequencies. The alternative is that the re-emission has a probability distribution, and is correlated to, but not determined by, the absorbed frequency. For example, if the process is regarded as one of the transition between two energy levels, which are not sharp but are composed of narrow bands of energy , the atom is not likely to return to the precise spot in the lower level from which it started, and the re-emission will not be the exact reverse of the absorption . In that case the line can only be studied as a whole. Modern attempts to interpret the contours of the absorption lines assume (rightly or wrongly) that there is no such redistribution of frequencies." He further put the remark in this regard as footnote in the same paper,"If the above assumption is untrue, the usual treatment of the line contours is entirely unsound".

Besides the redistribution of the kind mentioned in his quotation, he noticed another departure from the simple case, known as interlocking of lines.

When two or more lower (sub states or) lines in a spectrum posses a common upper state, the atom can be excited to that state by absorption in either lines (i.e.' any of the lower sub states); but the re-emission will take place according to the transition probability, regardless of the path by which the excitation was made. Thus the absorption from a certain sub state of the lower state in a certain frequency ν has a non-zero probability of the returning to another lower sub state emitting in frequency different from ν giving rise to non-coherent scattering. Similar situation will arise when the numbers of upper sub states will posses a common lower state. This phenomenon is called interlocking of lines without redistribution. The lines are said to form doublet, triplet, quartet or multiplet according to the number of such interlocked lines viz. two, three, four or many. As for examples, some of the interlocked multiplets are $n^3S - 2^3P$, the triplets of the alkali earths, and $2^2P - 2^2S$, the doublets of the alkali metals, which have the same upper states, and lower states which are not themselves the upper states of strong lines (vide Woolley[227]).

In the Mg triplet, at $5167A$, $5173A$, and $5184A$, taken from Grotrain[80](vide Woolley[227]), there are no other transitions connected with the 2^3S. The 2^3P_1 and 2^3P_3 state are not the upper state of any transitions, but 2^3P_2 is the upper state of a fairly weak line $1^1S - 2^3P_2$ at $4571A$.

The interlocking of one member of a multiplet with the other members may be regarded in the following way. The formation of any one line at any point in the atmosphere is governed by the number of atoms per c.c. of the atmosphere at that point in the two atomic states connected with the line. The atmosphere is not exactly in the thermodynamical equilibrium with radiation, and the number of atoms in any state will depend on the intensities of those lines involving transitions in which it is the upper state, the equilibrium condition being that the total number of transitions, per c.c. per sec., into the state must equal the number of transitions away from it, since there can be no secular increase in the number of atoms in any particular state. There is not, however, detailed balancing in each of the separate transitions (vide Woolley[227]).

The equations of formation of the lines are not independent but contain cross-terms.

The equation for the intensity in a particular frequency of a spectral line might then, in general, contain an infinite set of terms involving the intensities of other frequencies in the same line as well as terms involving the intensities in a finite number of other lines in the same spectrum. Fortunately, these difficulties do not arise in some important cases, namely principal lines in spectra, in which the ground state (or metastable state) is sharp. The reason for this is that the distribution of energy levels within a state depends on the life of the state. The spread of energy in the ground (or metastable) state can be ignored.

After coherent scattering, the next simplest case is that of interlocking of the principle lines, for $p\,(\nu, \nu')$ takes a small number only of the non-zero values. Examples of this are the principal lines of Al, ${}^2S_{\frac{1}{2}} -{}^2 P_{\frac{3}{2}}$, at $\lambda 3,962A$ and ${}^2S_{\frac{1}{2}} -{}^2 P_{\frac{1}{2}}$, at$\lambda 3,944A$, in which ${}^2P_{\frac{1}{2}}$ is the ground state and ${}^2S_{\frac{3}{2}}$ metastable; and the principal triplet of Mg, ${}^3S_1 -{}^3 P_2$ at $\lambda 5,184A$, ${}^3S_1 -{}^3 P_1$ at $\lambda 5,173A$ and ${}^3S_1 -{}^3 P_0$ at $\lambda 5,167A$.

In this case $^3P_2$ and $^3P_0$ are metastable and $^3P_1$ is linked by an inter combination line to the ground state $^1S_0$.

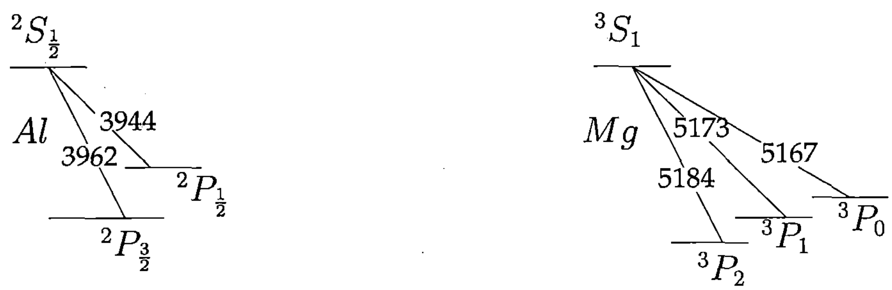


Figure 1.1: Interlocked principal lines of Al and Mg

Interlocking which is the core word of this thesis is associated with **"Transport Theory"** of Astrophysics. Transport theory (Neutron Transport or Radiative Transfer) is such a subject whose study is a must for studying the physics of the distant celestial bodies. Mathematically, it is the underlying physical phenomenon in many astrophysical problems and its study has a great importance as radiation field which is not only the root cause of the change of the structure and dynamics of the medium it propagates through, but also, is practically the only source of information about distant celestial objects serves as an important diagnostic tool in establishing their properties.

A major area of the study of the subject Radiative transfer concerned with the derivation of the distribution function (or the specific Intensity) for a given scattering function from an equation of transfer (usually an integro-differential equation) which is constructed by assuming the physics of the source wherefrom the photons (in case of radiative transfer) or neutrons (in case of neutron transfer) emerge out is known and the scattering laws of the medium through which the photons (or neutrons) proceed are also known, subject to two-point boundary conditions which depend on the nature of the source and the medium.

No one will disagree to admit that an integro-differential equation of transfer with two-point boundary conditions is a very difficult mathematical problem and in many cases it becomes a challenging one. An integro-differential equation of transfer related with interlocking problem is much more difficult. Only a few problems of interlocking have been solved till now. So far interlocking problem has been solved in isotropic medium only. This thesis is the first step in anisotropically scattering media for the case of interlocking problems.

As the notion of plane parallelism is so common to so many stars and other physical situations (vide Collons II[53]), we shall confine our study in determining the solution of the radiative transfer equation for plane parallel atmosphere only.

Before developing the general equation of radiative transfer equation for interlocked multiplet lines in anisotropically scattering medium, we shall write a few words about the phase function which makes a distinction between an equation in isotropic scattering atmosphere and one in anisotropic scattering atmosphere .

Phase Functions :

A phase function $p\ (\mu,\mu')$ expresses the ratio of energy propagated in direction μ compared to the energy coming from direction μ'. It satisfies two important properties that result directly from physics of light. First, due to the Helmholtz Reciprocity Rule, $p(\mu, \mu')$ is symmetric relative to μ and μ' :

$$\forall\, \mu \in \nu \; and \, \forall\, \mu' \in \nu, \;\; p(\mu, \mu') = p(\mu', \mu) \tag{1.1}$$

Second, due to the Energy Conservation Law, $p(\mu, \mu')$ has to fulfill the normalization condition :

$$\forall\, \mu \in \nu, \;\; \frac{1}{4\pi} \int_{\mu' \in \nu} p(\mu, \mu')\, d\mu' = 1 \tag{1.2}$$

Moreover $p(\mu, \mu')$ is usually symmetric around the incident direction of light and so $p(\mu, \mu')$ depends only on the angle Θ between μ and μ'. Therefore Equation (1.2) can be written :

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} p(\Theta, \gamma)\, sin\Theta d\Theta d\gamma = \frac{1}{2} \int_0^{\pi} p(\Theta)\, sin\Theta d\Theta = 1 \tag{1.3}$$

Finally $t = cos\Theta$ gives the following normalization condition:

$$\int_{-1}^{+1} p(t)\, dt = 2 \tag{1.4}$$

The following are some of the phase functions which we have used in our work:

1. Planetary phase function:

$$p(\cos\Theta) = \varpi_0 (1 + \varpi \cos\Theta) \;\; (-1 \le \varpi \le 1)$$

2. Rayleigh phase function :

$$p(\cos\Theta) = \frac{3}{4} \left(1 + \cos^2\Theta\right)$$

3. Pomraning phase function :

$$p(\cos\Theta) = \frac{3}{4}\left(1+\lambda\cos^2\Theta\right) \quad ; \quad \lambda = \frac{5\varpi_0}{5-3\varpi_0}$$

4. Three term scattering indicatix:

$$p(\cos\Theta) = 1 + \varpi_1 p_1(\cos\Theta) + \varpi_2 p_2(\cos\Theta)$$

where ϖ_1 and ϖ_2 are constants.

Planck Function:

The functional form of the Planck-function $B_\nu(T)$ follows immediately from Bose-Einstein quantum statistics which is given by

$$B_\nu(T) = \left(2h\nu^2/c^2\right)\left[e^{h\nu/kT} - 1\right]^{-1}$$

where $h =$ Planck constant, $k =$ Boltzmamn constant, $\nu =$ frequency, $c =$ speed of light and T is the temperature which, in a model stellar atmosphere in radiative equilibrium, is determined as a function of height τ, in the atmosphere.

Different authors used the different approximate forms of the Planck function $B_\nu(T)$. Some of them are

(i). **Linear form:**
$B_\nu(T) = b_0 + b_1\tau$, where b_0 and b_1 are constants.

(ii). **Non-linear forms:**

(a) **Exponential form:**
$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$, where β, b_0 and b_1 are constants.

(b) $B_\nu(T) = b_0 + b_1\tau + b_2E_2(\tau)$ where b_0, $b_1$and b_2 are constants and $E_2(\tau)$ is the function

$$E_2(\tau) = \int_1^\infty \frac{e^{-\tau x}}{x^2} dx$$

1.1 Development of the equation:-

Woolley and Stibbs[229] applied the theory of absorption lines by coherent scattering to the case of interlocking without redistribution to deduce the equation of transfer for interlocked triplets in the Milne-Eddigton model and they solved the problem by Eddington approximation method by making some assumptions which are stated below:

(I). No distribution in frequency takes place other than due to interlocking;

(II). The lines are so closed together that variations of the continuous absorptions coefficient and of the Planck-function with wave-length may be neglected. This also means that the lower states are nearly equal in excitation potential and they have the same classical damping constant. Then the ratios of the line absorption co-efficients to the continuous absorption co-efficients are proportional to the transition probabilities for spontaneous emission from the upper states to the respective lower states;

(III). The ratio of the line absorption co-efficients to the continuous absorption co-efficients are independent of the depth.;

(IV). The Planck-function and the co-efficient, which is introduced to allow for the thermal emission associated with the absorption, are independent of the both frequency and depth.

With same logic and assumption, we attempt here to give the derivation of the general form of the equations of radiative transfer for the case of interlocking of multiplets of order m.

Let ν_i, $(i = 1, 2, \cdots \cdots, m)$ be the central frequencies of the m number of multiplet lines and when a quantum of frequencies $\nu_i + \Delta\nu_i$, $(i = 1, 2, \cdots \cdots, m)$ is absorbed the energy of the upper state will be $E_0 + h\Delta\nu$ and the subsequent re-emission is any of $\nu_i + \Delta\nu_i$, $(i = 1, 2, \cdots \cdots, m)$ These 'm' frequencies are interlocked with each other but with no other frequencies. For each value of $\Delta\nu$ there are 'm' simultaneous equations of which the rth interlocked line is:

$$\cos\vartheta \frac{dI_r(z,\vartheta,\phi)}{\rho dz} = (k_r + l_r) I_r(z,\vartheta,\phi) + (k_r + \varepsilon_r l_r) B(\nu_r, T) + (1-\varepsilon_r) \sum_{s=1}^{m} p(r,s) l_r J_r \tag{1.5}$$

where ϑ denotes the polar angle which the direction considered makes with the outward normal to an element of area $d\sigma$ (across which the dE_ν amount of radiant energy in the frequency interval $(\nu, \nu + \Delta\nu)$ is transported), ϕ the azimuthal angle referred to a suitably chosen $x-$axis and ρ is the density of the material through which a pencil of radiation is propagated.

Introducing the normal optical thickness

$$\tau = \int_z^\infty k\rho dz$$

measured , in terms of the scattering co-efficients k, from the boundary inward

and

$$\mu = \cos\vartheta$$

we have

$$\mu\frac{dI_r(\tau,\mu,\phi)}{d\tau} = (k_r + l_r)\, I_r(\tau,\mu,\phi)$$

$$+ (k_r + \varepsilon_r l_r)\, B(\nu_r, T) + (1-\varepsilon_r)\sum_{s=1}^{m} p(r,s)\, l_r J_r \tag{1.6}$$

We now have to evaluate the quantities $p(r,s)$. To do this we note that the number of transitions per c.c.. from the m number of lower states to a band of the upper states lying within $E_0 + h\Delta\nu$ to $E_0 + h(\Delta\nu + \delta\nu)$ is

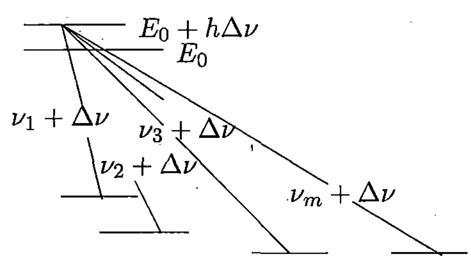


Figure 1.2: The Formation of m number of Interlocked lines with a common upper state

$$\rho dx \delta\nu \left\{ \sum_{s=1}^{m} \frac{l(\nu_s + \Delta\nu)\, J(\nu_s + \Delta\nu)}{h(\nu_s + \Delta\nu)} \right\}$$

This must be equal to the number of transitions leaving the upper sub-state into the m-lines per c.c. per sec.

Let the population of the upper state with energies between $E_0 + h\Delta\nu$ and $E_0 + h(\Delta\nu + \delta\nu)$ be $N_u(\Delta\nu)\,\delta\nu$; then the number of transition is

$$N_u(\Delta\nu)\,\delta\nu \left\{A_{u1} + A_{u2} + \cdots + A_{um}\right\}$$

The secular equilibrium of the sub-state gives

$$N_u(\Delta\nu)\,\delta\nu\,\{A_{u1}+A_{u2}+\cdots$$
$$+A_{um}\}=\rho dx\left\{\sum_{s=1}^{m}\frac{l(\nu_s+\Delta\nu)\,J(\nu_s+\Delta\nu)}{h(\nu_s+\Delta\nu)}\right\} \tag{1.7}$$

The energy emitted in the first line is $N_u A_{u1} h(\nu_1+\Delta\nu)$, and similarly for the other lines. Accordingly

$$p(r,s)=\frac{\nu_r+\Delta\nu}{\nu_s+\Delta\nu}\cdot\frac{A_{ur}}{A_{u1}+A_{u2}+\cdots A_{um}} \tag{1.8}$$

The equations can be written more simply if we suppose that the 'm' number of lines are so close together that we may ignore differences in the frequencies,
and

$$k_1=k_2=\cdots=k_m=k$$

and we take

$$B(\nu_1,T)=B(\nu_2,T)=\cdots=B(\nu_m,T)=B(\nu,T)=B_\nu(T)$$

So, the expression for $p(r,s)$ may be written as

$$p(r,s)=\frac{A_{ur}}{\sum\limits_{s=1}^{m}A_{us}} \tag{1.9}$$

Since p does not involve $'s'$ we set $p(r,s)=\alpha_r$,

we notice that

$$\sum_{r=1}^{m}\alpha_r=1$$

Then with

$$\varepsilon_1=\varepsilon_2=\cdots=\varepsilon_m=\varepsilon,$$

the equations can be written as

$$\mu \frac{dI_r(\tau,\mu,\phi)}{d\tau} = (k + l_r)\, I_r(\tau,\mu,\phi)$$
$$+ (k + \varepsilon l_r)\, B_\nu(T) + (1-\varepsilon) \sum_{s=1}^{m} p(r,s)\, l_r J_r \tag{1.10}$$

In these equations references to a particular $\Delta\nu$ have been omitted for the sake of clarity.

From Boltmann's equation we have

$$\frac{N_1}{q_1} = \frac{N_2}{q_2} = \cdots = \frac{N_m}{q_m} \tag{1.11}$$

where the q_i's are the statistical weights.

Now, we have

$$\eta_\nu = \frac{l_\nu}{k}$$

where l_ν= line absorption coefficient and k = the coefficient of continuous absorption.

But, the line absorption coefficient l_ν is related with the concentration N of the atoms forming line absorption is as follows:

$$l_\nu = \frac{N\alpha(\nu)_D}{\rho}$$

where $\alpha(\nu)_D$ is the atomic line absorption co-efficient modified by Doppler effect due to thermal motion of the atom and ρ is the density of the atmosphere.

So, we can write from above

$$\eta_\nu = \frac{N\alpha(\nu)_D}{k\rho} \tag{1.12}$$

From the equations (1.11) and (1.12), we obtain $\eta_n = const. \times q_n f$, the oscillator strength f being related to the downward transition probability, namely

$$f = \frac{1}{3} \cdot \frac{q_u}{q_n} \cdot \frac{A_{un}}{\gamma_n},$$

where γ_n is the classical damping constant $8\pi^2 e^2 \nu_0^2 / 3mc^3$, where m is the mass of photon. Since γ is the same for m numbers of lines,

$$\eta_1/A_{u1} = \eta_2/A_{u2} = \cdots = \eta_m/A_{um}$$

for all $\Delta\nu$, and from the equation (1.9), which defines α_n,we obtain

$$\frac{\eta_1}{\alpha_1} = \frac{\eta_2}{\alpha_2} = \cdots = \frac{\eta_m}{\alpha_m} \tag{1.13}$$

Hence

$$\alpha_r \left(\sum_{s=1}^{m} \eta_s \right) = \left(\sum_{s=1}^{m} \alpha_s \right) \eta_r = \eta_r \tag{1.14}$$

which gives

$$\alpha_r = \eta_r / \sum_{s=1}^{m} \eta_s \tag{1.15}$$

and

$$\sum_{r=1}^{m} \alpha_r = 1 \,, \qquad r = 1, 2, \cdots, m \tag{1.16}$$

and the equation (1.6)becomes

$$\mu \frac{dI_r(\tau, \mu, \phi)}{d\tau} = (1 + \eta_r) I_r(\tau, \mu, \phi) - (1 + \varepsilon\eta_r) B_\nu(T) + (1 - \varepsilon) \alpha_r \sum_{s=1}^{m} \eta_s J_s \tag{1.17}$$

But the source function $J_s(\vartheta, \phi)$ or equivalently $J_s(\mu, \phi)$ is given by Chandrasekhar[45]

$$J_s = \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} p(\mu, \phi; \mu', \phi')\, I_s(\tau, \mu, \phi)\, d\mu' d\phi', \quad \mu = \cos\vartheta \qquad (1.18)$$

where integration is taken over all directions (ϑ', ϕ')

So, the equation of transfer becomes

$$\begin{aligned}\mu\frac{dI_r(\tau, \mu, \phi)}{d\tau} &= (1+\eta_r)\, I_r(\tau, \mu, \phi) - (1+\varepsilon\eta_r)\, B_\nu(T) \\ &+ (1-\varepsilon)\,\alpha_r \left\{\sum_{s=1}^{m}\eta_s \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} p(\mu, \phi; \mu', \phi') \times \right. \\ &\left. \times\, I_s(\tau, \mu, \phi)\, d\mu' d\phi'\right\} \qquad (1.19)\end{aligned}$$

It is evident that for the type of the problem we have formulated, solutions of the equation of transfer must be sought which exhibits axial symmetry about z-axis. The intensity and the source function must therefore be azimuth independent, and the equation of transfer (1.19) becomes

$$\begin{aligned}\mu\frac{dI_r(\tau, \mu)}{d\tau} &= (1+\eta_r)\, I_r(\tau, \mu) - (1+\varepsilon\eta_r)\, B_\nu(T) \\ &-\frac{1}{2}(1-\varepsilon)\,\alpha_r \left\{\sum_{s=1}^{m}\eta_s \int_{-1}^{+1} p(\mu, \mu')\, I_s(\tau, \mu)\, d\mu'\right\} \qquad (1.20)\end{aligned}$$

In the above equation (1.20), τ denotes the optical depth and $\eta_r = l_r/k$, l_r denotes the absorption co-efficient for the rth interlocked line and k denotes the continuous absorption which is supposed to be constant for each line. ε, the co-efficient, which is introduced to allow for thermal emission associated with the line absorption, and $B_\nu(T)$,the Planck-function, are considered to be constant for each line.

1.1.1 Boundary Conditions:

The boundary conditions for solving the equation (1.20)

$$I_r(0,-\mu) = 0; \;(0 < \mu \leq 1) \tag{1.21}$$

$$I_r(\tau,\mu) \cdot \; e^{-\tau\mu} \rightarrow 0 \qquad as \;\; \tau \rightarrow \infty$$

i.e.

$$I_r(\tau,\mu) \text{ is at most linear in } \tau \text{ as } \tau \text{ tends to infinity} \tag{1.22}$$

Another form of the equation (1.20):

In the equation(1.17),

$$\alpha_r\left\{\sum_{s=1}^{m}\eta_s J_s\right\} = \alpha_r\left\{\sum_{s=1}^{m}\eta_s\left(J_s - B\right)\right\} = \eta_r\left\{\sum_{s=1}^{m}\alpha_s\left(J_s - B\right)\right\} + \eta_r B$$

The equation (1.17) now take the form

$$\begin{aligned}\mu\frac{dI_r\left(\tau,\mu,\phi\right)}{d\tau} &= \left(1+\eta_r\right)\left\{I_r\left(\tau,\mu,\phi\right) - B\right\}\\ &\quad - \left(1-\varepsilon\right)\eta_r\left\{\sum_{s=1}^{m}\alpha_s\left(J_s - B\right)\right\}\end{aligned} \tag{1.23}$$

The form (1.23) of the equation of transfer of interlocking lines for the case of triplet is used by Woolley and Stibbs[229] for obtaining the solution by Eddington's approximation. This form is used nowhere in this thesis.

1.2 Discrete-ordinate method.

The method used in this thesis was first extensively used by Chandrasekhar for solving a problem of stellar and atmospheric radiation and is popularly known as **Chandrasekhar's discrete-ordinate method.**

The method begins with the replacement of the source function $S_\nu(\tau_\nu, \mu)$ of the radiative transfer equation

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu, \mu) \tag{1.24}$$

by the mean intensity J, given by

$$S_\nu(\tau_\nu, \mu) = J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu')\, d\mu' \tag{1.25}$$

converting the resulting equation into an integro-differential equation

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau, \mu) - \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu')\, d\mu' \tag{1.26}$$

It is noted that the integro-differential equation (1.26) is converted in terms of specific intensity I_ν alone which, a function of two variables μ and τ, appears differentiated with respect to τ and integrated over μ.

In this method, the radiation field and as such the total solid angle is divided into a finite number of discrete-ordinates (directions). The intensity of each discrete ordinate represents the whole intensity of the corresponding small section of solid angle. The integral in the integro-differential equation associated with any problem of radiative or heat or neutron transfer is replaced by a quadrature, such as Gaussian, Lobatto, or Chebyshev. The radiative transfer equations for the set of discrete directions are then solved and the solution set is used to construct the solution of the main problem.

Chandrasekhar divided the radiation field into 2n number of discrete-ordinates in the directions μ_i, $(i = \pm 1, \cdots, \pm n)$, sub diving the interval $[-1, +1]$ of μ into 2n points μ_{-n}, $\mu_{-(n-1)}$, $\cdots$, μ_{-1}, μ_{+1}, $\cdots$, μ_{n-1}, μ_n so that the points are the 2n non-zero zeros of the even Legendre's polynomial $P_{2n}(\mu)$. Chandrasekhar replaced the definite integral of the integro-differential equation of radiative transfer by the Gaussian sums of numerical quadrature as follows:

$$\int_{-1}^{+1} I(\tau, \mu')\, d\mu' = \sum_{j=-n}^{j=+n} a_j I(\tau, \mu_j) \tag{1.27}$$

where μ_j's with $(j = \pm 1, \pm 2, \cdots, \pm n)$ are the $2n$ zeros of the Legendre polynomial $P_{2n}(\mu)$ of order $2n$ and the $a_j's$ are the weight factors, given by

$$a_j = \frac{1}{P'_{2n}(\mu_j)} \int_{-1}^{+1} \frac{P_{2n}(\mu)}{\mu - \mu_j} d\mu \tag{1.28}$$

The quadrature constants a_j and μ_j can be chosen in a variety of ways. Wick[224] suggested that the best choice for constants are those of the Gaussian quadrature formula. Gauss himself has shown that for a given number of divisions the best representation of an integral is obtained when the spacing of the division points is symmetrical about the mid-point of the range of integration, the interval being divided according to the zeros μ_j of the Legendre polynomial $P_{2n}(\mu)$.

Furthermore, μ_i's and a_j's follow the following properties:

$$\mu_i = -\mu_i \tag{1.29a}$$

$$a_j = a_{-j} \tag{1.29b}$$

by Abramowitz and Stegun[5] (vide Peraiah[159])

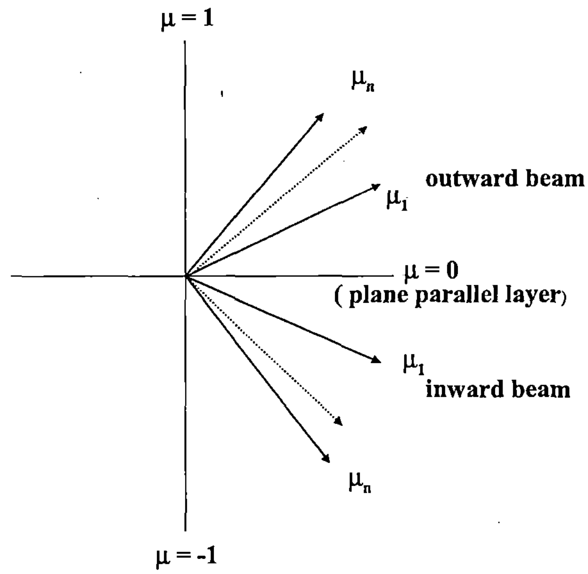


Figure 1.3: Gaussian points μ_j's

It is noticed that the Gaussian sum would give exact values of the integral, if $I\ (\mu)$ could be written down as a polynomial in μ of degree less than or equal to $4n - 1$.

He then discretized the differential equation so formed into 2n ordinary differential equations along 2n directions μ_i, $(i = \pm 1, \cdots, \pm n)$ and solved each individual equations and combined them to form the solution of the desire integro-differential equation.

On the other hand, some workers, like Siewert,[186] Barichello and Siewert,[20,21] forming the 2N ordinary differential equations in the same manner as Chandrasekhar, expressed them into a matrix form. They, in lieu of taking the points μ_i, $(i = \pm 1, \cdots, \pm n)$ as the zeros of an even Legendre polynomial of order 2n, assume that the points as the eigen value of a matrix. This method is identified by a few authors as **matrix form of discrete ordinate method**.

There are also some other authors who formed the ordinary differential equations for each direction μ_i by dividing the interval $[-1,+1]$ of μ of the definite integral involved in the integro-differential equation following Chandrasekhar's method, but they used neither Chandrasekhar's form nor matrix form of the method. The equations so formed are known as **discrete ordinate equations**.

Here we shall now present a solution of the simplest form of a transfer equation which Chandrasekhar solved by his discrete-ordinate method.

1.2.1 Basic Radiative Transfer Equation, the Boundary Conditions and the use of Discrete Ordinate Method:

1.2.1.1 Basic Radiative Transfer Equation:

Simplest form of a transfer equation which Chandrasekhar solved by his discrete-ordinate method is

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{+1} I(\tau,\mu')\,d\mu' \tag{1.30}$$

where I is the intensity, μ is the cosine of the incident angle ϑ made by the incident ray coming from any star to the surface at which intensity I is calculated with the outward normal and τ is the normal optical depth, given by, $\tau = \int_z^\infty k\rho dz$, z being the linear distances normal to the plane of stratification of a plane parallel atmosphere.

1.2.1.2 Boundary Conditions for Solving the Transfer Equation (1.30):

$$I(0,-\mu)=0\ ,\ \ (0<\mu\leq 1) \tag{1.31a}$$

and $$I_r(\tau,\mu)\cdot\ e^{-\tau\mu}\rightarrow 0$$

i.e. $I_r(\tau,\mu)$ is at most linear in τ as τ tends to infinity (1.31b)

1.2.2 Chandrasekhar's solution:

1.2.2.1 Use of Discrete-ordinate Method to Solve the Equation (1.30), Subject to the Boundary Conditions (1.31a) and (1.31b):

Dividing the radiation field into 2n streams in the direction μ_i, $(i=\pm 1,\ \cdots,\pm n)$, we can replace the equation of transfer (1.30) by the system of 2n linear differential equations:

$$\mu_i\frac{dI_i}{d\tau}=I_i-\frac{1}{2}\sum_j a_j I_j,\quad (i=\pm 1,\cdots,\pm n) \tag{1.32}$$

where the μ_i's, $(i=\pm 1,\cdots,\pm n\ and\ \mu_{-i}=-\mu_i)$ are the zeros of the Legendre Polynomial $P_{2n}(\mu)$ and the a_j's $(j=\pm 1,\cdots,\pm n$ and $a_{-j}=a_i)$ are the corresponding Gaussian Weights. Further I_i is used for $I(\tau,\mu_i)$.

First we observe that the equation (1.32) admits a solution of the form:

$$I_i=g_i e^{-k\tau},\quad (i=\pm 1,\cdots,\pm n\ ) \tag{1.33}$$

where g_i and k are constants.

Introducing the equation (1.33) in the equation (1.32), we obtain a relation:

$$g_i\,(1+\mu_i k) = \frac{1}{2}\sum_j a_j g_j, \quad (i = \pm 1, \cdots, \pm n\ ) \tag{1.34}$$

Hence,

$$g_i = \frac{\text{constant}}{(1+\mu_i k)}, \quad ,(i = \pm 1, \cdots, \pm n\ ) \tag{1.35}$$

where the 'constant' is independent of i . Substituting the foregoing form in the equation(1.34) , we obtain the characteristic equation:

$$1 = \frac{1}{2}\sum_j \frac{a_j}{(1+\mu_j k)} \tag{1.36}$$

Remembering that $\mu_{-i} = -\mu_i$ and $a_{-j} = a_j$, we can write the characteristic equation(1.36) in the form:

$$1 = \frac{1}{2}\sum_{j=1}^{n} \frac{a_j}{(1-\mu_j^2 k^2)} \tag{1.37}$$

which has two roots, each equal to zero, because $\sum_{j=1}^{n} a_j = 1$ and other $(2n-2)$ non zero distinct roots.

With $(2n-2)$ distinct non-zero roots $\pm\ k_\alpha$, $\alpha = 1, 2, \cdots, n-1$, we can establish a relation with the zeros $\pm\ \mu_i$, $i = 1, 2, \cdots, n$ of the Legendre Polynomial $P_{2n}\,(\mu)$ which is

$$k_1 \cdots k_{n-1} \cdot \mu_1 \cdots \mu_n = \frac{1}{\sqrt{3}} \tag{1.38}$$

With these distinct non-zero roots which are numerically greater than 1, we can show that the general solution of the system of equation (1.32) is of the form:

$$I_i = b\left(\sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1+\mu_i k} + \sum_{\alpha=1}^{n-1} \frac{L_{-\alpha} e^{+k_\alpha \tau}}{1+\mu_i k} + \tau + \mu_i + Q\right), \tag{1.39}$$

$$(i = \pm 1, \pm 2, \cdots, \pm n)$$

where $L_{\pm\alpha}$ and b, which can be connected with the flux of radiation πF normal to the plane of stratification of the plane-parallel scattering atmosphere by the relation

$$F = \frac{4}{3}b \tag{1.40}$$

and Q, satisfying the relation,

$$Q = \sum_{i=1}^{n} \mu_i - \sum_{\alpha=1}^{n-1} \frac{1}{k_\alpha} \tag{1.41}$$

are arbitrary constants of integration.

We have already mentioned the two boundary conditions (1.31a) and (1.31b) for this problem. By virtue of the boundary condition (1.31b) which gives that none of the I_i's increase more rapidly than e^τ as $\tau \to \infty$, we omit all the terms in $e^{+k_\alpha \tau}$, thus leaving

$$I_i = b\left(\sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1+\mu_i k} + \tau + \mu_i + Q\right), \tag{1.42}$$
$$(i = \pm 1, \pm 2, \cdots, \pm n)$$

Next, the boundary condition (1.31a) implies that there is no radiation incident on $\tau = 0$. The absence of any radiation in the directions $-1 \leq \mu < 0$ at $\tau = 0$ gives in our present approximation that

$$I_i = 0 \ \text{ at } \tau = 0 \text{ and } i = 1, 2. \cdots, n \tag{1.43}$$

Hence, by (1.42), we get

$$\sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1-\mu_i k_\alpha} - \mu_i + Q = 0 \quad (i = 1, 2, \cdots, n) \tag{1.44}$$

which are the n relations which determine the n constants of integration L_α, $(\alpha = 1, 2, \cdots, n)$ and Q . The constant b is left arbitrary and is related to the assigned constant net flux of the radiation through the atmosphere given by (1.40).

1.2.2.2 Closed form of emergent intensity $I\,(0,\mu)$

Letting

$$S\,(\mu)=\sum_{\alpha=1}^{n-1}\frac{L_\alpha}{1-\mu k}-\mu+Q\,, \tag{1.45}$$

the boundary conditions (1.31b) can be expressed as:

$$S\,(\mu_i)=0\quad,\quad i=1,2,\cdots,n \tag{1.46}$$

and the angular distribution of the emergent radiation is expressible as

$$I\,(0,\ \mu)=\frac{3}{4}FS\,(-\mu) \tag{1.47}$$

Now, multiplying $S\,(\mu)$ by $R\,(\mu)$, given by

$$R\,(\mu)=\prod_{\alpha=1}^{n-1}(1-k_\alpha\mu) \tag{1.48}$$

we get a polynomial in μ of degree n which vanishes for $\mu=\mu_i$, helping us to conclude that the polynomial $S\,(\mu)\,R\,(\mu)$ must be identical with the polynomial $P\,(\mu)$, given by,

$$P\,(\mu)=\prod_{i=1}^{n}(\mu-\mu_i) \tag{1.49}$$

and so, the co-efficients of each term of the two polynomials must coincide and therefore, we shall get

$$S\,(\mu)\,R\,(\mu)=(-1)^n\,k_1k_2\cdots k_{n-1}P\,(\mu)$$

producing the relation:

$$S\,(-\mu)=k_1k_2\cdots k_{n-1}\mu_1\cdots\mu_{n-1}H\,(\mu) \tag{1.50}$$

where

$$H\,(\mu)=\frac{1}{\mu_1\cdots\mu_{n-1}}\frac{\prod_{i=1}^{n}(\mu+\mu_i)}{\prod_{\alpha=1}^{n-1}(1+k_\alpha\mu)} \tag{1.51}$$

So, from the equation (1.47), using the equation (1.50), we can write

$$I\,(0,\ \mu) = \frac{3}{4}F\,k_1k_2\cdots k_{n-1}\mu_1\cdots\mu_{n-1}H\,(\mu) \tag{1.52}$$

Therefore, using the equation (1.38), we can express the emergent radiation in terms of H-function $H\,(\mu)$ as

$$I\,(0,\ \mu) = \frac{\sqrt{3}}{4}FH\,(\mu) \tag{1.53}$$

1.3 Works done so far

1.3.1 Works done on discrete-ordinate method

Discrete-ordinates method for the radiative transfer and the neutron transport is not a new, but has a long history. Though the method is an old one, even then it doesn't lose the importance. In the language of Atanacković-Vukmanović[12] mentioned in an invited review paper of radiative transfer ,"In 1940s and 1950s several powerful methods for solving RT problems were developed: the method of discrete ordinates by Chandrasekhar, Ambarzumian's method based on the invariance principle, Sobolev's escape probability method (1957), etc. Their importance is twofold: on one hand, they are the bases of many modern techniques and, on the other, their "exact" solutions to simplified transfer problems serve as a reliable test of accuracy of new numerical methods."

Many workers worked on this method time to time. Among them, Chandrasekhar is noteworthy. Some authors gave the identity of the discrete ordinate method, which is well known as **Chandrasekhar's discrete ordinate method**, by using the term **Wick-Chandrasekhar's**

discrete ordinate method, though neither Wick nor Chandrasekhar presented the method first.

The method was first brought to the transport theory from the Kinetic theory of gases as developed by Joule (vide Peraiah[159]) in a rather primitive form (two parallel and opposite intensities) by Schuster[178] and Schwarzsohild,[180] and Milne[147] (vide Kourganoff and Busbridge[129]).

In Kinetic theory of gases , the molecules in a box are presumed to be moving in three equal pairs of streams, parallel to length, breadth, and depth of the box in which the gas is situated and directed opposite direction to each other. (vide Peraiah[159]). Same treatment was done by Schuster and Schwarzsohild in transport theory. The transfer equation in plane parallel stratification

$$\mu\frac{dI\left(\tau,\mu\right)}{d\tau}=I\left(\tau,\mu\right)-\frac{1}{2}\int_{-1}^{+1}I\left(\tau,\mu'\right)d\mu'$$

is replaced by a pair of equations for I_+ and I_-, the outward and inward intensities, thus

$$+\frac{1}{2}\frac{dI_+}{d\tau}=I_+-\frac{1}{2}\left(I_++I_-\right) \tag{1.54a}$$

$$-\frac{1}{2}\frac{dI_-}{d\tau}=I_--\frac{1}{2}\left(I_++I_-\right) \tag{1.54b}$$

(vide Chandrasekhar[45] and Peraiah[159]).

The factor $\frac{1}{2}$ on the LHS is chosen arbitrarily as in the kinetic theory of gases.(vide Peraiah[159])

The method was generalized first by Wick[224] (vide Kourganoff and Busbridge[129] and Woolley and Stibbs[229]), in connection with a diffusion problem, by replacing the integral of the equation of transfer(1.24) by the Gaussian sums of numerical quadrature as in equation(1.27).

Chandrasekhar[45] developed one dimensional mathematical models of radiative transfer and discussed the time independent problems at a length by using this technique.

Considering the theory of diffuse reflection and transmission by a plane-parallel atmosphere of finite optical thickness under conditions of (I) isotropic scattering with an albedo $\tilde{\omega}_0 \leq 1$, (II) scattering in accordance with Rayleigh's Phase function (II) scattering in accordance with the phase function $\lambda\,(1 + x cos\theta)$, and (V) Rayleigh scattering with proper allowance for the polarization of radiation field, Chandrasekhar[43] showed it is possible to eliminate the constants of integration (which are twice as many as in the case of semi infinite atmospheres) and expressed the solutions for the reflected and transmitted radiations in closed forms in general n^{th} approximation. He also showed a pair of functions $X(\mu)$ and $Y(\mu)$ which depends only on the roots of a characteristic equation and the optical thickness of the atmosphere play the same basic role in the theory as $H(\mu)$ does in the theory of semi-infinite atmospheres making possible the passage to the limit of infinite approximation and the determination of the exact laws of diffuse reflection and transmission.

Chandrasekhar[45] applied his method of discrete ordinates first to solve the transfer equation for coherent scattering in the stellar atmosphere with Planck's function as a linear function of optical depth (viz.,$B_\nu\,(T) = b_0 + b_1\tau$).

He discussed the equations of Radiative transfer for an electron scattering atmosphere and gave the solution of the equation by his method of discrete ordinates (vide Chandrasekhar[45] and Woolley and Stibbs[229]).

A moving atmosphere, postulated for Cephied variables, is sometimes suspected of giving rise to irregular asymmetries and displacement of lines in spectra of supergaints. Underhill[206] applied the Chandrasekhar's theory, as explained in Chandrasekhar's paper[40] and his successive papers, of transfer of radiation through a Schuster- Schwarzschild model atmosphere to formulate the problem of a uniformly expanding atmosphere and applied conveniently the Chandrasekhar's discrete ordinate method to solve the transfer equation for the case.

Rybicki[173] wrote in a review paper about Chandrasekhar's works on the method of discrete ordinates, " Chandrasekhar's numerical comparison of low order results with the exact analytical result

$$H(\mu) = (1+\mu)\exp\left\{-\frac{\mu}{\pi}\int_{0}^{\pi/2}\frac{\log\left[(1-\phi\cot\phi)/\sin^2\phi\right]}{\cos^2\phi+\mu^2\sin^2\phi}d\phi\right\} \tag{1.55}$$

convinced him that the approximation results of the discrete ordinate method would converge to the exact results in the limit $n \rightarrow \infty$. Only much later was this convergence proved mathematically" (vide Anselone[10,11]). He also wrote,"Chandrasekhar also applied the method of discrete ordinates to the problem of diffuse reflection, in which radiation is incident on the medium at angle μ_0, and one is required to find the radiation emergent at angle μ. This relationship is given in terms of a scattering function $S(\mu, \mu_0)$. It had previously been shown by Hopf [Equation (191), of Hopf[86]] that the scattering function is related to the above H-function (1.55) by means of the relation

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(\mu,\mu_0) = \varpi_0 H(\mu)\,H(\mu_0) \tag{1.56}$$

Thus, the function $S(\mu, \mu_0)$ of two variables, can be simply expressed in terms of a single variable, the same H-function that appears in the solution for the radiative equilibrium problem.

When Chandrasekhar applied the discrete ordinate method to the semi-infinite diffuse reflection problem (cf §26 of Chandrasekhar[45]), he found a result of the same form as equation (1.56), where the H-function

$$H(\mu) = \frac{\prod_{i=1}^{n} (1 + \mu/\mu_i)}{\prod_{\alpha=1}^{n} (1 + \mu k_\alpha)} \tag{1.57}$$

were precisely the same as that for the discrete ordinate solution (1.57) to the radiative equilibrium problem".

Rybicki[173] further wrote,"For the case of a finite medium, besides the diffuse scattering function $S(\mu, \mu_0)$ there is also a diffuse transmission function $T(\mu, \mu_0)$ to be determined. These functions satisfy the extended relations, given by Ambarstsumian[8]

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\mu, \mu_0) = \varpi_0 H(\mu) H(\mu_0) \tag{1.58a}$$

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\mu, \mu_0) = \varpi_0 H(\mu) H(\mu_0) \tag{1.58b}$$

where the functions * X and Y were the solutions to certain functional equations. After seeing these forms in Ambartsumian's paper, Chandrasekhar[43] was able to put the discrete ordinate solution for this problem into the same form, where the X- and Y- functions were also expressible in closed form in terms of the roots of the characteristic function. ....".

He further pointed out that "The transfer problems Chandrasekhar considered had already been simplified by making a number of physical assumptions and approximations; e.g., plane-parallel geometry, coherent scattering, and single-scattering albedo independent of depth. In a sense, choosing discrete angles is just one more simplifying

* Ambartsumian used the notation ϕ and ψ for these functions

approximation, on a par with the others made. Then the crucial questions to ask are whether these simplified equations are of practical use, can increase our mathematical or physical understanding, or satisfy some criterion of mathematical beauty. I believe for the discrete ordinates method the answer is 'yes' to each of these questions.

As to the practicality of the method, remember that Chandrasekhar was not acting solely as a mathematical physicist, but as an astrophysicist attempting to find answers to practical problems in stellar atmospheres and planetary atmospheres......... The discrete ordinate method gave him highly accurate solutions in a completely straightforward way.

As to the method's relation to mathematical and physical understanding, Chandrasekhar was obviously delighted when he continually found many results of this method that were perfectly consistent with exact requirements of the theory. Many known analytically exact results are obeyed precisely to all orders of approximation in the discrete ordinate method, for example, the Hopf-Bronstein relation $J(0) = \sqrt{3}F/4$ in the Milne problem, the structure of the diffuse scattering and transmission functions as given in equations (1.56) and (1.58a,1.58b), and their reciprocity relations. Often he was able to determine the form of the exact solutions only after he had solved the discrete ordinate equations first. These circumstances convinced Chandrasekhar that the discrete ordinate method was more than just a convenient numerical method; it also preserved essential mathematical and physical characteristics of the problem being investigated."

The above discussion noted from the Rybicki's review paper[173] gives us a clear picture and the usefulness about the Chandrasekhar discrete ordinate method.

The method of discrete ordinates developed by Wick[224] and Chandrasekhar[45] constitutes a powerful technique for the solution of transfer problems. The accuracy of the method is strongly dependent on the particular choice of finite stream quadrature formula used to represent continuous radiation field. Chandrasekhar's[45] use of a Gaussian quadrature formula was criticized by Kourganoff and Pecker[130] and Kourganoff and Busbridge.[129]

Kourganoff and Pecker[130] produced a paper on the choice of numerical integration formulae in the solution of the integro-differential equations of transfer (radiation, neutrons) by the "method of discrete-ordinates". In their paper they commented that the method of Gaussian subdivisions and the characteristic roots used by Chandrasekhar in his treatment of radiative equilibrium is not necessarily the most effective one. To establish their comment they performed the calculations of the 1st and 4th approximations in the "standard problem" of isotropic diffusion in the Newton-Cotes and Tchebycheff formulae and found that the former gives more accurate results in the 2nd and 4th approximations than were obtained by Chandrasekhar. This result was explained by discussing the distribution of the Gaussian subdivisions.

Sykes[201] obtained highly accurate results from the discrete ordinate method by splitting the interval and fitting the Gaussian formula separately to the upward and downward hemispherical stream i.e. separately over the ranges $(-1, 0)$ and $(0, 1)$.

Kourganoff[128] extended Chandrasekhar's limiting process on the n^{th} approximation obtained by the method of discrete beams for emergent intensity to functions describing the internal state of the atmosphere. He showed how, by a suitable interpretation of the constant of integration , the solutions in the n^{th} approximation for the source function $\Im\tau$ can also be transformed into a form suitable for effecting the passage to infinite approximation and giving the exact $\Im\tau$ for the problem of isotropic scattering in a gray atmosphere and for a line formation in a non gray atmosphere.

Sen[181] solved the equation of transfer of radiation in a spherically scattering symmetric atmosphere for non-conservative isotropic scattering by the method of Chandrasekhar in which the integrals are replaced by corresponding Gaussian sums, and first approximation results have been fitted to two boundary conditions, one of no incident radiation and other of a very weak radiation field penetrating from outside.

Sen[182] solved the problem of softening of radiation by multiple compton scattering in stellar atmosphere containing free electrons in the first approximation (in Chandrasekhar's method of solution by Gaussian approximation) by the method of trigonometric series and calculated the intensity distribution at the outer surface by retaining the first- and second-order terms of Taylor's expansion of scattering intensity.

Horak[89] considered the transfer of radiation by a plane parallel atmosphere containing a uniform distribution of emission sources for the cases: (a) scattering according to the Rayleigh phase function and (b)Rayleigh scattering and derived the exact expression for the emergent intensity for finite atmospheres. He also gave the solution in n^{th} approximation for any depth τ and calculated polarization for the atmosphere of optical thickness $\tau = 0.20$.

King[120] developed radiative transfer theory for band-absorbing semi-infinite atmospheres possessing line structure by extending the formalism of Chandrasekhar to an integration over frequency space as well as over μ space and obtained the solution of the monochromatic equation of transfer for a plane-parallel atmosphere in local thermodynamic equilibrium with the aid of the method of discrete ordinates .

Under radiative equilibrium the emission co-efficient of a narrow band-absorbing gas is equal to the frequency integral of the average intensity and the band-absorption co-efficient. King[119] developed a two-point Gaussian quadrature formula for the frequency integration, by using the Elsasser band of equally spaced, equally intense, Lorentz-broadened lines as the absorption model. By the use of that formula, he extended the Chandrasekhar method of discrete-ordinates to include an integration over frequency space as well as μ-space.

Krook[131] translated the Wick-Chandrasekhar method of discrete ordinate into an equivalent moment procedure for the approximate solution of the equation of transfer for a plane-stratified gray atmosphere in radiative equilibrium and used it to discuss the relation between various methods that are also based on infinitely approximating to the angular distribution of intensity.

Lenoble,[138] applying the Chandrasekhar's method of approximations to the diffusion of radiation from sun and sky in a plane homogeneous scattering layer of large particles (fog or sea), gave notation and principle of the method and established the equations which were applied to two cases: (1) for sky radiation only and no solar radiation; (2) for an infinite layer with solar radiation only.

Piotrowski[161,162] (vide Busbridge and Orchard[32]) found the asymptotic solution for the phase function $1+\varpi\cos\theta$ using the method of discrete ordinates as developed by Chandrasekhar[41,45]

King[121] derived the exact form of the source function for a finite gray atmosphere in radiative equilibrium (planetary thermal problem) forming the solution in n^{th} approximation by using the method of discrete ordinates.

King[122] developed the transport theory of non-gray atmosphere of finite thickness and treated this planetary thermal problem by using both an invariance attack and the method of discrete ordinates.

Lenoble[139] used the Chandrasekhar method to calculate the illumination

(I) in the sea, for a uniform sky and for the sun at 59^0 from the zenith, assuming the absorption co-efficient to be half the scattering co-efficient and

(II) below, a mist of large drops, there being no absorption and a uniform sky.

Lenoble[140] gave the equation governing the penetration, summarizing Chandrasekhar's approximate method of resolution, applied it to a layer of haze and to the sea and discussed the approximations.

Jefferies and Thomas[99] obtained an algebraic solution for the depth variation of the source function $S_L(\tau)$ for resonance and strong subordinate lines by using Eddington approximation and the method of discrete ordinates.

Sen and Lee[183] solved the problem of broadening of spectral lines by the Doppler effect due to thermal motion of electrons in an axially symmetric, plane-parallel electron atmosphere, scattering according to Rayleigh's phase matrix and taking into account the polarization of the radiation field in the first approximation of Chandrasekhar's method of discrete ordinates.

King and Florance[123] demonstrated the physical basis underlying Sykes' choice of a double-Gauss method in providing the optimum fit of the kernel in the Schwartzchild-Milne integral equation by an exponential function series.

Abhyankar[1] presented a numerical method for computing absorption-line profiles in a plane-parallel stratified moving atmosphere by extending the Rottenberg's[171] idea of dividing the atmosphere into many thin layers, of course not for the spherical layers as done by Rottenberg, but for plane parallel layers and retaining the equation of transfer in its discrete ordinate form.

King, Sillars and Harrison[124] expressed the Hopf q-function in the equilibrium gray-atmosphere problem in the discrete-ordinate approximation to attain its extreme accurate value.

Hummer[94,95] (vide Rybicki and Hummer[174]) solved the radiative transfer equation for spectral line formation by non-coherent scattering in inhomogeneous plane-parallel media by using a generalization of Chandrasekhar's discrete ordinate method.

Avrett and Hummer[14] used the generalization of the Wick-Chandrasekhar discrete-ordinate method in the theory of line formation to find the expression for the source function $S\,(\tau)$

Samuelson[176] extended the method of discrete ordinates to describe the steady-state distribution of thermal radiation and the corresponding depth-dependent thermal structure of a plane parallel semi-infinite particulate medium in radiative equilibrium.

Samuelson[177] used the method of discrete ordinates to investigate the outgoing thermal radiation field at the top of cloudy atmospheres as a function of the scattering and thermal properties of the atmosphere.

Black[27] applied the method of discrete ordinates to calculate the diffusely reflected and transmitted spectral line profiles for uniform non coherent scattering media onto which radiation of frequency near that of a resonance line of the medium incident.

Blerkom and Hummer[28] obtained numerical solutions of high accuracy for the ionization balance in an isothermal, plane-parallel hydrogen model nebulae of various optical thickness, using generalization of the Wick-Chandrasekhar discrete-ordinate method .

Rybicki and Hummer[174] gave the discrete-ordinate representation of the radiative transfer equation for spectral line formation by non-coherent scattering in inhomogeneous plane-parallel media casting it into matrix form and derived the Reccati-transformation for finite atmosphere.

Assuming uniform velocity in each layer, Kulander[130] solved the Eddington approximation to transfer equation by a discrete ordinate method for a semi-infinite, isothermal atmosphere with a constant density of particles having only two discrete energy level.

The DOM, described by Chandrasekhar in 1950,[45] has been deeply studied by Lathrop and Carlson[34] (vide Joseph, Coelho, Cuenot and Hafi[101]) in 60-70s and by Truelove, Fiveland and Jamaluddin in the 80s (vide Joseph, Coelho, Cuenot and Hafi[101]). Significant improvements have been achieved in the last decade aiming at the reduction of the ray effects and false scattering, more accurate quadratures and the extension to complex geometries (vide Joseph Coelho Cuenot and Hafi[101])

Hummer[96] used a generalized discrete ordinate method to obtain accurate numerical solutions of the line transfer problem in which the scattering is described by a redistribution function.

Code[47] solved the time-dependent equation of radiative transfer for a plane-parallel isotropic scattering medium by the method of discrete ordinates.

Be[23] developed a method for solving the one-dimensional multigroup transport equation in a homogeneous semi-infinite medium with anisotropic scattering and used a variational treatment to enable the method to be applied to finite slabs where only the emergent angular fluxes are of interest. He showed the method which is not limited by any restriction on the number of spatial mesh intervals used to be competitive in computing time with conventional discrete ordinate techniques.

Considering the problem of the radiation field in a plane-parallel multilayer system, whose outer boundary is irradiated by parallel rays, Barkov[22] studied the case of non-isotropic radiation and gave a formal solution of the problem and using this, constructed the spatial angular distribution function of the intensity of the diffused radiation by applying the method of discrete ordinates.

Liou[142] developed theoretically the discrete-ordinate method for radiative transfer introduced originally by Chandrasekhar and verified numerically for use in solving the transfer of both solar and thermal infrared radiation through cloudy and hazy atmospheres.

Liou[143] derived explicitly the analytic equations in closed forms for the cases of two-stream and four-stream approximation from the exact solutions provided by Liou.[142]

Hansen and Travis[81] said in their review paper that an advantage of the discrete ordinate method is that it yields the internal field as well as the reflection and transmission. A disadvantage is that considerable algebra is required prior to numerical computations. However, at least for azimuth-independent term, the discrete ordinate method can give rather accurate results (within a percent or so) already for n = 3 or 4, so it is efficient procedure when accuracies of that order are sufficient (cf Weinman and Guetter,[221]Liou[142]). Liou[143] has given a quasi-analytic solution for n=2 (4-stream approximation) which might be sufficiently accurate for computations of the flux in many applications.

Roux and Smith[170] approximated the equation for one-dimensional, axisymmetric radiative transfer in an absorbing, emitting, and isotropically scattering medium by the method of discrete ordinates. Homogeneous and particular solutions are derived from the discrete ordinate form of the radiative transport equation.

Using a discrete ordinates method, Cram[55] solved the radiative transfer equation in a gray atmosphere subject to a specific distribution of mechanical heating and determine the resulting changes in LTE and non LTE conditions.

Nelson Jr. and Victory Jr.[157] compared the Nyström discrete ordinates method and interpolatory discrete ordinates method used in linear transport equation in the simple case of monoenergetic transport in azimuthally symmetric one dimensional slab geometry.

Zasova and Ustinov[233] applied the method of discrete ordinates, by developing it for making applicable to an inhomogeneous atmosphere of large optical depth, to the solution of the transfer equation in the case of an inhomogeneous planetary atmosphere.

Discussing the difficulties inherent in the conventional numerical implementation of the discrete ordinate method (following the Chandrasekhar's prescription) for solving the radiative transfer equation, Stamnes and Swanson[197] developed a matrix formulation to overcome the difficulties. Stamnes and Dale[196] extended the method to compute the full azimuthal dependence of the intensity.

Khalil, Shultis and Lesste[118] developed a plan systematic, gray model of coal particle suspension to test the accuracy of the low-order discrete-ordinates and flux method and of the differential approximation for calculating the radiant energy transport in multiply scattering and heat generating media bounded by diffusely reflecting surfaces and compared the results obtained by these three approximate techniques with those computed by a high order discrete-ordinates method.

The standard discrete-ordinates method is a deterministic(non stochastic) method for solving the linearized Bolzmann transport equation. It is commonly applied to neutron and photon transport problems. Finding its applicability superior to Monte Carlo methods for one dimensional problems in electron transport, Morel and Wienke[151] reviewed briefly the history of discrete-ordinates electron transport methods, described the state-of-art at that time and suggested directions of further works.

One of the dominant numerical approximation methods for the integro-differential equation for neutron transport is the discrete ordinate method. In this method one collocates the equation at preselected angular directions which are the quadrature points of the integral scattering term (the "discrete ordinates"), and then solves the resulting linear hyperbolic system by a variety of difference schemes. The problem with this hybrid collocation difference procedure lies in connecting the spatial differencing with the angular collocation. The possible mismatch can lead to distortions in the angular flux solution. This have led to numerous meliorative procedures, and an extensive literature. The method of collocation is well established among the numerical approximation methods for ordinary and partial differential equations and integral equations. Despite its very limited application in integro-differential equations, Grossman[79] felt it seemed of interest to apply a full collocation scheme to neutron transport equation; building in the required continuity and coupling between space and direction variables through suitable multidimensional spline basis functions and showed how this is done for a simple mono energetic one dimensional form of the neutron transport equation indicating its possible extensions of the method.

Mengüc and Viskanta[146] examined critically the accuracy of the two-flux, spherical harmonic and discrete ordinates method for predicting radiative transfer in a planar highly-forward scattering and absorbing medium.

Karp[117] showed that similar relations like the azimuth-averaged component of the intensity computed from the spherical harmonic method for solving the equation of the radiative transfer is 'exact' at Gaussian quadrature points holds for higher terms in the Fourier expansion of the intensity, but that result is 'exact' at the zeros of the associated Legendre polynomials. The relationship between discrete ordinates and spherical harmonics methods follows from the discussion. A discrete ordinates quadrature scheme, based on the zeros of the associated Legendre polynomials was shown to maintain the correspondence of the methods for those problems as well as providing a better set of points than the other methods in use.

Bergmann, Houf and Incropera[25] performed calculations, based on discrete-ordinate forward scattering and three-flux methods of solving the equation of transfer, to determine the effect of the scattering distribution, which had been systematically varied by changing the asymmetry factor used in the Heney-Greenstein form of the phase function, on radiative transfer in absorbing-scattering liquid which is irradiated across an air interface.

Larsen[136] introduced a parameter ϵ into the discrete ordinates equations in such a way that as ϵ tends to zero, the solution of these equations tends to the solution of the standard diffuse equation and then studied the behaviour of the spatial differencing scheme of the discrete ordinates equations for fixed spatial and angular meshes, in the limit as ϵ tends to zero.

Abhyankar and Bhatia[2] gave a definition of the effective depth of line formation which incorporates its dependence on the angle of emergence as well as on the position of the line and obtained the solution for isotropic scattering in the third approximation of discrete ordinates for various points on the disc of a planet viewed at different phase angles.

Marshak[145] studied the one dimensional transport equation in slab geometry with periodic boundary conditions , reduced it to the integral equation of the Peierls type and estimated the spectral radius of the integral operator. He analyzed the discrete ordinates algorithm for estimating the solution.

Nakajima and Tanka[156] presented matrix formulations for the discrete ordinate and matrix operator methods for solving the transfer of solar radiation in plane-parallel scattering atmosphere introducing eigenspace transformations of the symmetric matrices into the method of Stames and Swanson instead of using the decomposition of an asymmetric matrix. They gave the representations of the reflection and transmission matrices in the matrix operator method and the solutions of the discrete-ordinates method for inhomogeneous sublayers through the addition technique of the matrix operator method.

Nakajima and Tanka showed that the algebraic eigenvalue problem occurring in the discrete-ordinate and matrix operator methods can be reduced to finding eigenvalues and eigenvectors of the product of two symmetric matrices, one of which is positive definite. Stamnes Tsay and Nakajima[198] showed that cholesky decomposition of this positive definite matrix can be used to convert the eigenvalue problem into one involving a symmetric matrix and established, by a careful comparison of Nakajima and Tanka procedure, Cholesky decomposition method of Stamnes Tsay and Nakajima and the original procedure of Stamnes and Swanson, that Stamnes and Swanson prescription is still the most accurate because it avoids round-off errors due to matrix multiplications needed to symmetrize the matrix in the two other procedures.

Stamnes, Tsay, Wiscombe, and Jayaweera[199] summarized an advanced, thoroughly documented, and quite general purpose discrete ordinate algorithm for time-independent transfer calculations in vertically inhomogeneous, nonisothermal, plane-parallel media and made some progresses, in both formulation and numerical solution, in the algorithm.

Myneri, Asrar and Kanemasu[154] discussed a finite element discrete ordinates method for solving the radiative transfer equation in non-rotationally invariant scattering media and the application of the method to the leaf canopy problem.

Cefus and Larsen[36] describing the non-linear "quasi diffusion" method developed by Gol'din and the "second moment" method proposed by Lewis and Miller for obtaining iterative solutions of discrete-ordinate problems, showed that the methods reduce to almost the same linear method for a special class of problems and performed a Fourier stability analysis of the two methods for these special problems.

Yavuz and Larsen[232] proposed a spatial domain decomposition method for modifying the Source Iteration (SI) and Diffusion Synthetic

Acceleration (DSA) algorithms for solving discrete ordinates problems which consists of subdividing the spatial domain of the problem and performing the transport sweeps independently on each subdomain, has the advantage of being parallelizable because the calculations in each subdomain can be performed on separate processors.

Ben Jaffel and Vidal-Madjar[24] modified the discrete ordinate method developed by Wehrse[219] and Schmidt and Wehrse[179] for the resolution of the radiative transfer equation and showed that the construction of a quasi analytical solution to the corresponding matrix diagonalization problem reduces the time calculation and allows the use of more dense discrete frequency and angle grids.

Viik[211] solved a vector equation of the radiative transfer for conservative as well as non-conservative planetary atmospheres using the method of discrete ordinates.

Viik[212] solved another vector equation of the radiative transfer for non-conservative homogeneous plane parallel planetary atmosphere using the method of discrete ordinates.

Gouttebroze[78] extended the discrete ordinate method of Wick-Chandrasekhar to the case of radiative transfer equation of infinitely long cylinders, in Eddington approximation, by replacing the exponentials by modified Bessel's functions.

Wang[218] gave a systematic extensions of Chandrasekhar's work to three dimensions including discussions of specular and diffused parts, reciprocity, solutions and approximations.

Tsay and Stamnes[205] verified a reliable and efficient discrete ordinate method for multiple scattering, radiative transfer calculations in vertically inhomogeneous, non-isothermal atmospheres in local thermodynamic equilibrium.

Helliwell, Sullivan Macdonald and Voss[83] developed a finite difference discrete-ordinate iterative method to solve the three dimensional radiative transfer equation which is applicable to a volume of ocean with position dependent volumetric absorption and scattering coefficients. Input quantities include Sun position and sky radiation distribution, scattering phase function and absorbing, reflecting or emitting objects within the ocean volume. A solution of the one dimensional radiative transfer equation was used to provide boundary values for the 3D solutions.

Larsen[135] showed the distributional solutions of the transport equation to be a certain weak limit of regular solutions of the discrete ordinates equations as N, the order of the angular quadrature set, tends to infinity.

Viik[213] described a method based on the method of discrete ordinates by Chandrasekhar[45] to calculate the $X-$, $Y-$ and $H-$ matrices for molecular scattering in a homogeneous plane parallel atmosphere.

Ganguly, Allen and Victory, Jr.[76] suggested a new approach to discrete-ordinates neutron transport in plane geometry.

Jin and Levermore[100] studied the discrete ordinate method in these limits and found formulae for the resulting diffusion equation and its boundary conditions .

Karanjai and Deb[111] obtained the solution of a transfer equation for coherent scattering in a stellar atmosphere with Planck's function as a nonlinear function of optical depth (viz., $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$) by the method of discrete ordinates originally due to Chandrasekhar.[45]

Kobayashi[126] presented the discrete-ordinate solutions for a multidimensional radiative transfer equation for a collimated source and to demonstrate the effect of atmospheric heterogeneity on radiative flux.

Kylling[133] solved the transfer equation for normal waves in finite, inhomogeneous and plane-parallel megnetoactive media using discrete ordinate method developed by Chandrasekhar[45] as well as Stammes, Tsay, Wiscombe, Jayaweera[199]

Wehrse and Hof[220] studied the transfer of gamma rays by means of numerically stable method that yields all emergent intensities as well as the energy converted to heat which were determined by solving the equations with the discrete-ordinate-matrix-exponential method.

Weng[223] established a theory for discretizing the vector integral differential radiative equation in which phase matrix was derived from averaging the scattering matrix over poly disperse particles and then making a linear transformation of the averaged scattering matrix according to spherical trigonometry. The phase matrix and radiative vector in the vector radiative transfer equation were both expanded into Fourier-cosine and sine series. The complete set of solutions for the discrete matrix equations for cosine and sine modes of the radiative vectors was obtained by solving for the eigenvalues and eigenvectors and particular solutions. The integration co-efficients in the solutions were determined through the continuity conditions at vertically layered interface and the top and bottom boundaries.

Weng[222] applied a multi-layer discrete ordinates method for vector radiative transfer in vertically inhomogeneous, emitted and scattering atmosphere and compared the upwelling radiance from the vector radiative transfer model already established by himself with Chandrasekhar's analytic solutions for a conservative Rayleigh scattering atmosphere.

Shibata and Uchiyama[185] so incorporated the thermal infrared radiation with the discrete ordinate method that it becomes usable in climate models.

Yavuz[231] proposed a simplified discrete-ordinates (S_N) method completely free from all spatial truncation errors for the solution of one-group and isotropic source plane-geometry transport problems with an arbitrary anisotropic scattering of order $L\,(= N-1)$. The method is based on the expansion of the angular flux in spherical harmonic (P_{N-1}) solutions. The analytic expression for the angular flux for each discrete-ordinates direction depends on the exponential functions, arbitrary constants and interior source.

Barichello and Siewert[18] established the equivalence between the discrete ordinates method and the spherical harmonics method in the works concerning steady-state radiative transfer calculations in plane-parallel media. i.e. established that the choice for a quadrature scheme for the discrete ordinates method as the zeros of the associated Legendre polynomials and the use of generalized Mark boundary conditions in spherical harmonics method for standard radiative transfer problems without the imposed restriction of the azimuthal symmetry give the identical result for the radiation intensity.

Barichello and Siewert[20] used the discrete ordinate method to develop the solution to a class of non-gray problems in the theory of radiative transfer. The model considered allows for scattering with completely frequency redistribution (completely non-coherent scattering) and continuum absorption. Some numerical aspects and the use of this discrete ordinates solution were discussed. The classical X and Y functions were also computed by using the solution.

Barichello and Siewert[19] used the discrete-ordinates method to develop a solution to a class of polarization problems in the theory of radiative transfer.

Mitra and Churnside[149] estimate the optical signal for an oceanographic lidar from the one-dimensional transient (time dependent) radiative transfer equation using the discrete ordinates method.

Viik[215] presented accurate numerical solution for both the internal and external radiation field of a nonconservative plane-parallel semi-infinite Rayleigh-Cabannes scattering atmosphere using the method of discrete -ordinates of Chandrasekhar.

Sharp and Allen[184] solved the time-dependent transport equation for both rod and plane geometries using the discrete-ordinate method.

Barichello, Garcia and Siewert[16] developed a full-range orthogonality relation and used it to construct the infinite-medium Green's function for a general form of the discrete ordinates approximation to the transport equation in plane geometry. The Green's function is then used to define a particular solution that is required in the solution of inhomogeneous version of the discrete ordinates equations.

Siewert[187] used a discrete ordinate method along with elementary numerical Linear Algebra technique to establish an accurate solution for all components in a Fourier representation of the Stokes vector basic to the scattering of polarized light.

Siewert[188] used a discrete ordinate method along with elementary numerical Linear Algebra technique to establish an efficient and accurate solution to a class of multigroup transport problems for which up scattering is an important aspect of the model. The problems considered are defined for finite-plane parallel media, and anisotropic scattering from any group to any group is included in the formulation.

Galinsky[74] modified the forward discrete ordinate method, based on an expansion of the direct beam source term, similar to the gradient correction method used already by Galinsky[75] for diffusion approximation to include effects of a weak inhomogeneity of a medium.

Elaloufi, Carminati and Greffet[73] solved the time-dependent radiative transfer equation in the space-frequency domain by using a standard discrete-ordinate method to study the propagation of light pulses through scattering media.

Aboughantous[4] revisited the structure of the discrete ordinates set with a new approach and built a new set based on Gauss-Legendre (GL) quadrature. The new set comprises only positive direction cosines for all specific intensities. The new set of discrete ordinates enabled transcribing the transfer equation into a complete set of equations i.e. into the set of equations which is closed (N equations in N unknowns) and conservative (the solution satisfies the conservation relation).

A non-physical feature of the transfer equation in spherical geometry is that it is singular at the center of the sphere. This feature is intrinsic to the transfer equation in its native form as an abstract mathematical equation in spherical geometry. He cured this problem by an appropriate transformation of the frame of reference.

The solutions for the discrete ordinates equations and the diffusion equations are presented in two forms: continuous in r and end-points form, and tested quantitatively. The end-points solution is particularly attractive in numerical computations in optically thick media.

Spurr, Kurosu and Chance[195] carried out an internal perturbation analysis of the complete discrete ordinate solution in a plane-parallel multi-layered multiply-scattering atmosphere.

The discrete ordinates method fails in treating specular reflection at the boundary because the quadratures on the sphere do not assume any analytic representation of a function under integration and, therefore, the intensity of the specularly reflected beam is undetermined. Rukolaine and Yuferev[172] presented a new approach to the construction of quadrature schemes to solve this problem.

Spurr[194] solved the radiative transfer equation in a multi-layer multiply-scattering atmosphere using discrete ordinate method and evaluated explicitly all the partial derivatives of the intensity field.

Lemonnier and Dez[137] derived the radiative transfer equation (RTE) in both conservative and non-conservative forms for a plane slab made of an absorbing-emitting material with a continuous transverse variation of the refractive index. The RTE was set in a form which displays an angular redistribution term analogous to what appears in curvilinear media with uniform index. Numerical solutions were provided by means of discrete ordinates method.

Barichello, Rodrigues and Siewert[17] used a discrete ordinate method along with Hermite cubic splines and Newton's method to solve a class of coupled nonlinear radiation-conduction heat transfer problems in a solid cylinder.

Ray effects and false scattering are two major sources of inaccuracy of the discrete ordinates method. High order schemes may reduce false scattering, and the modified discrete ordinates method may mitigate ray effects. Although the origin of the two errors is different, there is an interaction between them, since they tend to compensate each other. Coelho[48] showed that decreasing of one of the errors while keeping the other unchanged in the standard discrete ordinates method may decrease the solution accuracy because the compensation effect disappears and the modified discrete ordinates method does not decrease ray effects caused by sharp gradients of the temperature of the medium. He proposed a new version that successfully mitigates ray effects in that case.

Coelho[49] applied the discrete ordinates and discrete transfer methods to the numerical simulation of radiative heat transfer from non-gray gases in three-dimensional enclosures.

Qin, Jupp and Box[166] extended an accurate and efficient algorithm, the discrete ordinate method, to solve the radiative transfer problem of plane parallel scattering atmosphere illuminated by a parallel beam, an idealized case of the sun, from above the atmosphere so that radiative problem of more general sources such as parallel surface source that illuminated with a parallel beam in any direction and any vertical position, and general surface sources that illuminate continuously in a hemisphere, can be solved.

Lacroix, Parent, Asllanaj, and Jeandel[134] solved the radiative heat transfer equation (RTE) using a S_8 quadrature and a discrete ordinate method.

Collin, Boulet, Lacroix and Jeandel[52] used 2-D discrete ordinate method, formulated by Lacroix, Parent, Asllanaj, and Jeandel,[134] to solve the radiative transfer equation to stimulate the radiation propagation from the heat source through water spray curtains.

van Oss and Spurr[209] derived the homogeneous and particular solutions for the general discrete-ordinate model, noting especially the factor of 2 reduction that allows analytic solutions to be written down for the 4=6 stream cases. The equation of radiative transfer is solved for a vertically inhomogeneous atmosphere by assuming a division into a number of optically uniform adjacent sub-layers.

Silva, Andraud, Charron, Stout and Lafait[56] presented the model based on the resolution of the radiative transfer equation by the discrete ordinate method in steady state domain.

Chalhoub[37] used the discrete-ordinates method to solve radiative transfer problems, in plane-parallel media and presented a generalized analytical discrete-ordinates model for solving single and multi-region problems in which internal sources, reflecting and emitting boundaries, incident distribution of radiation on each surface and a beam incident on one surface are included.

Coelho[50] proposed a new modified discrete ordinates method (NMDOM) to overcome the shortcomings of the standard discrete ordinate method (SDOM) and modified discrete ordinate method (MDOM). The standard discrete ordinates method suffers from two major sources of inaccuracy, the ray effects and false scattering. False scattering were significantly reduced using high order discretization schemes, while ray effects originated from abrupt changes of wall temperatures were mitigated by modified discrete ordinates method (MDOM).

Qin, Box and Jupp[167] presented two methods that can be used to derive the particular solution of the discrete-ordinate method for an arbitrary source in a plane-parallel atmosphere, which allows us to solve the transfer equation 1218 % faster in the case of a single beam source and is even faster for the atmosphere thermal emission source.

An equation of radiative transfer is more accurate than a diffusion equation for the widely employed frequency-domain case. Ren, Abdoulaev, Bal, and Hielscher[168] presented an algorithm by discretizing the equation of radiative transfer by a combination of discrete-ordinate and finite-volume methods that provides a frequency-domain solution of the equation of radiative transfer for heterogeneous media of arbitrary shape to present two numerical simulations.

Biological tissue is a turbid medium that both scatters and absorbs photons. An accurate model for the propagation of photons through tissue can be adopted from transport theory, and its diffusion approximation can be applied to predict the imaging signal around the biological tissue (vide Cong, Wang and Wang[54]).The use of short pulse laser for minimally invasive detection scheme has become an indispensable tool in the technology arsenal of modern medicine and biomedical engineering. Trevedi, Basu and Mitra[204] used a time-resolved technique to detect tumors/ inhomogeneities in tissue by measuring transmitted and reflected scattered temporal optical signals when a short pulse laser source is incident on tissue phantoms and validated the experimental measurements obtained by a parametric study involving different scattering and absorption coefficients of tissue phantoms and inhomogeneities, size of inhomogeneity as well as the detector position with a numerical solution of the transient radiative transport equation obtained by using discrete ordinates method.

Considering the processes of the solar radiation extinction in deep layers of the Venus atmosphere in a wavelength range from 0.44 to 0.66 μm and using the spectra of the solar radiation scattered in the atmosphere of Venus at various altitudes above the planetary surface measured by the Venera-11 entry probe in December 1978 as observational data, Maiorov, Ignat'ev, Moroz, Zasova, Moshkin, Khatuntsev and Ekonomov[144] solved the problem of the data analysis by selecting an atmospheric model applying the discrete-ordinate method in calculations.

Elaloufi and Arridge[72] used the discrete ordinate method to solve the radiative transfer equation (RTE) for slab geometry, taking into account rigorously the interfaces. The important role of interfaces in the ballistic regime and also the diffuse regime were underlined.

Radiative transfer theory considers radiation in turbid media and is used in a wide range of applications. Edström[70] outlined a problem formulation and a solution method for the radiative transfer problem in multilayer scattering and absorbing media using discrete ordinate model geometry.

Pimenta de Abreu[160] derived nonstandard layer-edge conditions for efficient solution of multislab atmospheric radiative transfer problems. Defining a local radiative transfer problem on the lowermost layer of a multislab model atmosphere , he considered a standard discrete ordinates version of this local problem.

Mishra Roy and Misra[148] suggested a new quadrature scheme to make discrete ordinate method computationally more attractive by making the complicated mathematics for determination of direction cosines and weights simple and lucid.

Zorzano, Mancho and Vázquez[234] considered the radiation transfer problem in the discrete-ordinate, plane-parallel approach and introduced two benchmark problems with exact known solutions and show that for strongly non-homogeneous media the homogeneous layers approximation can lead to errors of 10% in the estimation of intensity.

Pozzo, Brandi, Giombi, Baltanás and Cassano[165] determined volumetric optical properties (spectral absorption, scattering and extinction coefficients) of differently expanded narrow-path fluidized beds (FB) of photocatalyst obtained by plasma-CVD deposition of titania onto quartz sand, relavent for photoreactor design purposes by using an unidirectional and unidimensional model for solution of the radiative transfer equation(RTE). They used two simplified approaches: a Kubelka-Munk type of solution by which the RTE was transformed into a pair of ordinary differential equations and a discrete ordinate method by which the complete RTE was transformed into an algebraic system .

Hua, Flamant, Lu and Gauthier[93] developed a model to predict the bed-to-wall radiative heat transfer coefficient in the upper dilute zone of circulating fluidized bed (CFB) combustors and solved the radiative transfer equation by the discrete ordinate method.

Li[141] developed an easy-to-use and comprehensive method, named multi-rays method, on the basis of discrete ordinates scheme with (an) infinitely small weight(s) to calculate total, direct and medium intensities in arbitrary specified directions. In doing this, for each of the specified directions, three identical discrete directions with infinitely small weights are employed to represent the three intensities.

Chalhoub[38] used discrete ordinates method to solve uncoupled multi-wavelength radiative transfer problems in multi-region plane-parallel media. They presented a generalized analytical discrete-ordinates formulation that includes internal sources, as well as reflecting and emitting boundaries, incident distribution of radiation on each surface and a beam incident on one surface, as boundary conditions.

Trabelsi, Sghaier and Sifaoui[203] used a modified discrete ordinates method in a spherical participating media. By breaking up the radiative intensity into two components of which one component was traced back to the enclosure's source, called direct intensity and the other component was rather traced back to the contribution of the medium itself, called diffused intensity, they solved the diffuse RTE numerically using discrete ordinates method.

Klose, Ntziachristos and Hielscher[125] applied the ERT (equation of radiative transfer)-based forward model for light propagation in biological tissue using a finite-difference discrete-ordinates method.

An, Ruan, Qi and Liu[9] proposed a finite element method for simulation of radiative heat transfer in absorbing, emitting and anisotropic scattering. They developed the simulation on the basis of discrete ordinates method and the theories of finite element.

Box and Qin[29] presented an extension to the standard discrete-ordinate method (DOM) to consider generalized sources including: beam sources which can be placed at any (vertical) position and illuminate in any direction, thermal emission from the atmosphere and angularly distributed sources which illuminate from a surface as continuous functions of zenith and azimuth angles.

Banerjee, Ogale and Mitra[15] experimentally determined the information content of lightning optical emissions through clouds in the laboratory and they compared the experimental results with a transient radiative transfer formulation solved using the discrete ordinate method.

To perform a comprehensive experimental and numerical analysis of the shortpulse laser interaction with a tissue medium with the goal of tumor-cancer diagnostics, Pal, Basu, Mitra, and Vo-Dinh[158] formulated a numerical model using the discrete ordinates technique for solving the radiative transport equation associated with the problem.

Rozanov and Kokhanovsky[171] converted Siewert's[187] form of vector radiative transfer equation for a 'homogeneous isotropic symmetric plane-parallel light scattering slab to a nice form in which the discrete ordinate technique was used comfortably to solve.

Spurr, Haan, van Oss and Vasilko[193] demonstrated that the discrete-ordinate radiative transfer(RT) equations may be solved analytically in a multi-layer multiple scattering atmosphere in the presence of rotational Raman scattering (RRS) treated as a first-order perturbation

Coelho[51] presented a comparison of discretization schemes required to evaluate the radiation intensity at the cell faces of a control volume in differential solution methods of the radiative transfer equation and compared several schemes developed using the normalized variable diagram and the total variation diminishing formalisms along with essentially non-oscillatory schemes and genuinely multidimensional schemes. The calculations were carried out using the discrete ordinates method, but the analysis is found to be equally valid for the finite-volume method.

Kokhanovsky[127] carried out the numerical calculations of the halo brightness and contrast using the discrete ordinate method of the integro-differential radiative transfer equation solution for a typical phase function of crystalline clouds exhibiting halo at 22 and $46A^0$.

Abhiram, Deiveegan, Balaji and Venkateshan[3] presented a multilayer differential discrete ordinate method to solve the radiative transfer equation for an absorbing, emitting and scattering inhomogeneous plane parallel medium.

1.3.2 Works done on interlocking problems.

From the observational point of view, Houtgast[92] first noticed the importance of non coherent scattering for the interpretation of strong absorption lines. He showed that the behaviour of strong fraunhorfer lines across the Sun's disc can only be interpreted under the assumption of non coherent scattering. Spitzer[192] discussed the general characteristics of non coherency and concluded from physical arguments that non coherent scattering is important in stellar atmosphere and coherent scattering is comparatively rare there in. Theoretical treatments of the problem for the case of interlocking have been given by Spitzer[191] and Woolley.[228]

Eddington[68] derived the general method of calculating the contour of an absorption line when the number of atoms in the upper state has been disturbed by interlocking.

Woolley[226] discussed a case of two interlocked absorption lines. A direct solution was made of the simultaneous differential equations, obtained by making simplifying approximations as nearly as possible similar to those ordinarily made in the treatment of principle lines, and it was found that the widths of the lines were not appreciably affected by the interlocking.

Woolley[227] considered the case of triplet (or doublet) of lines and made the conclusion that the measurement of line width at the points for which $\frac{H'_\nu}{H} > \frac{7}{10}$, where $H = \frac{1}{4\pi}\int J(\theta)cos\theta d\omega$ and $H'_\nu = \frac{1}{4\pi}\int J(\theta)'cos\theta d\omega$ in which $J(\theta)'d\nu$ is the flow of radiation of frequency ν to $\nu + d\nu$ within the line in a direction making angle θ with the outward direction, $J(\theta)d\nu$ is the corresponding flow just outside the line and $d\omega$ is the element of solid angle, can be interpreted exactly as if the lines were not interlocked with each other.

Woolley and Stibbs[229] considered the problem of interlocking without redistribution in details and gave the integro-differential equations for triplets along with an approximate solution obtained by applying Eddington's approximation. To illustrate the effect of interlocking,they calculate the quantity

$$\frac{1}{2}\omega = \int_{\eta=\infty}^{0} (1-r)\, d\eta^{-\frac{1}{2}}$$

for doublet and triplet lines in a region of the spectrum where $b = \frac{3}{2}a$, and with $\epsilon = 0$ and drew conclusion that interlocking has an effect on the curve of growth(the relation between the equivalent width and the number of oscillators) which should be appreciable but not markedly so.

Considering the linear form of Planck-function, Busbridge and Stibbs[33] solved the radiative transfer equation for interlocked multiplets by the method of principal of invariance governing the law of diffuse reflection with a slight modification and calculated three hypothetical line profiles for doublets.

Busbridge[30] obtained the solution for interlocked multiplet lines by a mathematical method which was obtained by Busbridge and Stibbs[33] by the principle of invariance.

Miyamoto[150] investigated abnormally high residual intensities and very large Doppler core widths of Infrared Ca II multiplets in solar spectrum. The characteristic features of this multiplets are the metastability of the lower level and the strong interlocking with the resonance H and K lines through the upper level. By virtue of the metastability of the lower level, the nature of the line formation was found closer to absorption rather than scattering. This being combined with the strong interlocking with resonance line, explains an abnormally high residual intensities.

Siewert and Özişik[190] developed a matrix form of the equations of transfer of the lines for the interlocking multiplets of order N from the equation of transfer of Busbridge and Stibbs[33] for the interlocking multiplets of order k by making a suitable substitution and produced a rigorous solution to the equation of transfer for interlocking doublets by the use of the normal modes and the methods of solution introduced by Siewert & Zweifel.[189]

Karanjai[103] profitably used the approximate form of H-function[102] to minimize to a great extent the computational labour that involves in the calculation of H-function

An exact solution of the equation of transfer was given by Das Gupta[59] by his modified form of Wiener-Hopf technique.

Deb[65] used the following various approximate forms of the H-function, studied by Karanjai[102] and Karanjai and Sen[116] to calculate the value of H-function and residual intensities for doublets as well as triplet lines.

Dasgupta and Karanjai[63] solved the radiative transfer equation for interlocked multiplets without redistribution with the Planck-function, linear in τ, by applying Sobolev's probabilistic method.

Chamberlain and Wallace[39] also studied the case of a multiplet with common lower state with the assumption of monochromatic scattering in each line.

Nagirner and Shneivais[155] analyzed the formation of lines with a common upper level in a semi-infinite medium and used an analytical method developed for two-level atom to study the problem of radiation transfer with the assumptions of a Boltzmann distribution of atoms over sublevels of the lower level and of complete frequency distribution of a radiation within each line. They expressed the intensity of the radiation in the lines through the H-function, obtained the asymptotic and approximate equation for the H-function for the Doppler and Voigt coefficients of absorption and calculated the Doppler H-function estimating the accuracy of the asymptotic forms for the case of the 0.1 resonance triplet.

Das Gupta[61] also obtained an exact solution of the transfer equation with the Planck-function, linear in τ, for non-coherent scattering arising from interlocking principal lines by Laplace transform and the Wiener-Hopf technique using a new representation of the H-function obtained by Das Gupta.[60]

Karanjai and Barman[107] solved same problem by using the extension of the method of discrete-ordinates.

Karanjai[105] calculated Mg b line contours with the solution obtained by Dasgupta and Karanjai[63] and showed that his calculated lines have a good agreement with the observation.

Karanjai and Karanjai[114] solved the equation of transfer for interlocked multiplets with the Planck function as a non-linear function of optical depth following the method used by Das Gupta.[61] They considered two non-linear forms of Planck function viz.,

(a). an exponential atmosphere,(vide Degl'Innocenti,[66] equation(1.11)),

(b). an atmosphere (vide Busbridge[30]) in which

$B_\nu(\tau) = B(\tau) = b_0 + b_1\tau + b_2 E_2(\tau)$

Deb, Biswas and Karanjai[64] solved the radiative transfer equation for interlocked multiplets with non-linear Planck-function by using the extension of the method of discrete-ordinates and Deb and Karanjai[110] solved the same problem with the help of the the method of Busbridge and Stibbs.[33]

Mukherjee and Karanjai[152] used the double-ordinate spherical harmonic method presented by Wilson and Sen[225] to solve the equation of radiative transfer in the Milne-Eddington model for interlocked doublets. Solutions have been obtained in the first and second approximation in a particular case $r = 1$.

Chapter 2

Interlocked Multiplet in Anisotropically Scattering Medium with Planetary Phase Function

2.1 Introduction

Planetary Phase Function:

Planetary phase function is given by

$$P(\cos\Theta) = \varpi_0 (1 + \varpi \cos\Theta), \quad -1 \leq \varpi \leq 1$$

which is equivalent to

$$p(\mu, \varphi, \mu' \varphi') = \varpi_0 \left[1 + \varpi\mu\mu' + \varpi\left(1-\mu^2\right)^{\frac{1}{2}}\left(1-\mu'^2\right)^{\frac{1}{2}}\cos(\varphi - \varphi')\right]$$

Its azimuth independent form is

$$p(\mu, \mu') = 1 + \varpi\mu\mu'$$

Chandrasekhar[44,45] obtained the emergent intensity in semi infinite atmosphere with no incident radiation for scattering in accordance with the phase function P $(\cos\Theta) = \varpi_0\,(1+\varpi\cos\Theta)\,, -1 \leq \varpi \leq 1$

Horak[87] used the planetary phase function of the form $\varpi_0\,(1 + x cos\Theta)$ with two sets of different values of ϖ_0 and x, for yellow light $\varpi_0 = 0.985$ and $x = 0.9$ and blue light $\varpi_0 = 0.925$ and $x = 0.65$.

The calculations involved in any given application are very laborious. Horak and Little[90] wrote programmes in FORTRAN for calculations of the diffuse reflection by a semi-infinte atmospheres with different phase functions. One of them is the planetary phase function which he termed as Euler's phase function, given by $p\,(cos\theta) = \varpi_0 + \varpi_1 P_1\,(\cos\theta)\,;\ (\,\varpi_0, \varpi_1 =\ [0.95,\ 0.95],\ [0.95, -0.95],\ [0.95, 0.475]\ [0.95, -0.475])$.

Considering the transfer of radiation by a plane-parallel atmosphere containing a uniform distribution of emission source for the case scattering according to the asymmetric phase function of the form $\varpi_0\,(1 + x cos\Theta)$, Horak[88] derived the exact expression for the emergent intensity for both semi-infinite and finite atmosphere by applying method of Chandrasekhar's principle of invariance.

Mullikin[153] gave the formulae for the reflection and transmission of light incident by a plane-parallel atmosphere of arbitrary thickness with a phase function $1 + x cos\Theta$ for the conservative azimuth-independent case.

Busbridge and Orchard[31] derived simple asymptotic formula for the azimuth independent case of thick atmosphere scattering with a conservative phase function $1 + \varpi\cos\Theta$

Busbridge and Orchard[32] dealt with the reflection and transmission of light incident from an arbitrary direction on a non-absorbing plane-parallel atmosphere of large optical thickness bounded by the planes $x = 0$, $x = \tau$ with a conservative phase function $1 + \varpi \cos\Theta$, considering azimuth-independent terms only for deriving simple asymptotic formulae.

Piotrowski[161,162] (vide Busbridge and Orchard[32]) used the phase function $1 + \varpi \cos\Theta$ to find the asymptotic solution for it.

To apply a self introduced new method, combined with Laplace transform and the Wiener-Hopf technique, in a simpler problem of diffuse reflection by a plane-parallel atmosphere with axial symmetry scattering radiation with moderate anisotropy, Das Gupta[62] used planetary phase function of the form $p^0(\mu, \mu') = \varpi_0 + \varpi_1\mu\mu'$, where $0 < \varpi_0 \leq 1$ and $|\varpi_1| < 3\varpi_0$.

In order to design cosmetics producing the optical properties that are required for a beautiful skin, Yamada, Kawamura, Miura, Takata and Ogawa[230] investigated numerically the radiative transfer within the human skin by using a radiative transfer equation with scattering phase function p as $p(\Omega' \to \Omega) = 1 + a_0 cos\theta_0$.

Considering the transport equation for radiative transfer to a problem in semi- infinite atmosphere with no incident radiation in which scattering takes place according to planetary phase function, Karanjai and Deb[112] determined emergent intensity and the intensity at any optical depth by using Laplace transform and the Wiener-Hopf technique.

Busbridge and Stibbs[33] solved their problem on the basis of some assumptions. All those assumptions are kept unchanged here also. This chapter is devoted fully on derivation of the solution of the radiative transfer equation of interlocked multiplets in anisotropically scattering media with planetary phase function by Chandrasekhar's method of discrete-ordinates. In section-2.3, a linear form of

Planck-function is used and in the section-2.4, one of its exponential form $B_\nu(T) = B(\tau) = b_0 + b_1 e^{-\beta\tau}$, given by Degl'Innocenti,[67] is used.

2.2 The Equation of Transfer and the Boundary Conditions

2.2.1 The Equation of Transfer

The equation of transfer for the rth interlocked line is

$$\mu \frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)B_\nu(T)$$
$$-\frac{1}{2}(1-\varepsilon)\alpha_r \sum_{s=1}^{m} \eta_s \int_{-1}^{1} p\,(\mu,\acute{\mu}\,)I_s(\tau,\acute{\mu}\,)d\acute{\mu} \tag{2.1}$$

where α_r 's $(r = 1, \ldots, m)$ are of the form:

$$\alpha_r = \eta_r / \sum_{s=1}^{m} \eta_s \tag{2.2}$$

so that

$$\sum_{r=1}^{m} \alpha_r = 1 \tag{2.3}$$

the Planck-function, $B_\nu(T)$, considered in this case, is of the form:

$$B_\nu(T) = B(\tau) = b_0 + b_1\tau \qquad \text{[Linear form]} \tag{2.4a}$$
$$B_\nu(T) = B(\tau) = b_0 + b_1 e^{-\beta\tau} \qquad \text{[Exponential form]} \tag{2.4b}$$

b_0 and b_1, being positive constants and the (azimuth independent) planetary phase function $p(\mu, \mu')$, taken here, is given by

$$p\,(\mu, \mu') = 1 + \varpi P_1\,(\mu)\,P_1\,(\mu') = 1 + \varpi\mu\mu' \tag{2.5}$$

and ε, the co-efficient, is introduced to allow for thermal emission associated with the line absorption.

Using the relation (2.5) in the equation(2.1), we get

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)B_\nu\,(T)$$
$$-\frac{1}{2}(1-\varepsilon)\alpha_r\sum_{s=1}^{m}\eta_s\int_{-1}^{1}(1+\varpi\mu\mu')I_s(\tau,\mu')d\mu' \tag{2.6}$$

2.2.2 The Boundary Conditions

The boundary conditions for solving the equation (2.6) are

$$I_r(0,-\mu) = 0;\ (0 < \mu \leq 1) \tag{2.7}$$

$$I_r(\tau,\mu)\cdot e^{-\tau/\mu} \rightarrow 0$$

i.e. $I_r(\tau,\mu)$ is at most linear in τ as τ tends to infinity (2.8)

2.3 The Equation of Radiative Transfer with Linear Form of Planck Function

Using the relation (2.4a) in the equation (2.6), we get

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)\,(b_0+b_1\tau)$$
$$-\frac{1}{2}(1-\varepsilon)\alpha_r\sum_{s=1}^{m}\eta_s\int_{-1}^{1}(1+\varpi\mu\mu')I_s(\tau,\mu')d\mu' \tag{2.9}$$

2.3.1 Reduction of the Equation (2.9) to a Simpler Form

We assume, following Busbridge and Stibbs,[33] that the solution of the equation (2.9) is

$$I_r(\tau, \mu) = b_0 + b_1\left(\tau + \frac{\mu}{1+\eta_r}\right) + I_r^*(\tau, \mu) \tag{2.10}$$

which consists of two parts, the first part being the solution for an infinite unbounded atmosphere as τ tends to infinity and the second part $I_r^*(\tau, \mu)$ being the departure of the asymptotic solution from the value $I_r(\tau, \mu)$ as we approach the boundary $\tau = 0$

So, the equation (2.6), by virtue of the equations (2.4a) and (2.5), reduces to the form :

$$\begin{aligned}\frac{\mu}{1+\eta_r}\cdot\frac{dI_r^*(\tau, \mu)}{d\tau} &= I_r^*(\tau, \mu) - \frac{1}{2}\cdot\frac{(1-\varepsilon)\eta_r}{1+\eta_r}\cdot\frac{\alpha_r}{\eta_r}\sum_{s=1}^{m}\eta_s\int_{-1}^{1}(1\\ &+\varpi\mu\mu')I_s^*(\tau, \mu')\,d\mu' - \frac{1}{3}\cdot\frac{(1-\varepsilon)\eta_r}{1+\eta_r}\cdot\frac{\alpha_r}{\eta_r}\varpi b_1\mu\left(\sum_{s=1}^{m}\frac{\eta_s}{1+\eta_s}\right)\end{aligned} \tag{2.11}$$

Now, writing

$$\zeta_r = \frac{1}{1+\eta_r} \tag{2.12}$$

and

$$\omega_r = \frac{(1-\varepsilon)\,\eta_r}{1+\eta_r} \tag{2.13}$$

the equation (2.11) can be reduced to the form:

$$\zeta_r\mu \cdot \frac{dI_r^*(\tau,\mu)}{d\tau} = I_r^*(\tau,\mu) - \frac{1}{2}\cdot\omega_r\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\left\{\sum_{s=1}^{m}\eta_s\int_{-1}^{1}(1 + \varpi\mu\mu')\, I_s^*(\tau,\mu')d\mu'\right\} - \frac{1}{3}\cdot\omega_r\cdot\varpi_1\mu\left\{\frac{\sum\limits_{s=1}^{m}\left(\frac{\eta_s}{1+\eta_s}\right)}{\sum\limits_{s=1}^{m}\eta_s}\right\} \tag{2.14}$$

and the boundary conditions (2.7) and (2.8) can be reduced to the form:

$$I_r^*(0,-\mu) = b_1\mu\zeta_r - b_0, \quad \text{where } 0 < \mu \leq 1 \tag{2.15}$$

and

$$I_r^*(\tau,\mu) \text{ is almost linear in } \tau \text{ as } \tau \text{ tends to infinity.} \tag{2.16}$$

2.3.2 Solution of the Equation with Linear Form of Planck Function

2.3.2.1 Solution of the Equation of Radiative Transfer in the n^{th} Approximation

In the n^{th} approximation, we replace the integro-differential equation (2.14) by the system of 2n differential equations:

$$\zeta_r\mu_{(r)i}\cdot\frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{1}{2}\cdot\omega_r\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\left\{\sum_{s=1}^{m}\eta_s\sum_j(1+\varpi\mu_{(r)i}\mu_{(s)j})I^*_{(s)j}a_j\right\} - \frac{1}{3}\cdot\omega_r\cdot\varpi_1\mu_{(r)i}\left\{\frac{\sum\limits_{s=1}^{m}\left(\frac{\eta_s}{1+\eta_s}\right)}{\sum\limits_{s=1}^{m}\eta_s}\right\} \tag{2.17}$$

where $\mu_{(r)i}$'s ($i = \pm 1, \pm 2, \cdots, \pm n$; having the property that $\mu_{(r)-i} = -\mu_{(r)i}$) are the zeros of the Legendre polynomial $P_{2n}(\mu)$ which are independent on the lines of interlocking and a_j's ($j = \pm 1, \pm 2, \cdots, \pm n$; having the property that $a_{-j} = a_j$) are the corresponding Gaussian weights. However it is to be noted that there is no term with $j = 0$. and the boundary conditions (2.15) and (2.16) will take the new forms as

$$I^*_{(r)-i} = b_1 \mu_{(r)i} \zeta_r - b_0, \quad \text{where } 0 < \mu_{(r)i} \leq 1 \tag{2.18}$$

and

$$I^*_{(r)i} \text{ is almost linear in } \tau \text{ as } \tau \text{ tends to infinity.} \tag{2.19}$$

For the simplicity we have used here

$$I^*_{(r)i} \text{ for } I^*_r(\tau, \mu_{(r)i}) \tag{2.20}$$

Solution of the homogeneous part of the equation(2.17)

Now, we shall try to get the solution of the equation:

$$\zeta_r \mu_{(r)i} \cdot \frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{1}{2} \cdot \omega_r \cdot \frac{1}{\sum\limits_{s=1}^{m} \eta_s} \left\{ \sum_{s=1}^{m} \eta_s \sum_j \Big(1 + \varpi \mu_{(r)i} \mu_{(s)j} \Big) I^*_{(s)j} a_j \right\} \tag{2.21}$$

The system of the equation (2.21) admits integrals of the form:

$$I^*_{(r)i} = g_{(r)i} \, e^{-k\tau} \, ; \; i = \pm 1, \cdots, \pm n \tag{2.22}$$

So, the equation (2.21) gives

$$- k\zeta_r \mu_{(r)i} g_{(r)i}\, e^{-k\tau} = g_{(r)i}\, e^{-k\tau} - \frac{1}{2}\cdot\omega_r\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\left\{\sum_{s=1}^{m}\eta_s\sum_j\left(1 + \varpi\mu_{(r)i}\mu_{(s)j}\right) g_{(s)i}\, e^{-k\tau} a_j\right\}$$

i.e.

$$g_{(r)i}\left\{1 + k\zeta_r\mu_{(r)i}\right\} = \frac{1}{2}\cdot\omega_r\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\left\{\sum_{s=1}^{m}\eta_s\sum_j\left(1 + \varpi\mu_{(r)i}\mu_{(s)j}\right) g_{(s)i}\, e^{-k\tau} a_j\right\} \tag{2.23}$$

Hence,

$$g_{(r)i} = \omega_r\frac{\rho_1\mu_{(r)i} + \rho}{1 + k\zeta_r\mu_{(r)i}} \tag{2.24}$$

where ρ and ρ_1 are constants which are independent of $\mu_{(r)i}$.

If we use the equation (2.24) once again in the equation (2.23), then we get

$$\rho_1\mu_{(r)i} + \rho = \frac{1}{2}\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\left\{\left(\rho_1\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu_{(s)j}}{1 + k\zeta_s\mu_{(s)j}} + \rho\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j}{1 + k\zeta_s\mu_{(s)j}}\right) + \varpi\mu_{(r)i}\left(\rho_1\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu_{(s)j}^2}{1 + k\zeta_s\mu_{(s)j}} + \rho\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu_{(s)j}}{1 + k\zeta_s\mu_{(s)j}}\right)\right\} \tag{2.25}$$

Defining

$$D_\ell(x) = \sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu_{(s)j}^{\ell}}{1 + \mu_{(s)j}\, x} \tag{2.26}$$

and analysing the proof done in Section.3 by Busbridge and Stibbs,[33] we can write the above equation in the following compact form:

$$\rho_1\mu_{(r)i} + \rho = \frac{1}{2C}\Big[\{\rho_1 D_1(k\zeta_s) + \rho D_0(k\zeta_s)\} \\ +\varpi\mu_{(r)i}\{\rho_1 D_2(k\zeta_s) + \rho D_1(k\zeta_s)\}\Big] \qquad (2.27)$$

where

$$C = \sum_{s=1}^{m} \eta_s \qquad (2.28)$$

For the ease of calculation, denoting

$$D_\ell(k\zeta_s) \text{ by } D_\ell \qquad (2.29)$$

we get, from the equation (2.27), the following two equations:

$$(2C - \varpi D_2)\rho_1 - \varpi\rho D_1 = 0 \qquad (2.30a)$$

$$\rho_1 D_1 + (D_0 - 2C)\rho = 0 \qquad (2.30b)$$

which, on elimination of ρ and ρ_1, give

$$2CD_0 - \varpi D_0 D_2 - 4C^2 + 2C\varpi D_2 + \varpi D_1^2 = 0 \qquad (2.31)$$

By virtue of the relation (II.20d) of Appendix II, the equation (2.31) takes the form:

$$2CD_0 - \varpi D_0 D_2 - 4C^2 + 2C\varpi D_2 - \varpi\psi_0 D_2 + \varpi D_0 D_2 = 0$$

i.e.

$$2CD_0 + \varpi(2C - \psi_0)D_2 = 4C^2 \qquad (2.32)$$

which is the **characteristic equation** and is equivalent to

$$D_0 = \frac{4C^2k^2\zeta_s^2 + \varpi(2C - \psi_0)\psi_0}{2Ck^2\zeta_s^2 + \varpi(2C - \psi_0)} \qquad (2.33)$$

Therefore,

$$\psi_0 - D_0 = \frac{2Ck^2\zeta_s^2\left(\psi_0 - 2C\right)}{2Ck^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)} \tag{2.34}$$

Therefore,

$$D_1 = \frac{1}{k\zeta_s}\left(\psi_0 - D_0\right) = \frac{2Ck\zeta_s\left(\psi_0 - 2C\right)}{2Ck^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)} \tag{2.35}$$

Again,

$$2C - D_0 = 2C - \frac{4C^2k^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)\psi_0}{2Ck^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)}$$

i.e.

$$2C - D_0 = \frac{\varpi\left(2C - \psi_0\right)^2}{2Ck^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)} \tag{2.36}$$

Now, from the equation(2.30b), using the equations (2.35) and (2.36),we get

$$\left(\frac{\varpi\left(2C - \psi_0\right)^2}{2Ck^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)}\right)\rho = \rho_1\left(\frac{2Ck\zeta_s\left(\psi_0 - 2C\right)}{2Ck^2\zeta_s^2 + \varpi\left(2C - \psi_0\right)}\right)$$

i.e.

$$\rho_1 = \pm\frac{\varpi\left(\psi_0 - 2C\right)}{2Ck\zeta_s}\rho$$

i.e.

$$\rho_1 = +\frac{\varpi\left(\psi_0 - 2C\right)}{2Ck\zeta_s}\rho, \text{ taking the positive sign.}$$

Therefore, the equation(2.24) becomes

$$g_{(r)i} = \omega_r\rho\,\frac{2Ck\zeta_s + \varpi\left(\psi_0 - 2C\right)\ \mu_{(r)i}}{2C\left(1 + k\zeta_r\mu_{(r)i}\right)k\zeta_s} \tag{2.37}$$

Now we observe that the characteristic equation (2.32) is expressible as

$$\frac{1}{2C}D_0 + \varpi \cdot \frac{1}{2C}\left(\frac{2C - \psi_0}{2C}\right) D_2 = 1$$

i.e.

$$\sum_{s=1}^{m} \eta_s \omega_s \sum_{j} \frac{a_j}{1 + k\zeta_s \mu_{(s)j}} \left[\frac{1}{2C}\left\{1 + \varpi\left(\frac{2C - \psi_0}{2C}\right)\mu^2_{(s)j}\right\}\right] = 1 \tag{2.38}$$

This is the **characteristic equation** which is an equation in k of order 2n and it will give 2n distinct non-zero roots which occur in pair as $\pm k_\iota$, $(\iota = 1, 2, \cdots, n)$, if $\omega_r < 1$

Therefore, the equation(2.21) admits 2n independent integrals of the form:

$$I^*_{(r)i} = \omega_r \rho \frac{\pm\, 2Ck_\iota\zeta_s + \varpi\left(\psi_0 - 2C\right)\, \mu_{(r)i}}{2C\left(1 \pm k_\iota\zeta_r\mu_{(r)i}\right)\left(\pm\, k_\iota\zeta_s\right)}\, e^{\mp k_\iota \tau}$$

i.e.

$$I^*_{(r)i} = \omega_r \rho\, \frac{k_{)\alpha}\zeta_s \mp \varpi\left(\frac{2C - \psi_0}{2C}\right)\, \mu_{(r)i}}{k_\iota\zeta_s\left(1 \pm k_\iota\zeta_r\mu_{(r)i}\right)}\, e^{\mp k_\iota \tau}$$
$$\alpha = 1.2. \cdots, n\, ;\ i = \pm 1, \pm 2, \cdots, \pm n \tag{2.39}$$

Particular integral of the equation (2.17)

To get the complete solution of the equation (2.17), we require a particular integral which can obtained as follows:

we set

$$I^*_{(r)i} = \omega_r A h_{(r)i}\mu_{(r)i}\, ;\ i = \pm 1, \pm 2, \cdots, \pm n \tag{2.40}$$

where

$$-A = \frac{1}{3}b_1\varpi \cdot \left\{\frac{\sum\limits_{s=1}^{m}\left(\frac{\eta_s}{1+\eta_s}\right)}{\sum\limits_{s=1}^{m}\eta_s}\right\} \tag{2.41}$$

Then the equation (2.17) gives

$$h_{(r)i}\mu_{(r)i} = \frac{1}{2C}\cdot\left\{\sum_{s=1}^{m}\eta_s\omega_s\sum_j(1+\varpi\mu_{(r)i}\mu_{(s)j})h_{(s)j}\mu_{(s)j}a_j\right\} - \mu_{(r)i} \tag{2.42}$$

i.e.

$$h_{(r)i}\mu_{(r)i} = \left(\sigma\mu_{(r)i}+\sigma_1\right) - \mu_{(r)i}$$

i.e.

$$h_{(r)i} = (\sigma - 1) + \frac{\sigma_1}{\mu_{(r)i}} \tag{2.43}$$

where σ and σ_1 are connected by the relations obtained by putting $h_{(r)i}$ from the equation (2.43) to the the equation (2.42) i.e.

$$2C\left\{\sigma\mu_{(r)i}+\sigma_1\right\} = (\sigma-1)\,\psi_1 + \sigma_1\psi_0 + \left\{(\sigma-1)\,\varpi\psi_2 + \varpi\sigma_1\psi_1\right\}\mu_{(r)i}$$

i.e.

$$2C\sigma\mu_{(r)i} + 2C\sigma_1 = \frac{1}{3}(\sigma-1)\,\varpi\psi_0\mu_{(r)i} + \sigma_1\psi_0 \tag{2.44}$$

$$\left[\begin{array}{l}\text{Since we have already obtained}\\ \qquad \psi_0 = 2\sum\limits_{s=1}^{m}\eta_s\omega_s\ ,\quad \psi_1 = 0\\ \text{and }\ \psi_2 = \frac{1}{3}\psi_0\end{array}\right.$$

Since the equation (2.44) is valid for all i. So, we must have

$$2C\sigma = \frac{1}{3}(\sigma-1)\,\varpi\psi_0\ ;\ 2C\sigma_1 = \sigma_1\psi_0$$

i.e.

$$\frac{\sigma}{\sigma-1} = \frac{\frac{1}{3}\varpi\psi_0}{2C}\ ; \tag{2.45a}$$

$$(2C-\psi_0)\,\sigma_1 = 0 \tag{2.45b}$$

From the relation (2.45a),we get

$$\sigma - 1 = -\frac{2C}{2C - \frac{1}{3}\varpi\psi_0}$$

and from the equation (2.45b)

$$(2C - \psi_0)\,\sigma_1 = 0$$

But, $\psi_0 \neq 2C$. So, $\sigma_1 = 0$
Therefore,

$$\sigma_1 = 0 \;\; ; \;\; \sigma - 1 = -\frac{2C}{2C - \frac{1}{3}\varpi\psi_0} \tag{2.46}$$

Hence, the required particular integral [putting the values of σ_1 and $\sigma - 1$ from the equation (2.46) in the equation (2.43) and then putting the values of in the equation(2.40)] is

$$I^*_{(r)i} = \frac{1}{3}\omega_r b_1 \varpi \cdot \left\{ \frac{\sum\limits_{s=1}^{m} \left(\frac{\eta_s}{1+\eta_s} \right)}{\sum\limits_{s=1}^{m} \eta_s \left(1 - \frac{1}{3}\varpi\omega_s \right)} \right\} \mu_{(r)i} \; ; \qquad i = \pm 1, \pm 2, \cdots, \pm n \tag{2.47}$$

Complete solution of the equation (2.17)

The general solution (2.39) of the equation (2.21) together with the particular integral (2.47) of the equation (2.17) will constitute the **complete solution** of the equation (2.17).

According to the Chandrasekhar,[45] the solution of the equation

(2.17) satisfying the boundary conditions (2.18) can be put in the form:

$$I^*_{(r)i} = \frac{1}{3}\omega_r b_1 \left[\sum_{\iota=1}^{n} \left\{ k_\iota \zeta_s - \varpi \left(\frac{2C-\psi_0}{2C} \right) \mu_{(r)i} \right\} \times \right.$$

$$\left. \times \frac{L_{(r)\iota} e^{-k_\iota \tau}}{k_\iota \zeta_s \left(1 + k_\iota \zeta_r \mu_{(r)i}\right)} + \varpi \mu_{(r)i} \cdot \left\{ \frac{\sum_{s=1}^{m} \left(\frac{\eta_s}{1+\eta_s} \right)}{\sum_{s=1}^{m} \eta_s \left(1 - \frac{1}{3}\varpi\omega_s\right)} \right\} \right]$$

$$\iota = 1.2. \cdots, n\, ;\; i = \pm 1, \pm 2, \cdots, \pm n \tag{2.48}$$

where k_ι's $(\iota = 1, 2, \cdots, n)$ are the positive roots of the characteristic equation (2.38) and $L_{(r)\iota}$'s are the constants of integration to be determined by the boundary conditions (2.18).

Writing

$$M = \left\{ \sum_{s=1}^{m} \eta_s \left(1 - \omega_s\right) \right\} / \left(\sum_{s=1}^{m} \eta_s \right) = \frac{2C - \psi_0}{2C} \tag{2.49}$$

and

$$N = \sum_{s=1}^{m} \left(\frac{\eta_s}{1+\eta_s} \right) / \sum_{s=1}^{m} \eta_s \left(1 - \frac{1}{3}\varpi\omega_s \right) \tag{2.50}$$

the equation (2.48) can be put in a shorter form:

$$I^*_{(r)i} = \frac{1}{3}\omega_r b_1 \left\{ \sum_{\iota=1}^{n} \frac{\left(k_\iota \zeta_s - \varpi M \mu_{(r)i}\right) L_{(r)\iota} e^{-k_\iota \tau}}{k_\iota \zeta_s \left(1 + k_\iota \zeta_r \mu_{(r)i}\right)} + \varpi N \mu_{(r)i} \right\}$$

$$\iota = 1.2. \cdots, n\, ;\; i = \pm 1, \pm 2, \cdots, \pm n \tag{2.51}$$

2.3.2.2 Relation between the Characteristic Equation and Zeros of the Legendre Polynomial

Let $p_{2\ell}$ be the co-efficient of $\mu^{2\ell}$ in the Legendre polynomial $P_{2n}(\mu)$.

Then

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = \sum_{\ell=0}^{n} p_{2\ell} \sum_{s=1}^{k} \eta_s \omega_s \sum_{j} \frac{a_j \mu_{(s)j}^{2\ell}}{1+\zeta_s k \mu_{(s)j}},$$

using the definition (2.26)

i.e.

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = \sum_{s=1}^{k} \eta_s \omega_s \sum_{j} \frac{a_j}{1+\zeta_s k \mu_{(s)j}} \sum_{\ell} p_{2\ell}\, \mu_{(s)j}^{2\ell} \tag{2.52}$$

Since, $\mu_{(s)j}$'s are the zeros of the Legendre polynomial $P_{2n}(\mu)$.

So,

$$\sum_{\ell} p_{2\ell}\, \mu_{(s)j}^{2\ell} = 0 \tag{2.53}$$

and therefore, the equation (2.52) becomes

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = 0$$

i.e.

$$p_{2n} D_{2n}(\zeta_s k) + \cdots + p_0 D_0(\zeta_s k) = 0$$

i.e.

$$p_{2n}\left\{-\frac{1}{k^{2n}\zeta_s^{2n}}(\psi_0 - D_0) - \frac{1}{k^{2n-2}\zeta_s^{2n-2}}\psi_2 + \cdots\right\} + \cdots + p_0 D_0 = 0,$$

using the equation (II.1) of Appendix II, which, on using the equation (2.34) becomes

$$p_{2n}\left\{2C\left(2C-\psi_0\right)-\frac{1}{3}\psi_0\varpi\left(2C-\psi_0\right)\right\}\frac{1}{\left(\zeta_s^2k^2\right)^n}+\cdots+p_0\left(4C^2\right)=0 \tag{2.54}$$

which is an equation in $\frac{1}{\left(\zeta_s^2k^2\right)}$ of degree n.

Since the roots of the equation(2.54) are $\frac{1}{\left(\zeta_s^2k_1^2\right)},\ldots,\frac{1}{\left(\zeta_s^2k_n^2\right)}$

Therefore, we get

$$\frac{1}{\left(\zeta_s^2k_1^2\right)}\cdots\frac{1}{\left(\zeta_s^2k_n^2\right)}=\text{the product of the roots of the equation (2.54)}$$

$$=(-1)^n\cdot\frac{p_0\left(4C^2\right)}{\left\{2C\left(2C-\psi_0\right)-\frac{1}{3}\psi_0\varpi\left(2C-\psi_0\right)\right\}p_{2n}}$$

Therefore, we get

$$\frac{1}{\left(\zeta_s^2k_1^2\right)}\cdots\frac{1}{\left(\zeta_s^2k_n^2\right)}=(-1)^n\cdot\frac{p_0\left(4C^2\right)}{\left\{2C\left(2C-\psi_0\right)-\frac{1}{3}\psi_0\varpi\left(2C-\psi_0\right)\right\}p_{2n}}$$

This, by using the equation (2.49), can be put in the form:

$$\left(\zeta_sk_1\cdots\zeta_sk_n\right)^2=(-1)^n\left\{\left(\frac{2C-\psi_0}{2C}\right)-\frac{1}{3}\varpi\left(\frac{2C-\psi_0}{2C}\right)\frac{\psi_0}{2C}\right\}\frac{p_{2n}}{p_0}$$

i.e.

$$\left(\zeta_sk_1\cdots\zeta_sk_n\right)^2=(-1)^n\left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}\frac{p_{2n}}{p_0} \tag{2.55}$$

Again, $\mu_{(s)1}^2$, ... , $\mu_{(s)n}^2$ are the zeros of the Legendre polynomial $\sum_{\ell=0}^{n}p_{2\ell}\,\mu_{(s)j}^{2\ell}$ i.e. $p_{2n}\left(\mu_{(s)j}^2\right)^2+\cdots+p_0$

So, we get

$$\mu_{(s)1}^2\mu_{(s)2}^2\cdots\mu_{(s)n}^2 = \text{the product of the zeros of } \sum_{\ell=0}^{n} p_{2\ell}\,\mu_{(s)j}^{2\ell}$$

$$= (-1)^n \frac{p_0}{p_{2n}}$$

i.e.

$$\left(\mu_{(s)1}\mu_{(s)2}\cdots\mu_{(s)n}\right)^2 = (-1)^n \frac{p_0}{p_{2n}} \tag{2.56}$$

Multiplying the equation (2.55) and (2.56) together, we get

$$\left(\zeta_s k_1\cdots\zeta_s k_n\cdot\mu_{(s)1}\mu_{(s)2}\cdots\mu_{(s)n}\right)^2 = \left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}$$

i.e.

$$\zeta_s k_1\cdots\zeta_s k_n\cdot\mu_{(s)1}\mu_{(s)2}\cdots\mu_{(s)n} = \left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} \tag{2.57}$$

2.3.2.3 The Elimination of Constants and the Expression of the Law of Diffuse Reflection in Closed Form

From the equation (2.51), we can write

$$I_r^*\left(\tau,\mu\right) = \frac{1}{3}\omega_r b_1\left\{\sum_{\iota=1}^{n}\frac{\left(k_\iota\zeta_s-\varpi M\mu\right)L_{(r)\iota}e^{-k_\iota\tau}}{k_\iota\zeta_s\left(1+k_\iota\zeta_r\mu\right)}+\varpi N\mu\right\} \tag{2.58}$$

Now, we define

$$S_r\left(\mu\right) = \frac{1}{T_\iota}\left\{\sum_{\iota=1}^{n}\frac{\left(k_\iota\zeta_s+\varpi M\mu\right)L_{(r)\iota}}{k_\iota\zeta_s\left(1-k_\iota\zeta_r\mu\right)}-\varpi N\mu-\frac{3\zeta_r\mu}{\omega_r}+\frac{3b_0}{\omega_r b_1}\right\} \tag{2.59}$$

where

$$T_\iota = \frac{1}{k_\iota\zeta_s}\left(k_\iota\zeta_s+\frac{\varpi M}{k_\iota\zeta_r}\right) = 1+\frac{\varpi M}{k_{(s)\iota}^2\zeta_r\zeta_s} \tag{2.60}$$

Then the boundary conditions (2.18) are expressible as

$$\frac{1}{T_\iota}\left\{\sum_{\iota=1}^{n}\frac{\left(k_\iota\zeta_s+\varpi M\mu_{(r)i}\right)L_{(r)\iota}}{k_\iota\zeta_s\left(1-k_\iota\zeta_r\mu_{(r)i}\right)}-\varpi N\mu_{(r)i}+\frac{3\mu_{(r)i}\zeta_r}{\omega_r}+\frac{3b_0}{\omega_r b_1}\right\}=0,$$

i.e.

$$S_r\left(\mu_{(r)i}\right)=0 \quad i=1,2,\cdots,n \tag{2.61}$$

Again, we can express $I_r^*(0,\mu)$ in terms of $S_r(\mu)$ as follows:

$$\frac{1}{3}\omega_r b_1 T_\iota S_r(-\mu)=\frac{1}{3}\omega_r b_1\left\{\sum_{\iota=1}^{n}\frac{\left(k_\iota\zeta_s-\varpi M\mu\right)L_{(r)\iota}}{k_\iota\zeta_s\left(1+k_\iota\zeta_r\mu\right)}+\varpi N\mu\right\}+b_1\zeta_r\mu-b_0$$

i.e.

$$I_r^*(0,\mu)=\frac{1}{3}\omega_r b_1\left\{T_\iota S_r(-\mu)-\frac{3\zeta_r\mu}{\omega_r}-\frac{3b_0}{\omega_r b_1}\right\} \tag{2.62}$$

Now, by virtue of the boundary conditions (2.61), we can write that $\mu_{(r)i}\ ;\ i=1,2,\cdots,n$ will make the value of any polynomial obtained by multiplying $S_r(\mu)$ by an expression zero.

Now, we define

$$P_r(\mu)=\prod_{i=1}^{n}\left(\mu-\mu_{(r)i}\right) \tag{2.63}$$

and

$$R_r(\mu)=\prod_{i=1}^{n}\left(1-k_i\zeta_r\mu\right) \tag{2.64}$$

Then

$$S_r(\mu)R_r(\mu)=\frac{1}{T_\iota}\left\{\sum_{\iota=1}^{n}\frac{\left(k_\iota\zeta_s-\varpi M\mu\right)L_{(r)\iota}}{k_\iota\zeta_s\left(1+k_\iota\zeta_r\mu\right)}+\varpi N\mu+\frac{3\zeta_r\mu}{\omega_r}+\frac{3b_0}{\omega_r b_1}\right\}\prod_{i=1}^{n}\left(1-k_i\zeta_r\mu\right)$$

is a polynomial of degree $(n+1)$ in μ in which the co-efficient of μ^{n+1} in $S_r(\mu)\,R_r(\mu)$ is

$$\frac{(-1)^{n+1}}{T_\iota}\left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r$$

Clearly $\mu_{(r)1}$, $\mu_{(r)2}$, $\cdots$, $\mu_{(r)n}$ are the zeros of the polynomial $S_r(\mu)\,R_r(\mu)$. In addition to these zeros $S_r(\mu)\,R_r(\mu)$ has another zero, say, ξ_r. Also, $\mu_{(r)1}, \mu_{(r)2}, \cdots, \mu_{(r)n}$ are the zeros of the polynomial $P_r(\mu)$ in which the co-efficient of μ^n is 1.

So, the polynomials $S_r(\mu)\,R_r(\mu)$ and $(\mu - \xi_r)\,P_r(\mu)$ has exactly the same zeros, viz., $\mu_{(r)1}, \mu_{(r)2}, \cdots, \mu_{(r)n}, \xi_r$ whose co-efficient of μ^{n+1} are

$$\frac{(-1)^{n+1}}{T_\iota}\left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \;\; and \;\; 1$$

respectively.

So,

$$S_r(\mu) = \frac{(-1)^{n+1}}{T_\iota}\left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \frac{P_r(\mu)}{R_r(\mu)}(\mu - \xi_r) \tag{2.65}$$

where ξ_r is a constant and $P_r(\mu)$ and $R_r(\mu)$ are given the equation (2.63) and the equation (2.64).

Moreover, we observe from the equation (2.59) that

$$\lim_{\mu \to (k_\iota\zeta_r)^{-1}} (1 - k_\iota\zeta_r\mu)\left[\frac{1}{T_\iota}\left\{\sum_{\iota=1}^{n}\frac{(k_\iota\zeta_s + \varpi M\mu)\,L_{(r)\iota}}{k_\iota\zeta_s(1 - k_\iota\zeta_r\mu)} - \varpi N\mu - \frac{3\zeta_r\mu}{\omega_r} + \frac{3b_0}{\omega_r b_1}\right\}\right] = L_{(r)\iota}$$

i.e.

$$L_{(r)\iota} = \lim_{\mu \to (k_\iota\zeta_r)^{-1}} (1 - k_\iota\zeta_r\mu)\,S_r(\mu) \tag{2.66}$$

Therefore, using the equation(2.65) in the equation(2.66), we get

$$L_{(r)\iota} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \frac{(-1)^{n+1}}{T_\iota} \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \frac{P_r(\mu)}{R_{(r)\iota}(\mu)} (\mu - \xi_r)$$

i.e.

$$L_{(r)\iota} = \frac{(-1)^{n+1}}{T_\iota} \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \times \\ \times \frac{P_r\left(\frac{1}{k_\iota\zeta_s}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota\zeta_s}\right)} \left(\frac{1}{k_\iota\zeta_s} - \xi_r\right) \tag{2.67}$$

where,

$$R_{(r)\iota}(\mu) = \prod_{\beta(\neq\iota)=1}^{n} (1 - k_\beta\zeta_r\mu) \quad 1 \le \iota \le n \tag{2.68}$$

Summing up both sides of the equation (2.67) over ι, we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \times \\ \times \sum_{\iota=1}^{n} \frac{P_r\left(\frac{1}{k_\iota\zeta_s}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota\zeta_s}\right)} \cdot \frac{1}{T_\iota} \left(\frac{1}{k_\iota\zeta_s} - \xi_r\right)$$

i.e.

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r f_r(0) \tag{2.69}$$

where

$$f_r(x) = \sum_{\iota=1}^{n} \frac{P_r\left(\frac{1}{k_\iota\zeta_s}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota\zeta_s}\right)} \cdot \frac{1}{T_\iota} \left(\frac{1}{k_\iota\zeta_s} - \xi_r\right) R_{(r)\iota}(x) \tag{2.70}$$

Now, we notice that the polynomial $f_r(x)$ is a degree $(n-1)$ in x which takes the value

$$\frac{1}{T_\iota}P_r\left(\frac{1}{k_\iota\zeta_s}\right)\left(\frac{1}{k_\iota\zeta_s}-\xi_r\right) \text{ for } x=\frac{1}{k_\iota\zeta_s} \quad \iota=1,2,\cdots,n$$

So,

$$\left(1+\epsilon_{rs}\varpi Mx^2\right)f_r(x)-P_r(x)(x-\xi_r)=0 \text{ for } x=\frac{1}{k_\iota\zeta_s} \quad \iota=1,2,\cdots,n \tag{2.71}$$

This helps us to conclude that the polynomial on the left hand side of the equation (2.71) must be divisible by the polynomial $R_r(x)$. Hence, we get the following relation:

$$\left(1+\epsilon_{rs}\varpi Mx^2\right)f_r(x)-P_r(x)(x-\xi_r)=R_r(x)(A_rx+B_r) \tag{2.72}$$

Now, assuming $\varpi \neq 0$, let us put $x = +\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}$ and $x=-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}$ in the equation (2.72) to get

$$\frac{iA_r}{\sqrt{\epsilon_{rs}\varpi M}}+B_r=\frac{P_r\left(+\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}{R_r\left(+\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}\left(\xi_r-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right) \tag{2.73}$$

and

$$-\frac{iA_r}{\sqrt{\epsilon_{rs}\varpi M}}+B_r=\frac{P_r\left(-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}{R_r\left(-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}\left(\xi_r+\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right) \tag{2.74}$$

Then adding the equation (2.73) and the equation (2.74),

$$B_r = (-1)^n \left(\xi_r a'_r + \frac{i}{\sqrt{\epsilon_{rs} \varpi M}} b'_r \right) \mu_{(r)1} \mu_{(r)2} \cdots \mu_{(r)n} \tag{2.75}$$

and subtracting the equation (2.74) from the equation (2.73), we get

$$A_r = (-1)^{n+1} \left(a'_r - i \xi_r \sqrt{\epsilon_{rs} \varpi M}\, b'_r \right) \mu_{(r)1} \mu_{(r)2} \cdots \mu_{(r)n} \tag{2.76}$$

where

$$a'_r = \frac{1}{2} \left\{ H_r \left(+ \frac{i}{\sqrt{\epsilon_{rs} \varpi M}} \right) + H_r \left(- \frac{i}{\sqrt{\epsilon_{rs} \varpi M}} \right) \right\} \tag{2.77}$$

and

$$b'_r = \frac{1}{2} \left\{ H_r \left(+ \frac{i}{\sqrt{\epsilon_{rs} \varpi M}} \right) - H_r \left(- \frac{i}{\sqrt{\epsilon_{rs} \varpi M}} \right) \right\} \tag{2.78}$$

and H-function is given by

$$H_r(\mu) = \frac{1}{\mu_{(r)1} \mu_{(r)2} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} \left(\mu + \mu_{(r)i} \right)}{\prod_{i=1}^{n} \left(1 + k_i \zeta_r \mu \right)} \tag{2.79}$$

Now, putting $x = 0$ in the the equation(2.72), we get

$$f_r(0) = -\xi_r P_r(0) + R_r(0) B_r$$

which on using the equation (2.75) will yield

$$f_r(0) = (-1)^n \left\{ \frac{i}{\sqrt{\epsilon_{rs} \varpi M}} b'_r - \xi_r \left(1 - a'_r \right) \right\} \mu_{(r)1} \mu_{(r)2} \cdots \mu_{(r)n} \tag{2.80}$$

So, using the equation (2.80) in the the equation (2.69), we have

$$\sum_{\iota=1}^{n} L_{(r)\iota} = \left(\varpi N + \frac{3 \zeta_r}{\omega_r} \right) \left\{ \xi_r \left(1 - a'_r \right) - \frac{i}{\sqrt{M \varpi}} b'_r \right\} \times \\ \times \mu_{(r)1} \mu_{(r)2} \cdots \mu_{(r)n} k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \tag{2.81}$$

But the roots of the characteristic equation (2.38) follow the relation:

$$\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}k_1\zeta_r\cdot k_2\zeta_r\cdots k_n\zeta_r=\left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} \tag{2.82}$$

Applying the relation(2.82) in the equation(2.81), we, therefore, get

$$\sum_{\iota=1}^{n}L_{(r)\iota}=\left(\varpi N+\frac{3\zeta_r}{\omega_r}\right)\left\{\xi_r\ \left(1-a'_r\right)\right.$$
$$\left.-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\,b'_r\right\}\left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} \tag{2.83}$$

Again, by virtue of the result (2.82), the the equation (2.65) can be written as:

$$S_r\left(\mu\right)=\frac{\left(-1\right)^{n+1}}{T_\iota}\left(\varpi N+\frac{3\zeta_r}{\omega_r}\right)k_1\zeta_r\cdot k_2\zeta_r\cdots k_n\zeta_r\left(\mu-\xi_r\right)\times$$
$$\times\left(-1\right)^n\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\cdot H_r\left(-\mu\right)$$

i.e.

$$S_r\left(\mu\right)=-\frac{1}{T_\iota}\left(\varpi N+\frac{3\zeta_r}{\omega_r}\right)\left(\mu\right.$$
$$\left.-\xi_r\right)\left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}}\cdot H_r\left(-\mu\right) \tag{2.84}$$

So,

$$S_r\left(0\right)=\frac{\xi_r}{T_\iota}\left(\varpi N+\frac{3\zeta_r}{\omega_r}\right)\left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} \tag{2.85}$$

Again, from the equation(2.59),

$$S_r\left(0\right)=\frac{1}{T_\iota}\left\{\sum_{\iota=1}^{n}L_{(r)\iota}+\frac{3b_0}{\omega_r b_1}\right\} \tag{2.86}$$

So, on comparing the equation (2.85) and the equation (2.86), we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} = \xi_r \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) \left\{M \left(1 - \frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} - \frac{3b_0}{\omega_r b_1} \tag{2.87}$$

Using the value of $\sum_{\iota=1}^{n} L_{(r)\iota}$ from the equation(2.87) in the equation(2.83), we get

$$\xi_r \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) \left\{M \left(1 - \frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} - \frac{3b_0}{\omega_r b_1} = \left(\varpi N + \frac{3\zeta_r}{\omega_r}\right) \left\{\xi_r \left(1 - a'_r\right) - \frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\, b'_r\right\} \left\{M \left(1 - \frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}}$$

i.e.

$$\xi_r = \frac{b_0}{a'_r b_1 \left(\zeta_r + \frac{1}{3}\omega_r\varpi N\right) \left\{M \left(1 - \frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}}} - \frac{i}{\sqrt{\epsilon_{rs}\varpi M}} \cdot \frac{b'_r}{a'_r}$$

i.e.

$$\xi_r = \frac{b_0}{a'_r\, G_r} - \frac{i}{\sqrt{\epsilon_{rs}\varpi M}} \cdot \frac{b'_r}{a'_r} \tag{2.88}$$

where

$$G_r = b_1 \left(\zeta_r + \frac{1}{3}\omega_r\varpi N\right) \left\{M \left(1 - \frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}} \tag{2.89}$$

So,the equation(2.84) will give the value of $S_r(\mu)$ where ξ_r is given by the equation(2.88).

Now, from the equation(2.62), using the equation(2.84), we get

$$I_r^*(0,\mu) = \frac{1}{3}\omega_r b_1 \left[T_\iota \left\{ -\frac{1}{T_\iota} \left(\varpi N + \frac{3\zeta_r}{\omega_r} \right) (-\mu \right.\right.$$
$$\left.\left. -\xi_r) \left\{ M \left(1 - \frac{1}{3} \varpi (1-M) \right) \right\}^{\frac{1}{2}} \cdot H_r(+\mu) \right\} - \frac{3\zeta_r \mu}{\omega_r} - \frac{3b_0}{\omega_r b_1} \right] \tag{2.90}$$

i.e.

$$I_r^*(0,\mu) = b_1 \left(\zeta_r + \frac{1}{3}\omega_r \varpi N \right) \left\{ M \left(1 - \frac{1}{3} \varpi (1-M) \right) \right\}^{\frac{1}{2}} \times$$
$$\times (\mu + \xi_r) \cdot H_r(\mu) - \zeta_r \mu - b_0$$

i.e.

$$I_r^*(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) - \zeta_r \mu b_1 - b_0 \tag{2.91}$$

where ξ_r and G_r are given by the equations(2.88) and (2.89) respectively.

Now, from the equation (2.10), we get

$$I_r(0,\mu) = b_0 + \zeta_r \mu b_1 + I_r^*(0,\mu)$$

i.e.

$$I_r(0,\mu) = b_0 + \zeta_r \mu b_1 + G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) - \zeta_r \mu b_1 - b_0$$

i.e.

$$I_r(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) \tag{2.92}$$

The solution(2.92) is a desired solution of the equation (2.9) in n^{th} approximation, when linear form of Planck-function is used.

2.3.2.4 The Exact Diffusely Reflected Intensity and The Exact Solution for The Emergent Intensity.

Following Busbridge and Stibbs,[33] we change the variable $\zeta_r\mu$ and $\zeta_s\mu'$ to x and x' respectively [and consequently $\zeta_r\mu_{(r)i}$ and $\zeta_s\mu_{(s)j}'$ to x_i and $x'_j{}'$ respectively] to get from the equation (2.38)

$$\sum_{s=1}^{m}\eta_s\omega_s\sum_j \frac{a_j}{1+k\zeta_s\mu_{(s)j}}\left\{\frac{1}{2C}\left(1+\varpi M\mu_{(s)j}^2\right)\right\}=1$$

i.e.

$$\sum_j \frac{a'_j}{1+kx_j}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left\{\frac{1}{2C}\left(1+\frac{1}{\zeta_s^2}\varpi M x_j^2\right)\right\}=1$$

i.e.

$$\sum_j \frac{a'_j\Psi(x_j)}{1+kx_j}=1 \text{ with } a'_j=\zeta_r a_j \tag{2.93}$$

where, assuming that

$$\eta_1>\eta_2>\cdots>\eta_m \tag{2.94}$$

so that

$$0\le\zeta_1<\zeta_2<\cdots<\zeta_m\le 1 \tag{2.95}$$

$$\Psi(x')=\begin{cases}\sum\limits_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left\{\frac{1}{2C}\left(1+\frac{1}{\zeta_s^2}\varpi M x'^2\right)\right\}, & \text{if } 0\le x'\le\zeta_1\\[2ex] \sum\limits_{s=r+1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left\{\frac{1}{2C}\left(1+\frac{1}{\zeta_s^2}\varpi M x'^2\right)\right\}, & \text{if } \zeta_r\le x'\le\zeta_{r+1}\\[2ex] 0, & \text{if } \zeta_m\le x'\le 1\end{cases} \tag{2.96}$$

Then

$$\int_0^1 \Psi(x')\,dx' < \frac{1}{2} \tag{2.97}$$

Therefore, following the theory of H-function, developed by Chandrasekhar,[45] we shall able to show, in the present case, that $H(x)$, where $x = \zeta_r\mu$ or equivalently $[H(\zeta_r\mu) =]\ H_r(\mu)$, given by the equation(2.79) satisfies, in the limit of infinite approximation, the non-integral equation:

$$H(x) = 1 + xH(x)\int_0^1 \frac{\Psi(x')\,H(x)}{x + x'}dx'$$

i.e.

$$H_r(\mu) = 1 + \zeta_r\mu H_r(\mu)\int_0^1 \frac{\zeta_s\Psi(\zeta_s\mu')}{\zeta_r\mu + \zeta_s\mu'}\,H_s(\mu')\,d\mu' \tag{2.98}$$

which is bounded in the entire half plane $R(x) \geq 0$.

The characteristic function $\Psi(x')$ in the equation(2.98) satisfies the necessary condition:

$$\int_0^1 \Psi(x')\,dx' \leq \frac{1}{2} \tag{2.99}$$

Now, we allow n to tend to infinity for both the equations(2.91) and (2.92) to get the exact diffusely reflected intensity and the exact emergent intensity, given by:

$$I_r^*(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) - \zeta_r\mu b_1 - b_0 \tag{2.100}$$

and

$$I_r(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) \tag{2.101}$$

where ξ_r and G_r are given by the equations(2.88) and (2.89) respectively and $H_r(\mu)$ is the solution of the equation(2.98).

2.3.3 Derivation of Results of Previous Workers

If we put $\varpi = 0$ in the equation(2.9), then the equation reduces to the form of the equation of the transfer for the r^{th} interlocked line in the case of isotropically scattering media which was solved by Busbridge and Stibbs[33] by the principle of invariance governing the law of diffuse reflection with a slight modification and by Karanjai and Barman[106] by using the extension of the method of discrete-ordinates. Here, we have solved the equation by using the extension of the method of discrete-ordinates and therefore, we can hope for getting the diffusely reflected intensity $I_r^*(0, \mu)$ and the emergent intensity $I_r^*(0, \mu)$ for isotropically scattering media in the n^{th} approximation, given by the equations (41) and (42) of Karanjai and Barman,[106] from the equations(2.91) and (2.92) of section: 2.3.2.3 by substituting $\varpi = 0$. But the form of the equations (2.91) and (2.92) do not allow us to put $\varpi = 0$ directly. Of course, if we put $\varpi = 0$ directly in our solution(2.51) of Section: 2.3.2.1, then it becomes

$$I^*_{(r)i} = \frac{1}{3}\omega_r b_1 \sum_{\iota=1}^{n} \frac{L_{(r)\iota} e^{-k_\iota \tau}}{1 + k_\iota \zeta_r \mu_{(r)i}} \tag{2.102}$$

which, subject to the boundary conditions (2.18), is equivalent to the equation (19) of Karanjai and Barman,[106] subject to their boundary conditions (20).

The equations (2.91) and (2.92) cannot be studied by putting $\varpi = 0$ directly, but can be studied in the limiting case by allowing ϖ to tend to zero.

Using the equation (2.88) and (2.89) in the equation(2.92), we get

$$I_r(0,\mu) = b_1\left(\zeta_r + \frac{1}{3}\omega_r\varpi N\right)\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}}\left\{\left(\mu - \frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\cdot\frac{b'_r}{a'_r}\right) + \frac{b_0}{a'_r}\right\}\cdot H_r(\mu) \tag{2.103}$$

where a'_r and b'_r are given by the equations (2.77)and (2.78) respectively.

Now, before allowing ϖ to tend to zero, we see first the limiting values of a'_r and $i\frac{1}{\sqrt{\epsilon_{rs}\varpi M}}b'_r$ as ϖ tends to zero.

Denoting the sum of the product of the roots $\mu_{(r)i}; i = 1, 2, \cdots, n$ taken t, where $t \in \{1, 2, \cdots, n\}$, at a time by the symbol p_t and the sum of the product of the roots $\zeta_r k_\iota; \iota = 1, 2, \cdots, n$ taken t at a time by the symbol p'_t, we have,

$$a'_r = \frac{1}{\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}}}\cdot\left\{\frac{1+\cdots+p_n/p'_n\left(\sqrt{\epsilon_{rs}\varpi M}\right)^{2n}}{1+\cdots+1/p'^2_n\left(\sqrt{\epsilon_{rs}\varpi M}\right)^{2n}}\right\}$$

which $\rightarrow \frac{1}{M^{1/2}}$ as $\varpi \rightarrow 0$

Again,

$$i\frac{1}{\sqrt{\epsilon_{rs}\varpi M}}b'_r = \frac{1}{\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}}}\times$$

$$\times\left[\frac{\left(p_1 - \frac{p'_{n-1}}{p'_n}\right)+\cdots+\left(\frac{p_n p'_1 - p_{n-1}}{p'_n}\right)\left(\sqrt{\epsilon_{rs}\varpi M}\right)^{2n-2}}{1+\cdots+p'^2_n\left(\sqrt{\epsilon_{rs}\varpi M}\right)^{2n}}\right]$$

$$\text{which} \to \frac{1}{M^{1/2}}\left(p_1 - \frac{p'_{n-1}}{p'_n}\right) \text{ as } \varpi \to 0$$

i.e.

$$i\frac{1}{\sqrt{\epsilon_{rs}\varpi M}}\, b'_r \to \frac{1}{\sqrt{M}}\left(\sum_{i=1}^{n}\mu_{(r)i} - \sum_{\iota=1}^{n}\frac{1}{\zeta_r k_\iota}\right) \quad \text{as} \quad \varpi \to 0$$

i.e.

$$I_r(0,\mu) \to \sqrt{M}\left[b_1\zeta_r\left\{\mu - \left(\sum_{i=1}^{n}\mu_{(r)i} - \sum_{\iota=1}^{n}\frac{1}{\zeta_r k_\iota}\right)\right\} + b_0\right] H_r(\mu)$$

i.e., in the limiting position as ϖ approaches zero,

$$I_r(0,\mu) = \sqrt{M} H_r(\mu)\left\{b_0 + b_1\zeta_r\mu + b_1\zeta_r\left(\sum_{\iota=1}^{n}\frac{1}{\zeta_r k_\iota} - \sum_{i=1}^{n}\mu_{(r)i}\right)\right\} \tag{2.104}$$

which is identical with the equation (42) due to Karanjai and Barman,[106] if we substitute their value of C_r from the equation (40) in the equation (42). Like emergent intensity, we can show that the diffusely reflected intensity for isotropically scattering media can also be derived from the anisotropically scattering media.

The exact diffusely reflected intensity $I_r^*(0,\mu)$ and exact emergent intensity $I_r(0,\mu)$ for isotropically scattering media can also be derived from the anisotropically scattering media associated with the planetary phase function by allowing ϖ of the phase function (2.5)to tend to zero. The characteristic function $\Psi(x')$ of the H-function $H_r(\mu)$ for isotropically scattering media is to be taken from the equation (2.96) by putting $\varpi = 0$ directly which is given by

$$\Psi\left(x'\right) = \begin{cases} \sum\limits_{s=1}^{m} \frac{\eta_s \omega_s}{2C\zeta_s}, & \text{if } 0 \leq x' \leq \zeta_1 \\ \sum\limits_{s=r+1}^{m} \frac{\eta_s \omega_s}{2C\zeta_s}, & if\ \zeta_r \leq x' \leq \zeta_{r+1} \\ 0, & \text{if } \zeta_m \leq x' \leq 1 \end{cases} \tag{2.105}$$

2.4 The Equation of Radiative Transfer with Exponential Form of Planck Function

Using the relation (2.4b) in the equation (2.6), we get

$$\begin{aligned} \mu \frac{dI_r(\tau,\mu)}{d\tau} &= (1+\eta_r) I_r(\tau,\mu) - (1+\varepsilon\eta_r)\left(b_0 + b_1 e^{-\beta\tau}\right) \\ &-\frac{1}{2}(1-\varepsilon)\alpha_r \sum_{s=1}^{m} \eta_s \int_{-1}^{1} (1+\varpi\mu\mu')I_s(\tau,\mu')d\mu' \end{aligned} \tag{2.106}$$

2.4.1 Reduction of the Equation (2.106) to a Simpler Form

We suppose, following Busbridge and Stibbs,[33] that one of the solution of the equation (2.106) is

$$I_r(\tau,\mu) = b_0 + b_1 \left(1 - \frac{(1-\varepsilon)\,\eta_r U + \beta\mu}{1+\eta_r+\beta\mu}\right) e^{-\beta\tau} + I_r^*(\tau,\mu) \tag{2.107}$$

where U is given by

$$U = \frac{1 - \frac{1}{2\beta}\sum\limits_{s=1}^{m} \alpha_s \left(1+\eta_s\right) \log\left(\frac{1+\eta_s+\beta}{1+\eta_s-\beta}\right)}{1 - \frac{1}{2\beta}\left(1-\varepsilon\right)\sum\limits_{s=1}^{m} \alpha_s \eta_s \log\left(\frac{1+\eta_s+\beta}{1+\eta_s-\beta}\right)} \tag{2.108}$$

The equation(2.107) consists of two parts , the first part being the solution for an infinitely unbounded atmosphere as τ tends to infinity and the second part $I_r^*(\tau, \mu)$ being the departure of the asymptotic solution from the value $I_r(\tau, \mu)$ as we approach the boundary $\tau = 0$

Now, introducing the three symbols, given by the equations (2.12), (2.13) and (2.28) and using the equation (2.4b), we can write the equation (2.6) in the form:

$$\zeta_r \mu \frac{dI_r(\tau, \mu)}{d\tau} = I_r(\tau, \mu) - (1 - \omega_r)\left(b_0 + b_1 e^{-\beta\tau}\right) - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \int_{-1}^{1} (1 + \varpi \mu \acute{\mu}) I_s(\tau, \acute{\mu}) d\acute{\mu} \tag{2.109}$$

and since (2.107) is one of the solutions of the equation (2.109), it can be put in the form:

$$- \beta \zeta_r \mu b_1 (1 - \omega_r U) \frac{e^{-\beta\tau}}{1 + \beta\mu\zeta_r} + \zeta_r \mu \frac{dI_r^*(\tau, \mu)}{d\tau} = b_0 + b_1 (1 - \omega_r U) \frac{e^{-\beta\tau}}{1 + \beta\mu\zeta_r} + I_r^*(\tau, \mu) - (1 - \omega_r)\left(b_0 + b_1 e^{-\beta\tau}\right) - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \int_{-1}^{1} (1 + \varpi \mu \acute{\mu}) \left\{ b_0 + b_1 (1 - \omega_s U) \frac{e^{-\beta\tau}}{1 + \beta\acute{\mu}\zeta_s} + I_s^*(\tau, \acute{\mu}) \right\} d\acute{\mu}$$

$\Rightarrow$

$$\zeta_r \mu \frac{dI_r^*(\tau,\mu)}{d\tau} - I_r^*(\tau,\mu) = b_0 \left\{ 1 - (1-\omega_r) - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \int_{-1}^{1} d\mu' \right\}$$
$$+ b_1 e^{-\beta\tau} \left\{ \frac{(1-\omega_r U)}{1+\beta\mu\zeta_r} (1+\beta\zeta_r\mu) - (1-\omega_r) \right.$$
$$\left. - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s (1-\omega_s U) \left(\int_{-1}^{1} \frac{d\mu'}{1+\beta\mu'\zeta_s} + \varpi\mu \int_{-1}^{1} \frac{\mu' \, d\mu'}{1+\beta\mu'\zeta_s} \right) \right\}$$
$$- \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \int_{-1}^{1} (1+\varpi\mu\mu') I_s^*(\tau,\mu') d\mu'$$

i.e.

$$\zeta_r \mu \frac{dI_r^*(\tau,\mu)}{d\tau} - I_r^*(\tau,\mu) - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \int_{-1}^{1} (1+\varpi\mu\mu') I_s^*(\tau,\mu') d\mu'$$
$$= b_1 e^{-\beta\tau} \omega_r \left[(1-U) - \frac{1}{2C} \cdot \sum_{s=1}^{m} \eta_s (1-\omega_s U) \left\{ \frac{1}{\beta\zeta_s} \log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s} \right) \right. \right.$$
$$\left. \left. + \frac{\varpi\mu}{\beta\zeta_s} \left(2 - \frac{1}{\beta\zeta_s} \log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s} \right) \right) \right\} \right]$$

i.e.

$$\zeta_r \mu \frac{dI_r^*(\tau,\mu)}{d\tau} - I_r^*(\tau,\mu) - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \int_{-1}^{1} (1+\varpi\mu\mu') I_s^*(\tau,\mu') d\mu'$$
$$= b_1 e^{-\beta\tau} \omega_r \left[\left(1 - \frac{1}{2\beta} \cdot \sum_{s=1}^{m} \frac{\eta_s \log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\zeta_s} \frac{\alpha_s}{\eta_s} \right) \right.$$
$$- U \left(1 - \frac{1}{2\beta} \cdot \sum_{s=1}^{m} \frac{\eta_s \omega_s \log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\zeta_s} \frac{\alpha_s}{\eta_s} \right)$$
$$\left. - \frac{1}{2\beta} \cdot \frac{\varpi\mu}{C} \cdot \sum_{s=1}^{m} \frac{\eta_s (1-\omega_s U)}{\zeta_s} \left(2 - \frac{\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\beta\zeta_s} \right) \right]$$

i.e.

$$\zeta_r\mu\ \frac{dI_r^*(\tau,\mu)}{d\tau} - I_r^*(\tau,\mu) - \frac{\omega_r}{2C}\cdot\sum_{s=1}^{m}\eta_s\int_{-1}^{1}(1+\varpi\mu\mu')I_s^*(\tau,\mu')d\mu'$$

$$= b_1 e^{-\beta\tau}\omega_r\left[\left(1-\frac{1}{2\beta}\cdot\sum_{s=1}^{m}\frac{\alpha_s\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\zeta_s}\right)\right.$$

$$-\left(\frac{1-\frac{1}{2\beta}\sum_{s=1}^{m}\alpha_s\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{1-\frac{1}{2\beta}\sum_{s=1}^{m}\frac{\alpha_s\omega_s\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\zeta_s}}\right)\times$$

$$\times\left(1-\frac{1}{2\beta}\cdot\sum_{s=1}^{m}\frac{\alpha_s\omega_s\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\zeta_s}\right)$$

$$\left.-\frac{1}{2\beta}\cdot\frac{\varpi\mu}{C}\cdot\sum_{s=1}^{m}\frac{\eta_s\left(1-\omega_s U\right)}{\zeta_s}\left(2-\frac{\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)}{\beta\zeta_s}\right)\right]$$

i.e.

$$\zeta_r\mu\frac{dI_r^*(\tau,\mu)}{d\tau} = I_r^*(\tau,\mu) - \frac{\omega_r}{2C}\cdot\sum_{s=1}^{m}\eta_s\int_{-1}^{1}(1+\varpi\mu\mu')I_s^*(\tau,\mu')d\mu'$$

$$-\frac{1}{2}b_1\omega_r\varpi\cdot\frac{1}{C}\sum_{s=1}^{m}\frac{\eta_s\left(1-\omega_s U\right)}{\beta\zeta_s}\left\{2-\frac{1}{\beta\zeta_s}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\right\}\mu e^{-\beta\tau} \tag{2.110}$$

and the boundary conditions (2.7) and (2.8) can be converted into

$$0 = I_r(0,-\mu) = b_0 + b_1\left(1-\omega_r U\right)\frac{1}{1-\beta\mu\zeta_r} + I_r^*(0,-\mu),$$

$$0<\mu\leq 1$$

i.e.

$$I_r^*(0,-\mu) = -b_0 - b_1\left(\frac{1-\omega_r U}{1-\beta\mu\zeta_r}\right) \quad , \quad 0 < \mu \le 1 \tag{2.111}$$

and

$$I_r^*(\tau,\mu) \text{ is atmost linear in } \tau \text{ as } \tau \text{ tends to infinity} \tag{2.112}$$

2.4.2 Construction of a System of 2n Integro-differential Equations from the Equation (2.110) Following Chandrasekhar[45]

Making n^{th} approximation, the integro-differential equation (2.110) is replaced by the system of 2n linear differential equations:

$$\begin{aligned}\zeta_r\mu_{(r)i}\frac{dI^*_{(r)i}}{d\tau} &= I^*_{(r)i} - \frac{\omega_r}{2C}\cdot\sum_{s=1}^{m}\eta_s\sum_j(1+\varpi\mu_{(r)i}\mu_{(r)j})I^*_{(s)j}a_j \\ &\quad -\frac{1}{2}b_1\omega_r\varpi\cdot\frac{1}{C}\sum_{s=1}^{m}\frac{\eta_s(1-\omega_s U)}{\beta\zeta_s}\Big\{2 \\ &\quad -\frac{1}{\beta\zeta_s}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\Big\}\mu_{(r)i}e^{-\beta\tau}\end{aligned} \tag{2.113}$$

and the boundary conditions (2.111) and (2.112) can be put in the following shorter form:

$$I^*_{(r)-i} = -b_0 - b_1\left(\frac{1-\omega_r U}{1-\beta\mu_{(r)i}\zeta_r}\right) \quad , \quad 0 < \mu_{(r)i} \le 1 \tag{2.114}$$

and

$$I^*_{(r)i} \text{ is atmost linear in } \tau \text{ as } \tau \text{ tends to infinity} \tag{2.115}$$

where the notations $I^*_{(r)i}$, $\mu_{(r)i}$ etc. have the same meaning as they are used for the equation (2.17).

2.4.3 Solution of the Equation with Exponential Form of Planck Function

2.4.3.1 Diffusely Reflected Intensity and Emergent Intensity in n^{th} Approximation

The Solution of the Associated Homogeneous System of the Equation (2.113)

The associated homogeneous part of the equation (2.113)

$$\zeta_r \mu \frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{\omega_r}{2C} \cdot \sum_{s=1}^{m} \eta_s \sum_j (1 + \varpi \mu_{(r)i} \mu_{(r)j}) I^*_{(s)j} a_j \quad (2.116)$$

which is same as the equation (2.21). So, its solution will be same as the solution (2.39) of the equation (2.21) i.e.

$$I^*_{(r)i} = \omega_r \rho \frac{k_\iota \zeta_s \mp \varpi \left(\frac{2C - \psi_0}{2C}\right) \mu_{(r)i}}{k_\iota \zeta_s \left(1 \pm k_\iota \zeta_r \mu_{(r)i}\right)} e^{\mp k_\iota \tau}$$
$$\iota = 1, 2, \cdots, n \,;\; i = \pm 1, \pm 2, \cdots, \pm n$$

i.e.

$$I^*_{(r)i} = \omega_r \rho \frac{k_\iota \zeta_s \mp \varpi \; M \; \mu_{(r)i}}{k_\iota \zeta_s \left(1 \pm k_\iota \zeta_r \mu_{(r)i}\right)} e^{\mp k_\iota \tau}$$
$$\iota = 1.2. \cdots, n \,;\; i = \pm 1, \pm 2, \cdots, \pm n \quad (2.117)$$

where $\pm k_\iota$; $\iota = 1, 2, \cdots, n$ are the zeros of the **characteristic equation** (2.38) in k i.e.

$$\sum_{s=1}^{m} \eta_s \omega_s \sum_j \frac{a_j \left(1 + \varpi \; M \mu^2_{(s)j}\right)}{2C \left(1 + k \zeta_s \mu_{(s)j}\right)} = 1 \quad (2.118)$$

Particular integral of the equation (2.113)

To constitute the complete solution of the equation (2.113), we have to add one more expression, known as a particular integral, to the solution (2.117) which is obtained as follows:

We take

$$I^*_{(r)i} = -\omega_r\, b_1 \varpi L\ \ h_{(r)i}\mu_{(r)i}e^{-\beta\tau} \tag{2.119}$$

where

$$L = \sum_{s=1}^{m} \frac{\eta_s\,(1-\omega_s U)}{\beta\zeta_s}\left\{1 - \frac{1}{2\beta\zeta_s}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\right\} / \sum_{s=1}^{n}\eta_s \tag{2.120}$$

So, the equation (2.113) gives

$$\begin{aligned} h_{(r)i}\mu_{(r)i}\left\{1+\zeta_r\beta\mu_{(r)i}\right\} = \frac{1}{2C}\cdot\sum_{s=1}^{m}\eta_s\omega_s\sum_{j}(1 \\ +\varpi\mu_{(r)i}\mu_{(r)j}\ )\ h_{(s)j}\mu_{(s)j}a_j + \mu_{(r)i} \end{aligned} \tag{2.121}$$

i.e.

$$h_{(r)i}\mu_{(r)i}\left(1+\zeta_r\beta\mu_{(r)i}\right) = \left(\acute{\rho} + \acute{\rho}_1\mu_{(r)i}\right) + \mu_{(r)i}$$

i.e.

$$h_{(r)i} = \frac{\acute{\rho} + \left(\acute{\rho}_1 + 1\right)\mu_{(r)i}}{\left(1+\zeta_r\beta\mu_{(r)i}\right)\mu_{(r)i}} \tag{2.122}$$

where $\acute{\rho}$ and $\acute{\rho}_1$ are constants (free from $\mu_{(r)i}$).

Substituting $h_{(r)i}$ from the equation (2.122) to the equation (2.121)

$$\begin{aligned}\rho' + \rho'_1\mu_{(r)i} = \frac{1}{2C}\Bigg\{ \rho' \cdot \sum_{s=1}^{m}\eta_s\omega_s \sum_j \frac{a_j}{1+\zeta_s\beta\mu_{(s)j}} \\ + (\rho'_1+1)\cdot \sum_{s=1}^{m}\eta_s\omega_s \sum_j \frac{a_j\mu_{(s)j}}{1+\zeta_s\beta\mu_{(s)j}} \\ +\varpi\mu_{(r)i}\Bigg(\rho' \cdot \sum_{s=1}^{m}\eta_s\omega_s \sum_j \frac{a_j\mu_{(s)j}}{1+\zeta_s\beta\mu_{(s)j}} \\ + (\rho'_1+1)\cdot \sum_{s=1}^{m}\eta_s\omega_s \sum_j \frac{a_j\mu^2_{(s)j}}{1+\zeta_s\beta\mu_{(s)j}}\Bigg)\Bigg\}\end{aligned}$$

i.e.

$$\begin{aligned}\rho' + \rho'_1\mu_{(r)i} = \frac{1}{2C}\Bigg\{\Big(\rho' \, D_0(\beta\zeta_S) + (\rho'_1+1)\, D_1(\beta\zeta_S)\Big) \\ +\varpi\mu_{(r)i}\Big(\rho' \cdot D_1(\beta\zeta_S) + (\rho'_1+1)\cdot D_2(\beta\zeta_S)\Big)\Bigg\}\end{aligned}$$

i.e.

$$\begin{aligned}2C\rho' &= \rho' \, D_0(\beta\zeta_S) + \rho'_1 D_1(\beta\zeta_S) + D_1(\beta\zeta_S) \\ 2C\rho'_1 &= \varpi\rho' \cdot D_1(\beta\zeta_S) + \varpi(\rho'_1+1)\cdot D_2(\beta\zeta_S)\end{aligned}$$

i.e.

$$\left(2C - \bar{D}_0\right)\rho' - \rho'_1\bar{D}_1 = \bar{D}_1 \quad ; \quad -\varpi\rho' \cdot \bar{D}_1 + \rho'_1\left(2C - \varpi \cdot \bar{D}_2\right) = \varpi \cdot \bar{D}_2$$

where $\bar{D}_\ell = D_\ell(\beta\zeta_S)$.

Solving the above two equations, we get

$$\rho' = \frac{\bar{D}_1}{\left(2C - \bar{D}_0\right) - \varpi M\bar{D}_2} \quad ; \quad \rho'_1 = \frac{\varpi M\bar{D}_2}{\left(2C - \bar{D}_0\right) - \varpi M\bar{D}_2}$$

Now, from the equations (II.5a) and (II.5b) of Appendix II, we can establish that

$$\bar{D}_1 = D_1(\beta\zeta_s) = \frac{1}{\beta\zeta_s}(\psi_0 - D_0(\beta\zeta_s)) = \frac{1}{\beta\zeta_s}(\psi_0 - \bar{D}_0)$$

and

$$\bar{D}_2 = D_2(\beta\zeta_s) = -\frac{1}{\beta^2\zeta_s^2}(\psi_0 - D_0(\beta\zeta_s)) = -\frac{1}{\beta^2\zeta_s^2}(\psi_0 - \bar{D}_0)$$

But,

$$\bar{D}_0 = D_0(\beta\zeta_s) = \sum_{s=1}^{m} \frac{\eta_s\omega_s}{\beta\zeta_s}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)$$

So,

$$\psi_0 - \bar{D}_0 = 2\sum_{s=1}^{m}\eta_s\omega_s\left(1 - \frac{1}{2\beta\zeta_s}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\right)$$

and

$$2C - \bar{D}_0 = 2\sum_{s=1}^{m}\eta_s\left(1 - \frac{\omega_s}{2\beta\zeta_r}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\right)$$

Therefore

$$\rho' = \frac{\beta\zeta_s(\psi_0 - \bar{D}_0)}{(2C - \bar{D}_0)\beta^2\zeta_s^2 + \varpi M(\psi_0 - \bar{D}_0)};$$
$$\rho'_1 = -\frac{\varpi M(\psi_0 - \bar{D}_0)}{(2C - \bar{D}_0)\beta^2\zeta_s^2 + \varpi M(\psi_0 - \bar{D}_0)}$$

So, the equation (2.122) becomes

$$h_{(r)i} = \frac{\beta\zeta_s L' + \beta^2\zeta_s^2 L''\mu_{(r)i}}{(L''\beta^2\zeta_s^2 + \varpi M L')(1 + \zeta_r\beta\mu_{(r)i})\mu_{(r)i}} \tag{2.123}$$

where

$$L' = \sum_{s=1}^{m}\eta_s\omega_s\left(1 - \frac{1}{2\beta\zeta_s}\log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\right) \tag{2.124}$$

and

$$L'' = \sum_{s=1}^{m} \eta_s \left(1 - \frac{\omega_s}{2\beta\zeta_s} \log\left(\frac{1+\beta\zeta_s}{1-\beta\zeta_s}\right)\right) \tag{2.125}$$

So, the equations (2.113) have particular integrals of the form:

$$I^*_{(r)i} = -\frac{\omega_r b_1 \varpi L \left(\beta\zeta_s L' + \beta^2\zeta_s^2 L'' \mu_{(r)i}\right)}{\left(L''\beta^2\zeta_s^2 + \varpi M L'\right)\left(1 + \zeta_r\beta\mu_{(r)i}\right)\mu_{(r)i}} \mu_{(r)i}\, e^{-\beta\tau} \tag{2.126}$$

The Complete Solution Of The Equation (2.113)

The complete solution of the equation (2.113) is given by the sum of the solution (2.117) of the equation (2.116) and the particular integral of (2.126) of the equation (2.113).

Following Chandrasekhar,[45] we can write the solution of the equation (2.113) in the form :

$$I^*_{(r)i} = \omega_r b_1 \sum_{\iota=1}^{n} \left\{ \frac{k_\iota\zeta_s - \varpi\, M\, \mu_{(r)i}}{k_\iota\zeta_s\left(1 + k_\iota\zeta_r\mu_{(r)i}\right)} L'_{(r)\iota} e^{-k_\iota\tau} \right.$$
$$\left. - \frac{\varpi L \left(\beta\zeta_s L' + \beta^2\zeta_s^2 L''\mu_{(r)i}\right)}{\left(L''\beta^2\zeta_s^2 + \varpi M L'\right)\left(1 + \zeta_r\beta\mu_{(r)i}\right)\mu_{(r)i}} \mu_{(r)i}\, e^{-\beta\tau} \right\}$$
$$\iota = 1.2.\cdots, n\, ;\; i = \pm1, \pm2, \cdots, \pm n \tag{2.127}$$

where k_ι's ($\iota = 1, 2, \cdots, n$) are the positive roots of the characteristic equation(2.38) and $L_{(r)\iota}$'s are the constants of integration to be determined by the boundary conditions (2.114).

2.4.3.2 Elimination of Constants and the Diffusely Reflected Intensity and the Solution for the Emergent Intensity in Closed Form.

From the equation (2.127), we get

$$I_r^*(\tau,\mu)=\omega_r b_1\sum_{\iota=1}^{n}\left\{\frac{k_\iota\zeta_s-\varpi\,M\,\mu}{k_\iota\zeta_s\,(1+k_\iota\zeta_r\mu)}\,L'_{(r)\iota}e^{-k_\iota\tau}\right.$$
$$\left.-\frac{\varpi L\,(\beta\zeta_s L'+\beta^2\zeta_s^2L'\mu)}{(L'\beta^2\zeta_s^2+\varpi ML')\,(1+\zeta_r\beta\mu)\,\mu}\mu\,e^{-\beta\tau}\right\}$$
$$\iota=1.2.\cdots,n\,;\; i=\pm1,\pm2,\cdots,\pm n \tag{2.128}$$

We define,

$$S_r(\mu)=\sum_{\iota=1}^{n}\left(\frac{(\,k_\iota\zeta_s+\varpi\,M\,\mu)\,L'_{(r)\iota}}{k_\iota\zeta_s\,(1-k_\iota\zeta_r\mu)}-\frac{\varpi L\,(\beta\zeta_s L'-\beta^2\zeta_s^2L'\mu)}{(L'\beta^2\zeta_s^2+\varpi ML')\,(1-\zeta_r\beta\mu)}\right)$$
$$+\frac{b_0}{\omega_r b_1}+\frac{1-\omega_r U}{\omega_r\,(1-\beta\mu\zeta_r)} \tag{2.129}$$

Now,

$$I_r^*(0;\mu)=\omega_r b_1\sum_{\iota=1}^{n}\left\{\frac{(\,k_\iota\zeta_s-\varpi\,M\,\mu)\,L'_{(r)\iota}}{k_\iota\zeta_s\,(1+k_\iota\zeta_r\mu)}\right.$$
$$\left.-\frac{\varpi L\,(\beta\zeta_s L'+\beta^2\zeta_s^2L'\mu)}{(L'\beta^2\zeta_s^2+\varpi ML')\,(1+\zeta_r\beta\mu)}\right\}$$
$$\iota=1,2,\cdots,n\,;\; i=\pm1,\pm2,\cdots,\pm n$$

i.e.,

$$I_r^*(0,\mu)=\omega_r b_1\cdot S_r(-\mu)-b_0-\frac{b_1\,(1-\omega_r U)}{(1+\beta\mu\zeta_r)}$$
$$\iota=1,2,\cdots,n\,;\; i=\pm1,\pm2,\cdots,\pm n \tag{2.130}$$

Now, the boundary conditions (2.114) are expressible as

$$I_r^*(0, -\mu_{(r)i}) = -b_0 - \frac{b_1\,(1-\omega_r U)}{1-\beta\mu_{(r)i}\zeta_r}$$

i.e.

$$\omega_r b_1 \sum_{\iota=1}^{n} \left\{ \frac{\left(k_\iota\zeta_s + \varpi\; M\; \mu_{(r)i}\right) L'_{(r)\iota}}{k_\iota\zeta_s\left(1-k_\iota\zeta_r\mu_{(r)i}\right)} \right.$$

$$\left. - \frac{\varpi L\left(\beta\zeta_s L' - \beta^2\zeta_s^2 L'\mu_{(r)i}\right)}{\left(L'\beta^2\zeta_s^2 + \varpi M L'\right)\left(1-\zeta_r\beta\mu_{(r)i}\right)} \right\} = -b_0 - \frac{b_1\,(1-\omega_r U)}{1-\beta\mu_{(r)i}\zeta_r}$$

i.e.

$$\omega_r b_1 \left\{ \sum_{\iota=1}^{n} \left(\frac{\left(k_\iota\zeta_s + \varpi\; M\; \mu_{(r)i}\right) L'_{(r)\iota}}{k_\iota\zeta_s\left(1-k_\iota\zeta_r\mu_{(r)i}\right)} \right. \right.$$

$$\left. \left. - \frac{\varpi L\left(\beta\zeta_s L' - \beta^2\zeta_s^2 L'\mu_{(r)i}\right)}{\left(L'\beta^2\zeta_s^2 + \varpi M L'\right)\left(1-\zeta_r\beta\mu_{(r)i}\right)} \right) + \frac{b_0}{\omega_r b_1} + \frac{1-\omega_r U}{\omega_r\left(1-\beta\mu_{(r)i}\zeta_r\right)} \right\} = 0$$

$$S_r\left(\mu_{(r)i}\right) = 0 \;\;,\;\; i = 1, 2, \cdots, n \qquad (2.131)$$

Now, the boundary conditions (2.131) help us to draw the conclusion that $\mu_{(r)i}\,;\, i = 1, 2, \cdots, n$ will be the zeros of any polynomial when it is multiplied by $S_r\,(\mu)$.

Now, we have

$$S_r\,(\mu)\, R_r\,(\mu)\,(1-\beta\mu\zeta_r) =$$

$$\left\{ \sum_{\iota=1}^{n} \left(\frac{\left(k_\iota\zeta_s + \varpi\; M\; \mu\right) L'_{(r)\iota}}{k_\iota\zeta_s\,(1-k_\iota\zeta_r\mu)} - \frac{\varpi L\,(\beta\zeta_s L' - \beta^2\zeta_s^2 L'\mu)}{(L'\beta^2\zeta_s^2 + \varpi M L')\,(1-\zeta_r\beta\mu)} \right) \right.$$

$$\left. + \frac{b_0}{\omega_r b_1} + \frac{1-\omega_r U}{\omega_r\,(1-\beta\mu\zeta_r)} \right\} \prod_{i=1}^{n} \left(1 - k_{(s)i}\zeta_r\mu\right)(1-\beta\mu\zeta_r)$$

is polynomials in μ of degree of $(n+1)$ in which the co-efficient of μ^{n+1} is

$$(-1)^{n+1}\cdot\left(\frac{b_0}{\omega_r b_1}+\varpi M\beta\zeta_r\sum_{s=1}^{n}\frac{L'_{(r)\iota}}{k_\iota\zeta_s}-\frac{\varpi\beta^2\zeta_s^2 LL'}{(L'\beta^2\zeta_s^2+\varpi ML')}\right)\cdot\beta\zeta_r\times \\ \times k_1\zeta_r\cdot k_2\zeta_r\cdots k_n\zeta_r$$

whose $(n+1)$ roots are $\mu_{(r)i}$; $i=1,2,\cdots,n$, and an additional root, say ξ'_r. Again $(\mu-\xi'_r)\,P_r(\mu)$ is also polynomials in μ of same degree $(n+1)$ in which the co-efficient of μ^{n+1} is 1.

So, we shall get

$$S_r(\mu)=(-1)^{n+1}\cdot q'_r\cdot\beta\zeta_r\,k_1\zeta_r\cdot k_2\zeta_r\cdots k_n\zeta_r\frac{1}{(1-\beta\mu\zeta_r)}\times \\ \times\frac{P_r(\mu)}{R_r(\mu)}(\mu-\xi'_r)\quad(2.132)$$

where ξ'_r is a constant, $P_r(\mu)$ and $R_r(\mu)$ are given by the equations (2.63) and (2.64) and q'_r is given by

$$q'_r=\frac{b_0}{\omega_r b_1}+\varpi M\beta\zeta_r\sum_{s=1}^{n}\frac{L'_{(r)\iota}}{k_\iota\zeta_s}-\frac{\varpi\beta^2\zeta_s^2 LL'}{(L'\beta^2\zeta_s^2+\varpi ML')}\qquad(2.133)$$

Moreover, we observe from the equation (2.129) that

$$\lim_{\mu\to(k_\iota\zeta_r)^{-1}}\frac{1}{T_\iota}(1-k_\iota\zeta_r\mu)\,S_r(\mu)$$

$$=\lim_{\mu\to(k_\iota\zeta_r)^{-1}}(1-k_\iota\zeta_r\mu)\left[\frac{1}{T_\iota}\left\{\sum_{\iota=1}^{n}\left(\frac{(k_\iota\zeta_s+\varpi\,M\,\mu)\,L'_{(r)\iota}}{k_\iota\zeta_s\,(1-k_\iota\zeta_r\mu)}\right.\right.\right. \\ \left.-\frac{\varpi L\,(\beta\zeta_s L'-\beta^2\zeta_s^2 L'\mu)}{(L'\beta^2\zeta_s^2+\varpi ML')\,(1-\zeta_r\beta\mu)}\right) \\ \left.\left.+\frac{b_0}{\omega_r b_1}+\frac{1-\omega_r U}{\omega_r\,(1-\beta\mu\zeta_r)}\right\}\right]$$

$$=L'_{(r)\iota}$$

where T_ι is given by the equation (2.60).

i.e.

$$L'_{(r)\iota} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \frac{1}{T_\iota} (1 - k_\iota \zeta_r \mu) \, S_r(\mu) \tag{2.134}$$

Therefore, using the equation(2.132) in the equation(2.134), we get

$$\begin{aligned} L'_{(r)\iota} = & \lim_{\mu \to (k_\iota \zeta_r)^{-1}} (1 - k_\iota \zeta_r \mu) (-1)^{n+1} \cdot \frac{1}{T_\iota} q'_r \cdot \beta \zeta_r \times \\ & \times k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \cdot \frac{1}{(1 - \beta \mu \zeta_r)} \cdot \frac{P_r(\mu)}{R_r(\mu)} (\mu - \xi'_r) \end{aligned}$$

i.e.

$$\begin{aligned} L'_{(r)\iota} = & \frac{(-1)^{n+1}}{T_\iota} \cdot q'_r \cdot k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \beta \zeta_r \cdot \times \\ & \times \frac{1}{\left(1 - \frac{\beta \zeta_r}{k_\iota \zeta_s}\right)} \cdot \frac{P_r\left(\frac{1}{k_\iota \zeta_s}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_s}\right)} \left(\frac{1}{k_\iota \zeta_s} - \xi'_r\right) \end{aligned} \tag{2.135}$$

where $R_{(r)\iota}(\mu)$ is given by the equation (2.68).

Summing up both sides of the equation (2.135) over ι, we get

$$\begin{aligned} \sum_{\iota=1}^{n} L'_{(r)\iota} = & (-1)^{n+1} q'_r \cdot k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \beta \zeta_r \cdot \times \\ & \times \sum_{\iota=1}^{n} \frac{P_r\left(\frac{1}{k_\iota \zeta_s}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_s}\right)} \cdot \frac{1}{\left(1 - \frac{\beta \zeta_r}{k_\iota \zeta_s}\right)} \cdot \frac{1}{T_\iota} \left(\frac{1}{k_\iota \zeta_s} - \xi_r\right) \end{aligned}$$

i.e.

$$\sum_{\iota=1}^{n} L'_{(r)\iota} = (-1)^{n+1} q'_r \cdot k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \beta \zeta_r g_r(0) \tag{2.136}$$

where $g_r(x)$ is given by the equation

$$g_r(x) = \sum_{\iota=1}^{n} \frac{P_r\left(\frac{1}{k_\iota \zeta_s}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_s}\right)} \cdot \frac{1}{\left(1 - \frac{\beta \zeta_r}{k_\iota \zeta_s}\right)} \cdot \frac{1}{T_\iota}\left(\frac{1}{k_\iota \zeta_s} - \xi_r\right) R_{(r)\iota}(x) \tag{2.137}$$

Now, we notice that the polynomial $g_r(x)$ is a polynomial of degree $(n-1)$ in x which takes the value

$$\frac{1}{T_\iota} P_r\left(\frac{1}{k_\iota \zeta_s}\right)\left(\frac{1}{k_\iota \zeta_s} - \xi_r\right) \cdot \frac{1}{\left(1 - \frac{\beta \zeta_r}{k_\iota \zeta_s}\right)}$$

$$\text{for } x = \frac{1}{k_\iota \zeta_s}, \ \iota = 1, 2, \cdots, n$$

So,

$$\left(1 + \xi_{rs} \varpi M x^2\right)\left(1 - \beta \zeta_r x\right) g_r(x) - P_r(x)\left(x - \xi_r\right) = 0$$
$$\text{for } x = \frac{1}{k_\iota \zeta_s}, \ \iota = 1, 2, \cdots, n \tag{2.138}$$

This helps us to conclude that the polynomial on the left hand side of the equation (2.138) must be divisible by the polynomial $R_r(x)$. Hence, we get the following relation:

$$\left(1 + \epsilon_{rs} \varpi M x^2\right)\left(1 - \beta \zeta_r x\right) g_r(x) - P_r(x)\left(x - \xi_r\right)$$
$$= R_r(x)\left(A'_r x^2 + B'_r x + C'_r\right) \tag{2.139}$$

Now, assuming $\varpi \neq 0$, let us put $x = +\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}$ and $x = -\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}$ in the equation (2.139) to get

$$\frac{P_r\left(+\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}\right)}{R_r\left(+\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}\right)}\left(\xi'_r - \frac{i}{\sqrt{\epsilon_{rs} \varpi M}}\right) = -\frac{A'_r}{\epsilon_{rs} \varpi M} + \frac{iB'_r}{\sqrt{\epsilon_{rs} \varpi M}} + C_r' \tag{2.140}$$

and

$$\frac{P_r\left(-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}{R_r\left(-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}\left(\xi'_r+\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)=-\frac{A'_r}{\epsilon_{rs}\varpi M}-\frac{iB'_r}{\sqrt{\epsilon_{rs}\varpi M}}+C_r' \tag{2.141}$$

Then adding the equation (2.140) and the equation (2.141),

$$-\frac{A'_r}{\epsilon_{rs}\varpi M}+C_r'=(-1)^n\left(\xi_r a'_r+\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}b'_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.142}$$

Again subtracting the equation (2.141) from the equation (2.140), we get

$$B'_r=(-1)^{n+1}\left(a'_r-i\xi'_r\sqrt{\epsilon_{rs}\varpi M}\,b'_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.143}$$

where a'_r and b'_r are respectively given by the equations (2.77) and (2.78) and H-function is given by the equation (2.79).

Again, putting $x = \frac{1}{\beta\zeta_r}$ in the equation(2.139) and using the equation (2.143), we get

$$A'_r\frac{1}{\beta^2\zeta_r^2}+C'_r = (-1)^n\cdot\left\{H_r\left(-\frac{1}{\beta\zeta_r}\right)\left(\xi_r'-\frac{1}{\beta\zeta_r}\right)+\frac{1}{\beta\zeta_r}\left(i\xi_r b'_r\sqrt{\epsilon_{rs}\varpi M}-a'_r\right)\right\}\cdot\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.144}$$

Now multiplying the equation (2.144) by the term $\frac{1}{\epsilon_{rs}\varpi M}$ and the equation (2.142) by the term $\frac{1}{\beta\zeta_r}$ and then adding, we get

$$C'_r=\frac{(-1)^n}{\beta^2\zeta_r^2+\epsilon_{rs}\varpi M}\left[\epsilon_{rs}\varpi M\left(\xi_r' a'_r+\frac{ib'_r}{\sqrt{\epsilon_{rs}\varpi M}}\right)\right.$$
$$\left.+\beta^2\zeta_r^2\left\{H_r\left(-\frac{1}{\beta\zeta_r}\right)\left(\xi_r'-\frac{1}{\beta\zeta_r}\right)-\frac{1}{\beta\zeta_r}\left(i\xi_r b'_r\sqrt{\epsilon_{rs}\varpi M}-a'_r\right)\right\}\right]\times$$
$$\times\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.145}$$

Now, putting $x = 0$ in the the equation (2.139), we get

$$g_r(0) = (-1)^{n+1}\,\xi'_r\,\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} + 1\cdot C_r'$$

Therefore, using the equation (2.145), we get

$$g_r(0) = \frac{(-1)^n}{\beta^2\zeta_r^2 + \epsilon_{rs}\varpi M}\Bigg[\xi_r'\left\{a_r'\epsilon_{rs}\varpi M - ib_r'\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M} - 1 + H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta^2\zeta_r^2\right\} - \left\{\beta\zeta_r H_r\left(-\frac{1}{\beta\zeta_r}\right) - a_r'\beta\zeta_r - ib_r'\sqrt{\epsilon_{rs}\varpi M}\right\}\Bigg]\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.146}$$

So, using the the equation (2.146) in the the equation (2.136) and applying the relation (2.57), we get, we have

$$\sum_{\iota=1}^{n} L'_{(r)\iota} = q_r'\frac{\beta\zeta_r}{\beta^2\zeta_r^2 + \epsilon_{rs}\varpi M}\Bigg[-\xi_r'\left\{a_r'\epsilon_{rs}\varpi M - ib_r'\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M} - 1 + H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta^2\zeta_r^2\right\} + \left\{\beta\zeta_r H_r\left(-\frac{1}{\beta\zeta_r}\right) - a_r'\beta\zeta_r - ib_r'\sqrt{\epsilon_{rs}\varpi M}\right\}\Bigg]\cdot\left\{M\left(1 - \frac{1}{3}\varpi(1-M)\right)\right\}^{1/2} \tag{2.147}$$

Now, putting $\mu = 0$ in the equation (2.129)

$$S_r(0) = \sum_{\iota=1}^{n} L'_{(r)\iota} - \frac{\varpi L L'\beta\zeta_s}{(L'\beta^2\zeta_s^2 + \varpi M L')} + \frac{b_0}{\omega_r b_1} + \frac{1-\omega_r U}{\omega_r} \tag{2.148}$$

Again from the equation (2.132)

$$S_r(\mu) = -\cdot\frac{\beta\zeta_r}{(1-\beta\mu\zeta_r)}\cdot q_r'\cdot k_1\zeta_r\cdot k_2\zeta_r\cdots k_n\zeta_r\cdot\times \times\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\cdot H_r(-\mu)(\mu - \xi'_r)$$

i.e. using the relation(2.57),

$$S_r(\mu) = -\frac{\beta\zeta_r}{(1-\beta\mu\zeta_r)} \cdot q'_r \cdot \left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}} \cdot H_r(-\mu)(\mu - \xi'_r) \tag{2.149}$$

So, putting $\mu = 0$, we get

$$S_r(0) = \left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}} \frac{1}{T_\iota} \cdot \beta\zeta_r \cdot q'_r \xi'_r \tag{2.150}$$

Now, comparing the equation (2.148) and the equation (2.150), we get

$$\sum_{\iota=1}^{n} L'_{(r)\iota} - \left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}} \cdot \beta\zeta_r \cdot q'_r \xi'_r$$
$$= \frac{\varpi L L' \beta\zeta_s}{(L'\beta^2\zeta_s^2 + \varpi M L')} - \frac{b_0}{\omega_r b_1} - \frac{1-\omega_r U}{\omega_r}$$

Therefore, using the equation (2.147),

$$\left[\beta\zeta_r\left\{\beta\zeta_r H_r\left(-\frac{1}{\beta\zeta_r}\right) - a'_r\beta\zeta_r - ib'_r\sqrt{\epsilon_{rs}\varpi M}\right\} \times\right.$$
$$\left.\times\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}\right] q'_r - \left[\beta\zeta_r\left\{a'_r\epsilon_{rs}\varpi M\right.\right.$$
$$\left.-ib'_r\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M} - 1 + H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta^2\zeta_r^2\right\} + \beta\zeta_r\cdot\left(\beta^2\zeta_r^2\right.$$
$$\left.\left.+\epsilon_{rs}\varpi M\right)\right]\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2} q'_r\xi'_r$$
$$= \frac{\varpi L L'\beta\zeta_s\left(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M\right)}{(L'\beta^2\zeta_s^2+\varpi M L')} - \frac{b_0\left(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M\right)}{\omega_r b_1}$$
$$-\frac{(1-\omega_r U)\left(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M\right)}{\omega_r} \tag{2.151}$$

Now, proceeding as we have proceed to get the equation (2.134), we can get the equation:

$$\frac{L'_{(r)\iota}}{k_\iota \zeta_r} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \frac{1}{T'_\iota} (1 - k_\iota \zeta_r \mu) S_r(\mu) \tag{2.152}$$

where T'_ι is given by

$$T'_\iota = k_\iota \zeta_s + \frac{\varpi M}{k_\iota \zeta_r} \tag{2.153}$$

Now, we notice that the polynomial $h_r(x)$ is a polynomial of degree $(n-1)$ in x which takes the value

$$\frac{1}{T'_\iota} P_r\left(\frac{1}{k_\iota \zeta_s}\right)\left(\frac{1}{k_\iota \zeta_s} - \xi_r\right) \cdot \frac{1}{\left(1 - \frac{\beta \zeta_r}{k_\iota \zeta_s}\right)}$$

$$\text{for } x = \frac{1}{k_\iota \zeta_s}, \ \iota = 1, 2, \cdots, n$$

So,

$$\left(1 + \xi_{rs} \varpi M x^2\right)(1 - \beta \zeta_r x) h_r(x) - P_r(x) x (x - \xi_r) = 0$$
$$\text{for } x = \frac{1}{k_\iota \zeta_s}, \ \iota = 1, 2, \cdots, n \tag{2.154}$$

This helps us to write the following equation:

$$\left(1 + \epsilon_{rs} \varpi M x^2\right)(1 - \beta \zeta_r x) h_r(x) - P_r(x) x (x - \xi_r)$$
$$= R_r(x)\left(A''_r x^2 + B''_r x + C''_r\right) \tag{2.155}$$

Let us put $x = +\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}$ and $x = -\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}$ in the equation (2.155) to get

$$\frac{P_r\left(+\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}\right)}{R_r\left(+\frac{i}{\sqrt{\epsilon_{rs} \varpi M}}\right)}\left(\frac{i\, \xi'_r}{\sqrt{\epsilon_{rs} \varpi M}} + \frac{1}{\epsilon_{rs} \varpi M}\right) = -\frac{A''_r}{\epsilon_{rs} \varpi M} + \frac{i B''_r}{\sqrt{\epsilon_{rs} \varpi M}} + C''_r \tag{2.156}$$

and

$$\frac{P_r\left(-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}{R_r\left(-\frac{i}{\sqrt{\epsilon_{rs}\varpi M}}\right)}\left(\frac{i\,\xi_r'}{\sqrt{\epsilon_{rs}\varpi M}}-\frac{1}{\epsilon_{rs}\varpi M}\right)=\frac{A_r''}{\epsilon_{rs}\varpi M}+\frac{iB_r''}{\sqrt{\epsilon_{rs}\varpi M}}-C_r'' \tag{2.157}$$

Then adding the equation (2.156) and the equation (2.157),

$$B_r''=(-1)^n\left(\xi_r'a_r'+ib_r'\sqrt{\epsilon_{rs}\varpi M}\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.158}$$

Again subtracting the equation (2.157) from the equation (2.156), we get

$$-\frac{A_r''}{\epsilon_{rs}\varpi M}+C_r''=(-1)^n\left(\frac{1}{\epsilon_{rs}\varpi M}a_r'-\frac{i\,\xi_r'}{\sqrt{\epsilon_{rs}\varpi M}}b_r'\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{2.159}$$

Again, putting $x=\frac{1}{\beta\zeta_r}$ in the equation(2.155), we get

$$A_r''\frac{1}{\beta^2\zeta_r^2}+B_r''\frac{1}{\beta\zeta_r}+C_r''=(-1)^n\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\cdot H_r\left(-\frac{1}{\beta\zeta_r}\right)\left(\frac{\xi_r}{\beta\zeta_r}-\frac{1}{\beta^2\zeta_r^2}\right)$$

Now, using the equation (2.158),

$$\begin{aligned}A_r''\frac{1}{\beta^2\zeta_r^2}+C_r'' &=(-1)^n\left\{H_r\left(-\frac{1}{\beta\zeta_r}\right)\left(\frac{\xi_r}{\beta\zeta_r}-\frac{1}{\beta^2\zeta_r^2}\right)\right.\\ &\left.-\left(\xi_r'a_r'+ib_r'\sqrt{\epsilon_{rs}\varpi M}\right)\frac{1}{\beta\zeta_r}\right\}\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\end{aligned} \tag{2.160}$$

Now multiplying the equation (2.160) by the term $\frac{1}{\xi_{rs}\varpi M}$ and the equation (2.159) by the term $\frac{1}{\beta\zeta_r}$ and then adding, we get

$$\begin{aligned}C_r''=\frac{(-1)^n}{\xi_{rs}\varpi M+\beta^2\zeta_r^2}&\left[\left\{H_r\left(-\frac{1}{\beta\zeta_r}\right)(\xi_r'\beta\zeta_r-1)\right.\right.\\ &\left.-\beta\zeta_r\left(\xi_r'a_r'+ib_r'\sqrt{\epsilon_{rs}\varpi M}\right)\right\}+(a_r'\\ &\left.-i\,\xi_r'\sqrt{\epsilon_{rs}\varpi M}\,b_r')\right]\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\end{aligned} \tag{2.161}$$

Now, putting $x = 0$ in the the equation (2.155), we get

$$h_r(0) = R_r(0)\, C_r'' = C_r'' \tag{2.162}$$

i.e.

$$\begin{aligned} h_r(0) = \frac{(-1)^n}{\xi_{rs}\varpi M + \beta^2\zeta_r^2} \Bigg[\Bigg\{ & H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta\zeta_r \\ & -i\sqrt{\epsilon_{rs}\varpi M}\, b_r' - \beta\zeta_r a_r' \Bigg\} \xi_r' + \Bigg\{ a_r' - i b_r' \beta\zeta_r \sqrt{\epsilon_{rs}\varpi M} \\ & \qquad -H_r\left(-\frac{1}{\beta\zeta_r}\right)\Bigg\}\Bigg] \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \end{aligned}$$

So, from the equation (2.152), we get

$$\begin{aligned} \sum_{\iota=1}^{n} \frac{L'_{(r)\iota}}{k_\iota \zeta_r} = \frac{q_r' \beta\zeta_r}{\xi_{rs}\varpi M + \beta^2\zeta_r^2} \Bigg[\Bigg\{ & \beta\zeta_r a_r' - H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta\zeta_r \\ & + i\sqrt{\epsilon_{rs}\varpi M}\, b_r' \Bigg\} \xi_r' + \Bigg\{ H_r\left(-\frac{1}{\beta\zeta_r}\right) - a_r' \\ & + i b_r' \beta\zeta_r \sqrt{\epsilon_{rs}\varpi M} \Bigg\}\Bigg] \left\{ M(1 - \frac{1}{3}\varpi(1-M)) \right\}^{1/2} \end{aligned}$$

Applying this to the equation(2.133), we get

$$\begin{aligned} & \frac{\varpi M \beta^2 \zeta_r^2}{\xi_{rs}\varpi M + \beta^2\zeta_r^2} \Bigg[\Bigg\{ \beta\zeta_r a_r' - H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta\zeta_r \\ & \qquad + i\sqrt{\epsilon_{rs}\varpi M}\, b_r' \Bigg\} q_r' \xi_r' + \Bigg\{ H_r\left(-\frac{1}{\beta\zeta_r}\right) - a_r' \\ & + i b_r' \beta\zeta_r \sqrt{\epsilon_{rs}\varpi M} - \left\{ M(1 - \frac{1}{3}\varpi(1-M)) \right\}^{-1/2} \Bigg\} q_r' \Bigg] \times \\ & \times \left\{ M(1 - \frac{1}{3}\varpi(1-M)) \right\}^{1/2} = \frac{\varpi\beta^2\zeta_s^2 L L'}{(L'\beta^2\zeta_s^2 + \varpi M L')} - \frac{b_0}{\omega_r b_1} \end{aligned} \tag{2.163}$$

Solving the equations (2.151) and (2.163), we get

$$\xi_r' = \frac{N_1}{D_1} \tag{2.164}$$

and

$$q_r' = \frac{N_2}{D_2} \tag{2.165}$$

with

$$\begin{aligned}
N_1 \;=\; & \left\{M(1-\frac{1}{3}\varpi\,(1-M))\right\}^{1/2}\left[\left\{H_r\left(-\frac{1}{\beta\zeta_r}\right)-a_r'\right.\right. \\
& \left.+ib_r'\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M}-\left\{M(1-\frac{1}{3}\varpi\,(1-M))\right\}^{-1/2}\right\}\times \\
& \times\left\{\frac{\varpi LL'\beta\zeta_s\,(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M)}{(L'\beta^2\zeta_s^2+\varpi ML')}-\frac{b_0\,(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M)}{\omega_r b_1}\right. \\
& \left.-\frac{(1-\omega_r U)\,(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M)}{\omega_r}\right\}-\beta\zeta_r\left\{\beta\zeta_r H_r\left(-\frac{1}{\beta\zeta_r}\right)\right. \\
& \left.\left.-a_r'\beta\zeta_r-ib_r'\sqrt{\epsilon_{rs}\varpi M}\right\}\left(\frac{\varpi\beta^2\zeta_s^2LL'}{(L'\beta^2\zeta_s^2+\varpi ML')}-\frac{b_0}{\omega_r b_1}\right)\right]
\end{aligned}$$

$$\begin{aligned}
D_1 \;=\; & N_2=\left\{M\left(1-\frac{1}{3}\varpi\,(1-M)\right)\right\}^{1/2}[[-\beta\zeta_r\,\{a_r'\epsilon_{rs}\varpi M \\
& +ib_r'\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M}+1-H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta^2\zeta_r^2\right\}-\beta\zeta_r\cdot(\beta^2\zeta_r^2 \\
& +\epsilon_{rs}\varpi M)]\left\{\frac{\varpi\beta^2\zeta_s^2LL'}{(L'\beta^2\zeta_s^2+\varpi ML')}-\frac{b_0}{\omega_r b_1}\right\}-\frac{\varpi M\beta^2\zeta_r^2}{\xi_{rs}\varpi M+\beta^2\zeta_r^2}[\{\beta\zeta_r a_r' \\
& -H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta\zeta_r+i\sqrt{\epsilon_{rs}\varpi M}\,b_r'\}]\left(\frac{\varpi LL'\beta\zeta_s\,(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M)}{(L'\beta^2\zeta_s^2+\varpi ML')}\right. \\
& \left.-\frac{b_0\,(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M)}{\omega_r b_1}-\frac{(1-\omega_r U)\,(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M)}{\omega_r}\right)]
\end{aligned}$$

$$
\begin{aligned}
D_2 = & \left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}[[-\beta\zeta_r\left\{a'_r\epsilon_{rs}\varpi M\right. \\
& \left.+ib'_r\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M}+1-H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta^2\zeta_r^2\right\} \\
& -\beta\zeta_r\cdot\left(\beta^2\zeta_r^2+\epsilon_{rs}\varpi M\right)]\left\{H_r\left(-\frac{1}{\beta\zeta_r}\right)-a'_r\right. \\
& \left.+ib'_r\beta\zeta_r\sqrt{\epsilon_{rs}\varpi M}-\left\{M(1-\frac{1}{3}\varpi\left(1-M\right))\right\}^{-1/2}\right\} \\
& -\frac{\varpi M\beta^2\zeta_r^2}{\xi_{rs}\varpi M+\beta^2\zeta_r^2}\left[\left\{\beta\zeta_r a'_r-H_r\left(-\frac{1}{\beta\zeta_r}\right)\beta\zeta_r\right.\right. \\
& \left.\left.+i\sqrt{\epsilon_{rs}\varpi M}\,b'_r\right\}\right]\beta\zeta_r\left\{\beta\zeta_r H_r\left(-\frac{1}{\beta\zeta_r}\right)\right. \\
& \left.-a'_r\beta\zeta_r-ib'_r\sqrt{\epsilon_{rs}\varpi M}\right\}]
\end{aligned}
$$

Again, from the equation (2.149)

$$
\begin{aligned}
S_r\left(-\mu\right) = & \cdot q'_r\cdot\beta\zeta_r\cdot\left\{M\left(1-\frac{1}{3}\,\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}}\frac{1}{\left(1+\beta\mu\zeta_r\right)}\times \\
& \left(\mu+\xi'_r\right)\,H_r\left(\mu\right) \qquad (2.166)
\end{aligned}
$$

$$
S_r\left(-\mu\right)=\frac{q'_r\cdot\beta\zeta_r}{\left(1+\beta\mu\zeta_r\right)}\left\{M\left(1-\frac{1}{3}\,\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}}\cdot H_r\left(\mu\right)\left(\mu+\xi'_r\right)
$$

So, we get from the equation(2.130), the diffusely reflected intensity as

$$
\begin{aligned}
I_r^*(0,\mu) = & \frac{q'_r\cdot\omega_r b_1\beta\zeta_r}{\left(1+\beta\mu\zeta_r\right)}\left\{M\left(1-\frac{1}{3}\,\varpi\left(1-M\right)\right)\right\}^{\frac{1}{2}}\cdot H_r\left(\mu\right)\left(\mu+\xi'_r\right) \\
& -b_0-\frac{b_1\left(1-\omega_r U\right)}{\left(1+\beta\mu\zeta_r\right)}
\end{aligned}
$$

i.e.

$$I_r^*(0,\mu) = G_r \cdot (\mu + \xi'_r) \cdot H_r(\mu) - b_0 - \frac{b_1(1-\omega_r U)}{(1+\beta\mu\zeta_r)} \tag{2.167}$$

where ξ'_r is given by the equation (2.164) and G_r is given by

$$G_r = \frac{q'_r \cdot \omega_r b_1 \beta \zeta_r}{(1+\beta\mu\zeta_r)} \left\{ M\left(1 - \frac{1}{3}\varpi(1-M)\right)\right\}^{\frac{1}{2}}$$

in which q'_r is given by the equation(2.165).

Again, from the equation (2.107)

$$I_r(0,\mu) = b_0 + \frac{b_1(1-\omega_r U)}{1+\beta\mu\zeta_r} + I_r^*(\tau,\mu)$$

i.e.

$$I_r(0,\mu) = G_r \cdot (\mu + \xi'_r) \cdot H_r(\mu) \tag{2.168}$$

2.4.4 Discussion

If we put $\varpi = 0$, then the equation (2.106) will be converted into the equation of Radiative transfer for the case of isotropic scattering medium. We shall derive here the result for the case, from our result obtained above, by substituting $\varpi = 0$. We observe that $a' \to \frac{1}{M^{1/2}}$ and $i\frac{1}{\sqrt{M\varpi}}b'_r \to \frac{1}{\sqrt{M}}\left(\sum_{i=1}^{n}\mu_{(r)i} - \sum_{\iota=1}^{n}\frac{1}{\zeta_r k_\iota}\right)$ as $\varpi \to 0$.

Now, allowing $\varpi \to 0$ in the equations (2.164)and (2.165), we get

$$\xi'_r = \frac{\beta\zeta_r\left[b_0 + b_1(1-\omega_r U)\left\{2 - M^{1/2}H_r\left(-\frac{1}{\beta\zeta_r}\right)\right\}\right]}{b_0 M^{1/2}\left[\left\{1 - \beta^2\zeta_r^2 H_r\left(-\frac{1}{\beta\zeta_r}\right)\right\} + \beta^2\zeta_r^2\right]} \tag{2.169}$$

and

$$q'_r = \frac{b_0}{\omega_r b_1\left\{M^{1/2}H_r\left(-\frac{1}{\beta\zeta_r}\right) - 2\right\}} \tag{2.170}$$

Putting $\varpi = 0$ in the equation (2.167) and the equation(2.168)

$$I_r^*(0,\mu) = \frac{b_0\beta\zeta_r\sqrt{M}\,(\mu+\xi_r')\,H_r\,(\mu)}{(1+\beta\mu\zeta_r)\left\{\sqrt{M}H_r\,(-1/\beta\zeta_r)-2\right\}} - b_0 - \frac{b_1\,(1-\omega_r U)}{(1+\beta\mu\zeta_r)}$$

and

$$I_r(0,\mu) = \frac{b_0\beta\zeta_r\sqrt{M}\,(\mu+\xi_r')\,H_r\,(\mu)}{(1+\beta\mu\zeta_r)\left\{\sqrt{M}H_r\,(-1/\beta\zeta_r)-2\right\}}$$

which give the diffusely reflected intensity and emergent intensity for the case of interlocked multiplet lines in isotropically scattering medium with an exponential atmosphere. This results were also obtained by Deb, Biswas and Karanjai.[64]

Chapter 3

Interlocked Multiplets with Rayleigh and Rayleigh Like Phase Functions

Rayleigh Like Phase Functions

This chapter is devoted mainly to Rayleigh Phase Function. In 1871 Lord Rayleigh introduced a phase function[45] which is the simplest and in some ways the most important example of a physical law of light scattering, while in searching for the cause of the blueness of the sky. However it has wider applications. We have used this scattering law of light to find the emergent intensity from the radiative transfer equation in section -I of this chapter.

In 1998, Pomraning[164] introduced a new phase function which appears to be Rayleigh like differing only with one more parameter λ, representing albedo, from that given by Rayleigh. This scattering law is used to derive the emergent intensity from a radiative transfer equation in section - 3.2.2.1 of this chapter.

We observe that by substituting albedo $\lambda = 1$ in the phase function which we call as Pomraning phase function, we can derive the Rayleigh phase function. As the manipulation involved in the problem related with Pomraning phase function we have dealt with here is complicated enough, the same problem with Rayleigh phase function is discussed in the section-3.1.2

3.1 Rayleigh Phase Function

3.1.1 Introduction

Rayleigh's law of scattering of light states that when a pencil of natural light of wave length λ , intensity I, and solid angle $d\omega$, is incident on a particle of polaizability α, energy at the rate

$$\frac{128\pi^5}{3\lambda^4}\alpha^2 I d\omega \times \frac{3}{4}\left(1 + cos^2\Theta\right)\frac{d\omega'}{4\pi}$$

is scattered in the direction making an angle Θ with the direction of the incidence and in a solid angle $d\omega'$.

The Rayleighy's phase function, normalized to unity so that it becomes a case of perfect scattering, is

$$p\left(cos\Theta\right) = \frac{3}{4}\left(1 + cos^2\Theta\right) \tag{3.1}$$

In conservative cases of perfect scattering, net-flux is constant and due to this constancy in net flux, we meet with the type of problem of a semi-infinite plane parallel atmospheres with no incident radiation and with a constant net flux, πF,of radiation which is provided by the deep interior of the atmosphere flowing through it normal to the plane of stratification. For this type of problem for the axial symmetry about the z-axis, the intensity and the source function are azimuth independent and therefore, the phase function involved in the problem must be azimuth independent.

The Rayleigh's phase function is

$$p(\mu, \phi; \mu', \phi') = \frac{3}{4}\left[1 + \mu^2\mu'^{\,2} + \left(1 - \mu^2\right)\left(1 - \mu'^{\,2}\right) cos^2(\phi - \phi')\right.$$
$$\left. +2\mu\mu'\left(1 - \mu^2\right)^{\frac{1}{2}}\left(1 - \mu'^{\,2}\right)^{\frac{1}{2}} cos(\phi - \phi')\right]$$

If it is azimuth independent

$$p^{(0)}(\mu, \mu') = \frac{3}{4}\left[1 + \mu^2\mu'^{\,2} + \frac{1}{2}\left(1 - \mu^2\right)\left(1 - \mu'^{\,2}\right)\right]$$

or,

$$p^{(0)}(\mu, \mu') = \frac{3}{8}\left[3 - \mu^2 + (3\mu' - 1)\,\mu'^{\,2}\right]$$

El-Wakil, Degheidy and Sallah[71] considered the problem of time-dependent radiation transfer in a semi-infinite plane-parallel random medium with Rayleigh scattering phase function including polarization.

Karanjai and Biswas[108] solved the equation of radiative transfer with scattering according to Rayleigh's phase function by a modified version of the spherical-harmonic method proposed by Wan Wilson and Sen.[217]

Saad, El-Wakil and Haggag[175] calculated diffuse reflection and transmission coefficients in a plane parallel medium for a Rayleigh phase function averaged over polarization and Rayleigh polarized phase function by embedding the finite medium into a semi-infinite scattering and absorbing medium.

Constructing the solution of the equation of transfer in a semi-infinite atmosphere with no incident radiation for Rayleigh's phase function by the method of the "Principles of Invariance" and using the law of diffuse reflection, Karanjai and Barman[106] applied it to find the laws of darkening for Rayleigh's phase function .

Sweigart[200] solved numerically the Chandrasekhar's integral equations of radiative transfer for both conservative and non-conservative atmosphere in which the scattering is governed by Rayleigh phase function, polarization of the scattered radiation not being treated.

Das[58] used Chandrasekhar[45]'s equation of radiative transfer for diffuse reflection and transmission with Rayleigh's phase function in a finite atmosphere to have the solution in terms of Chandrasekhar's X-Y equations exactly for emergent distributions from the bounding faces by employing the Laplace transform and the theory of singular operators, as outlined by Das[57] himself.

Pomraning[163] showed that the albedo is insensitive to whichever of the three descriptions of scattering such as an isotropic phase function, or Rayleigh phase function averaged over polarization, or Rayleigh scattering properly accounting for polarization, is used in the half-space.

Chandrasekhar and Breen[46] tabulated the various H-functions which occur in the solutions of the transfer problems involving Rayleigh's phase function and Rayleigh's scattering (including the state of polarization of the scattered radiation) and determined numerically the H-functions as the solutions of the exact functional equations which they satisfy.

Here, attempt has been made to derive the emergent intensity and diffusely reflected intensity from the radiative transfer equation of interlocked multiplet lines with Rayleigh's phase function.

3.1.2 Formulation of the Problem

3.1.2.1 The Equation of Radiative Transfer with Rayleigh Phase Function

The equation transfer for the rth interlocked line is

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)B_\nu(T)$$
$$-\frac{1}{2}(1-\varepsilon)\alpha_r\sum_{s=1}^{m}\eta_s\int_{-1}^{1} p\,(\mu,\mu')I_s(\tau,\mu')d\mu' \qquad (3.2)$$

where $\alpha_r's$ $(r=1,\ldots,m)$ are of the form:

$$\alpha_r = \eta_r/\sum_{s=1}^{m}\eta_s \qquad (3.3)$$

so that

$$\sum_{r=1}^{m}\alpha_r = 1 \qquad (3.4)$$

the Planck-function, $B_\nu(T)$, considered in this case, is of the form:

$$B_\nu(T) = B(\tau) = b_0 + b_1\tau \qquad \text{[Linear form]} \qquad (3.5)$$

b_0 and b_1, being positive constants and the (azimuth independent) Rayleigh phase function $p(\mu, \mu')$, taken here, is given by

$$\begin{aligned} p(\mu, \mu') = 1 + \frac{1}{2} P_2(\mu) P_2(\mu') &= \frac{3}{4}\left[1 + \mu^2\mu'^2 + \frac{1}{2}\left(1 - \mu^2\right)\left(1 - \mu'^2\right)\right] \\ &= \frac{3}{8}\left[\left(3 - \mu^2\right) + \left(3\mu^2 - 1\right)\mu'^2\right] \qquad (3.6) \end{aligned}$$

and ε, the co-efficient, is introduced to allow for thermal emission associated with the line absorption.

Using the relations (3.6) and (3.5) in the equation (3.2), we get

$$\begin{aligned} \mu \frac{dI_r(\tau, \mu)}{d\tau} &= (1 + \eta_r) I_r(\tau, \mu) - (1 + \varepsilon\eta_r)(b_0 + b_1\tau) \\ &\quad - \frac{3}{16}(1 - \varepsilon)\alpha_r \sum_{s=1}^{m} \eta_s \int_{-1}^{1} \left\{\left(3 - \mu^2\right)\right. \\ &\quad \left. + \left(3\mu^2 - 1\right)\mu'^2\right\} I_s(\tau, \mu') d\mu' \qquad (3.7) \end{aligned}$$

3.1.2.2 The Boundary Conditions

The boundary conditions for solving the equation (3.7) are

$$I_r(0, -\mu) = 0; \; (0 < \mu \leq 1) \qquad (3.8)$$

and

$$I_r(\tau, \mu) \cdot e^{-\tau/\mu} \to 0$$

i.e. $I_r(\tau, \mu)$ is at most linear in τ as τ tends to infinity . (3.9)

3.1.2.3 Reduction of the Equation of the Transfer to a Simpler Form

As Busbridge and Stibbs[33] assumed for solving an equation of transfer of the type (3.7), we assume that one of the solution of the equation (3.7) is

$$I_r(\tau,\mu) = b_0 + b_1\left(\tau + \frac{\mu}{1+\eta_r}\right) + I_r^*(\tau,\mu) \tag{3.10}$$

which consists of two parts, the first part being the solution of for an infinite unbounded atmosphere as τ tends to infinity and the second part $I_r^*(\tau,\mu)$ being the departure of the asymptotic solution from the value $I_r(\tau,\mu)$ as we approach the boundary $\tau = 0$.

Then we shall get

$$\frac{\mu}{1+\eta_r}\cdot\frac{dI_r^*(\tau,\mu)}{d\tau} = I_r^*(\tau,\mu) - \frac{3}{16}\cdot\frac{(1-\varepsilon)\eta_r}{1+\eta_r}\cdot\frac{\alpha_r}{\eta_r}\sum_{s=1}^{m}\eta_s\int_{-1}^{1}\{(3 - \mu^2) + \left(3\mu^2 - 1\right)\mu'^2\}\, I_s^*(\tau,\mu')d\mu' \tag{3.11}$$

Now, writing

$$\zeta_r = \frac{1}{1+\eta_r} \tag{3.12}$$

$$\omega_r = \frac{(1-\varepsilon)\,\eta_r}{1+\eta_r} \tag{3.13}$$

and

$$C = \frac{\eta_r}{\alpha_r} = \sum_{s=1}^{m}\eta_s \tag{3.14}$$

the equation (3.11) can be written as :

$$\zeta_r\mu\cdot\frac{dI_r^*(\tau,\mu)}{d\tau} = I_r^*(\tau,\mu) - \frac{3}{16C}\cdot\omega_r\cdot\sum_{s=1}^{m}\eta_s\int_{-1}^{1}\{\left(3-\mu^2\right) + \left(3\mu^2 - 1\right)\mu'^2\}\, I_s^*(\tau,\mu')d\mu' \tag{3.15}$$

The boundary conditions for solving the equation (3.15) will be

$$I_r^*(0,-\mu) = b_1\zeta_r\mu - b_0;\ (0<\mu\leq 1) \tag{3.16}$$

$I_r^*(\tau,\mu)$ is at most linear in τ as τ tends to infinity. (3.17)

[From the equations (3.8) and (3.9)

3.1.3 Solution of the Equation

3.1.3.1 Diffusely Reflected Intensity and Emergent Intensity in nth Approximation

In the n^{th} approximation, we replace the integro-differential equation (3.15) by the system of 2n linear differential equations:

$$\zeta_r\mu_{(r)i}\cdot\frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{3}{16C}\cdot\omega_r\cdot\sum_{s=1}^{m}\eta_s\sum_j\left\{\left(3-\mu^2_{(r)i}\right)\right. \\ \left. + \left(3\mu^2_{(r)i}-1\right)\mu^2_{(s)j}\right\} I^*_{(s)j}a_j \tag{3.18}$$

where $\mu_{(r)i}$'s, $(i=\pm1,\pm2,\cdots,\pm n)$; [having the property that $\mu_{(r)-i}=-\mu_{(r)i}$] are the zeros of the Legendre polynomial $P_{2n}(\mu)$ which are independent of the lines of interlocking and a_j's $(j=\pm1,\pm2,\cdots,\pm n)$; [having the property that $a_{-j}=a_j$] are the corresponding Gaussian weights . However it is to be noted that there is no term with $j=0$. For the simplicity we have used

$$I^*_{(r)i}\ \text{for}\ I_r^*(\tau,\mu_{(r)i}) \tag{3.19}$$

in the equation (3.18).

The system of equations (3.18) admits integrals of the form:

$$I^*_{(r)i} = g_{(r)i}\, e^{-k\tau}\ ;\ i = \pm 1, \cdots, \pm n \tag{3.20}$$

So, use of the equations (3.20) in the equation (3.18) gives

$$g_{(r)i}\left\{1 + k\zeta_r\mu_{(r)i}\right\} = \frac{3\omega_r}{16C}\sum_{s=1}^{m}\eta_s\left\{\sum_j\left(3 - \mu^2_{(s)j}\right) + \mu^2_{(r)i}\sum_j\left(3\mu^2_{(s)j} - 1\right)\right\} g_{(s)j}a_j \tag{3.21}$$

Hence,

$$g_{(r)i} = \omega_r\frac{\rho + \rho_1\mu^2_{(r)i}}{1 + k\zeta_r\mu_{(r)i}} \tag{3.22}$$

where ρ and ρ_1 are constants which are independent of $\mu_{(r)i}$.

If we use the equation (3.22) once again in the equation (3.21), then we get

$$\begin{aligned}
\rho + \rho_1\mu^2_{(r)i} = \\
\frac{3}{16C}\Bigg[\Bigg\{&\rho\left(3\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j}{1 + k\zeta_s\mu_{(s)j}} - \sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu^2_{(s)j}}{1 + k\zeta_s\mu_{(s)j}}\right)\\
&+\rho_1\left(3\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu^2_{(s)j}}{1 + k\zeta_s\mu_{(s)j}} - \sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu^4_{(s)j}}{1 + k\zeta_s\mu_{(s)j}}\right)\Bigg\}\\
&+\Bigg\{\rho\left(3\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j}{1 + k\zeta_s\mu_{(s)j}}\mu^2_{(s)j} - \sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j}{1 + k\zeta_s\mu_{(s)j}}\right)\\
&+\rho_1\left(3\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu^4_{(s)j}}{1 + k\zeta_s\mu_{(s)j}} - \sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu^2_{(s)j}}{1 + k\zeta_s\mu_{(s)j}}\right)\Bigg\}\mu^2_{(r)i}\Bigg]
\end{aligned}$$

Now, defining

$$D_\ell(x) = \sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j\mu^\ell_{(s)j}}{1 + \mu_{(s)j}\,x} \tag{3.23}$$

and analysing the proof done in subsection. 3 by Busbridge and Stibbs,[33] we can write the above equation as

$$\rho+\rho_1\mu_{(r)i}^2=\frac{3}{16C}\left[\left\{\rho\left(3D_0\left(\zeta_s k\right)-D_2\left(\zeta_s k\right)\right)+\rho_1\left(3D_2\left(\zeta_s k\right)-D_4\left(\zeta_s k\right)\right)\right\}+\left\{\rho\left(3D_2\left(\zeta_s k\right)-D_0\left(\zeta_s k\right)\right)+\rho_1\left(3D_4\left(\zeta_s k\right)-D_2\left(\zeta_s k\right)\right)\right\}\mu_{(r)i}^2\right] \tag{3.24}$$

Since the relation (3.24) is valid for all $\mu_{(r)i}$, we must, therefore, have two relations

$$\left(16C-9D_0+3D_2\right)\rho+\left(3D_4-9D_2\right)\rho_1 = 0 \tag{3.25a}$$

and

$$\left(3D_0-9D_2\right)\rho+\left(16C-9D_4+3D_2\right)\rho_1 = 0 \tag{3.25b}$$

in which notation D_ℓ's is used for $D_\ell\left(k\zeta_s\right)$.

Now, eliminating ρ and ρ_1 in between the relations (3.25a) and (3.25b), we get

$$256C^2-48C\left(3D_0-2D_2+3D_4\right)+72\left(D_0D_4-D_2^2\right) \tag{3.26}$$

Therefore, using the equations (II.20b) and (II.20g),

$$9\left(2C-\psi_0\right)D_4+3\left(\psi_0-4C\right)D_2+18CD_0=32C^2 \tag{3.27}$$

Using the equation (II.3b) and (II.3d), we can write the above equation as

$$D_0=\frac{9\left(\psi_0-2C\right)\psi_0+6C\psi_0k^2\zeta_s^2-32C^2k^4\zeta_s^4}{9\left(\psi_0-2C\right)+3\left(4C-\psi_0\right)k^2\zeta_s^2-18Ck^4\zeta_s^4} \tag{3.28}$$

Again, the equation (3.27) is reducible to the form : -

$$\sum_{s=1}^{m} \eta_s \omega_s \sum_{j} \frac{a_j}{1 + k\zeta_s \mu_{(s)j}} \left\{ \frac{9M}{16C} \mu_{(s)j}^4 - \frac{3}{16C} (1 + M) \mu_{(s)j}^2 + \frac{9}{16C} \right\} = 1 \quad (3.29)$$

where

$$M = \left(\sum_{s=1}^{k} \eta_s (1 - \eta_s) \right) / \sum_{s=1}^{k} \eta_s = \frac{2C - \psi_0}{2C} \quad (3.30)$$

The equation (3.29) is the **characteristic equation** which, being an equation in k of order 2n, will give 2n distinct roots occurring in pair as $\pm k_\iota$, $(\iota = 1, 2, \cdots, n)$.

Now, from the relations (3.25a) and (3.25b)

$$\rho_1 = -\frac{(8C - 3\psi_0) + 3\left(1 + \frac{1}{k^2\zeta_s^2}\right)(\psi_0 - D_0)}{\left(8C + \frac{\psi_0}{k^2\zeta_s^2}\right) + 3\left(\frac{1}{k^4\zeta_s^4} + \frac{1}{k^2\zeta_s^2}\right)(\psi_0 - D_0)} \rho \quad (3.31)$$

But,

$$\psi_0 - D_0 = \frac{3(2C - \psi_0)\psi_0 k^2\zeta_s^2 - 2C(9\psi_0 - 16C)k^4\zeta_s^4}{9(\psi_0 - 2C) + 3(4C - \psi_0)k^2\zeta_s^2 - 18Ck^4\zeta_s^4} \quad (3.32)$$

So, using the relation (3.32) in the equation (3.31), we get

$$\rho_1 = -\frac{k^4\zeta_s^4 - 4Mk^2\zeta_s^2 + 3M^2}{3Ck^4\zeta_s^4 - 4Mk^2\zeta_s^2 + M^2} \rho$$

i.e.

$$\rho_1 = \frac{3M - k^2\zeta_s^2}{3k^2\zeta_s^2 - M} \rho \quad (3.33)$$

So, from the equation (3.22), using the equation (3.33), we get

$$g_{(r)i} = \omega_r \rho \frac{(3k^2\zeta_s^2 - M) + \mu_{(r)i}^2 (3M - k^2\zeta_s^2)}{(3k^2\zeta_s^2 - M)\left(1 + k\zeta_r \mu_{(r)i}\right)} \tag{3.34}$$

Therefore the equation (3.18) admits 2n- independent integrals of the form:

$$I_{(r)i}^* = \omega_r \rho \frac{(3k_\iota^2\zeta_s^2 - M) + \mu_{(r)i}^2 (3M - k_\iota^2\zeta_s^2)}{(3k_\iota^2\zeta_s^2 - M)\left(1 \pm k_\iota\zeta_r \mu_{(r)i}\right)} e^{\mp k_\iota \tau}, \quad \iota = 1, 2, \cdots, n \tag{3.35}$$

According to Chandrasekhar,[45] the general solution of the system of equations (3.18) can be written in the form :

$$I_r^*(\tau, \mu) = \omega_r b_1 \sum_{\iota=1}^{n} \left\{ \frac{(3k_\iota^2\zeta_s^2 - M) + \mu_{(r)i}^2 (3M - k_\iota^2\zeta_s^2)}{(3k_\iota^2\zeta_s^2 - M)\left(1 + k_\iota\zeta_r \mu_{(r)i}\right)} \right\} L_{(r)\iota}\, e^{-k_\iota \tau} \tag{3.36}$$

where k_ι's, $(\iota = 1, 2, \cdots, n)$, are the positive roots of the Characteristic Equation (3.29) and $L_{(r)\iota}$'s are the constants of integration to be determined by the boundary conditions (3.16) i.e.

$$I_{(r)-i}^* = b_1 \zeta_r \mu_{(r)i} - b_0; \; (0 < \mu_{(r)i} \leq 1) \tag{3.37}$$

3.1.3.2 Relation between the roots of the characteristic equation with the zeros of an even order Legendre-polynomial:

The proof of the result (3.43) is established below:

If $p_{2\ell}$ be the co-efficient of $\mu^{2\ell}$ in the Legendre polynomial $P_{2n}(\mu)$. Then we have

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = \sum_{\ell=0}^{n} p_{2\ell} \sum_{s=1}^{k} \eta_s \omega_s \sum_j \frac{a_j \mu_{(s)j}^{2\ell}}{1 + \zeta_s k \mu_{(s)j}}$$

by the use of the definition (3.23)

i.e.

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}\left(\zeta_s k\right) = \sum_{s=1}^{k} \eta_s \omega_s \sum_{j} \frac{a_j}{1+\zeta_s k \mu_{(s)j}} \sum_{\ell} p_{2\ell}\, \mu_{(s)j}^{2\ell} \tag{3.38}$$

Since, $\mu_{(s)j}$'s are the zeros of the Legendre polynomial $P_{2n}(\mu)$.

So,

$$\sum_{\ell} p_{2\ell}\, \mu_{(s)j}^{2\ell} = 0 \tag{3.39}$$

and therefore, the equation (3.38) becomes

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}\left(\zeta_s k\right) = 0$$

i.e.

$$p_{2n}\left\{-\frac{1}{k^{2n}\zeta_s^{2n}}\left(\psi_0 - D_0\right) - \frac{1}{k^{2n-2}\zeta_s^{2n-2}}\psi_2 + \cdots\right\} + \cdots + p_0 D_0 = 0$$

using the equation (II.1) of Appendix II,

which, on using the equation (II.4c) and (3.28) becomes

$$p_{2n}\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}\Upsilon^{n} + \cdots + p_0 = 0 \tag{3.40}$$

which is an equation in $\Upsilon = \frac{1}{\left(\zeta_s^2 k^2\right)}$ of degree n.

Since the roots of the equation (3.40) are $\frac{1}{\left(\zeta_s^2 k_1^2\right)}, \ldots, \frac{1}{\left(\zeta_s^2 k_n^2\right)}$

Therefore, we get

$$\left(\zeta_s k_1 \cdots \zeta_s k_n\right)^2 = (-1)^n \left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}\frac{p_{2n}}{p_0} \tag{3.41}$$

Again, $\mu_{(s)1}^2, \ldots, \mu_{(s)n}^2$ are the zeros of the Legendre polynomial $\sum_{\ell=0}^{n} p_{2\ell}\, \mu_{(s)j}^{2\ell}$ i.e. $p_{2n}\left(\mu_{(s)j}^2\right)^2 + \cdots + p_0$

So, we get

$$\left(\mu_{(s)1}\mu_{(s)2}\cdots\mu_{(s)n}\right)^2 = (-1)^n \frac{p_0}{p_{2n}} \tag{3.42}$$

Multiplying the equation (3.41) and (3.42) together, we get

$$\zeta_s k_1 \cdots \zeta_s k_n \cdot \mu_{(s)1}\mu_{(s)2}\cdots\mu_{(s)n} = \left\{ M \left(1 - \frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}} \tag{3.43}$$

3.1.3.3 The Elimination of Constants and the Expression of the Law of Diffuse Reflection in Closed Form

We define

$$S_r(\mu) = \sum_{\iota=1}^{n} \left\{ \frac{(3k_\iota^2\zeta_s^2 - M) + \mu^2 (3M - k_\iota^2\zeta_s^2)}{(3k_\iota^2\zeta_s^2 - M)(1 + k_\iota\zeta_r\mu)} \right\} L_{(r)\iota} - \frac{\zeta_r\mu}{\omega_r} + \frac{b_0}{b_1\omega_r} \tag{3.44}$$

So, the boundary conditions (3.16) can be expressed as

$$S_r\left(\mu_{(r)i}\right) = 0 \ ; \ i = 1, 2, \cdots, n \ ; \ \left(0 < \mu_{(r)i} \leq 1\right) \tag{3.45}$$

Now, we can express $I_r^*(0, \mu)$ in terms of $S_r(\mu)$ as follows:

$$I_r^*(\tau, \mu) = \omega_r b_1 \left\{ S_r(-\mu) - \frac{\zeta_r\mu}{\omega_r} - \frac{b_0}{b_1\omega_r} \right\} \tag{3.46}$$

Now, by the boundary conditions (3.45), we can write that $\mu_{(r)i}$; $i = 1, 2, \cdots, n$, are the zeros of the polynomial $S_r(\mu)$ which is a polynomial of degree $(n+1)$. Now, defining $R_r(\mu)$ as $R_r(\mu) = \prod_{\iota=1}^{n}(1 - \zeta_r k_\iota \mu)$, we find that $S_r(\mu)\, R_r(\mu)$ is a polynomial of degree $(n+1)$ in μ which vanishes for $\mu = \mu_{(r)i}$; $i = 1, 2, \cdots, n$. So, $\mu_{(r)1}$, $\mu_{(r)2}$, ..., $\mu_{(r)n}$ are the zeros of the polynomial $S_r(\mu)\, R_r(\mu)$. It has one more zero, say, ξ_r.

Also, we see that the polynomial $P_r(\mu) = \prod_{i=1}^{n} \left(\mu - \mu_{(r)i}\right)$ is a polynomial of degree n having its zeros as $\mu_{(r)1}, \mu_{(r)2}, \ldots, \mu_{(r)n}$.

So, the polynomials $S_r(\mu)\, R_r(\mu)$ and $P_r(\mu)\,(\mu - \xi_r)$ have the same zeros $\mu_{(r)1}, \mu_{(r)2}, \ldots, \mu_{(r)n}$ and ξ_r.

But, the co-efficient of μ^{n+1} in $S_r(\mu)\ R_r(\mu)$ is $q_r \zeta_r k_1 \cdots \zeta_r k_n$, where $q_r = (-1)^{n-1} \left(\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{Q_\iota \zeta_r k_\iota} + \frac{\zeta_r}{\omega_r} \right)$

Again $P_r(\mu)$ is a polynomial of degree n and the co-efficient of μ^n in it is 1 so that the co-efficient of μ^{n+1} in $P_r(\mu)\,(\mu - \xi_r)$ is also 1.

Hence,

$$S_r(\mu) = q_r \zeta_r k_1 \cdots \zeta_r k_n \frac{P_r(\mu)}{R_r(\mu)} (\mu - \xi_r) \tag{3.47}$$

where

$$P_r(\mu) = \prod_{i=1}^{n} \left(\mu - \mu_{(r)i}\right) \tag{3.48}$$

$$R_r(\mu) = \prod_{\iota=1}^{n} (1 - \zeta_r k_\iota \mu) \tag{3.49}$$

and

$$q_r = (-1)^{n-1} \left(\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{Q_\iota \zeta_r k_\iota} + \frac{\zeta_r}{\omega_r} \right) \tag{3.50}$$

in which

$$Q_\iota = \frac{3k_\iota^2 \zeta_s^2 - M}{3M - k_\iota^2 \zeta_s^2} \tag{3.51}$$

Again, we can write the equation (3.47) in the form:

$$S_r(\mu) = (-1)^n q_r \zeta_r k_1 \cdots \zeta_r k_n\, \mu_{(r)1} \cdots \mu_{(r)n} (\mu - \xi_r)\, H(-\mu) \tag{3.52}$$

where

$$H_r(\mu) = \frac{1}{\mu_{(r)1} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} \left(\mu + \mu_{(r)i}\right)}{\prod_{\iota=1}^{n} \left(1 + k_\iota \zeta_r \mu\right)} \tag{3.53}$$

So, using the relation (3.43), the equation (3.52) is equivalent to

$$S_r(\mu) = (-1)^n q_r \left\{ M \left(1 - \frac{1}{10}(1 - M)\right) \right\}^{\frac{1}{2}} (\mu - \xi_r) H_r(-\mu) \tag{3.54}$$

Again, we observe that

$$L_{(r)\iota} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \frac{1}{T_\iota} (1 - \zeta_r k_\iota \mu) S_r(\mu) \tag{3.55}$$

provided that

$$T_\iota = \frac{3k_\iota^4 \zeta_s^2 \zeta_r^2 - \left(k_{(s)\iota}^2 \zeta_s^2 + M k_{(s)\iota}^2 \zeta_r^2\right) + 3M}{3k_\iota^4 \zeta_s^2 \zeta_r^2 - M k_{(s)\iota}^2 \zeta_r^2} \tag{3.56}$$

Now, using the equation (3.47) in the equation (3.55), we get

$$L_{(r)\iota} = \frac{1}{T_\iota} q_r \zeta_r k_1 \cdots \zeta_r k_n \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \tag{3.57}$$

where

$$R_{(r)\iota}(x) = \prod_{\beta(\neq \iota)=1}^{n} (1 - \zeta_r k_\beta x) \quad 1 \leq \iota \leq n \tag{3.58}$$

Summing up to both sides, we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} = q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r f_r(0) \tag{3.59}$$

where

$$f_r(x) = \sum_{\iota=1}^{n} \frac{1}{T_\iota} \cdot \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \cdot \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) R_{(r)\iota}(x) \qquad (3.60)$$

so that

$$f_r(0) = \sum_{\iota=1}^{n} \frac{1}{T_\iota} \cdot \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \cdot \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right)$$

Now, $f_r(x)$, defined as in equation (3.60), is a polynomial of degree $(n-1)$ in x which takes the value

$$\frac{1}{T_\iota} P_r\left(\frac{1}{k_\iota \zeta_r}\right)\left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \text{ for } x = \frac{1}{k_\iota \zeta_r} \quad \iota = 1, 2, \cdots, n$$

So, we get

$$\frac{\left(3 - M\,\epsilon_{rs}^2 x^2\right)(x - \xi_r)}{3 - \left(1 + M\,\epsilon_{rs}^2\right)x^2 + 3M\,\epsilon_{rs}^2 x^4} P_r(x) = f_r(x) \quad ; \quad \epsilon_{rs} = \zeta_r/\zeta_s$$

$$\text{for } x = (k_\iota \zeta_r)^{-1} \quad \iota = 1, 2, \cdots, n$$

Therefore,

$$\left(3 - \left(1 + M\epsilon_{rs}^2\right)x^2 + 3M\epsilon_{rs}^2\,x^4\right) f_r(x) - \left(3 - M\,\epsilon_{rs}^2 x^2\right)(x - \xi_r)\,P_r(x) = 0 \text{ for } x = (k_\iota \zeta_r)^{-1}\;,\; \iota = 1, 2, \cdots, n \qquad (3.61)$$

The left hand side of the equation (3.61) must, therefore, be exactly divisible by the polynomial $R_r(x)$. So, we must, accordingly, have a relation of the form:

$$\left\{3 - \left(1 + M\epsilon_{rs}^2\right)x^2 + 3M\epsilon_{rs}^2\,x^4\right\} f_r(x) - \left(3 - M\,\epsilon_{rs}^2 x^2\right)(x - \xi_r)\,P_r(x) = R_r(x)\left(A_{(r)1}x^3 + A_{(r)2}x^2 + A_{(r)3}x + A_{(r)4}\right) \qquad (3.62)$$

where $A_{(r)1}$, $A_{(r)2}$, $A_{(r)3}$ and $A_{(r)4}$ are constants.

Now the equation

$$3-\left(1+M\epsilon_{rs}^2\right)x^2+3M\epsilon_{rs}^2\,x^4=0 \tag{3.63}$$

will give four roots of the type $\pm V_1$ and $\pm V_2$.

Now, putting $x=+V_1$ and $x=-V_1$ in the equation (3.62), and then adding we get

$$A_{(r)2}V_1^2+A_{(r)4}=(-1)^n\left(3-M\,\epsilon_{rs}^2V_1^2\right)\left(b_{(r)1}V_1\right.$$
$$\left.+a_{(r)1}\xi_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{3.64}$$

where

$$a_{(r)1}=\frac{1}{2}\left\{H_r\left(+V_1\right)+H_r\left(-V_1\right)\right\} \tag{3.65}$$

and

$$b_{(r)1}=\frac{1}{2}\left\{H_r\left(+V_1\right)-H_r\left(-V_1\right)\right\} \tag{3.66}$$

Similarly, for $x=+V_2$ and $x=-V_2$, we shall get,

$$A_{(r)2}V_2^2+A_{(r)4}=(-1)^n\left(3-M\,\epsilon_{rs}^2V_2^2\right)\left(b_{(r)2}V_2\right.$$
$$\left.+a_{(r)2}\xi_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{3.67}$$

where

$$a_{(r)2}=\frac{1}{2}\left\{H_r\left(+V_2\right)+H_r\left(-V_2\right)\right\} \tag{3.68}$$

and

$$b_{(r)2}=\frac{1}{2}\left\{H_r\left(+V_2\right)-H_r\left(-V_2\right)\right\} \tag{3.69}$$

Multiplying the equation (3.64) by V_2^2 and the equation (3.67) by V_1^2 and subtracting, we get

$$\begin{aligned}A_{(r)4} = & \frac{(-1)^n\,\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}}{\left(V_1^2-V_2^2\right)}\left[\xi_r\left\{3\left(V_1^2a_{(r)2}-V_2^2a_{(r)1}\right)\right.\right.\\ & \left.+M\,\epsilon_{rs}^2V_1^2V_2^2\left(a_{(r)1}-a_{(r)2}\right)\right\}+\left\{3V_1V_2\left(V_1b_{(r)2}-V_2b_{(r)1}\right)\right.\\ & \left.\left.+M\,\epsilon_{rs}^2V_1^2V_2^2\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}\right]\end{aligned} \tag{3.70}$$

Now, putting $x = 0$ in the the equation (3.62), we get

$$f_r(0) = (-1)^{n+1} \xi_r \, \mu_{(r)1}\mu_{(r)2} \cdots \mu_{(r)n} + \frac{1}{3} \cdot A_{(r)4} \tag{3.71}$$

So, the equation (3.59), by using the equation (3.71) becomes

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \frac{q_r}{3} k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \, \mu_{(r)1}\mu_{(r)2} \cdots \mu_{(r)n} \left(3\xi_r + \frac{(-1)^{n+1} A_{(r)4}}{\mu_{(r)1}\mu_{(r)2} \cdots \mu_{(r)n}} \cdot\right)$$

which, on using the equation (3.43), becomes

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \frac{q_r}{3} \left\{ M \left(1 - \frac{1}{10}(1 - M) \right) \right\}^{\frac{1}{2}} \left(3\xi_r + \frac{(-1)^{n+1} A_{(r)4}}{\mu_{(r)1}\mu_{(r)2} \cdots \mu_{(r)n}} \cdot\right) \tag{3.72}$$

Now, putting $\mu = 0$ in the the equations (3.44) and (3.54), we respectively get

$$S_r(0) = \sum_{\iota=1}^{n} L_{(r)\iota} + \frac{b_0}{b_1\omega_r}$$

and

$$S_r(0) = (-1)^n q_r \left\{ M \left(1 - \frac{1}{10}(1 - M) \right) \right\}^{\frac{1}{2}} H(0)(-\xi_r)$$
$$= (-1)^{n+1} q_r\xi_r \left\{ M \left(1 - \frac{1}{10}(1 - M) \right) \right\}^{\frac{1}{2}}$$

which, on comparing the two expressions for $S_r(0)$, will produce

$$\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + M\, \epsilon_{rs}^2 V_1^2 V_2^2 \left(a_{(r)1} - a_{(r)2}\right)\right\} q_r\xi_r + \left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + M\, \epsilon_{rs}^2 V_1^2 V_2^2 \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} q_r = \frac{(-1)^{n+1} 3\left(V_1^2 - V_2^2\right) b_0}{b_1\omega_r \left\{ M \left(1 - \frac{1}{10}(1 - M) \right) \right\}^{\frac{1}{2}}} \tag{3.73}$$

Again to derive another relation involving q_r and ξ_r from the relations (3.50) and (3.51), we proceed as follows:-

We observe that

$$\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \left\{ \frac{1}{T'_\iota} (1 - \zeta_r k_\iota \mu) S_r(\mu) \right\} \tag{3.74}$$

where

$$T'_\iota = \frac{3k_\iota^4 \zeta_r^4 - \left(1 + M\epsilon_{rs}^2\right) k_{(s)\iota}^2 \zeta_r^2 + 3M\epsilon_{rs}^2}{k_\iota \zeta_r \left(3M\epsilon_{rs}^2 - k_\iota^2 \zeta_r^2\right)} \tag{3.75}$$

Therefore, using the equation (3.47) in the equation (3.74), we get,

$$\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} = \frac{1}{T'_\iota} q_r \zeta_r k_1 \cdots \zeta_r k_n \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \tag{3.76}$$

Summing up both sides of the equation (3.76) over ι, we get

$$\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} = q_r \zeta_r k_1 \cdots \zeta_r k_n g_r(0) \tag{3.77}$$

where

$$g_r(x) = \sum_{\iota=1}^{n} \frac{1}{T'_\iota} \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) R_\iota(x) \tag{3.78}$$

Now, from the equation (3.77)

$$q_r = (-1)^{n-1} \left(q_r \zeta_r k_1 \cdots \zeta_r k_n g_r(0) + \frac{\zeta_r}{\omega_r} \right) \tag{3.79}$$

But, $g_r(x)$ is a polynomial of degree $(n-1)$ and it assumes the values

$$\frac{1}{T'_\iota} P_r\left(\frac{1}{k_\iota \zeta_r}\right) \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \text{ for } x = \frac{1}{k_\iota \zeta_r} \quad \iota = 1, \cdots, n$$

So,

$$\left\{3M\epsilon_{rs}^2 x^2 - \left(1 + M\epsilon_{rs}^2\right) x^2 + 3\right\} g_r\left(x\right) + x\left(1 - 3M\epsilon_{rs}^2 x^2\right)\left(x - \xi_r\right) P_r\left(x\right) = 0 \text{ for } x = \frac{1}{k_\iota \zeta_r}\ ;\ \iota = 1, \cdots, n \tag{3.80}$$

Observing the equation (3.80), we can conclude that the polynomial on the left hand side of the equation (3.80) must be divisible by $R_r\left(x\right)$. So, we must have the relation:

$$\left\{3M\epsilon_{rs}^2 x^2 - \left(1 + M\epsilon_{rs}^2\right) x^2 + 3\right\} g_r\left(x\right) = x\left(3M\epsilon_{rs}^2 x^2 - 1\right)\left(x - \xi_r\right) P_r\left(x\right) + R_r\left(x\right)\left(B_{(r)1}x^4 + B_{(r)2}x^3 + B_{(r)3}x^2 + B_{(r)4}x + B_{(r)5}\right) \tag{3.81}$$

where $B_{(r)1}$, $B_{(r)2}$, $B_{(r)3}$, $B_{(r)4}$ and $B_{(r)5}$ are constant so that

$$g_r\left(0\right) = \frac{1}{3}B_{(r)5} \tag{3.82}$$

Now, comparing the co-efficients of x^{n+1} from both sides of the equation (3.81), we get

$$B_{(r)1} = \left(-1\right)^{n+1}\frac{3M}{\zeta_r k_1 \cdots \zeta_r k_n} \tag{3.83}$$

But, we have assumed that the roots of the equation (3.63) are $\pm V_1$ and $\pm V_2$.

Now, putting $x = +V_1$ and $x = -V_1$, by turns, in the equation (3.81) and adding we get

$$B_{(r)1}V_1^4 + B_{(r)3}V_1^2 + B_{(r)5} = \left(-1\right)^n V_1\left(1 - 3M\epsilon_{rs}^2 V_1^2\right)\left\{V_1 a_{(r)1} + \xi_r b_{(r)1}\right\}\mu_{(r)1}\cdots\mu_{(r)n} \tag{3.84}$$

Again, for $x = +V_2$ and $x = -V_2$, proceeding exactly as above, we shall get

$$B_{(r)1}V_2^4 + B_{(r)3}V_2^2 + B_{(r)5} = \left(-1\right)^n V_2\left(1 - 3M\epsilon_{rs}^2 V_2^2\right)\left\{V_2 a_{(r)2} + \xi_r b_{(r)1}\right\}\mu_{(r)2}\cdots\mu_{(r)n} \tag{3.85}$$

Multiplying the equation (3.84) by $V_2^2$and the equation (3.85) by V_1^2 and subtracting and then using the equation (3.83) in the result, we get

$$B_{(r)5} = (-1)^{n+1}\left[\frac{3M\epsilon_{rs}^2 V_1^2 V_2^2}{\zeta_r k_1\cdots\zeta_r k_n \mu_{(r)1}\cdots\mu_{(r)n}} + \frac{V_1^2V_2^2}{V_1^2-V_2^2}\left\{\left(a_{(r)1}-a_{(r)2}\right)\right.\right.$$
$$\left.-3M\epsilon_{rs}^2\left(V_1^2 a_{(r)1}-V_2^2 a_{(r)2}\right)\right\} + \xi_r\cdot\frac{V_1V_2}{V_1^2-V_2^2}\left\{\left(V_2 b_{(r)1}-V_1 b_{(r)2}\right)\right.$$
$$\left.\left.-3M\epsilon_{rs}^2 V_1V_2\left(V_1 b_{(r)1}-V_2 b_{(r)2}\right)\right\}\right]\mu_{(r)1}\cdots\mu_{(r)n}$$

i.e.

$$B_{(r)5} = (-1)^{n+1}\left[\frac{3M\epsilon_{rs}^2 V_1^2 V_2^2}{\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}} + \frac{V_1^2V_2^2}{V_1^2-V_2^2}\left\{\left(a_{(r)1}-a_{(r)2}\right)\right.\right.$$
$$\left.-3M\epsilon_{rs}^2\left(V_1^2 a_{(r)1}-V_2^2 a_{(r)2}\right)\right\} + \xi_r\cdot\frac{V_1V_2}{V_1^2-V_2^2}\left\{\left(V_2 b_{(r)1}-V_1 b_{(r)2}\right)\right.$$
$$\left.\left.-3M\epsilon_{rs}^2 V_1V_2\left(V_1 b_{(r)1}-V_2 b_{(r)2}\right)\right\}\right]\mu_{(r)1}\cdots\mu_{(r)n}$$

using the equation (3.43)

i.e.

$$\begin{aligned} B_{(r)5} \;=\; & (-1)^{n+1}\frac{V_1V_2}{V_1^2-V_2^2}\left[\xi_r\cdot\left\{\left(V_2 b_{(r)1}-V_1 b_{(r)2}\right)-3M\epsilon_{rs}^2 V_1V_2\times\right.\right. \\ & \times\left.\left(V_1 b_{(r)1}-V_2 b_{(r)2}\right)\right\} + V_1V_2\left\{\left(a_{(r)1}-a_{(r)2}\right)-3M\epsilon_{rs}^2\left(V_1^2 a_{(r)1}\right.\right. \\ & \left.\left.\left.-V_2^2 a_{(r)2}\right)\right\} + \frac{3M\epsilon_{rs}^2 V_1V_2\left(V_1^2-V_2^2\right)}{\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}}\right]\mu_{(r)1}\cdots\mu_{(r)n} \qquad (3.86) \end{aligned}$$

Now, from the equation (3.79), using the equation (3.82) and the equation (3.86), we get

$$q_r = \frac{1}{3}\,q_r\frac{V_1V_2}{V_1^2-V_2^2}\left[\xi_r\cdot\left\{\left(V_2 b_{(r)1}-V_1 b_{(r)2}\right)-3M\epsilon_{rs}^2 V_1V_2\left(V_1 b_{(r)1}\right.\right.\right.$$
$$\left.\left.-V_2 b_{(r)2}\right)\right\} + V_1V_2\left\{\left(a_{(r)1}-a_{(r)2}\right)-3M\epsilon_{rs}^2\left(V_1^2 a_{(r)1}-V_2^2 a_{(r)2}\right)\right\}$$
$$\left.+\frac{3M\epsilon_{rs}^2 V_1V_2\left(V_1^2-V_2^2\right)}{\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}}\right]\mu_{(r)1}\cdots\zeta_r k_1\cdots\zeta_r k_n\mu_{(r)n} + (-1)^{n-1}\frac{\zeta_r}{\omega_r}$$

which, on using the equation (3.43), can be written as

$$q_r\xi_r \cdot \; V_1V_2\left\{\left(V_2b_{(r)1}-V_1b_{(r)2}\right)-3M\epsilon_{rs}^2V_1V_2\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}$$
$$+q_rV_1^2V_2^2\left\{\left(a_{(r)1}-a_{(r)2}\right)-3M\epsilon_{rs}^2\left(V_1^2a_{(r)1}-V_2^2a_{(r)2}\right)\right\}$$
$$+\frac{3\left(M\epsilon_{rs}^2V_1^2V_2^2-1\right)\left(V_1^2-V_2^2\right)}{\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}}q_r=(-1)^n\frac{3\zeta_r\left(V_1^2-V_2^2\right)}{\omega_r\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}}$$

But V_1^2 and V_2^2 are the roots of the equations (3.63). Therefore,

$$M\epsilon_{rs}^2V_1^2V_2^2=1 \text{ and } V_1^2+V_2^2=\frac{1+M\epsilon_{rs}^2}{3M\epsilon_{rs}^2}$$

So, the above relation is equivalent to

$$q_r\xi_r\left\{V_1V_2\left(V_2b_{(r)1}-V_1b_{(r)2}\right)-3\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}$$
$$+q_r\left\{V_1^2V_2^2\left(a_{(r)1}-a_{(r)2}\right)-3\left(V_1^2a_{(r)1}-V_2^2a_{(r)2}\right)\right\}$$
$$=(-1)^n\frac{3\zeta_r\left(V_1^2-V_2^2\right)}{\omega_r\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}} \tag{3.87}$$

Solving the equation (3.73) and (3.87), we get

$$q_r=(-1)^{n+1}\frac{3\left(V_1^2-V_2^2\right)}{\omega_r\left\{M\left(1-\frac{1}{10}(1-M)\right)\right\}^{\frac{1}{2}}}\times$$
$$\times\frac{\left[\frac{b_0}{b_1}\left\{V_1V_2\left(V_2b_{(r)1}-V_1b_{(r)2}\right)-3\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}+\zeta_r\left\{3\left(V_1^2a_{(r)2}-V_2^2a_{(r)1}\right)+\left(a_{(r)1}-a_{(r)2}\right)\right\}\right]}{\begin{aligned}&\left[\left\{3\left(\frac{V_2^2}{M\epsilon_{rs}^2}+V_1^2\right)-\frac{10}{M\epsilon_{rs}^2}\right\}a_{(r)1}^2+\left\{3\left(\frac{V_1^2}{M\epsilon_{rs}^2}+V_2^2\right)\right.\right.\\&\left.-\frac{10}{M\epsilon_{rs}^2}\right\}a_{(r)2}^2-\left\{3\left(\frac{V_2^2}{M\epsilon_{rs}^2}+V_1^2\right)-\frac{10}{M\epsilon_{rs}^2}\right\}b_{(r)1}^2\\&-\left\{3\left(\frac{V_1^2}{M\epsilon_{rs}^2}+V_2^2\right)-\frac{10}{M\epsilon_{rs}^2}\right\}b_{(r)2}^2-\frac{16}{M\epsilon_{rs}^2}a_{(r)1}a_{(r)2}\\&\left.+\frac{8V_1V_2}{3M\epsilon_{rs}^2}\left(1+M\epsilon_{rs}^2\right)b_{(r)1}b_{(r)2}\right]\end{aligned}} \tag{3.88}$$

and

$$\xi_r = \frac{\left[\begin{array}{l} b_0 \left\{3\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - V_1^2 V_2^2 \left(a_{(r)1} - a_{(r)2}\right)\right\} \\ \quad + b_1 \zeta_r \left\{3 V_1 V_2 \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) + \left(V_2 b_{(r)2} - V_1 b_{(r)1}\right)\right\} \end{array}\right]}{\left[\begin{array}{l} b_0 \left\{V_1 V_2 \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} \\ \quad + b_1 \zeta_r \left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\} \end{array}\right]} \tag{3.89}$$

Now, using the equation (3.54) in the equation (3.46), we get

$$I_r^*(0,\mu) = G_r(\mu + \xi_r) H_r(\mu) - b_0 - b_1 \zeta_r \mu \tag{3.90}$$

with

$$G_r = \frac{\begin{array}{c} 3\left(V_1^2 - V_2^2\right)\left[b_0 \left\{V_1 V_2 \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right.\right. \\ \left. -3\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} + b_1 \zeta_r \left\{3\left(V_1^2 a_{(r)2}\right.\right. \\ \left.\left.\left. - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\right] \end{array}}{\left[\begin{array}{l} \left\{3\left(\frac{V_2^2}{M\epsilon_{rs}^2} + V_1^2\right) - \frac{10}{M\epsilon_{rs}^2}\right\} a_{(r)1}^2 + \left\{3\left(\frac{V_1^2}{M\epsilon_{rs}^2} + V_2^2\right)\right. \\ \left. - \frac{10}{M\epsilon_{rs}^2}\right\} a_{(r)2}^2 - \left\{3\left(\frac{V_2^2}{M\epsilon_{rs}^2} + V_1^2\right) - \frac{10}{M\epsilon_{rs}^2}\right\} b_{(r)1}^2 \\ - \left\{3\left(\frac{V_1^2}{M\epsilon_{rs}^2} + V_2^2\right) - \frac{10}{M\epsilon_{rs}^2}\right\} b_{(r)2}^2 - \frac{16}{M\epsilon_{rs}^2} a_{(r)1} a_{(r)2} \\ + \frac{8 V_1 V_2}{3 M \epsilon_{rs}^2}\left(1 + M\epsilon_{rs}^2\right) b_{(r)1} b_{(r)2} \end{array}\right]} \tag{3.91}$$

and ξ_r given by the equation (3.89) .

Now, from the equation(3.10), we get

$$I_r(0,\mu) = b_0 + \zeta_r \mu b_1 + I_r^*(0,\mu)$$

i.e.

$$I_r(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) \tag{3.92}$$

The solution(3.92) is a desired solution of the equation(3.7) in n^{th} approximation.

3.1.3.4 The Exact Diffusely Reflected Intensity and the Exact Solution For The Emergent Intensity.

Following Busbridge and Stibbs,[33] we change the variable $\zeta_r\mu$ and $\zeta_s\mu'$ to x and x' respectively [and consequently $\zeta_r\mu_{(r)i}$ and $\zeta_s\mu_{(s)j}'$ to x_i and $x'_j{}'$ respectively] to get from the equation (3.29)

$$\sum_j \frac{a'_j}{1+kx_j} \sum_{s=1}^{m} \frac{\eta_s\omega_s}{\zeta_s}\left[\frac{1}{16C}\left\{\frac{1}{\zeta_s^4}\cdot 9Mx_j^4 - \frac{1}{\zeta_s^2}\cdot 3(1+M)x_j^2 + 9\right\}\right] = 1$$

i.e.

$$\sum_j \frac{a'_j\Psi(x_j)}{1+kx_j} = 1 \text{ with } a'_j = \zeta_r a_j \tag{3.93}$$

where, assuming that

$$\eta_1 > \eta_2 > \cdots > \eta_m \tag{3.94}$$

so that

$$0 \leq \zeta_1 < \zeta_2 < \cdots < \zeta_m \leq 1 \tag{3.95}$$

$$\Psi\left(x'\right) = \begin{cases} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \left[\frac{1}{16C} \left\{ \frac{1}{\zeta_s^4} \cdot 9Mx'^{\,4} - \frac{1}{\zeta_s^2} \cdot 3\left(1+M\right) x'^{\,2} + 9 \right\} \right], \\ \qquad \text{if} \quad 0 \leq x' \leq \zeta_1 \\ \sum_{s=r+1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \left[\frac{1}{16C} \left\{ \frac{1}{\zeta_s^4} \cdot 9Mx'^{\,4} - \frac{1}{\zeta_s^2} \cdot 3\left(1+M\right) x'^{\,2} + 9 \right\} \right], \\ \qquad \text{if} \quad \zeta_r \leq x' \leq \zeta_{r+1} \\ 0, \\ \qquad \text{if} \quad \zeta_m \leq x' \leq 1 \end{cases} \tag{3.96}$$

Then

$$\int_0^1 \Psi\left(x'\right) dx' = \frac{1}{8}\left(M+5\right) \cdot \left(1-M\right)$$

which is less than or equal to $\frac{1}{2}$ provided that

$$|M+2| \geq 5$$

i.e.

$$M \geq -2+\sqrt{5} \text{ or } M \leq -2-\sqrt{5}$$

i.e.

$$\int_0^1 \Psi\left(x'\right) dx' \leq \frac{1}{2}$$
$$\text{provided that } M \geq -2+\sqrt{5} \text{ or } M \leq -2-\sqrt{5} \tag{3.97}$$

Therefore, following the theory of H-function, developed by Chandrasekhar,[45] we shall able to show, in the present case, that $H\left(x\right)$, where $x = \zeta_r \mu$ or equivalently $\left[H\left(\zeta_r \mu\right) =\right] H_r\left(\mu\right)$, given by the equation (3.53), satisfies, in the limit of infinite approximation, the non-integral equation:

$$H(x) = 1 + xH(x)\int_0^1 \frac{\Psi(x')\,H(x)}{x+x'}dx'$$

and

$$H_r(\mu) = 1 + \zeta_r \mu H_r(\mu)\int_0^1 \frac{\zeta_r \Psi(\zeta_r \mu')}{\zeta_r \mu + \zeta_r \mu'}\,H_s(\mu')\,d\mu' \tag{3.98}$$

which is bounded in the entire half plane $\mathbb{R}(x) \geq 0$.

The characteristic function $\Psi(x')$ in the equation (3.98) satisfies the necessary condition:

$$\int_0^1 \Psi(x')\,dx' \leq \frac{1}{2} \tag{3.99}$$

Now, we allow n to tend to infinity for both the equations (3.92) and (3.92) to get the exact diffusely reflected intensity and the exact emergent intensity, given by:

$$I_r^*(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) - \zeta_r \mu b_1 - b_0 \tag{3.100}$$

i.e.

$$I_r(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) \tag{3.101}$$

where ξ_r and G_r are given by the equations (3.89) and (3.91) respectively and $H_r(\mu)$ is the solution of the equation (3.98)

3.2 Pomraning Phase Function

3.2.1 Introduction

Pomraning[164] introduced a new phase function where the radiation is scattered according to the Rayleigh like phase function

$$p(\cos\Theta) = \frac{3}{4}\left(1 + \lambda\cos^2\Theta\right) \;\; ; \;\; \lambda = \frac{5\varpi_0}{5 - 3\varpi_0}$$

which we call the Pomraning phase function.

Its azimuth independent form is given by

$$\begin{aligned} p(\mu, \mu') &= 1 + \frac{\lambda}{2} p(\mu) p(\mu') = 1 + \frac{\lambda}{8}(3\mu^2 - 1)(3\mu'^2 - 1) \\ &= 1 + \frac{\lambda}{8}(9\mu^2\mu'^2 - 3\mu'^2 - 3\mu^2 - 1) \;\; ; \;\; \lambda = \frac{5\varpi_0}{5 - 3\varpi_0} \end{aligned}$$

Viik[216] used it to derive the intensities in a homogeneous plane-parallel optically semi-infinite atmosphere where there are sources of radiation infinitely deep in the atmosphere and where the radiation is scattered according to the Rayleigh like Pomraning phase function. He considered the accuracy of the phase function on the basis of the Milne problem in a homogeneous plane parallel atmosphere by solving the vector transfer equation using the Chandrasekhar's discrete ordinate method and the respective scalar equations by using Chandrasekhar-Ivanov principles of invariance to reduce the boundary value problem into a Cauchy initial-value problem.

By using the same phase function, Viik & McCormick[210] performed approximate polarized Rayleigh transfer calculations with a scalar radiative transfer equation.

Ghosh, Mukherjee and Karanjai[77] used some approximate forms of H-function already studied by Karanjai[102] and Karanjai & Sen[116] in case of anisotropically scattering atmosphere with Pomraning Phase function.

Islam, Mukherjee and Karanjai[97] used the Pomraning Phase function to solve the equation of radiative transfer in a plane semi-infinite atmosphere with axial symmetry by Laplace Transform and Wiener-Hopf technique with a non-linear source. They determined the emergent intensity in terms of Chandrasekhar's H-function and the intensity at any optical depth by inversion.

In this section, the radiative transfer equation for interlocked multiplet problem has been solved with Pomraning phase function.

3.2.2 Formulation of the Problem

3.2.2.1 The Equation of Radiative Transfer with Pomraning Phase Function

The equation transfer for the r^{th} interlocked line.:

The general equation of transfer is

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)B_\nu(T) \\ -\frac{1}{2}(1-\varepsilon)\alpha_r\sum_{s=1}^{m}\eta_s\int_{-1}^{1}p\,(\mu,\mu')I_s(\tau,\mu')d\mu' \tag{3.102}$$

Here , $\alpha_r's$ $(r=1,\ldots,m)$ are of the form:

$$\alpha_r = \eta_r/\sum_{s=1}^{m}\eta_s \tag{3.103}$$

and obey the relation:

$$\sum_{r=1}^{m} \alpha_r = 1 \tag{3.104}$$

τ is the optical depth given by

$$\tau = \int_z^{\infty} k\rho dz \tag{3.105}$$

measured, in terms of the scattering co-efficients k , from the boundary inward and

$$\mu' = \cos\vartheta \tag{3.106}$$

where ϑ denotes the polar angle which the direction considered makes with the outward normal to an element of area $d\sigma$ (across which the dE_ν amount of radiant energy in the frequency interval $(\nu, \nu + d\nu)$ is transported), $\eta_r = l_r : K$ (where l_r is the absorption co-efficient for the r^{th} interlocked line and K the continuous absorption), $B_\nu(T)$ is the Planck-function, considered in this case, is of the form:

$$B_\nu(T) = B(\tau) = b_0 + b_1\tau \tag{3.107}$$

b_0 and b_1 ,being positive constants and the (azimuth independent) Pomraning phase function $p(\mu, \mu')$, taken here, is given by

$$p(\mu, \mu') = 1 + \frac{\lambda}{2} p(\mu) p(\mu') = 1 + \frac{\lambda}{8}(3\mu^2 - 1)(3\mu'^2 - 1)$$

i.e.

$$p(\mu, \mu') = 1 + \frac{\lambda}{8}(9\mu^2\mu'^2 - 3\mu'^2 - 3\mu^2 - 1)\ ; \tag{3.108}$$

$$\lambda = \frac{5\varpi_0}{5 - 3\varpi_0}$$

and ε , the co-efficient, is introduced to allow for thermal emission associated with the line absorption.

Busbridge and Stibbs[33] solved their problem on the basis of some assumptions. All of those assumptions are also considered here .

Using the relations (3.107) and (3.108) in the equation (3.102), we get:

$$\mu \frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r) I_r(\tau,\mu) - (1+\varepsilon\eta_r)(b_0+b_1\tau) \\ -\frac{1}{2}(1-\varepsilon)\sum_{s=1}^{m}\eta_s \int_{-1}^{+1}\left\{1+\frac{\lambda}{8}\left(9\mu^2\mu'^2 \right.\right. \\ \left.\left. -3\mu'^2-3\mu^2-1\right)\right\} I_r(\tau,\mu')\,d\mu' \qquad (3.109)$$

3.2.2.2 The boundary conditions

The boundary conditions for solving the equation (3.109) are

$$I_r(0,-\mu) = 0;\ (0<\mu\leq 1) \qquad (3.110)$$

$$I_r(\tau,\mu)\cdot e^{-\tau/\mu} \to 0$$

i.e. $I_r(\tau,\mu)$ is at most linear in τ as τ tends to infinity (3.111)

3.2.3 Solution of the Equation

3.2.3.1 Solution in n^{th} approximation

We observe that if we assume, like Busbridge and Stibbs,[33] that one of the solution of the Equation (3.109) to be

$$I_r(\tau,\mu) = b_0 + b_1\left(\tau+\frac{\mu}{1+\eta_r}\right) + I_r^*(\tau,\mu) \qquad (3.112)$$

which consists of two parts , the first part being the solution for an infinitely unbounded atmosphere as τ tends to infinity and the second part $I_r^*(\tau,\mu)$ being the departure of the asymptotic solution from the value $I_r(\tau,\mu)$ as we approach the boundary $\tau=0$

Now, writing

$$\zeta_r = \frac{1}{1+\eta_r} \tag{3.113}$$

and

$$\omega_r = \frac{(1-\varepsilon)\,\eta_r}{(1+\eta_r)} \tag{3.114}$$

and using the equation (3.112), the equation (3.109) can be reduced to the form:

$$\zeta_r\mu\frac{dI_r}{d\tau} = b_1\zeta_r\mu + I_r^*(\tau,\mu) - \left(b_0 + b_1\tau\right)\omega_r$$
$$-\frac{1}{2}\cdot\omega_r\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\cdot\sum_{s=1}^{m}\eta_s\int_{-1}^{+1}\left\{1+\frac{\lambda}{8}\left(9\mu^2\mu'^2 - 3\mu'^2 - 3\mu^2 + 1\right)\right\}\times$$
$$\times\left\{b_0 + b_1\tau + b_1\zeta_s\mu' + I_s^*(\tau,\mu')\right\}d\mu' \tag{3.115}$$

and the boundary conditions to the forms:

$$I_r^*(0,-\mu) = b_0 - b_1\zeta_r\mu, \ \text{ where } 0 < \mu \leq 1 \tag{3.116}$$

and

$$I_r^*(\tau,\mu) \text{ is at most linear in } \tau \text{ as } \tau \text{ tends to infinity} \tag{3.117}$$

In the nth approximation , we replace the integro-differential equation (3.115) by the system of 2n linear differential equations :

$$\zeta_r\mu_{(r)i}\frac{dI_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{1}{2}\cdot\omega_r\cdot\frac{1}{\sum\limits_{s=1}^{m}\eta_s}\sum_{s=1}^{m}\eta_s\sum_j\{1$$
$$+\frac{\lambda}{8}\left(9\mu^2_{(r)i}\mu^2_{(s)j} - 3\mu^2_{(s)j} - 3\mu^2_{(r)i} + 1\right)\Big\}\, I^*_{(s)j}a_j \tag{3.118}$$

where the symbol $I^*_{(r)i}$, for brevity, is used for $I^*_r(\tau, \mu_{(r)i})$, $\mu_{(r)i}$'s ($i = 1, 2, \cdots, n$; assuming that $\mu_{(r)-i} = -\mu_{(r)i}$) are the zeros of the Legendre polynomial $P_{2n}(\mu)$ which are independent on the lines of interlocking and a_j's ($j = 1, 2, \cdots, n$; having the property that $a_{-j} = a_j$) are the corresponding Gaussian Weights . However it is to be noted that there is no term with $j = 0$.

Now, we try to get the solution of the equation (3.118). The system of Equation(3.118) admits integrals of the form:

$$I^*_{(r)i} = g_{(r)i} e^{-k\tau} \ , \ i = \pm 1, \pm 2, \cdots, \pm n \tag{3.119}$$

then using the Equation (3.119) in Equation (3.118) ,we get

$$g_{(r)i}\left\{1 + k\zeta_r \mu_{(r)i}\right\} = \frac{\omega_r}{2C} \sum_{s=1}^{m} \sum_{j} \left\{1 + \frac{\lambda}{8}\left(9\mu^2_{(r)i}\mu^2_{(s)j} - 3\mu^2_{(s)j} - 3\mu^2_{(r)i} + 1\right)\right\} g_{(s)j} a_j \tag{3.120}$$

where

$$C = \sum_{s=1}^{m} \eta_s \tag{3.121}$$

Hence,

$$g_{(r)i} = \omega_r \frac{\rho + \rho_1 \mu^2_{(r)i}}{1 + k\zeta_r \mu_{(r)i}} \tag{3.122}$$

where ρ and ρ_1 are constants which are independent of $\mu_{(r)i}$.

Now, defining

$$D_\ell(x) = \sum_{s=1}^{m} \eta_s \omega_s \sum_{j} \frac{a_j \mu^\ell_{(s)j}}{1 + \mu_{(s)j}\, x} \tag{3.123}$$

and, for simplification, replacing

$$D_\ell(k\zeta_s) \text{ by } D_\ell$$

we can write the equation, obtained by using the equation (3.122) in equation (3.120), in the compact form as

$$2C\left(\rho+\rho_1\mu_{(r)i}^2\right)=\left\{\rho\left(1+\frac{\lambda}{8}\right)D_0-\frac{3\rho\lambda}{8}D_2+\rho_1\left(1+\frac{\lambda}{8}\right)D_2\right.$$
$$\left.-\frac{3\rho_1\lambda}{8}D_4\ +\mu_{(r)i}^2\left(\frac{9\rho_1\lambda}{8}D_4+\frac{9\rho\lambda}{8}D_2-\frac{3\rho\lambda}{8}D_0-\frac{3\rho_1\lambda}{8}D_2\right)\right\} \tag{3.124}$$

This equation is true for all $\mu_{(r)i}$. So, it will produce the following two relations:

$$\left.\begin{aligned}&\left(16C-(8+\lambda)D_0+3\lambda D_2\right)\rho+\left(3\lambda D_4-(8+\lambda)D_2\right)\rho_1=0\\&\left(3\lambda D_0-9\lambda D_2\right)\rho+\left(16C-9\lambda D_4+3\lambda D_2\right)\rho_1=0\end{aligned}\right\} \tag{3.125}$$

Eliminating ρ and ρ_1 from the above two equations we get

$$72\lambda\left(D_0D_4-D_2^2\right)-144C\lambda D_4+96C\lambda D_2-128CD_0$$
$$-16C\lambda D_0+256C^2=0 \tag{3.126}$$

Using the equations (II.20b) and (II.20g) of Appendix-II, we get

$$72\lambda\left(\psi_0D_4-\psi_2D_2\right)-144C\lambda D_4+96C\lambda D_2-128CD_0$$
$$-16C\lambda D_0+256C^2=0$$

$$\Rightarrow\ \frac{1}{16C}(8+\lambda)D_0+\frac{3}{16C}\lambda\cdot\frac{\psi_0}{2C}D_2-\frac{3}{8C}\lambda D_2$$
$$-\frac{9}{16C}\lambda\cdot\frac{\psi_0}{2C}D_4+\frac{9}{16C}\lambda D_4=1 \tag{3.127}$$

Now, writing

$$M=\left(\sum_{s=1}^{m}\eta_s\left(1-\omega_s\right)\right)/\sum_{s=1}^{m}\eta_s=\frac{2C-\psi_0}{2C} \tag{3.128}$$

the equation (3.127) can be converted into the form:

$$\frac{1}{16C}\left\{(8+\lambda)\,D_0-3\lambda\cdot(1+M)\,D_2+9M\lambda\cdot D_4\right\}=1$$

which is equivalent to:

$$\sum_{s=1}^{m}\eta_s\omega_s\sum_j\frac{a_j}{16C\left(1+k\zeta_s\mu_{(s)j}\right)}\left\{(8+\lambda)-3\lambda\cdot(1+M)\,\mu_{(s)j}^2\right.$$
$$\left.+9M\lambda\mu_{(s)j}^4\right\}=1 \qquad (3.129)$$

The equation (3.129) is the **characteristic equation** which is an equation in k of order 2n and it will give 2n distinct non-zero roots which occur in pair as $\pm k_\iota$, $(\iota=1,2,\cdots,n)$, if $\omega_r<1$

From the first equation of the pair of equations (3.125), we get

$$\rho_1=\rho\cdot\frac{3\lambda D_0-9\lambda D_2}{9\lambda D_4-3\lambda D_2-16C}$$

which, by the use of the equations (II.5b) and (II.5d) of Appendix II and the relation(3.128), is equivalent to

$$\rho_1=\rho\cdot\frac{\left[6C\lambda\,(1-M)\,k^4\zeta_s^4+3\lambda k^2\zeta_s^2\,(3-k^2\zeta_s^2)\,(2C\,(1-M)-D_0)\right]}{\left[3\lambda\,(k^2\zeta_s^2-3)\,(2C\,(1-M)-D_0)-2Ck^2\zeta_s^2\Big(8k^2\zeta_s^2+3\lambda\,(1-M)\Big)\right]} \qquad (3.130)$$

Again the equation (3.129), by the use of the equations (II.5b) and (II.5d) of Appendix II, will produce

$$D_0=\frac{16Ck^4\zeta_s^4-6C\lambda\,(1-M)\,k^2\zeta_s^2+18CM\lambda\,(1-M)}{(8+\lambda)\,k^4\zeta_s^4-3\lambda\cdot(1+M)\,k^2\zeta_s^2+9M\lambda} \qquad (3.131)$$

So, the equation (3.130) is equivalent to

$$\rho_1 = \rho \cdot \frac{48C\lambda\left(k^2\zeta_s^2 - 3M\right)k^6\zeta_s^6}{16C\left\{3\lambda M - (8+\lambda)k^2\zeta_s^2\right\}k^6\zeta_s^6}$$

i.e.

$$\rho_1 = \rho \cdot \frac{3\lambda\left(k^2\zeta_s^2 - 3M\right)}{3\lambda M - (8+\lambda)k^2\zeta_s^2} \tag{3.132}$$

So, from the equation (3.122), using the equation (3.132), we get

$$g_{(r)i} = \omega_r\rho\frac{3\lambda M - (8+\lambda)k^2\zeta_s^2 + \left\{3\lambda\left(k^2\zeta_s^2 - 3M\right)\right\}\mu_{(r)i}^2}{\left\{3\lambda M - (8+\lambda)k^2\zeta_s^2\right\}\left(1 + k\zeta_r\mu_{(r)i}\right)} \tag{3.133}$$

Therefore, the equation (3.118) admits the 2n-independent integrals of the form:

$$I_{(r)i}^* = \omega_r\rho\frac{3\lambda M - (8+\lambda)k_\iota^2\zeta_s^2 + 3\lambda\left(k_\iota^2\zeta_s^2 - 3M\right)\mu_{(r)i}^2}{\left\{3\lambda M - (8+\lambda)k_\iota^2\zeta_s^2\right\}\left(1 \pm k_\iota\zeta_r\mu_{(r)i}\right)}e^{\mp k_\iota\tau}, \quad i = \pm1, \pm2, \cdots, \pm n \tag{3.134}$$

According to Chandrasekhar,[45] the general solution of the system of equations (3.118) can be written in the form:

$$I_{(r)i}^* = \omega_r b_1 \sum_{\iota=1}^{n}\left[\frac{1}{\left\{3\lambda M - (8+\lambda)k_\iota^2\zeta_s^2\right\}\left(1 + k_\iota\zeta_r\mu_{(r)i}\right)}\left\{3\lambda M - (8+\lambda)k_\iota^2\zeta_s^2 + 3\lambda\left(k_\iota^2\zeta_s^2 - 3M\right)\mu_{(r)i}^2\right\}\right]L_{(r)\iota}e^{-k_\iota\tau}, \tag{3.135}$$

where k_ι's (ι = 1, 2,$\cdots$, n) are the positive roots of the characteristic equation (3.129) and $L_{(r)\iota}$'s are the constants of the integration to be determined by the boundary condition (3.116) i.e.

$$I_{(r)-i}^* = b_1\zeta_r\mu_{(r)i} - b_0\ , \text{ where } 0 < \mu_{(r)i} \le 1 \tag{3.136}$$

3.2.3.2 Relation between the roots of the characteristic equation and zeros of the Legendre polynomial

Let $p_{2\ell}$ be the co-efficient of $\mu^{2\ell}$ in the Legendre polynomial $P_{2n}(\mu)$. Then

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = \sum_{\ell=0}^{n} p_{2\ell} \sum_{s=1}^{k} \eta_s \omega_s \sum_{j} \frac{a_j \mu_{(s)j}^{2\ell}}{1+\zeta_s k \mu_{(s)j}},$$

using the definition (3.123)

i.e.

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = \sum_{s=1}^{k} \eta_s \omega_s \sum_{j} \frac{a_j}{1+\zeta_s k \mu_{(s)j}} \times \\ \times \sum_{\ell} p_{2\ell}\, \mu_{(s)j}^{2\ell} \tag{3.137}$$

Since, $\mu_{(s)j}$'s are the zeros of the Legendre polynomial $P_{2n}(\mu)$.

So,

$$\sum_{\ell} p_{2\ell}\, \mu_{(s)j}^{2\ell} = 0 \tag{3.138}$$

and therefore, the equation (3.137) becomes

$$\sum_{\ell=0}^{n} p_{2\ell} D_{2\ell}(\zeta_s k) = 0$$

i.e.

$$p_{2n} D_{2n}(\zeta_s k) + \cdots + p_0 D_0(\zeta_s k) = 0$$

i.e.

$$p_{2n}\left\{-\frac{1}{k^{2n}\zeta_s^{2n}}\left(\psi_0-D_0\right)-\frac{1}{k^{2n-2}\zeta_s^{2n-2}}\psi_2+\cdots\right\}+ \\ \cdots+p_0D_0=0\,,$$

using the relations (II.1) and (II.2) of Appendix II,

which, on using the equation (3.131) i.e

$$\psi_0-D_0=\frac{2C\left[\{(1-M)\,\lambda-8M\}-3\lambda M\,(1-M)\left(\frac{1}{k^2\zeta_s^2}\right)\right]}{(8+\lambda)-3\lambda\cdot(1+M)\left(\frac{1}{k^2\zeta_s^2}\right)+9M\lambda\left(\frac{1}{k^4\zeta_s^4}\right)}$$

becomes an equation in $\frac{1}{\left(\zeta_s^2k^2\right)}$ of degree n of the form:

$$p_{2n}\left\{2C\left\{(1-M)\,\lambda+8M\right\}-2C\,(1-M)\,\lambda\cdot(1 \\ +M)\right\}\frac{1}{\left(\zeta_s^2k^2\right)^n}+\cdots+p_0\,(16C)=0 \qquad (3.139)$$

Since the roots of the equation(3.139) are $\frac{1}{\left(\zeta_s^2k_1^2\right)}, \ldots, \frac{1}{\left(\zeta_s^2k_n^2\right)}$.

Therefore, we get

$$\left(\zeta_sk_1\cdots\zeta_sk_n\right)^2=(-1)^n\left\{M\left(1-\frac{1}{10}\lambda\,(1-M)\right)\right\}\frac{p_{2n}}{p_0} \qquad (3.140)$$

Again, $\mu^2_{(s)1}$, ... , $\mu^2_{(s)n}$ are the zeros of the Legendre polynomial $\sum_{\ell=0}^{n} p_{2\ell}\,\mu^{2\ell}_{(s)j}$ .So, we get

$$\left(\mu_{(s)1}\mu_{(s)2}\cdots\mu_{(s)n}\right)^2=(-1)^n\,\frac{p_0}{p_{2n}} \qquad (3.141)$$

Multiplying the equation (3.140) and (3.141) together, we get

$$\left(\zeta_sk_1\cdots\zeta_sk_n\cdot\mu_{(s)1}\cdots\mu_{(s)n}\right)^2=\left\{M\left(1-\frac{1}{10}\lambda\,(1-M)\right)\right\}$$

i.e.

$$\zeta_s k_1 \cdots \zeta_s k_n \cdot \mu_{(s)1} \cdots \mu_{(s)n} = \left\{ M \left(1 - \frac{1}{10} \lambda \left(1 - M \right) \right) \right\}^{\frac{1}{2}} \quad (3.142)$$

3.2.3.3 The Elimination of the Constants and the Expression of the Law of Diffuse Reflection in Closed Form

Now, we define

$$S_r\left(\mu\right) = \sum_{\iota=1}^{n} \left[\frac{1}{\left\{3\lambda M - \left(8+\lambda\right) k_\iota^2 \zeta_s^2\right\} \left(1 - k_\iota \zeta_r \mu\right)} \left\{ 3\lambda M - \left(8+\lambda\right) k_\iota^2 \zeta_s^2 + 3\lambda \left(k_\iota^2 \zeta_s^2 - 3M\right) \mu^2 \right\} \right] L_{(r)\iota} - \frac{\zeta_r \mu}{\omega_r} + \frac{b_0}{b_1 \omega_r} \quad (3.143)$$

But, by the virtue of the definition of the function $S_r\left(\mu\right)$, given in (3.143), the set of boundary conditions (3.116) can alernatively expressed as

$$\sum_{\iota=1}^{n} \left[\frac{1}{\left\{3\lambda M - \left(8+\lambda\right) k_\iota^2 \zeta_s^2\right\} \left(1 - k_\iota \zeta_r \mu_{(r)i}\right)} \left\{ 3\lambda M - \left(8+\lambda\right) k_\iota^2 \zeta_s^2 + 3\lambda \left(k_\iota^2 \zeta_s^2 - 3M\right) \mu_{(r)i}^2 \right\} \right] L_{(r)\iota} = \frac{\zeta_r \mu_{(r)i}}{\omega_r} - \frac{b_0}{\omega_r b_1}$$

i.e.

$$S_r\left(\mu_{(r)i}\right) = 0 \text{ , where } 0 < \mu_{(r)i} \leq 1 \text{ , } i = 1, 2, \cdots, n \quad (3.144)$$

Now, to express $I_r^*\left(0, \mu\right)$ in terms of $S\left(\mu\right)$, we proceed as follows:

The equation (41) produces

$$I_r^*(0,\mu) = \omega_r b_1 \left[\sum_{\iota=1}^{n} \frac{1}{\{3\lambda M - (8+\lambda) k_\iota^2 \zeta_s^2\}(1+k_\iota \zeta_r \mu)} \left\{ 3\lambda M - (8+\lambda) k_\iota^2 \zeta_s^2 + 3\lambda \left(k_\iota^2 \zeta_s^2 - 3M\right) \mu^2 \right\} L_{(r)\iota} + \frac{\zeta_r \mu}{\omega_r} + \frac{b_0}{b_1 \omega_r} - \frac{\zeta_r \mu}{\omega_r} - \frac{b_0}{b_1 \omega_r} \right]$$

So,

$$I_r^*(0,\mu) = \omega_r b_1 S_r(-\mu) - \zeta_r \mu - b_0 \tag{3.145}$$

Now, the boundary conditions (3.144) make it clear that $\mu_{(r)i}\,;\, i = 1, 2, \cdots, n$ are the zeros of the polynomial $S_r(\mu)$ which is a polynomial of degree $n+1$.

Now, we define the function $R_r(\mu)$ as

$$R_r(\mu) = \prod_{i=1}^{n} (1 - k_i \zeta_r \mu) \tag{3.146}$$

Then we find that $S_r(\mu)\, R_r(\mu)$ is a polynomial of degree $(n+1)$ in μ which vanishes for $\mu = \mu_{(r)i}\,;\, i = 1, 2, \cdots, n$. So, $\mu_{(r)1}, \mu_{(r)2}, ..., \mu_{(r)n}$ are the zeros of the polynomial $S_r(\mu)\, R_r(\mu)$. It should have another zero, say ξ_r.

Also, we see that the

$$P_r(\mu) = \prod_{i=1}^{n} \left(\mu - \mu_{(r)i}\right) \tag{3.147}$$

is also a polynomial of degree n having its zeros as $\mu_{(r)1}, \mu_{(r)2}, ..., \mu_{(r)n}$.

So, the polynomials $P_r(\mu)(\mu - \xi_r)$ and $S_r(\mu)\, R_r(\mu)$ have the same zeros $\mu_{(r)1}, \mu_{(r)2}, ..., \mu_{(r)n}$ and ξ_r possessing the co-efficient of μ^{n+1} as 1 and $q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r$ respectively in which

$$q_r = (-1)^{n-1} \left(\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} + \frac{\zeta_r \mu}{\omega_r} \right) \tag{3.148}$$

Hence

$$S_r(\mu) = q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \frac{P_r(\mu)}{R_r(\mu)} (\mu - \xi_r) \tag{3.149}$$

which is equivalent to

$$S_r(\mu) = (-1)^n q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \cdot \mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n} \times \\ \times (\mu - \xi_r) H(-\mu) \tag{3.150}$$

where

$$H_r(\mu) = \frac{1}{\mu_{(r)1} \mu_{(r)2} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} \left(\mu + \mu_{(r)i}\right)}{\prod_{i=1}^{n} (1 + k_i \zeta_r \mu)} \tag{3.151}$$

Using the equation (3.142), we get

$$S_r(\mu) = (-1)^n q_r \left\{ M \left(1 - \frac{\lambda}{10} (1 - M) \right) \right\}^{\frac{1}{2}} (\mu - \xi_r) H_r(-\mu) \tag{3.152}$$

Again we observe that

$$L_{(r)\iota} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \frac{1}{T_\iota} (1 - \zeta_r k_\iota \mu) S_r(\mu) \tag{3.153}$$

provided that

$$T_\iota = \frac{(8+\lambda) k_\iota^4 \zeta_r^4 - 3\lambda (1 + M \epsilon_{rs}^2) k_\iota^2 \zeta_r^2 + 9 M \epsilon_{rs}^2 \lambda}{(8+\lambda) k_\iota^4 \zeta_r^4 - 3\lambda M \epsilon_{rs}^2 k_{(s)\iota}^2 \zeta_r^2}; \quad \epsilon_{rs} = \zeta_r / \zeta_s \tag{3.154}$$

Therefore, using the equation (3.149) in the equation (3.153), we get

$$L_{(r)\iota} = \frac{1}{T_\iota} q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r \times$$
$$\times \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \tag{3.155}$$

where

$$R_{(s)\iota}(x) = \prod_{\beta(\neq\iota)=1}^{n} (1 - k_\beta \zeta_r x) \ , \ 1 \leq \iota \leq n \tag{3.156}$$

Summing up both sides of the equation (3.155) over ι, we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} = q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r f_r(0) \tag{3.157}$$

with

$$f_r(x) = \sum_{\iota=1}^{n} \frac{1}{T_\iota} \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) R_{(r)\iota}(x) \tag{3.158}$$

Now, $f_r(x)$, defined as in the equation (3.158), is a polynomial of degree $(n-1)$ in x which takes the value

$$\frac{1}{T_\iota} P_r\left(\frac{1}{k_\iota \zeta_r}\right) \ \text{for} \ x = \frac{1}{k_\iota \zeta_r} \ , \ \iota = 1, 2, \cdots, n$$

So, we get

$$\left\{(8+\lambda) - 3\left(1 + M\epsilon_{rs}^2\right)\lambda \cdot x^2 + 9\lambda M \epsilon_{rs}^2 x^4\right\} f_r(x)$$
$$- \left((8+\lambda) - 3M\epsilon_{rs}^2 \lambda \cdot x^2\right)(x - \xi_r) P_r(x) = 0$$
$$\text{for} \ x = (k_\iota \zeta_r)^{-1} \ , \ \iota = 1, 2, \cdots, n \tag{3.159}$$

The left hand side of the equation (3.159) must, therefore, be exactly divided by $R_r(x)$. So, we must, accordingly, have a relation of the form:

$$\begin{aligned}\left\{(8+\lambda)-3\left(1+M\epsilon_{rs}^{2}\right)\lambda\cdot x^{2}+9\lambda M\epsilon_{rs}^{2}x^{4}\right\}f_r(x)&=((8+\lambda)\\ -3M\epsilon_{rs}^{2}\lambda\cdot x^{2}\right)(x-\xi_r)P_r(x)+R_r(x)\left(A_{(r)1}x^{3}\right.\\ \left.+A_{(r)2}x^{2}+A_{(r)3}x+A_{(r)4}\right)\text{ for }x=(k_\iota\zeta_r)^{-1}\ ,\ \iota=1,2,\cdots,n\end{aligned} \tag{3.160}$$

where $A_{(r)1}$, $A_{(r)2}$, $A_{(r)3}$ and $A_{(r)4}$ are constants.

Now the equation

$$(8+\lambda)-3\left(1+M\epsilon_{rs}^{2}\right)\lambda\cdot x^{2}+9\lambda M\epsilon_{rs}^{2}x^{4}=0 \tag{3.161}$$

will give four roots of the type $\pm V_1$ and $\pm V_2$.

Now, putting $x=+V_1$ and $x=-V_1$ in the equation (3.161), and then adding we get

$$\begin{aligned}A_{(r)2}V_1^2+A_{(r)4}&=(-1)^n\left((8+\lambda)-3M\epsilon_{rs}^{2}\lambda\cdot V_1^2\right)\left(b_{(r)1}V_1\right.\\ &\left.+a_{(r)1}\xi_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\end{aligned} \tag{3.162}$$

where

$$a_{(r)1}=\frac{1}{2}\left\{H_r(+V_1)+H_r(-V_1)\right\} \tag{3.163}$$

and

$$b_{(r)1}=\frac{1}{2}\left\{H_r(+V_1)-H_r(-V_1)\right\} \tag{3.164}$$

Again, putting $x=+V_2$ and $x=-V_2$, by turns, in the equation (3.160) and then adding, we get,

$$\begin{aligned}A_{(r)2}V_2^2+A_{(r)4}&=(-1)^n\left((8+\lambda)-3M\epsilon_{rs}^{2}\lambda\cdot V_2^2\right)\left(b_{(r)2}V_2\right.\\ &\left.+a_{(r)2}\xi_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\end{aligned} \tag{3.165}$$

where

$$a_{(r)2} = \frac{1}{2}\left\{H_r\left(+V_2\right) + H_r\left(-V_2\right)\right\} \tag{3.166}$$

and

$$b_{(r)2} = \frac{1}{2}\left\{H_r\left(+V_2\right) - H_r\left(-V_2\right)\right\} \tag{3.167}$$

Eliminating $A_{(r)2}$ between the equations (3.162) and (3.165), we get

$$\begin{aligned} A_{(r)4} = \frac{(-1)^n \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}}{\left(V_1^2 - V_2^2\right)} & \left[\xi_r \left\{(8+\lambda)\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right.\right. \\ +3\lambda M \epsilon_{rs}^2 V_1^2 V_2^2 \left(a_{(r)1} - a_{(r)2}\right)\Big\} & + \Big\{(8+\lambda) V_1 V_2 \left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) \\ & \left.\left. +3\lambda M \epsilon_{rs}^2 V_1^2 V_2^2 \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\right] \end{aligned} \tag{3.168}$$

Now, putting $x = 0$ in the the equation (3.160), we get

$$f_r(0) = (-1)^{n+1}\xi_r\, \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} + \frac{1}{8+\lambda}\cdot A_{(r)4}$$

So, by using the equation (3.168) it becomes

$$\begin{aligned} f_r(0) = (-1)^{n+1}\xi_r\, \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} + (-1)^n \frac{\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}}{(8+\lambda)\left(V_1^2 - V_2^2\right)} \times \\ \times \left[\xi_r\left\{(8+\lambda)\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + 3\lambda M\epsilon_{rs}^2 V_1^2 V_2^2\left(a_{(r)1} - a_{(r)2}\right)\right\}\right. \\ + \Big\{(8+\lambda) V_1 V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + 3\lambda M \epsilon_{rs}^2 V_1^2 V_2^2 \Big(V_1 b_{(r)1} \\ \left.\left. - V_2 b_{(r)2}\right)\right\}\Big] \end{aligned} \tag{3.169}$$

So, by using the equations (3.169) and (3.142), we get from the equation (3.157) that

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \frac{\left\{M\left(1-\frac{\lambda}{10}(1-M)\right)\right\}^{\frac{1}{2}}}{(8+\lambda)\left(V_1^2-V_2^2\right)} \Big[q_r\xi_r \ \{(8$$
$$+\lambda)\left(V_1^2-V_2^2\right) - (8+\lambda)\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)$$
$$-3\lambda M\epsilon_{rs}^2 V_1^2 V_2^2 \left(a_{(r)1} - a_{(r)2}\right)\} - q_r \left\{(8+\lambda) V_1 V_2 \left(V_1 b_{(r)2}\right.\right.$$
$$\left.\left.-V_2 b_{(r)1}\right) + 3\lambda M\epsilon_{rs}^2 V_1^2 V_2^2 \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\Big] \tag{3.170}$$

Now, putting $\mu = 0$ in the equation (3.143), we get

$$S_r(0) = \sum_{\iota=1}^{n} L_{(r)\iota} + \frac{b_0}{b_1\omega_r}$$

and, in the equation (3.152), we get

$$S_r(0) = (-1)^{n+1} q_r\xi_r \left\{M\left(1-\frac{\lambda}{8}(1-M)\right)\right\}^{\frac{1}{2}}$$

and then comparing the two expression for $S_r(0)$, we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} + \frac{b_0}{b_1\omega_r} = (-1)^{n+1} q_r\xi_r \left\{M\left(1-\frac{\lambda}{8}(1-M)\right)\right\}^{\frac{1}{2}} \tag{3.171}$$

Putting $\sum\limits_{\iota=1}^{n} L_{(r)\iota}$ from the equation (3.170) in the equation (3.171), we get

$$q_r\xi_r\left\{(8+\lambda)\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + 3\lambda M\epsilon_{rs}^2 V_1^2 V_2^2\left(a_{(r)1} - a_{(r)2}\right)\right\}$$
$$+q_r\left\{(8+\lambda) V_1 V_2 \left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + 3\lambda M\epsilon_{rs}^2 V_1^2 V_2^2 \left(V_1 b_{(r)1}\right.\right.$$
$$\left.\left.-V_2 b_{(r)2}\right)\right\} = \frac{(-1)^{n+1} b_0 (8+\lambda)\left(V_1^2 - V_2^2\right)}{b_1\omega_r\left\{M\left(1-\frac{\lambda}{8}(1-M)\right)\right\}^{\frac{1}{2}}} \tag{3.172}$$

Again, from the equation (3.148)

$$q_r = (-1)^{n-1}\left(\sum_{\iota=1}^{n}\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r}+\frac{\zeta_r \mu}{\omega_r}\right)$$

which will give the second relation involving q_r and ξ_r. To get that relation the term $\sum\limits_{\iota=1}^{n}\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r}$ is to be derived which can be done as follows:

We have

$$\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}}\left\{\frac{1}{T'_\iota}(1-\zeta_r k_\iota \mu)\, S_r(\mu)\right\} \tag{3.173}$$

if we choose

$$T'_\iota = \frac{(8+\lambda)\,k_\iota^4 \zeta_s^2 \zeta_r^2 - 3\lambda\left(k_{(s)\iota}^2 \zeta_s^2 + M k_{(s)\iota}^2 \zeta_r^2\right) + 9M\lambda}{3\lambda k_\iota \zeta_r \left(3M - k_\iota^2 \zeta_s^2\right)} \tag{3.174}$$

Therefore, using the equation (3.149) in the equation (3.173), we get,

$$\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} = \frac{1}{T'_\iota}\, q_r \zeta_r k_1 \cdots \zeta_r k_n \,\frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)}\left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \tag{3.175}$$

Summing up to both sides of the equation (3.175) over ι, we get

$$\sum_{\iota=1}^{n}\frac{L_{(r)\iota}}{Q_\iota k_\iota \zeta_r} = q_r \zeta_r k_1 \cdots \zeta_r k_n g_r(0)\ , \tag{3.176}$$

defining the function $g_r(x)$ as

$$g_r(x) = \sum_{\iota=1}^{n}\frac{1}{T'_\iota}\frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)}\left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) R_\iota(x) \tag{3.177}$$

Now, from the equation (3.176)

$$q_r = (-1)^{n-1}\left(q_r \zeta_r k_1 \cdots \zeta_r k_n g_r(0) + \frac{\zeta_r}{\omega_r}\right) \tag{3.178}$$

But, $g_r(x)$ is a polynomial of degree $(n-1)$ and it assumes the values

$$\frac{1}{T'_\iota} P_r\left(\frac{1}{k_\iota \zeta_r}\right)\left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \text{ for } x = \frac{1}{k_\iota \zeta_r}, \ \iota = 1, \cdots, n$$

so that

$$\left\{9M\lambda\epsilon_{rs}^2 x^4 - 3\lambda\left(1 + M\epsilon_{rs}^2\right)x^2 + (8+\lambda)\right\} g_r(x) \\ +3\lambda x\left(1 - 3M\epsilon_{rs}^2 x^2\right)(x - \xi_r) P_r(x) = 0 \tag{3.179}$$

$$\text{for } x = \frac{1}{k_\iota \zeta_r} \ ; \iota = 1, \cdots, n$$

Therefore, the roots of $R_r(x)$ satisfies the polynomial on left hand side of the equation (3.179). So, we can conclude that the polynomial on the left hand side of the equation (3.179) must be divisible by $R_r(x)$. So, we can write the relation:

$$\left\{9M\lambda\epsilon_{rs}^2 x^4 - 3\lambda\left(1 + M\epsilon_{rs}^2\right)x^2 + (8+\lambda)\right\} g_r(x) = 3\lambda x\left(3M\epsilon_{rs}^2 x^2 \right. \\ \left. -1\right)(x - \xi_r) P_r(x) + R_r(x)\left(B_{(r)1}x^4 + B_{(r)2}x^3 \right. \\ \left. +B_{(r)3}x^2 + B_{(r)4}x + B_{(r)5}\right) \tag{3.180}$$

where $B_{(r)1}$, $B_{(r)2}$, $B_{(r)3}$, $B_{(r)4}$ and $B_{(r)5}$ are constant so that

$$g_r(0) = \frac{1}{8+\lambda} B_{(r)5} \tag{3.181}$$

Now, comparing the co-efficients of x^{n+4} from both sides of the equation (3.180), we get

$$B_{(r)1} = (-1)^{n+1} \frac{9M\lambda\epsilon_{rs}^2}{\zeta_r k_1 \cdots \zeta_r k_n} \tag{3.182}$$

Keeping in mind that the roots of the equation (3.161) are $\pm V_1$ and $\pm V_2$, putting these roots in the equation (3.180), we shall be able to reach the result

$$B_{(r)5} = B_{(r)1}V_1^2V_2^2 + (-1)^{n+1}\, 3\lambda \left[\frac{V_1^2 V_2^2}{V_1^2 - V_2^2} \left\{ \left(a_{(r)1} - a_{(r)2}\right) \right.\right.$$
$$\left. -3M\epsilon_{rs}^2 \left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) \right\} + \xi_r \cdot \frac{V_1V_2}{V_1^2 - V_2^2} \left\{ \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) \right.$$
$$\left.\left. -3M\epsilon_{rs}^2 V_1V_2 \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right) \right\} \right] \mu_{(r)1} \cdots \mu_{(r)n}$$

in which using the equations (3.182) and (3.142), we shall get

$$B_{(r)5} = (-1)^{n+1}\, 3\lambda \frac{V_1V_2}{V_1^2 - V_2^2} \left[\xi_r \cdot \left\{ \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) \right.\right.$$
$$\left. -3M\epsilon_{rs}^2 V_1V_2 \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right) \right\} + V_1V_2 \left\{ \left(a_{(r)1} \right.\right.$$
$$\left.\left. -a_{(r)2}\right) - 3M\epsilon_{rs}^2 \left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) \right\}$$
$$\left. + \frac{3M\epsilon_{rs}^2 V_1V_2 \left(V_1^2 - V_2^2\right)}{\left\{ M \left(1 - \frac{\lambda}{10}(1-M)\right) \right\}^{\frac{1}{2}}} \right] \mu_{(r)1} \cdots \mu_{(r)n} \tag{3.183}$$

Now, from the equation (3.178), using the equation (3.181) and the equation (3.183), we get

$$q_r = \frac{3\lambda}{8+\lambda} q_r \frac{V_1V_2}{V_1^2 - V_2^2} \left[\xi_r \cdot \left\{ \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3M\epsilon_{rs}^2 V_1V_2 \left(V_1 b_{(r)1} \right.\right.\right.$$
$$\left.\left. -V_2 b_{(r)2}\right) \right\} + V_1V_2 \left\{ \left(a_{(r)1} - a_{(r)2}\right) - 3M\epsilon_{rs}^2 \left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) \right\}$$
$$\left. + \frac{3M\epsilon_{rs}^2 V_1V_2 \left(V_1^2 - V_2^2\right)}{\left\{ M \left(1 - \frac{\lambda}{10}(1-M)\right) \right\}^{\frac{1}{2}}} \right] \zeta_r k_1 \cdots \zeta_r k_n \cdot \mu_{(r)1} \cdots \mu_{(r)n} + (-1)^{n-1} \frac{\zeta_r}{\omega_r}$$

which, on using the equation (3.142), can be written as

$$
\begin{aligned}
3\lambda \cdot V_1V_2 &\left\{\left(V_2b_{(r)1} - V_1b_{(r)2}\right) - 3M\epsilon_{rs}^2V_1V_2\left(V_1b_{(r)1} - V_2b_{(r)2}\right)\right\} q_r\xi_r \\
&+3\lambda\, V_1^2V_2^2\left\{\left(a_{(r)1} - a_{(r)2}\right) - 3M\epsilon_{rs}^2\left(V_1^2a_{(r)1} - V_2^2a_{(r)2}\right)\right\} q_r \\
&+\frac{\{9\lambda M\epsilon_{rs}^2V_1V_2 - (8+\lambda)\}\left(V_1^2 - V_2^2\right)}{\left\{M\left(1 - \frac{\lambda}{10}(1-M)\right)\right\}^{\frac{1}{2}}} q_r \\
&= (-1)^n \frac{3\zeta_r(8+\lambda)\left(V_1^2 - V_2^2\right)}{\omega_r\left\{M\left(1 - \frac{\lambda}{10}(1-M)\right)\right\}^{\frac{1}{2}}}
\end{aligned}
$$

But V_1^2 and V_2^2 are the roots of the equations (3.161). Therefore,

$$9\lambda M\epsilon_{rs}^2V_1^2V_2^2 - (8+\lambda) = 0$$

and

$$V_1^2 + V_2^2 = \frac{3\left(1 + M\epsilon_{rs}^2\right)\lambda}{9\lambda M\epsilon_{rs}^2}$$

So, the above relation is equivalent to

$$
\begin{aligned}
3\lambda \cdot V_1V_2 &\left\{\left(V_2b_{(r)1} - V_1b_{(r)2}\right) - 3M\epsilon_{rs}^2V_1V_2\left(V_1b_{(r)1} - V_2b_{(r)2}\right)\right\} q_r\xi_r \\
&+3\lambda\, V_1^2V_2^2\left\{\left(a_{(r)1} - a_{(r)2}\right) - 3M\epsilon_{rs}^2\left(V_1^2a_{(r)1} - V_2^2a_{(r)2}\right)\right\} q_r \\
&= (-1)^n \frac{3\,\zeta_r(8+\lambda)\left(V_1^2 - V_2^2\right)}{\omega_r\left\{M\left(1 - \frac{\lambda}{10}(1-M)\right)\right\}^{\frac{1}{2}}}
\end{aligned}
\tag{3.184}
$$

Solving the equations (3.172) and (3.184) , we get

$$q_r = (-1)^{n+1}\left\{\frac{3\left(V_1^2 - V_2^2\right)}{b_1\varpi_r\left\{M\left(1-\frac{1}{10}\lambda\left(1-M\right)\right)\right\}^{1/2}}\right\}\times$$
$$\times\frac{\begin{array}{l}\Big\{b_1\zeta_r\left(8+\lambda\right)\left\{3\left(V_1^2a_{(r)2}-V_2^2a_{(r)1}\right)+\left(a_{(r)1}-a_{(r)2}\right)\right\}\\ +b_0\left\{9\lambda V_1V_2\left(V_2b_{(r)1}-V_1b_{(r)2}\right)-3\left(8+\lambda\right)\left(V_1b_{(r)1}\right.\right.\\ \left.\left.-V_2b_{(r)2}\right)\right\}\Big\}\end{array}}{\begin{array}{l}\Big[\left(8+\lambda\right)\left\{\left(\frac{V_2^2}{M\epsilon_{rs}^2}+V_1^2\right)-\frac{2\left(4+\lambda\right)}{3M\epsilon_{rs}^2}\right\}H_r\left(V_1\right)\cdot H_r\left(-V_1\right)\\ +\left(8+\lambda\right)\left\{\left(\frac{V_1^2}{M\epsilon_{rs}^2}+V_2^2\right)-\frac{2\left(4+\lambda\right)}{3M\epsilon_{rs}^2}\right\}H_r\left(V_2\right)\cdot H_r\left(-V_2\right)\\ -\left\{\frac{16\left(8+\lambda\right)}{3M\lambda\epsilon_{rs}^2}\right\}a_{(r)1}a_{(r)2}+\left\{8\left(\frac{1+M\epsilon_{rs}^2}{M\epsilon_{rs}^2}\right)V_1V_2\right\}b_{(r)1}b_{(r)2}\Big]\end{array}} \tag{3.185}$$

and

$$\xi_r = \frac{\begin{array}{l}\Big[b_0\left\{3\left(8+\lambda\right)\left(V_1^2a_{(r)1}-V_2^2a_{(r)2}\right)-9\lambda V_1^2V_2^2\left(a_{(r)1}\right.\right.\\ \left.\left.-a_{(r)2}\right)\right\}-b_1\zeta_r\left(8+\lambda\right)\left\{3V_1V_2\left(V_1b_{(r)2}-V_2b_{(r)1}\right)\right.\\ \left.+\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}\Big]\end{array}}{\begin{array}{l}\Big[b_0\left\{9\lambda V_1V_2\left(V_2b_{(r)1}-V_1b_{(r)2}\right)-3\left(8+\lambda\right)\left(V_1b_{(r)1}\right.\right.\\ \left.\left.-V_2b_{(r)2}\right)\right\}+b_1\zeta_r\left(8+\lambda\right)\left\{3\left(V_1^2a_{(r)2}-V_2^2a_{(r)1}\right)\right.\\ \left.+\left(a_{(r)1}-a_{(r)2}\right)\right\}\Big]\end{array}} \tag{3.186}$$

Now, from the equation (3.145), using the equation (3.152)

$$I_r^*(0,\mu) = G_r(\mu+\xi_r)H_r(\mu) - \zeta_r\mu - b_0 \tag{3.187}$$

where G_r and ξ_r are given by

$$G_r = \frac{\begin{array}{l}(V_1^2-V_2^2)\Big[b_1\zeta_r(8+\lambda)\left\{3\left(V_1^2a_{(r)2}-V_2^2a_{(r)1}\right)\right.\\ \left.+\left(a_{(r)1}-a_{(r)2}\right)\right\}+b_0\left\{9\lambda V_1V_2\left(V_2b_{(r)1}-V_1b_{(r)2}\right)\right.\\ \left.-3(8+\lambda)\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}\Big]\end{array}}{\begin{array}{l}\left[(8+\lambda)\left\{\left(\frac{V_2^2}{M\epsilon_{rs}^2}+V_1^2\right)-\frac{2(4+\lambda)}{3M\epsilon_{rs}^2}\right\}H_r(V_1)\cdot H_r(-V_1)\right.\\ +(8+\lambda)\left\{\left(\frac{V_1^2}{M\epsilon_{rs}^2}+V_2^2\right)-\frac{2(4+\lambda)}{3M\epsilon_{rs}^2}\right\}H_r(V_2)\cdot H_r(-V_2)\\ \left.-\left\{\frac{16(8+\lambda)}{3M\lambda\epsilon_{rs}^2}\right\}a_{(r)1}a_{(r)2}+\left\{8\left(\frac{1+M\epsilon_{rs}^2}{M\epsilon_{rs}^2}\right)V_1V_2\right\}b_{(r)1}b_{(r)2}\right]\end{array}} \tag{3.188}$$

and the equation (3.186)

Now, from the equation(3.112), we get

$$I_r(0,\mu) = b_0 + \zeta_r\mu b_1 + I_r^*(0,\mu)$$

and by virtue of the equation (3.187), we get

$$I_r(0,\mu) = G_r\cdot(\mu+\xi_r)\cdot H_r(\mu) \tag{3.189}$$

The solution (3.189) is a desired solution of the equation (3.109) in n^{th} approximation.

3.2.3.4 The Exact Diffusely Reflected Intensity and the Exact Solution for the Emergent Intensity.

As Busbridge and Stibbs[33] did, we change the variable $\zeta_r\mu$ and $\zeta_s\mu'$ to x and x' respectively [and consequently $\zeta_r\mu_{(r)i}$ and $\zeta_s\mu_{(s)j}'$ to x_i and $x'_j{}'$ respectively] in the equation (3.129) to get the characteristic equation as:

$$\sum_j \frac{a'_j}{1+kx_j}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left[\frac{1}{16C}\left\{(8+\lambda)-\frac{3\lambda}{\zeta_s^2}\cdot(1+M)\,x_j^2 + \frac{9M\lambda}{\zeta_s^4}x_j^4\right\}\right]=1$$

i.e.

$$\sum_j \frac{a'_j\Psi(x_j)}{1+kx_j}=1\ ,\ \text{with}\ a'_j=\zeta_r a_j \tag{3.190}$$

where, with the assumption that

$$\eta_1>\eta_2>\cdots>\eta_m \tag{3.191}$$

so that

$$0\le\zeta_1<\zeta_2<\cdots<\zeta_m\le 1 \tag{3.192}$$

$\Psi(x')$ is of the form

$$\Psi(x')=\begin{cases}\sum\limits_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left[\frac{1}{16C}\left\{(8+\lambda)-\frac{3\lambda}{\zeta_s^2}\cdot(1+M)\,x_j^2+\frac{9M\lambda}{\zeta_s^4}x_j^4\right\}\right], & \text{if}\ 0\le x'\le\zeta_1\\[2ex] \sum\limits_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left[\frac{1}{16C}\left\{(8+\lambda)-\frac{3\lambda}{\zeta_s^2}\cdot(1+M)\,x_j^2+\frac{9M\lambda}{\zeta_s^4}x_j^4\right\}\right], & \text{if}\ \zeta_r\le x'\le\zeta_{r+1}\\[2ex] 0, & \text{if}\ \zeta_m\le x'\le 1\end{cases} \tag{3.193}$$

Then

$$\int_0^1 \Psi(x')\,dx' < \frac{1}{2} \tag{3.194}$$

Therefore, by the theory of H-function, developed by Chandrasekhar,[45] $H(x)$, where $x = \zeta_r\mu$ or equivalently $[H(\zeta_r\mu) =]\, H_r(\mu)$, given by the equation (3.151) satisfies, in the limit of infinite approximation, the non-integral equation:

$$H(x) = 1 + xH(x)\int_0^1 \frac{\Psi(x')\,H(x)}{x+x'}dx'$$

i.e.

$$H_r(\mu) = 1 + \zeta_r\mu H_r(\mu)\int_0^1 \frac{\zeta_s\Psi(\zeta_s\mu')}{\zeta_r\mu+\zeta_s\mu'}H_s(\mu')\,d\mu' \tag{3.195}$$

which is bounded in the entire half plane $\mathbb{R}(x) \geq 0$.

The characteristic function $\Psi(x')$ in the equation (3.195) satisfies the necessary condition:

$$\int_0^1 \Psi(x')\,dx' \leq \frac{1}{2} \tag{3.196}$$

Now, we allow n to tend to infinity for both the equations (3.187) and (3.189) to get the exact diffusely reflected intensity and the exact emergent intensity, given by:

$$I_r^*(0,\mu) = G_r\cdot(\mu+\xi_r)\cdot H_r(\mu) - \zeta_r\mu b_1 - b_0 \tag{3.197}$$

and

$$I_r(0,\mu) = G_r\cdot(\mu+\xi_r)\cdot H_r(\mu) \tag{3.198}$$

where ξ_r and G_r are given by the equations (3.186) and (3.188) respectively and $H_r(\mu)$ is the solution of the equation (3.195).

3.2.4 Derivation of the Solution with Rayleigh Phase Function from the Solution with Pomraning Phase Function

If we put $\lambda = 1$ in the phase function (3.108), the Pomraning phase function reduces to the Rayleigh phase funcion . So, the solution (3.187) and (3.189) will be the solutions for the case of Rayleigh scattering if we substitute $\lambda = 1$ in ξ_r and G_r, given by the equations (3.186) and (3.188).

Now, $\lambda = 1$ reduces G_r to the form :

$$G_r = \frac{\begin{aligned}&3\left(V_1^2 - V_2^2\right)\Big[b_1\zeta_r\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right.\\ &\left.+\left(a_{(r)1} - a_{(r)2}\right)\right\} + b_0\left\{V_1V_2\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right.\\ &\left.-3\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\Big]\end{aligned}}{\begin{aligned}&\left[\left\{3\left(\frac{V_2^2}{M\epsilon_{rs}^2} + V_1^2\right) - \frac{10}{M\epsilon_{rs}^2}\right\} H_r\left(V_1\right)\cdot H_r\left(-V_1\right)\right.\\ &+\left\{3\left(\frac{V_1^2}{M\epsilon_{rs}^2} + V_2^2\right) - \frac{10}{M\epsilon_{rs}^2}\right\} H_r\left(V_2\right)\cdot H_r\left(-V_2\right)\\ &\left.-\left\{\frac{16}{M\epsilon_{rs}^2}\right\} a_{(r)1}a_{(r)2} + \left\{\frac{8}{3}\left(\frac{1+M\epsilon_{rs}^2}{M\epsilon_{rs}^2}\right) V_1V_2\right\} b_{(r)1}b_{(r)2}\right]\end{aligned}}$$

and ξ_r to the form :

$$\xi_r = \frac{\begin{aligned}&\Big[b_0\left\{3\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - V_1^2V_2^2\left(a_{(r)1} - a_{(r)2}\right)\right\}\\ &-b_1\zeta_r\left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\Big]\end{aligned}}{\begin{aligned}&\Big[b_0\left\{V_1V_2\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\\ &+b_1\zeta_r\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\Big]\end{aligned}}$$

which are the same as the equations(3.89) and (3.91) derived in the section - 3.1.2. Thus the emergent intensity , and the diffusely reflected intensity for Rayleigh scattering media can be derived.

Likewise the exact diffusely reflected intensity $I_r^*(0,\mu)$ and exact emergent intensity $I_r(0,\mu)$ for Rayleigh phase function can also be derived from the case of Pomraning phase function by putting $\lambda = 1$. The characteristic function $\Psi(x')$ of the H-function $H_r(\mu)$ for the case of Rayleigh scattering phase function is observed to be the same as the characteristic function, obtained by putting $\lambda = 1$ in the equation (3.193).

Chapter 4

Interlocked Multiplets with Three Term Scattering Indicatrix.

4.1 Introduction

The **three term scattering indicatrix** is

$$p\left(cos\Theta\right)=\varpi_0+\varpi_1P_1\left(cos\Theta\right)+\varpi_2P_2\left(cos\Theta\right)\ ,\ \left(\varpi_0\leq 1\right)\ ,\qquad(4.1)$$

where ϖ_0(albedo), ϖ_1 and ϖ_2 are constants P_1 and P_2 are Legendre polynomials.

Considering that parallel light of flux density πF_0 is incident on a plane-parallel, semi-infinite atmosphere which scatters light in accordance with phase function 4.1, Horak and Chandrasekhar[91] found the exact solution for the emergent radiation field by using the invariance-principle method.

With the phase function $P\left(\cos\theta\right)=\varpi+\varpi_1P_1\left(\cos\theta\right)+\varpi_2P_2\left(\cos\theta\right)$ in which the parameters $\widetilde{\varpi}_1=\varpi_1/\varpi$ and $\widetilde{\varpi}_2=\varpi_2/\varpi$ can be varied to get different shapes for the phase function, implying different angular dependence of the energy distribution of the scattered radiation, Bhatia and Abhyankar[26] observed how it affects the phase variation of equivalent widths.

Karanjai and Deb[109] used the three term scattering indicatrix to find an exact solution of the equation of transfer in an exponential atmosphere by the method of Laplace transform and Wiener-Hopf technique.

Karanjai and Karanjai[113] proposed a numerical method for finding the approximate value of H-function involved in the radiative transfer equation for anisotropically scattering medium with three term scattering indicatrix.

Here we obtain the expressions for diffusely reflected intensities and emergent intensities of interlocked multiplet lines in anisotropically scattering medium due to three term scattering indicatrix.

4.2 Formulation of the Problem

4.2.1 The Equation of Transfer for the r^{th} interlocked line and the Boundary Conditions:

4.2.1.1 The Equation of Transfer:

The equation of transfer for the r^{th} interlocked line is of the form:

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)B_\nu(T) - \frac{1}{2}(1-\epsilon)\alpha_r\sum_{s=1}^{m}\eta_s\int_{-1}^{1}p(\mu,\mu')I_s(\tau,\mu')d\mu' \qquad (4.2)$$

in which

$$\alpha_r = \eta_r / \sum_{s=1}^{m} \eta_s \tag{4.3}$$

so that

$$\sum_{r=1}^{m} \alpha_r = 1 \tag{4.4}$$

the Planck-function, $B_\nu(T)$, is a linear function of τ which is of the form

$$B_\nu(T) = B(\tau) = b_0 + b_1\tau \tag{4.5}$$

in which the terms b_0 and b_1 are positive constants.

The phase function $p(\mu, \mu')$ is

$$p(\mu, \mu') = \sum_{l=0}^{2} \varpi_l P_l(\mu) P_l(\mu') \ , \ \varpi_0 = 1$$

i.e.

$$p(\mu, \mu') = 1 + \varpi_1 \mu\mu' + \frac{1}{4} \cdot \varpi_2 \left(3\mu^2 - 1\right) \left(3\mu'^2 - 1\right) , \tag{4.6}$$

is three term scattering indicatrix for non-conservative scattering.

Here, τ denotes the optical depth. The term η_r, being the ratio of the absorption co-efficient l_r for the r^{th} interlocked line to the continuous absorption k; ε, the co-efficient, which is introduced to allow for the thermal emission associated with the line absorption, $B_\nu(T)$, the Planck-functions are considered to be constant for each line.

Placing $B_\nu(T)$ from the equation (4.5) and $p(\mu, \acute{\mu})$ from the equation (4.6) in the equation (4.2), we get

$$\zeta_r \mu \frac{dI_r(\tau,\mu)}{d\tau} = I_r(\tau,\mu) - (1-\omega_r)(b_0 + b_1\tau) - \frac{\omega_r}{2C} \sum_{s=1}^{m} \eta_s \int_{-1}^{1} \left\{ 1 \right.$$
$$\left. + \varpi_1 \mu\mu' + \frac{1}{4} \cdot \varpi_2 \left(3\mu^2 - 1\right)\left(3\mu'^2 - 1\right) \right\} I_s(\tau, \mu') d\mu' \tag{4.7}$$

where

$$C = \sum_{s=1}^{n} \eta_s \tag{4.8}$$

$$\zeta_r = \frac{1}{1+\eta_r} \tag{4.9}$$

and

$$\omega_r = \frac{(1-\epsilon)\,\eta_r}{1+\eta_r} \tag{4.10}$$

4.2.1.2 The Boundary Conditions:

We wish to get the solution of the equation (4.7), subject to the boundary conditions :

$$I_r(0,-\mu) = 0; \ (0 < \mu \leq 1) \tag{4.11}$$

$$I_r(\tau,\mu) \cdot \ e^{-\tau/\mu} \rightarrow 0 \text{ as } \tau \ \rightarrow 0$$

i.e. $I_r(\tau,\mu)$ is at most linear in τ as τ tends to infinity (4.12)

4.3 Solution of the Equation

4.3.1 Reduction of the Equation of Transfer and its Boundary Conditions to a Standard Form by using a Transformation:

4.3.1.1 Reduction of the Equation of Transfer and Formation of 2n Discrete Ordinate Equations:

We observe that if we assume, following Busbridge and Stibbs ,[33] that one of the solution of the equation (4.7) is

$$I_r\left(\tau,\mu\right)=b_0+b_1\left(\tau+\frac{\mu}{1+\eta_r}\right)+I_r^*(\tau,\mu) \tag{4.13}$$

in which there are two parts of which the first part is the solution for an infinitely unbounded atmosphere as τ tends to infinity and the second part $I_r^*(\tau,\mu)$ is the departure of the asymptotic solution from the value $I_r(\tau,\mu)$ as we approach the boundary $\tau=0$

Then, using the equation (4.13), we can write the equation (4.7) to the form:

$$\begin{aligned}\zeta_r\mu\frac{dI_r^*\left(\tau,\mu\right)}{d\tau} &= I_r^*\left(\tau,\mu\right)-\frac{\omega_r}{2C}\sum_{s=1}^{m}\eta_s\int_{-1}^{+1}\Big\{1+\varpi_1\mu\mu' \\ &+\frac{1}{4}\varpi_2\left(9\mu^2\mu'^2-3\mu^2-3\mu'^2+1\right)\Big\}I_s^*\left(\tau,\mu'\right)d\mu' \\ &-\frac{1}{3}\omega_r b_1\varpi_1\left\{\frac{\sum\limits_{s=1}^{m}\frac{\eta_s}{1+\eta_s}}{\sum\limits_{s=1}^{m}\eta_s}\right\}\mu\end{aligned}$$

In n^{th} approximation, we replace the above integro-differential equation by the system of $2n$ linear differential equations:

$$\zeta_r \mu_{(r)i} \frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{\omega_r}{2C} \sum_{s=1}^{m} \eta_s \sum_j \Bigg\{ 1 + \varpi_1 \mu_{(r)i} \mu_{(s)j} + \frac{1}{4} \varpi_2 \left(9\mu^2_{(r)i} \mu^2_{(s)j} - 3\mu^2_{(r)i} - 3\mu^2_{(s)j} + 1 \right) \Bigg\} I^*_{(s)j} a_j - \frac{1}{3} \omega_r b_1 \varpi_1 \left\{ \frac{\sum_{s=1}^{m} \left(\frac{\eta_s}{1+\eta_s} \right)}{\sum_{s=1}^{m} \eta_s} \right\} \mu_{(r)i} \tag{4.14}$$

Here, we have used, for brevity, the symbol

$$I^*_{(r)i} \text{ for } I^*_r \left(\tau, \mu_{(r)i} \right)$$

and $\mu_{(r)i}$'s $\left(i = \pm 1, \cdots \pm n \text{ ; having property that } \mu_{(r)-i} = -\mu_{(r)i} \right)$ are the zeros of the Legendre polynomial $P_{2n}(\mu)$ which are independent of interlocking and a_j's $\left(j = \pm 1, \cdots \pm n \text{ ; having property that } a_{-j} = a_j \right)$ are the corresponding Gaussian weights. However it is to be noted that there is no term with $j = 0$.

4.3.1.2 Reduction of boundary conditions and Discretization into 2n parts:

Again with the help of above mentioned symbols, the boundary conditions (4.11) and (4.12) can be written as

$$I^*_{(r)-i} = b_1 \zeta_r \mu_{(r)i} - b_0 \quad ; \ \left(0 < \mu_{(r)i} \leq 1 \right) \tag{4.15}$$

and

$$I^*_{(r)i} \cdot e^{-\tau/\mu_{(r)i}} \longrightarrow 0 \text{ as } \tau \longrightarrow 0$$

i.e.

$I^*_{(r)i}$ is at most linear in τ as τ tends to infinity (4.16)

4.3.2 Solution of the Discrete Ordinate Equations:

4.3.2.1 Solution of the Associated Homogeneous Parts:

Now, we shall solve the associated homogeneous part of the equation (4.14) i.e.

$$\zeta_r \mu_{(r)i} \frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{\omega_r}{2C} \sum_{s=1}^{m} \eta_s \sum_j \Big\{ 1 + \varpi_1 \mu_{(r)i} \mu_{(s)j}$$
$$+ \frac{1}{4} \varpi_2 \left(9\mu^2_{(r)i} \mu^2_{(s)j} - 3\mu^2_{(r)i} - 3\mu^2_{(s)j} + 1 \right) \Big\} I^*_{(s)j} a_j \tag{4.17}$$

The system of the equations (4.17) admits integrals of the form:

$$I^*_{(r)i} = g_{(r)i}\, e^{-k\tau} \;;\; (i = \pm 1, \cdots, \pm n) \tag{4.18}$$

Then, the equation (4.17) gives

$$g_{(r)i} \left\{ 1 + k\zeta_r \mu_{(r)i} \right\} = \frac{\omega_r}{2C} \sum_{s=1}^{m} \eta_s \sum_j \Big\{ 1 + \varpi_1 \mu_{(r)i} \mu_{(s)j}$$
$$+ \frac{1}{4} \varpi_2 \left(9\mu^2_{(r)i} \mu^2_{(s)j} - 3\mu^2_{(r)i} - 3\mu^2_{(s)j} + 1 \right) \Big\} g_{(s)j} a_j \tag{4.19}$$

which can be expressed as

$$g_{(r)i} \left\{ 1 + k\zeta_r \mu_{(r)i} \right\} = \frac{\omega_r}{2C} \Bigg\{ \left(\sum_{s=1}^{m} \eta_s \sum_j g_{(s)j} a_j + \frac{1}{4} \varpi_2 \sum_{s=1}^{m} \eta_s \sum_j g_{(s)j} a_j \right.$$
$$\left. - \frac{3}{4} \varpi_2 \sum_{s=1}^{m} \eta_s \sum_j g_{(s)j} a_j \mu^2_{(s)j} \right) + \left(\varpi_1 \sum_{s=1}^{m} \eta_s \sum_j g_{(s)j} a_j \mu_{(s)j} \right) \mu_{(r)i}$$
$$+ \left(\frac{9}{4} \varpi_2 \sum_{s=1}^{m} \eta_s \sum_j g_{(s)j} a_j \mu^2_{(s)j} - \frac{3}{4} \varpi_2 \sum_{s=1}^{m} \eta_s \sum_j g_{(s)j} a_j \right) \mu^2_{(r)i} \Bigg\}$$

i.e.

$$g_{(r)i} = \omega_r \cdot \frac{\rho + \rho_1 \mu_{(r)i} + \rho_2 \mu^2_{(r)i}}{1 + k\zeta_r \mu_{(r)i}} \tag{4.20}$$

where ρ, ρ_1 and ρ_2 are constants independent of $\mu_{(r)i}$.

Inserting $g_{(r)i}$ from the equation (4.20) in the equation (4.19), we get a relation

$$\begin{aligned}
8C\left(\rho + \rho_1 \mu_{(r)i} + \rho_2 \mu^2_{(r)i}\right) = \sum_{s=1}^{m} \eta_s \omega_s \sum_j \frac{a_j}{\left(1 + k\zeta_s \mu_{(s)j}\right)} \Bigg[\rho \Bigg\{ (4 + \varpi_2 \\
- 3\varpi_2 \mu^2_{(s)j}) + \left(4\varpi_1 \mu_{(s)j}\right) \mu_{(r)i} + \left(9\varpi_2 \mu^2_{(s)j} - 3\varpi_2\right) \mu^2_{(r)i} \Bigg\} \\
+ \rho_1 \Bigg\{ \left(4 + \varpi_2 - 3\varpi_2 \mu^2_{(s)j}\right) \mu_{(s)j} + \left(4\varpi_1 \mu_{(s)j}\right) \mu_{(s)j} \mu_{(r)i} \\
+ \left(9\varpi_2 \mu^2_{(s)j} - 3\varpi_2\right) \mu_{(s)j} \mu^2_{(r)i} \Bigg\} + \rho_2 \Bigg\{ \left(4 + \varpi_2 - 3\varpi_2 \mu^2_{(s)j}\right) \mu^2_{(s)j} \\
+ \left(4\varpi_1 \mu_{(s)j}\right) \mu^2_{(s)j} \mu_{(r)i} + \left(9\varpi_2 \mu^2_{(s)j} - 3\varpi_2\right) \mu^2_{(s)j} \mu^2_{(r)i} \Bigg\} \Bigg]
\end{aligned}$$

Defining

$$D_\ell(x) = \sum_{s=1}^{m} \eta_s \omega_s \sum_j \frac{a_j \mu^\ell_{(s)j}}{1 + \mu_{(s)j}\, x} \tag{4.21}$$

and analyzing the proof done in Section.3 by Busbridge and Stibbs,[33] we can write the above equation in the following compact form:

$$8C\left(\rho+\rho_1\mu_{(r)i}+\rho_2\mu^2_{(r)i}\right)=\rho\Big\{\left(\left(4+\varpi_2\right)D_0(k\zeta_s)-3\varpi_2D_2(k\zeta_s)\right)$$
$$+4\varpi_1D_1(k\zeta_s)\mu_{(r)i}+\left(9\varpi_2D_2(k\zeta_s)-3\varpi_2D_0(k\zeta_s)\right)\mu^2_{(r)i}\Big\}$$
$$+\rho_1\Big\{\left(\left(4+\varpi_2\right)D_1(k\zeta_s)-3\varpi_2D_3(k\zeta_s)\right)+4\varpi_1D_2(k\zeta_s)\mu_{(r)i}$$
$$+\left(9\varpi_2D_3(k\zeta_s)-3\varpi_2D_1(k\zeta_s)\right)\mu^2_{(r)i}\Big\}$$
$$+\rho_2\Big\{\left(\left(4+\varpi_2\right)D_2(k\zeta_s)-3\varpi_2D_4(k\zeta_s)\right)+4\varpi_1D_3(k\zeta_s)\mu_{(r)i}$$
$$+\left(9\varpi_2D_4(k\zeta_s)-3\varpi_2D_2(k\zeta_s)\right)\mu^2_{(r)i}\Big\}$$

Again, writing

$$D_\ell \text{ for } D_\ell(k\zeta_s)$$

we can write the above equation as

$$8C\left(\rho+\rho_1\mu_{(r)i}+\rho_2\mu^2_{(r)i}\right)=\Big\{\rho\left(\left(4+\varpi_2\right)D_0-3\varpi_2D_2\right)$$
$$+\rho_1\left(\left(4+\varpi_2\right)D_1-3\varpi_2D_3\right)+\rho_2\left(\left(4+\varpi_2\right)D_2-3\varpi_2D_4\right)\Big\}$$
$$+4\varpi_1\Big\{\rho D_1+\rho_1D_2+\rho_2D_3\Big\}\mu_{(r)i}+\Big\{\rho\left(9\varpi_2D_2-3\varpi_2D_0\right)$$
$$+\rho_1\left(9\varpi_2D_3-3\varpi_2D_1\right)+\rho_2\left(9\varpi_2D_4-3\varpi_2D_2\right)\Big\}\mu^2_{(r)i}$$

which is true for every $\mu_{(r)i}$. So, equating the co-efficient of $\mu^2_{(r)i}$, the co-efficient of $\mu_{(r)i}$ and the constant terms separately from both sides of the relation, we get

$$\begin{aligned}\rho\left(9\varpi_2 D_2-3\varpi_2 D_0\right)+\rho_1\left(9\varpi_2 D_3-3\varpi_2 D_1\right)&\\+\rho_2\left(9\varpi_2 D_4-3\varpi_2 D_2-8C\right)&=0 \qquad (4.22a)\\ \rho\varpi_1 D_1+\rho_1\left(\varpi_1 D_2-2C\right)+\rho_2\varpi_1 D_3&=0 \qquad (4.22b)\\ \rho\left(\left(4+\varpi_2\right)D_0-3\varpi_2 D_2-8C\right)+\rho_1\left(\left(4+\varpi_2\right)D_1\right.&\\ \left.-3\varpi_2 D_3\right)+\rho_2\left(\left(4+\varpi_2\right)D_2-3\varpi_2 D_4\right)&=0 \qquad (4.22c)\end{aligned}$$

Eliminating ρ, ρ_1, ρ_2 from the equations (4.22a), (4.22b) and (4.22c),using the equation (II.4c) of Appendix II, we get

$$\begin{aligned}&\varpi_1\varpi_2\left\{12\psi_0\left(D_2^2-D_1D_3\right)-4\psi_0\left(D_0D_2-D_1^2\right)+72C\left(D_2D_4-D_3^2\right)\right.\\&\quad\left.-36\psi_0\left(D_2D_4-D_3^2\right)+24C\left(D_1D_3-D_2^2\right)+12\psi_0\left(D_2^2-D_1D_3\right)\right\}\\&+\varpi_1\varpi_2^2\left\{3\psi_0\left(D_2^2-D_1D_3\right)-9\psi_0\left(D_2D_4-D_3^2\right)-\psi_0\left(D_0D_2-D_1^2\right)\right.\\&\quad\left.+3\psi_0\left(D_0D_4-D_1D_3\right)\right\}-\varpi_2\left\{72C\left(D_2^2-D_1D_3\right)+72C\left(D_1D_3\right.\right.\\&\left.\left.-D_0D_4\right)+16C^2D_0-96C^2D_2+144C^2D_4\right\}-32C\varpi_1\left(2C-\psi_0\right)D_2\\&\qquad-64C^2D_0+128C^3=0\end{aligned}$$

which, by using the equations (II.20b), (II.20d), (II.20e) and (II.20g) and the equation (II.4c) of Appendix II , can written in the form:

$$\begin{aligned}&64C^2D_0+32C\varpi_1\left(2C-\psi_0\right)D_2+16C^2\varpi_2\left(D_0-3D_2\right)\\&+24C\varpi_2\left(2C-\psi_0\right)\left(3D_4-D_2\right)+4\varpi_1\varpi_2\psi_0\left(2C-\psi_0\right)\times\\&\qquad\times\left(D_2-3D_4\right)=128C^3 \qquad (4.23)\end{aligned}$$

i.e.

$$\begin{aligned}\frac{1}{2C}\Bigg\{&\left(1+\frac{1}{4}\varpi_2\right)D_0+\left(M\varpi_1-\frac{3}{4}\varpi_2-\frac{3}{4}M\varpi_2\right.\\&\left.+\frac{1}{4}\cdot M\left(1-M\right)\varpi_1\varpi_2\right)D_2+\left(\frac{9}{4}\cdot M\varpi_2\right.\\&\left.-\frac{3}{4}\cdot M\left(1-M\right)\varpi_1\varpi_2\right)D_4\Bigg\}=1\end{aligned}$$

i.e.

$$\sum_{s=1}^{m} \eta_s \omega_s \sum_j \frac{a_j}{8C\left(1+k\zeta_s\mu_{(s)j}\right)} \left\{ (4+\varpi_2) + \left(4M\varpi_1 - 3(1+M)\varpi_2 \right.\right.$$
$$\left.\left. + M(1-M)\varpi_1\varpi_2\right)\mu^2_{(s)j} + M\left(9\varpi_2 - 3(1-M)\varpi_1\varpi_2\right)\mu^4_{(s)j} \right\} = 1 \tag{4.24}$$

where

$$M = \frac{2C - \psi_0}{2C} = \frac{\sum\limits_{s=1}^{m} \eta_s (1-\varpi_s)}{\sum\limits_{s=1}^{m} \eta_s} \tag{4.25}$$

The equation (4.24) is the **characteristic equation** which, being an equation in k of order $2n$, will give $2n$ distinct non-zero roots which occur in pair as $\pm k_\iota$, ($\iota = 1, 2, \cdots, n$), if $\varpi_r < 1$

Now, from the equation (4.23), using the equation (4.25), we get

$$(4+\varpi_2) D_0 + \left\{4\varpi_1 M - 3(1+M)\varpi_2 + \varpi_1\varpi_2(1 \right.$$
$$\left. -M) M\right\} D_2 + \left\{9\varpi_2 M - 3\varpi_1\varpi_2(1-M)M\right\} D_4 = 8C$$

Therefore, using the equations(II.5b), (II.5d) and (II.4c) of Appendix II,

$$\left[(4+\varpi_2) k^4\zeta_s^4 + \left\{4\varpi_1 M - 3(1+M)\varpi_2 + \varpi_1\varpi_2(1-M)M\right\} k^2\zeta_s^2 \right.$$
$$\left. +3\varpi_2\left\{3M - \varpi_1(1-M)M\right\}\right] (\psi_0 - D_0) = -8Ck^4\zeta_s^4 + (4$$
$$+\varpi_2)\psi_0 k^4\zeta_s^4 - \varpi_2\left\{3M - \varpi_1(1-M)M\right\}\psi_0 k^2\zeta_s^2$$

and hence by using the equation (4.25),

$$\Big[(4+\varpi_2)\,k^4\zeta_s^4+\Big\{4\varpi_1 M-3\,(1+M)\,\varpi_2+\varpi_1\varpi_2\,(1-M)\,M\Big\}k^2\zeta_s^2 \\ +3\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]\,(\psi_0-D_0)=2C\,\{-4M+\varpi_2\,(1 \\ -M)\}\,k^4\zeta_s^4-2C\,(1-M)\,\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}k^2\zeta_s^2 \tag{4.26}$$

i.e.

$$\psi_0-D_0=\frac{\Big[2C\,\{\varpi_2\,(1-M)-4M\}\,k^4\zeta_s^4-2C\,(1-M)\times \\ \times\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}k^2\zeta_s^2\Big]}{\Big[(4+\varpi_2)\,k^4\zeta_s^4+\Big\{4\varpi_1 M-3\,(1+M)\,\varpi_2+\varpi_1\varpi_2\,(1 \\ -M)\,M\Big\}k^2\zeta_s^2+3\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]} \tag{4.27}$$

Again, from the equation (4.26), using the equation (4.25),

$$\Big[(4+\varpi_2)\,k^4\zeta_s^4+\Big\{4\varpi_1 M-3\,(1+M)\,\varpi_2+\varpi_1\varpi_2\,(1-M)\,M\Big\}k^2\zeta_s^2 \\ +3\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]\,D_0=\Big[(4+\varpi_2)\,k^4\zeta_s^4+\Big\{4\varpi_1 M \\ -3\,(1+M)\,\varpi_2+\varpi_1\varpi_2\,(1-M)\,M\Big\}k^2\zeta_s^2+3\varpi_2\Big\{3M- \\ \varpi_1\,(1-M)\,M\Big\}\Big]\,2C\,(1-M)-\Big[2C\,\{\varpi_2\,(1-M)-4M\}\,k^4\zeta_s^4 \\ -2C\,(1-M)\,\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}k^2\zeta_s^2\Big]$$

i.e.

$$\Big[(4+\varpi_2)\,k^4\zeta_s^4+\Big\{4\varpi_1 M-3\,(1+M)\,\varpi_2+\varpi_1\varpi_2\,(1-M)\,M\Big\}k^2\zeta_s^2$$
$$+3\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]D_0=2C\Big[4k^4\zeta_s^4+\Big(4\varpi_1 M-3\varpi_2\Big)\times$$
$$\times\,(1-M)\,k^2\zeta_s^2+3\varpi_2\,(1-M)\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]$$

i.e.

$$D_0=\frac{2C\Big[4k^4\zeta_s^4+(4\varpi_1 M-3\varpi_2)\,(1-M)\,k^2\zeta_s^2+3\varpi_2\,(1-M)\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]}{\Big[(4+\varpi_2)\,k^4\zeta_s^4+\Big\{4\varpi_1 M-3\,(1+M)\,\varpi_2+\varpi_1\varpi_2\,(1-M)\,M\Big\}k^2\zeta_s^2+3\varpi_2\Big\{3M-\varpi_1\,(1-M)\,M\Big\}\Big]}$$

Now, from the equations (4.22b) and (4.22c), we get

$$\frac{\rho}{E_1}=\frac{\rho_1}{E_2}=\frac{\rho_2}{E_3}$$

where

$$\begin{aligned}
E_1 &= 4\varpi_1\left(D_2^2-D_1D_3\right)+\varpi_1\varpi_2\left\{\left(D_2^2-D_1D_3\right)\right.\\
&\quad \left.-3\left(D_2D_4-D_3^2\right)\right\}+2\varpi_2 C\,(3D_4-D_2)-8CD_2\\
&= \frac{16MC^2}{3Z}\{3-\varpi_1\,(1-M)\}\Big[M\varpi_2\,\{3-\varpi_1\,(1-M)\}\\
&\quad -(4+\varpi_2)\,k^2\zeta_s^2\Big]
\end{aligned}$$

$$\begin{aligned}
E_2 &= -\Big\{4\varpi_1\,(D_0D_3-D_1D_2)+\varpi_1\varpi_2\,(D_0D_3-D_1D_2)\\
&\quad +3\varpi_1\varpi_2\,(D_1D_4-D_2D_3)-8C\varpi_1 D_3\Big\}\\
&= \frac{64C^2M^2\varpi_1}{3Z}\{3-\varpi_1\,(1-M)\}\,k\zeta_s
\end{aligned}$$

and

$$
\begin{aligned}
E_3 &= \Big\{4\varpi_1\left\{2CD_2 - \left(D_0D_2 - D_1^2\right)\right\} + \varpi_1\varpi_2\left\{3\left(D_2^2 - D_1D_3\right)\right. \\
&\quad \left. - \left(D_0D_2 - D_1^2\right)\right\} + 2C\varpi_2\left(D_0 - 3D_2\right) + 8CD_0 - 16C^2\Big\} \\
&= \frac{16C^2}{Z} M\varpi_2\left\{3 - \varpi_1\left(1 - M\right)\right\}\left[k^2\zeta_s^2 - M\left\{3 - \varpi_1\left(1 - M\right)\right\}\right]
\end{aligned}
$$

in all of which the term Z is given by

$$
\begin{aligned}
Z = \left(4 + \varpi_2\right)k^4\zeta_s^4 + \Big\{4\varpi_1 M - 3\left(1 + M\right)\varpi_2 + \varpi_1\varpi_2\left(1\right. \\
\left. - M\right)M\Big\}k^2\zeta_s^2 + 3\varpi_2\Big\{3M - \varpi_1\left(1 - M\right)M\Big\}
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\frac{\rho}{M\varpi_2\left\{3 - \varpi_1\left(1 - M\right)\right\} - \left(4 + \varpi_2\right)k^2\zeta_s^2} &= \frac{\rho_1}{4M\varpi_1 k\zeta_s} \\
&= \frac{\rho_2}{3\varpi_2\left[k^2\zeta_s^2 - M\left\{3 - \varpi_1\left(1 - M\right)\right\}\right]}
\end{aligned}
\tag{4.28}
$$

i.e.

$$
\rho_1 = \frac{4\varpi_1 Mk\zeta_s}{M\varpi_2\left\{3 - \varpi_1\left(1 - M\right)\right\} - \left(4 + \varpi_2\right)k^2\zeta_s^2}\,\rho \tag{4.29a}
$$

$$
\rho_2 = \frac{3\varpi_2\left[k^2\zeta_s^2 - M\left\{3 - \varpi_1\left(1 - M\right)\right\}\right]}{M\varpi_2\left\{3 - \varpi_1\left(1 - M\right)\right\} - \left(4 + \varpi_2\right)k^2\zeta_s^2}\,\rho \tag{4.29b}
$$

With the values of ρ_1 and ρ_2, given by the equations (4.29a) and (4.29b), we shall get from the equation (4.20), the expression for $g_{(r)i}$ as

$$
\begin{aligned}
g_{(r)i} = \frac{\omega_r \rho}{\left(1 + k\zeta_r\mu_{(r)i}\right)\left[M\varpi_2\left\{3 - \varpi_1\left(1 - M\right)\right\} - \left(4 + \varpi_2\right)k^2\zeta_s^2\right]} \times \\
\times\Bigg[\left[M\varpi_2\left\{3 - \varpi_1\left(1 - M\right)\right\} - \left(4 + \varpi_2\right)k^2\zeta_s^2\right] + 4\varpi_1 Mk\zeta_s\mu_{(r)i} \\
+ 3\varpi_2\left[k^2\zeta_s^2 - M\left\{3 - \varpi_1\left(1 - M\right)\right\}\right]\mu_{(r)i}^2\Bigg]
\end{aligned}
$$

The equation (4.17), by virtue of that value of $g_{(r)i}$, will admit $2n$ independent integrals of the form:

$$I_{(r)i} = \omega_r \rho \cdot \frac{\left[\begin{array}{l} \{M\varpi_2 \{3 - \varpi_1(1-M)\} - (4+\varpi_2) k_\iota^2 \zeta_s^2\} \\ \quad \pm 4\varpi_1 M k_\iota \zeta_s \mu_{(r)i} + 3\varpi_2 \{k_\iota^2 \zeta_s^2 \\ \quad\quad - M\{3 - \varpi_1(1-M)\}\} \mu_{(r)i}^2 \end{array} \right] \cdot e^{\mp k_\iota \tau}}{\left(1 \pm k_\iota \zeta_r \mu_{(r)i}\right) \left[\begin{array}{l} M\varpi_2 \{3 - \varpi_1(1-M)\} \\ \quad - (4+\varpi_2) k_\iota^2 \zeta_s^2 \end{array} \right]};$$

$$\iota = 1, 2, \cdots, n \tag{4.30}$$

4.3.2.2 Determination of Particular Integrals:

Now, for obtaining the particular integral of the equation (4.14), we set

$$I_{(r)i} = -\frac{1}{3}\omega_r b_1 \varpi_1 \cdot \left\{ \frac{\sum\limits_{s=1}^{m} \left(\frac{\eta_s}{1+\eta_s} \right)}{\sum\limits_{s=1}^{m} \eta_s} \right\} h_{(r)i} \mu_{(r)i}\ ;\ i = \pm 1, \pm 2, \cdots, \pm n \tag{4.31}$$

Then, the equation (4.14) gives

$$\begin{aligned} h_{(r)i}\mu_{(r)i} &= \frac{1}{2C} \sum_{s=1}^{m} \eta_s \omega_s \sum_j \Big\{ 1 + \varpi_1 \mu_{(r)i}\mu_{(s)j} + \frac{1}{4}\varpi_2 \left(9\mu_{(r)i}^2 \mu_{(s)j}^2 \right. \\ &\quad \left. - 3\mu_{(r)i}^2 - 3\mu_{(s)j}^2 + 1 \right) \Big\} \cdot h_{(s)j}\mu_{(s)j} a_j + \mu_{(r)i} \end{aligned} \tag{4.32}$$

i.e.

$$\begin{aligned}h_{(r)i}\mu_{(r)i} = &\left(\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\sum_j a_j h_{(s)j}\mu_{(s)j} - \frac{3}{4}\varpi_2\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\times\right.\\ &\left.\times\sum_j a_j h_{(s)j}\mu^3_{(s)j} + \frac{1}{4}\varpi_2\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\sum_j a_j h_{(s)j}\mu_{(s)j}\right) + \left(1\right.\\ &\left.+\varpi_1\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\sum_j a_j h_{(s)j}\mu^2_{(s)j}\right)\mu_{(r)i} + \left(\frac{9}{4}\varpi_2\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\times\right.\\ &\left.\times\sum_j a_j h_{(s)j}\mu^3_{(s)j} - \frac{3}{4}\varpi_2\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\sum_j a_j h_{(s)j}\mu_{(s)j}\right)\mu^2_{(r)i}\end{aligned}$$

Therefore,

$$h_{(r)i} = \sigma_1 + \sigma_2\mu_{(r)i} + \frac{\sigma}{\mu_{(r)i}} \tag{4.33}$$

where σ, σ_1 and σ_2 are constants (w.r.t. $\mu_{(r)i}$) connected by the following relation obtained by putting $h_{(r)i}$ from the equation (4.33) in the equation (4.32)

$$\begin{aligned}30C\sigma + 30C\sigma_1\mu_{(r)i} + 30C\sigma_2\mu^2_{(r)i} = &\left(15\sigma\psi_0 - \varpi_2\sigma_2\psi_0 + 5\sigma_2\psi_0\right)\\ &+\left(5\varpi_1\sigma_1\psi_0 + 30C\right)\mu_{(r)i} + 3\varpi_2\sigma_2\psi_0\mu^2_{(r)i}\end{aligned}$$

Therefore,

$$30C\sigma = 15\sigma\psi_0 - \varpi_2\sigma_2\psi_0 + 5\sigma_2\psi_0 \tag{4.34a}$$

$$30C\sigma_1 = 5\varpi_1\sigma_1\psi_0 + 30C \tag{4.34b}$$

$$30C\sigma_2 = 3\varpi_2\sigma_2\psi_0 \tag{4.34c}$$

The equations (4.34a–4.34c) gives

$$\sigma = 0\ ;\ \sigma_1 = \frac{6C}{6C - \varpi_1\psi_0}\ ;\ \sigma_2 = 0$$

Putting these values of σ, σ_1 and σ_2 in the equation (4.33), we get

$$h_{(r)i} = \frac{6C}{6C - \varpi_1\psi_0}$$

Hence, the particular integral of the equation (4.14) is

$$I_{(r)i} = \frac{1}{3}\omega_r b_1 \varpi_1 \frac{\sum\limits_{s=1}^{m}\left(\frac{\eta_s}{1+\eta_s}\right)}{\sum\limits_{s=1}^{m}\eta_s\left(1-\frac{1}{3}\varpi_1\omega_s\right)}\mu_{(r)i};\ \ i=\pm1,\pm2,\cdots,\pm n$$

i.e.

$$I_{(r)i} = \frac{1}{3}\omega_r b_1 \varpi_1 N \mu_{(r)i};\ \ i=\pm1,\pm2,\cdots,\pm n \tag{4.35}$$

where

$$N = \frac{\sum\limits_{s=1}^{m}\left(\frac{\eta_s}{1+\eta_s}\right)}{\sum\limits_{s=1}^{m}\eta_s\left(1-\frac{1}{3}\varpi_1\omega_s\right)} \tag{4.36}$$

4.3.2.3 Complete Solutions

The complete solution of the equation (4.14) is the sum of the general solution (4.30) of the homogeneous equation (4.17) and the particular integral (4.35).

According to Chandrasekhar,[45] the solution of the system of equation (4.14), satisfying the boundary condition (4.15), can be put in the form:

$$\begin{aligned}I^*_{(r)i} = \frac{1}{3}\omega_r b_1 \Bigg[\sum_{\iota=1}^{n} & \frac{L_{(r)\iota}\cdot e^{-k_\iota \tau}}{\left(1+k_\iota\zeta_r\mu_{(r)i}\right)\left[M\varpi_2\left\{3-\varpi_1\left(1-M\right)\right\} - \left(4+\varpi_2\right)k_\iota^2\zeta_s^2\right]} \times \\ & \times\Big\{\left[M\varpi_2\left\{3-\varpi_1\left(1-M\right)\right\} - \left(4+\varpi_2\right)k_\iota^2\zeta_s^2\right] + 4\varpi_1 M k_\iota\zeta_s\mu_{(r)i} \\ & +3\varpi_2\left[k_\iota^2\zeta_s^2 - M\left\{3-\varpi_1\left(1-M\right)\right\}\right]\mu_{(r)i}^2\Big\} + \varpi_1 N\mu_{(r)i}\Bigg]\end{aligned} \tag{4.37}$$

where k_ι's $(\iota = 1, 2, \cdots, n)$ are the positive roots of the characteristic equation (4.24) and $L_{(r)\iota}$'s are the constants of integration to be determined by the boundary conditions (4.15).

4.3.3 Relation between the characteristic roots k_1, k_2, $\cdots$, k_n of the characteristic equation (4.24) and the zeros $\mu_{(r)1}$, $\mu_{(r)2}$, $\cdots$, $\mu_{(r)n}$ of the Legendre-polynomial $P_{2n}(\mu)$

If p_{2j} be the co-efficient of μ^{2j} in the Legendre polynomial $P_{2n}(\mu)$, then

$$\sum_{j=0}^{n} p_{2j} D_{2j}(k\zeta_s) = \sum_{j=0}^{n} p_{2j} \sum_{s=1}^{m} \eta_s \omega_s \sum_{i} \frac{a_i \mu_{(r)i}^{2j}}{1+\mu_{(r)i} k\zeta_s}$$

using the definition (4.21)

$$\sum_{j=0}^{n} p_{2j} D_{2j}(k\zeta_s) = \sum_{s=1}^{m} \eta_s \omega_s \sum_{i} \frac{a_i}{1+\mu_{(r)i} k\zeta_s} \sum_{j=0}^{n} p_{2j} \mu_{(r)i}^{2j} \tag{4.38}$$

Now, since $\mu_{(r)i}^{2j}$'s are the zeros of the Legendre polynomial $P_{2n}(\mu)$.

So,

$$\sum_{j=0}^{n} p_{2j} \mu_{(r)i}^{2j} = 0 \tag{4.39}$$

Therefore, the equation (4.38) becomes

$$\sum_{j=0}^{n} p_{2j} D_{2j}(k\zeta_s) = 0$$

i.e.

$$p_{2n}\left\{-\frac{1}{k^{2n}\zeta_s^{2n}}(\psi_0-D_0)-\frac{\psi_2}{k^{2n-2}\zeta_s^{2n-2}}-\frac{\psi_4}{k^{2n-4}\zeta_s^{2n-4}}\cdots-\frac{\psi_{2n-2}}{k^2\zeta_s^2}\right\}$$
$$+p_{2n-2}\left\{-\frac{1}{k^{2n-2}\zeta_s^{2n-2}}(\psi_0-D_0)-\frac{\psi_2}{k^{2n-4}\zeta_s^{2n-4}}-\cdots-\frac{\psi_{2n-4}}{k^2\zeta_s^2}\right\}$$
$$+\cdots+p_2\left\{-\frac{1}{k^2\zeta_s^2}(\psi_0-D_0)+\frac{\psi_1}{k\zeta_s}\right\}+p_0D_0=0$$

i.e

$$p_{2n}\cdot M\left(1-\frac{1}{3}\varpi_1(1-M)\right)\left(1-\frac{1}{5}\varpi_2(1-M)\right)\left(\frac{1}{k^2\zeta_s^2}\right)^n$$
$$+\cdots+p_0=0 \tag{4.40}$$

But, the equation (4.40) is an equation in $\left(\frac{1}{k^2\zeta_s^2}\right)$ of order n and so, it will give n non-zero roots $\left(\frac{1}{k_\iota^2\zeta_s^2}\right)$; $\iota=1,2,\cdots,n$.

Therefore, we get

$$\frac{1}{k_1^2\zeta_s^2}\cdot\frac{1}{k_2^2\zeta_s^2}\cdots\frac{1}{k_n^2\zeta_s^2}$$
$$=(-1)^n\frac{p_0}{p_{2n}\cdot M\left(1-\frac{1}{3}\varpi_1(1-M)\right)\left(1-\frac{1}{5}\varpi_2(1-M)\right)} \tag{4.41}$$

Again, $\mu_{(r)i}$'s being the roots of the equation (4.39) i.e.

$$p_{2n}\mu_{(r)i}^{2n}+\cdots+p_2\mu_{(r)i}^2=0 \tag{4.42}$$

will follow the relation:

$$\mu_{(r)1}^2\cdot\mu_{(r)2}^2\cdots\mu_{(r)n}^2=(-1)^n\frac{p_0}{p_{2n}} \tag{4.43}$$

Now, dividing the equation (4.41) by the equation (4.43), we get

$$k_1\zeta_s \cdot k_2\zeta_s \cdots k_n\zeta_s \mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n} = \left\{ M\left(1 - \frac{1}{3}\varpi_1 (1-M)\right)\left(1 - \frac{1}{5}\varpi_2 (1-M)\right)\right\}^{1/2} \quad (4.44)$$

4.3.4 The Elimination of the Constants and the Expression of the Law of Diffuse Reflection in Closed Form

We get from the equation (4.37)

$$I_r^*(\tau,\mu) = \frac{1}{3}\omega_r b_1 \left[\sum_{\iota=1}^{n} \frac{L_{(r)\iota} \cdot e^{-k_\iota \tau}}{(1+k_\iota\zeta_r\mu)\left[M\varpi_2\{3-\varpi_1(1-M)\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right]} \times \left\{ \left[M\varpi_2\{3-\varpi_1(1-M)\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right] + 4\varpi_1 M k_\iota \zeta_s \mu + 3\varpi_2\left[k_\iota^2\zeta_s^2 - M\{3-\varpi_1(1-M)\}\right]\mu^2 \right\} + \varpi_1 N\mu \right] \quad (4.45)$$

Now, we define a function $S_r(\mu)$ as

$$S_r(\mu) = \sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{\left[(1-k_\iota\zeta_r\mu)\left[M\varpi_2\{3-\varpi_1(1-M)\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right]\right]} \left\{ \left[M\varpi_2\{3 - \varpi_1(1-M)\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right] - 4\varpi_1 M k_\iota\zeta_s\mu + 3\varpi_2\left[k_\iota^2\zeta_s^2 - M\{3-\varpi_1(1-M)\}\right]\mu^2 \right\} - \varpi_1 N\mu - \frac{3\zeta_r\mu}{\varpi_r} + \frac{3b_0}{b_1\varpi_r} \quad (4.46)$$

Then we can express the boundary conditions (4.15) in terms of the function $S_r(\mu)$ as follows:

$$I_r^*\left(0, -\mu_{(r)i}\right) = b_1\zeta_r\mu_{(r)i} - b_0 \qquad ; \left(0 < \mu_{(r)i} \leq 1\right)$$

i.e.

$$\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{\left(1 - k_\iota\zeta_r\mu_{(r)i}\right)\left[M\varpi_2\left\{3 - \varpi_1(1-M)\right\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right]} \times$$
$$\times\left\{\left[M\varpi_2\left\{3 - \varpi_1(1-M)\right\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right] - 4\varpi_1 M k_\iota\zeta_s\mu_{(r)i}\right.$$
$$\left. + 3\varpi_2\left[k_\iota^2\zeta_s^2 - M\left\{3 - \varpi_1(1-M)\right\}\right]\mu_{(r)i}^2\right\} - \varpi_1 N\mu_{(r)i}$$
$$- \frac{3\zeta_r\mu_{(r)i}}{\omega_r} + \frac{3b_0}{\omega_r b_1} = 0 \qquad ; \left(-1 \leq \mu_{(r)i} < 0\right)$$

i.e.

$$S_r\left(\mu_{(r)i}\right) = 0 \quad ; \left(0 < \mu_{(r)i} \leq 1\right) \quad ; i = 1, 2, \cdots, n \tag{4.47}$$

Also, $I_r^*(0,\mu)$ can be expressed, from the equation (4.45), in term of $S_r(\mu)$ as follows:

$$I_r^*(0,\mu) = \frac{1}{3}\omega_r b_1\left[\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{(1 + k_\iota\zeta_r\mu)\left[M\varpi_2\left\{3 - \varpi_1(1-M)\right\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right]} \times\right.$$
$$\times\left\{\left[M\varpi_2\left\{3 - \varpi_1(1-M)\right\} - (4+\varpi_2)k_\iota^2\zeta_s^2\right] + 4\varpi_1 M k_\iota\zeta_s\mu\right.$$
$$\left. + 3\varpi_2\left[k_\iota^2\zeta_s^2 - M\left\{3 - \varpi_1(1-M)\right\}\right]\mu^2\right\} + \varpi_1 N\mu$$
$$\left. + \frac{3\zeta_r\mu}{\omega_r} + \frac{3b_0}{\omega_r}\right] - b_1\zeta_r\mu - b_0$$

i.e.

$$I_r^*(0,\mu) = \frac{1}{3}\omega_r b_1\left\{S_r(-\mu) - \frac{3\zeta_r\mu}{\omega_r} - \frac{3b_0}{b_1\omega_r}\right\} \tag{4.48}$$

Now, we define two polynomials

$$P_r(\mu) = \prod_{i=1}^{n} \left(\mu - \mu_{(r)i}\right) \tag{4.49}$$

and

$$R_r(\mu) = \prod_{\iota=1}^{n} (1 - \zeta_r k_\iota \mu) \tag{4.50}$$

and construct another polynomial $S_r(\mu)\, R_r(\mu)$

Obviously the polynomial $P_r(\mu)$ has n-zeros viz. $\mu_{(r)1}, \mu_{(r)2}, \cdots, \mu_{(r)n}$ and by virtue of the condition (4.47), the polynomial $S_r(\mu)\, R_r(\mu)$ has at least n-zeros $\mu_{(r)1}, \mu_{(r)2}, \cdots, \mu_{(r)n}$. But the polynomial $S_r(\mu)\, R_r(\mu)$ is a polynomial of degree $(n+1)$. So, this polynomial has another zero, say ξ_r. Hence, the polynomial $(\mu - \xi_r)\, P_r(\mu)$ and $S_r(\mu)\, R_r(\mu)$ have exactly the identical zeros which are $\mu_{(r)1}, \mu_{(r)2}, \cdots, \mu_{(r)n}$ and ξ_r. Now, we notice that the co-efficient of highest power of μ in $(\mu - \xi_r)\, P_r(\mu)$ is 1, but that of μ in $S_r(\mu)\, R_r(\mu)$ is

$$(-1)^{n-1} \left(\varpi_1 N + \frac{3\zeta_r}{\varpi_r} + \sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{Q_\iota \zeta_s k_\iota} \right) k_1 \zeta_s \cdot k_2 \zeta_s \cdots k_n \zeta_s$$

i.e.

$$q_r k_1 \zeta_s \cdot k_2 \zeta_s \cdots k_n \zeta_s$$

where

$$q_r = (-1)^{n-1} \left(\varpi_1 N + \frac{3\zeta_r}{\varpi_r} + \sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{Q_\iota \zeta_s k_\iota} \right) \tag{4.51}$$

and

$$Q_\iota = \frac{\left[M\varpi_2 \left\{3 - \varpi_1 (1 - M)\right\} - (4 + \varpi_2)\, k_\iota^2 \zeta_s^2 \right]}{3\varpi_2 \left[k_\iota^2 \zeta_s^2 - M \left\{3 - \varpi_1 (1 - M)\right\} \right]} \tag{4.52}$$

Hence,

$$S_r(\mu) = q_r k_1\zeta_s \cdot k_2\zeta_s \cdots k_n\zeta_s (\mu - \xi_r) \frac{P_r(\mu)}{R_r(\mu)} \tag{4.53}$$

i.e.

$$S_r(\mu) = q_r k_1\zeta_s \cdot k_2\zeta_s \cdots k_n\zeta_s \mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n} (\mu - \xi_r) \times$$
$$\times \frac{1}{\mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} (\mu - \mu_{(r)i})}{\prod_{\iota=1}^{n} (1 - \zeta_r k_\iota \mu)}$$

i.e.

$$S_r(\mu) = (-1)^n q_r k_1\zeta_s \cdot k_2\zeta_s \cdots k_n\zeta_s \mu_{(r)1} \times$$
$$\times \mu_{(r)2} \cdots \mu_{(r)n} (\mu - \xi_r) H_r(-\mu) \tag{4.54}$$

where

$$H_r(\mu) = \frac{1}{\mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} (\mu + \mu_{(r)i})}{\prod_{\iota=1}^{n} (1 + \zeta_r k_\iota \mu)} \tag{4.55}$$

But we have established the relation (4.44) between the characteristic roots $k_1, k_2, \cdots, k_n$ of the characteristic equation (4.24) and the zeros $\mu_{(r)1}, \mu_{(r)2}, \cdots, \mu_{(r)n}$ in the the Legendre-polynomial $P_{2n}(\mu)$ of section 4.3.3 which is given by:

$$k_1\zeta_s \cdot k_2\zeta_s \cdots k_n\zeta_s \mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n}$$
$$= \left\{ M \left(1 - \frac{1}{3}\varpi_1 (1 - M)\right) \left(1 - \frac{1}{5}\varpi_2 (1 - M)\right) \right\}^{1/2}$$

So, the equation (4.54) is equivalent to

$$S_r(\mu) = (-1)^n q_r \left\{ M \left(1 - \frac{1}{3}\varpi_1 (1 - M)\right) \left(1 - \frac{1}{5}\varpi_2 (1 - M)\right) \right\}^{1/2} \times$$
$$\times (\mu - \xi_r) H_r(-\mu) \tag{4.56}$$

Again, we observe that

$$L_{(r)\iota} = \lim_{\mu \to (k_\iota \zeta_r)^{-1}} \frac{1}{T_\iota} (1 - \zeta_r k_\iota \mu)\, S_r(\mu) \tag{4.57}$$

provided that

$$T_\iota = \frac{\begin{array}{r}\left[(4+\varpi_2)\, k_\iota^4 \zeta_r^4 - [M\varpi_2 \{3 - \varpi_1 (1-M)\}\, \epsilon_{rs}^2 - 4\varpi_1 M \epsilon_{rs} \right. \\ \left. +3\varpi_2]\, k_\iota^2 \zeta_r^2 + 3M\varpi_2 \{3 - \varpi_1(1-M)\}\, \epsilon_{rs}^2 \right]\end{array}}{k_\iota^2 \zeta_r^2 \left[(4+\varpi_2)\, k_\iota^2 \zeta_r^2 - M\varpi_2 \{3 - \varpi_1 (1-M)\}\, \epsilon_{rs}^2\right]} \; ; \; \epsilon_{rs} = \zeta_r / \zeta_s \tag{4.58}$$

Now, using the equation (4.53) in the equation (4.57), we get

$$L_{(r)\iota} = \frac{1}{T_\iota}\, q_r \zeta_r k_1 \cdots \zeta_r k_n \, \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \tag{4.59}$$

where

$$R_{(r)\iota}(x) = \prod_{\beta(\neq \iota)=1}^{n} (1 - \zeta_r k_\beta\, x) \quad, 1 \le \iota \le n \tag{4.60}$$

Summing up both sides of the equation (4.59) over ι, we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} = q_r k_1 \zeta_r \cdot k_2 \zeta_r \cdots k_n \zeta_r f_r(0) \tag{4.61}$$

where

$$f_r(x) = \sum_{\iota=1}^{n} \frac{1}{T_\iota} \cdot \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \cdot \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) R_{(r)\iota}(x) \tag{4.62}$$

Now, the polynomial $f_r(x)$, defined above as in equation (4.62), is a polynomial of degree $(n-1)$ in x assuming the values

$$\frac{1}{T_\iota} P_r\left(\frac{1}{k_\iota \zeta_r}\right)\left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \text{ for } x = \frac{1}{k_\iota \zeta_r}\,,\ \iota = 1, 2, \cdots, n$$

So, we get

$$\Big[(4+\varpi_2) - \left[M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 - 4\varpi_1 M\epsilon_{rs} + 3\varpi_2\right]x^2$$
$$+3M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 x^4\Big] f_r(x) - \left[(4+\varpi_2)\right.$$
$$\left.-M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 x^2\right](x-\xi_r)P_r(x) = 0$$
$$\text{for } x = (k_\iota \zeta_r)^{-1}\,,\ \iota = 1, 2, \cdots, n \tag{4.63}$$

The left hand side of the equation (4.63) must, therefore, be exactly divided by the polynomial $R_r(x)$. So, we must have a relation of the type:

$$\Big[(4+\varpi_2) - \left[M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 - 4\varpi_1 M\epsilon_{rs} + 3\varpi_2\right]x^2$$
$$+3M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 x^4\Big] f_r(x) = \left[(4+\varpi_2)\right.$$
$$\left.-M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 x^2\right](x-\xi_r)P_r(x)$$
$$+R_r(x)\left(A_{(r)1}x^3 + A_{(r)2}x^2 + A_{(r)3}x + A_{(r)4}\right) \tag{4.64}$$

where $A_{(r)1}$, $A_{(r)2}$, $A_{(r)3}$ and $A_{(r)4}$ are constants.

Now the equation

$$(4+\varpi_2) - \left[M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 - 4\varpi_1 M\epsilon_{rs} + 3\varpi_2\right]x^2$$
$$+3M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2 x^4 = 0 \tag{4.65}$$

will give four roots of the type $\pm V_1$ and $\pm V_2$.

The numbers V_1 and V_2 may be purely positive real numbers (if the quadratic equation (4.65) in x gives two positive real numbers V_1^2 and V_2^2). The numbers V_1 or V_2 or both are purely imaginary (if the quadratic equation (4.65) in x gives roots V_1^2 and V_2^2 so that V_1^2 or V_2^2 or both negative). The numbers V_1 and V_2 may also be conjugate complex numbers (if V_1^2 and V_2^2 are complex or V_1^2 and V_2^2 are purely imaginary differing in sign only.)

Now, putting $x = +V_1$ and $x = -V_1$ in the equation (4.64),

$$\begin{aligned}(-V_1+\xi_r)\left[(4+\varpi_2)-M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2V_1^2\right]P_r(+V_1)\\ = R_r(+V_1)\left(A_{(r)1}V_1^3+A_{(r)2}V_1^2+A_{(r)3}V_1+A_{(r)4}\right)\end{aligned} \tag{4.66}$$

and

$$\begin{aligned}(V_1+\xi_r)\left[(4+\varpi_2)-M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2V_1^2\right]P_r(-V_1)\\ = R_r(-V_1)\left(-A_{(r)1}V_1^3+A_{(r)2}V_1^2-A_{(r)3}V_1+A_{(r)4}\right)\end{aligned} \tag{4.67}$$

which, when added, will produce

$$\begin{aligned}A_{(r)2}V_1^2+A_{(r)4} = (-1)^n\left[(4+\varpi_2)-M\varpi_2\left\{3\right.\right.\\ \left.\left.-\varpi_1(1-M)\right\}\epsilon_{rs}^2V_1^2\right]\left(b_{(r)1}V_1+a_{(r)1}\xi_r\right)\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}\end{aligned} \tag{4.68}$$

where

$$a_{(r)1} = \frac{1}{2}\left\{H_r(+V_1)+H_r(-V_1)\right\} \tag{4.69}$$

and

$$b_{(r)1} = \frac{1}{2}\left\{H_r(+V_1)-H_r(-V_1)\right\} \tag{4.70}$$

Again, putting $x = +V_2$ and $x = -V_2$, by turns, in the equation (4.64) and then adding, we get,

$$A_{(r)2}V_2^2 + A_{(r)4} = (-1)^n \left[(4+\varpi_2) - M\varpi_2 \{3 \right. \\ \left. -\varpi_1 (1-M)\} \epsilon_{rs}^2 V_2^2\right] \left(b_{(r)2}V_2 + a_{(r)2}\xi_r\right) \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \tag{4.71}$$

where

$$a_{(r)2} = \frac{1}{2}\{H_r(+V_2) + H_r(-V_2)\} \tag{4.72}$$

and

$$b_{(r)2} = \frac{1}{2}\{H_r(+V_2) - H_r(-V_2)\} \tag{4.73}$$

The equations (4.68) and (4.71) will give

$$A_{(r)4} = \frac{(-1)^n \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}}{(V_1^2 - V_2^2)} \left[\xi_r \left\{(4+\varpi_2)\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) \right.\right. \\ \left. +M\varpi_2 \{3-\varpi_1(1-M)\} \epsilon_{rs}^2 V_1^2 V_2^2 \left(a_{(r)1} - a_{(r)2}\right)\right\} \\ + \left\{(4+\varpi_2) V_1 V_2 \left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) \right. \\ \left.\left. +M\varpi_2 \{3-\varpi_1(1-M)\} \epsilon_{rs}^2 V_1^2 V_2^2 \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\right] \tag{4.74}$$

Now, putting $x = 0$ in the equation (4.64), we get

$$f_r(0) = (-1)^{n+1} \xi_r \mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} + \frac{1}{4+\varpi_2} \cdot A_{(r)4} \tag{4.75}$$

So, the equation (4.61), by using the equation (4.75), becomes

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \frac{q_r}{4+\varpi_2} k_1\zeta_r \cdot k_2\zeta_r \cdots k_n\zeta_r \,\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n} \times \\ \times \left((4+\varpi_2)\xi_r + \frac{(-1)^{n+1} A_{(r)4}}{\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}} \cdot\right)$$

Applying the relation (4.44) to it, we get

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} \frac{q_r}{4+\varpi_2} \times$$
$$\times \left\{ M \left(1 - \frac{1}{3}\varpi_1 (1-M)\right) \left(1 - \frac{1}{5}\varpi_2 (1-M)\right) \right\}^{1/2} \times$$
$$\times \left((4+\varpi_2)\, \xi_r + \frac{(-1)^{n+1} A_{(r)4}}{\mu_{(r)1}\mu_{(r)2}\cdots\mu_{(r)n}} \cdot \right) \tag{4.76}$$

Now, putting $\mu = 0$ in the the equation (4.46), we get

$$S_r(0) = \sum_{\iota=1}^{n} L_{(r)\iota} + \frac{3b_0}{b_1 \varpi_r} \tag{4.77}$$

and in equation (4.56), we get

$$S_r(0) = (-1)^{n+1} q_r \xi_r \left\{ M \left(1 - \frac{1}{3}\varpi_1 (1-M)\right) \left(1 - \frac{1}{5}\varpi_2 (1-M)\right) \right\}^{1/2} \tag{4.78}$$

Therefore, comparing the equations (4.77) and (4.78),

$$\sum_{\iota=1}^{n} L_{(r)\iota} = (-1)^{n+1} q_r \xi_r \left\{ M \left(1 - \frac{1}{3}\varpi_1 (1-M)\right) \left(1 - \frac{1}{5}\varpi_2 (1-M)\right) \right\}^{1/2} - \frac{3b_0}{b_1 \varpi_r} \tag{4.79}$$

So, comparing the equation (4.76) and (4.79), and using the equation (4.74) we get

$$\begin{aligned}&\left[(4+\varpi_2)\left(V_1^2 a_{(r)2}-V_2^2 a_{(r)1}\right)+M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2\times\right.\\&\left.\times V_1^2V_2^2\left(a_{(r)1}-a_{(r)2}\right)\right]q_r\xi_r+\left[(4+\varpi_2)V_1V_2\left(V_1b_{(r)2}-V_2b_{(r)1}\right)\right.\\&\qquad\left.+M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2V_1^2V_2^2\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right]q_r\\&\qquad=\frac{(-1)^{n+1}3(4+\varpi_2)b_0\left(V_1^2-V_2^2\right)}{b_1\varpi_r\left\{M\left(1-\frac{1}{3}\varpi_1(1-M)\right)\left(1-\frac{1}{5}\varpi_2(1-M)\right)\right\}^{1/2}}\end{aligned} \tag{4.80}$$

which is a linear equation involving the variables $q_r\xi_r$ and q_r

Again, we observe that

$$\frac{L_{(r)\iota}}{k_\iota\zeta_rQ_\iota}=\lim_{\mu\to(k_\iota\zeta_r)^{-1}}\frac{1}{T_\iota'}\left(1-\zeta_rk_\iota\mu\right)S_r(\mu) \tag{4.81}$$

where

$$T_\iota'=\frac{\left[(4+\varpi_2)k_\iota^4\zeta_r^4-\left\{M\epsilon_{rs}^2\varpi_2\left\{3-\varpi_1(1-M)\right\}-4\varpi_1M\epsilon_{rs}+3\varpi_2\right\}k_\iota^2\zeta_r^2+3\varpi_2M\epsilon_{rs}^2\left\{3-\varpi_1(1-M)\right\}\right]}{3\varpi_2M\epsilon_{rs}^2\left\{3-\varpi_1(1-M)\right\}k_\iota\zeta_r-3\varpi_2k_\iota^3\zeta_r^3};\quad \epsilon_{rs}=\zeta_r/\zeta_s \tag{4.82}$$

Now, using the equation (4.53) in the equation (4.81) , we get

$$\frac{L_{(r)\iota}}{k_\iota\zeta_rQ_\iota}=\frac{1}{T_\iota'}q_rk_1\zeta_s\cdot k_2\zeta_s\cdots k_n\zeta_s\left(\frac{1}{k_\iota\zeta_r}-\xi_r\right)\frac{P_r\left(\frac{1}{k_\iota\zeta_r}\right)}{R_r\left(\frac{1}{k_\iota\zeta_r}\right)} \tag{4.83}$$

Summing up both sides of the equation (4.83) over ι, we get

$$\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{k_\iota \zeta_r Q_\iota} = q_r k_1 \zeta_s \cdot k_2 \zeta_s \cdots k_n \zeta_s \sum_{\iota=1}^{n} \frac{1}{T_\iota'} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_r\left(\frac{1}{k_\iota \zeta_r}\right)}$$

i.e.

$$\sum_{\iota=1}^{n} \frac{L_{(r)\iota}}{k_\iota \zeta_r Q_\iota} = q_r k_1 \zeta_s \cdot k_2 \zeta_s \cdots k_n \zeta_s \, g_r(0) \tag{4.84}$$

where

$$g_r(x) = \sum_{\iota=1}^{n} \frac{1}{T'_\iota} \frac{P_r\left(\frac{1}{k_\iota \zeta_r}\right)}{R_{(r)\iota}\left(\frac{1}{k_\iota \zeta_r}\right)} \left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) R_\iota(x) \tag{4.85}$$

Using the equation (4.84) in the equation (4.51), we get

$$q_r = (-1)^{n-1} \left(\varpi_1 N + \frac{3\zeta_r}{\varpi_r} + q_r k_1 \zeta_s \cdot k_2 \zeta_s \cdots k_n \, g_r(0)\right) \tag{4.86}$$

where $g_r(0)$ can be obtained from the function $g_r(x)$, given by (4.85).

But, $g_r(x)$ is a polynomial of degree $(n-1)$ and it takes the values

$$\frac{1}{T'_\iota} P_r\left(\frac{1}{k_\iota \zeta_r}\right)\left(\frac{1}{k_\iota \zeta_r} - \xi_r\right) \text{ for } x = \frac{1}{k_\iota \zeta_r}, \ \iota = 1, \cdots, n$$

So,

$$\Big[(4+\varpi_2) - \Big\{M\epsilon_{rs}^2 \varpi_2 \{3 - \varpi_1(1-M)\} - 4\varpi_1 M \epsilon_{rs} + 3\varpi_2\Big\} x^2$$
$$+ 3\varpi_2 M \epsilon_{rs}^2 \{3 - \varpi_1(1-M)\} x^4\Big] g_r(x) - \Big[3\varpi_2 M \epsilon_{rs}^2 \{3$$
$$- \varpi_1(1-M)\} x^3 - 3\varpi_2 x\Big] P_r(x)(x - \xi_r) = 0$$
$$\text{for } x = \frac{1}{k_\iota \zeta_r}, \ \iota = 1, \cdots, n \tag{4.87}$$

This helps us to reach the conclusion that the polynomial on the left hand side of the above equation must be divisible by $R(x)$ and therefore we can write that

$$\Big[(4+\varpi_2) - \Big\{M\epsilon_{rs}^2\varpi_2\{3-\varpi_1(1-M)\} - 4\varpi_1 M\epsilon_{rs} + 3\varpi_2\Big\}x^2 + 3\varpi_2 M\epsilon_{rs}^2\{3-\varpi_1(1-M)\}x^4\Big] g_r(x) = \Big[3\varpi_2 M\epsilon_{rs}^2\{3 - \varpi_1(1-M)\}x^3 - 3\varpi_2 x\Big] P_r(x)(x-\xi_r) + R_r(x)\left(B_{(r)1}x^4 + B_{(r)2}x^3 + B_{(r)3}x^2 + B_{(r)4}x + B_{(r)5}\right) \tag{4.88}$$

where $B_{(r)1}$, $B_{(r)2}$, $B_{(r)3}$, $B_{(r)4}$ and $B_{(r)5}$ are constants.

Putting $x = 0$ in the equation (4.88), we get

$$g_r(0) = \frac{1}{4+\varpi_2} B_{(r)5} \tag{4.89}$$

Since, $\pm V_2$ and $\pm V_2$ are the roots of the equation (4.65), we put $x = +V_1$ and $x = -V_1$ in the equation (4.88), and simplify to get

$$B_{(r)1}V_1^4 + B_{(r)3}V_1^2 + B_{(r)5} = (-1)^{n+1}\left[3\varpi_2 M\epsilon_{rs}^2\{3-\varpi_1(1-M)\}V_1^3 - 3\varpi_2 V_1\right]\left(V_1 a_{(r)1} + \xi_r b_{(r)1}\right)\mu_{(r)1}\cdots\mu_{(r)n} \tag{4.90}$$

Since, $+V_2$ and $-V_2$ are the roots of the equation (4.65), putting $x = +V_2$ and $x = -V_2$ in the equation (4.88), we get similarly

$$B_{(r)1}V_2^4 + B_{(r)3}V_2^2 + B_{(r)5} = (-1)^{n+1}\left[3\varpi_2 M\epsilon_{rs}^2\{3-\varpi_1(1-M)\}V_2^3 - 3\varpi_2 V_2\right]\left(V_2 a_{(r)2} + \xi_r b_{(r)2}\right)\mu_{(r)1}\cdots\mu_{(r)n} \tag{4.91}$$

Eliminating $B_{(r)3}$ between the equations (4.90) and (4.91) we can get,

$$\begin{aligned}B_{(r)5} = (-1)^n \frac{\mu_{(r)1}\cdots\mu_{(r)n}}{V_1^2 - V_2^2}\Bigg[(-1)^n \frac{V_1^2V_2^2\left(V_1^2 - V_2^2\right)}{\mu_{(r)1}\cdots\mu_{(r)n}} B_{(r)1} \\ +3\varpi_2 V_1^2V_2^2\left[M\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right)\right. \\ \left. -\left(a_{(r)1} - a_{(r)2}\right)\right] + 3\varpi_2 V_1V_2\xi_r\left[MV_1V_2\epsilon_{rs}^2\left\{3\right.\right. \\ \left.\left. -\varpi_1(1-M)\right\}\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right) - \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right]\Bigg]\end{aligned} \tag{4.92}$$

Again, comparing the co-efficients of x^{n+4} from both sides of the equation (4.88), we get

$$B_{(r)1} = (-1)^{n+1}\frac{3\varpi_2 M\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}}{\zeta_r k_1\cdots\zeta_r k_n} \tag{4.93}$$

So, using the equation (4.93) in the equation (4.92), we get

$$\begin{aligned}B_{(r)5} = (-1)^{n+1} 3\varpi_2 \frac{\mu_{(r)1}\cdots\mu_{(r)n}}{V_1^2 - V_2^2} \times \\ \times\Bigg[\frac{V_1^2V_2^2\left(V_1^2 - V_2^2\right)M\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}}{\mu_{(r)1}\cdots\mu_{(r)n}\cdot\zeta_r k_1\cdots\zeta_r k_n} \\ +V_1^2V_2^2\left[-M\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right)\right. \\ \left. +\left(a_{(r)1} - a_{(r)2}\right)\right] + V_1V_2\xi_r\left[-MV_1V_2\epsilon_{rs}^2\left\{3\right.\right. \\ \left.\left. -\varpi_1(1-M)\right\}\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right) + \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right]\Bigg]\end{aligned}$$

i.e.using the relation (4.44)

$$\begin{aligned}B_{(r)5} = (-1)^{n+1} 3\varpi_2 \frac{\mu_{(r)1}\cdots\mu_{(r)n}}{V_1^2 - V_2^2}\Bigg[\xi_r\cdot V_1V_2\left[\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right. \\ \left. -MV_1V_2\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right] \\ +V_1^2V_2^2\left[\left(a_{(r)1} - a_{(r)2}\right) - M\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}\left(V_1^2 a_{(r)1}\right.\right. \\ \left.\left. -V_2^2 a_{(r)2}\right)\right] + \frac{V_1^2V_2^2\left(V_1^2 - V_2^2\right)M\epsilon_{rs}^2\left\{3 - \varpi_1(1-M)\right\}}{\left\{M\left(1 - \frac{1}{3}\varpi_1(1-M)\right)\left(1 - \frac{1}{5}\varpi_2(1-M)\right)\right\}^{1/2}}\Bigg]\end{aligned} \tag{4.94}$$

Now, using the equations (4.89) and (4.94) in the equation (4.86), we get

$$3\varpi_2 \cdot V_1V_2\left[MV_1V_2\epsilon_{rs}^2\left\{3-\varpi_1(1-M)\right\}\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right.$$
$$\left.-\left(V_2b_{(r)1}-V_1b_{(r)2}\right)\right]q_r\xi_r+3\varpi_2V_1^2V_2^2\left[M\epsilon_{rs}^2\left\{3-\varpi_1(1\right.\right.$$
$$\left.\left.-M)\right\}\left(V_1^2a_{(r)1}-V_2^2a_{(r)2}\right)-\left(a_{(r)1}-a_{(r)2}\right)\right]q_r$$
$$+\frac{\left[(4+\varpi_2)-3\varpi_2V_1^2V_2^2M\epsilon_{rs}^2\left\{3-\varpi_1(1-M)\right\}\right]\left(V_1^2-V_2^2\right)}{\left\{M\left(1-\frac{1}{3}\varpi_1(1-M)\right)\left(1-\frac{1}{5}\varpi_2(1-M)\right)\right\}^{1/2}}q_r$$
$$=(-1)^{n-1}\frac{\left(\varpi_1N+\frac{3\zeta_r}{\varpi_r}\right)\left(V_1^2-V_2^2\right)(4+\varpi_2)}{\left\{M\left(1-\frac{1}{3}\varpi_1(1-M)\right)\left(1-\frac{1}{5}\varpi_2(1-M)\right)\right\}^{1/2}} \tag{4.95}$$

But V_1^2 and V_2^2 are the roots of the equations (4.65).

$$(4+\varpi_2)-\left[M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2-4\varpi_1M\epsilon_{rs}+3\varpi_2\right]x^2$$
$$+3M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2x^4=0$$

Therefore,

$$3M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2V_1^2V_2^2=(4+\varpi_2) \tag{4.96}$$

and

$$V_1^2+V_2^2=\frac{\left[M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2-4\varpi_1M\epsilon_{rs}+3\varpi_2\right]}{3M\varpi_2\left\{3-\varpi_1(1-M)\right\}\epsilon_{rs}^2} \tag{4.97}$$

By virtue of the relations (4.96) and (4.96), the equations (4.80) and (4.95) become respectively

$$\left[3\left(V_1^2a_{(r)2}-V_2^2a_{(r)1}\right)+\left(a_{(r)1}-a_{(r)2}\right)\right]q_r\xi_r$$
$$+\left[3V_1V_2\left(V_1b_{(r)2}-V_2b_{(r)1}\right)+\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right]q_r$$
$$=\frac{(-1)^{n+1}9b_0\left(V_1^2-V_2^2\right)}{b_1\varpi_r\left\{M\left(1-\frac{1}{3}\varpi_1(1-M)\right)\left(1-\frac{1}{5}\varpi_2(1-M)\right)\right\}^{1/2}} \tag{4.98}$$

and

$$
\begin{aligned}
&3\varpi_2 \cdot V_1 V_2 \left[M V_1 V_2 \epsilon_{rs}^2 \left\{3 - \varpi_1 (1 - M)\right\} \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right. \\
&\quad \left. - \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right] q_r \xi_r + 3\varpi_2 V_1^2 V_2^2 \left[M \epsilon_{rs}^2 \left\{3 - \varpi_1 (1\right.\right. \\
&\quad \left.\left. - M)\right\} \left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - \left(a_{(r)1} - a_{(r)2}\right)\right] q_r \\
&+ \frac{\left[(4 + \varpi_2) - 3\varpi_2 V_1^2 V_2^2 M \epsilon_{rs}^2 \left\{3 - \varpi_1 (1 - M)\right\}\right] \left(V_1^2 - V_2^2\right)}{\left\{M \left(1 - \frac{1}{3}\varpi_1 (1 - M)\right) \left(1 - \frac{1}{5}\varpi_2 (1 - M)\right)\right\}^{1/2}} q_r \\
&= (-1)^{n-1} \frac{\left(\varpi_1 N + \frac{3\zeta_r}{\varpi_r}\right) \left(V_1^2 - V_2^2\right) (4 + \varpi_2)}{\left\{M \left(1 - \frac{1}{3}\varpi_1 (1 - M)\right) \left(1 - \frac{1}{5}\varpi_2 (1 - M)\right)\right\}^{1/2}}
\end{aligned}
\tag{4.99}
$$

Solving the equations (4.98) and (4.99), we get

$$
\begin{aligned}
q_r = (-1)^{n+1} &\left\{ \frac{3\left(V_1^2 - V_2^2\right)}{b_1 \varpi_r \left\{M \left(1 - \frac{1}{3}\varpi_1 (1 - M)\right) \left(1 - \frac{1}{5}\varpi_2 (1 - M)\right)\right\}^{1/2}} \right\} \times \\
&\times \frac{\begin{aligned}&\left\{\left[b_1 \zeta_r (4 + \varpi_2) \left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\right.\right. \\
&+ b_0 \left\{9\varpi_2 V_1 V_2 \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3(4 + \varpi_2)\left(V_1 b_{(r)1}\right.\right. \\
&\left.\left. - V_2 b_{(r)2}\right)\right\} + \tfrac{1}{3}\varpi_1 b_1 \varpi_r N (4 + \varpi_2) \left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right. \\
&\left.\left.\left. + \left(a_{(r)1} - a_{(r)2}\right)\right\}\right]\right\}\end{aligned}}{\begin{aligned}&\left\{V_1^2 \left(3V_2^2 - 1\right) \left\{3\varpi_2 V_2^2 - (4 + \varpi_2)\right\} H_r(V_1) \cdot H_r(-V_1)\right. \\
&+ V_2^2 \left(3V_1^2 - 1\right) \left\{3\varpi_2 V_1^2 - (4 + \varpi_2)\right\} H_r(V_2) \cdot H_r(-V_2) \\
&+ \left\{(4 + \varpi_2)\left(3\left(V_1^4 + V_2^4\right) - \left(V_1^2 + V_2^2\right)\right) - 3\varpi_2 V_1^2 V_2^2 \left(3\left(V_1^2\right.\right.\right. \\
&\left.\left.\left. + V_2^2\right) - 2\right)\right\} a_{(r)1} a_{(r)2} + \left\{(4 + \varpi_2) V_1 V_2 \left(2 - 3\left(V_1^2 + V_2^2\right)\right)\right. \\
&\left.\left. + 3\varpi_2 V_1 V_2 \left(6V_1^2 V_2^2 - \left(V_1^2 + V_2^2\right)\right)\right\} b_{(r)1} b_{(r)2}\right\}\end{aligned}}
\end{aligned}
\tag{4.100}
$$

and

$$\xi_r = \frac{\begin{bmatrix} b_0\left\{3\left(4+\varpi_2\right)\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - 9\varpi_2 V_1^2 V_2^2\left(a_{(r)1}\right.\right. \\ \left.\left.-a_{(r)2}\right)\right\} - b_1\zeta_r\left(4+\varpi_2\right)\left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right)\right. \\ \left.+\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} - \frac{1}{3}\varpi_1 b_1 \varpi_r N\left(4+\varpi_2\right)\left\{3V_1V_2\times\right. \\ \left.\times\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} \end{bmatrix}}{\begin{bmatrix} b_0\left\{9\varpi_2 V_1 V_2\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3\left(4+\varpi_2\right)\left(V_1 b_{(r)1}\right.\right. \\ \left.\left.-V_2 b_{(r)2}\right)\right\} + b_1\zeta_r\left(4+\varpi_2\right)\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right. \\ \left.+\left(a_{(r)1} - a_{(r)2}\right)\right\} + \frac{1}{3}\varpi_1 b_1 \varpi_r N\left(4+\varpi_2\right)\left\{3\left(V_1^2 a_{(r)2}\right.\right. \\ \left.\left.-V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\} \end{bmatrix}} \tag{4.101}$$

Now, using the equation (4.56) in the equation (4.48), we get

$$I_r^*\left(0,\mu\right) = G_r\left(\mu+\xi_r\right) H_r\left(\mu\right) - b_0 - b_1\zeta_r\mu \tag{4.102}$$

with

$$G_r = \frac{\begin{matrix} \left(V_1^2 - V_2^2\right)\left\{b_1\zeta_r\left(4+\varpi_2\right)\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right.\right. \\ \left.+\left(a_{(r)1} - a_{(r)2}\right)\right\} + b_0\left\{9\varpi_2 V_1 V_2\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right. \\ \left.-3\left(4+\varpi_2\right)\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} + \frac{1}{3} b_1 \varpi_r \varpi_1 N\left(4+\varpi_2\right)\times \\ \left.\times\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\right\} \end{matrix}}{\begin{matrix} \left\{V_1^2\left(3V_2^2 - 1\right)\left\{3\varpi_2 V_2^2 - \left(4+\varpi_2\right)\right\} H_r\left(V_1\right)\cdot H_r\left(-V_1\right)\right. \\ +V_2^2\left(3V_1^2 - 1\right)\left\{3\varpi_2 V_1^2 - \left(4+\varpi_2\right)\right\} H_r\left(V_2\right)\cdot H_r\left(-V_2\right) \\ +\left\{\left(4+\varpi_2\right)\left(3\left(V_1^4 + V_2^4\right) - \left(V_1^2 + V_2^2\right)\right) - 3\varpi_2 V_1^2 V_2^2\left(3\left(V_1^2\right.\right.\right. \\ \left.\left.\left.+V_2^2\right) - 2\right)\right\} a_{(r)1} a_{(r)2} + \left\{\left(4+\varpi_2\right) V_1 V_2\left(2 - 3\left(V_1^2 + V_2^2\right)\right)\right. \\ \left.\left.+3\varpi_2 V_1 V_2\left(6V_1^2 V_2^2 - \left(V_1^2 + V_2^2\right)\right)\right\} b_{(r)1} b_{(r)2}\right\} \end{matrix}} \tag{4.103}$$

and ξ_r given by the equation (4.101).

Now, from the equation (4.13), using the equation (4.102), we get

$$I_r(0,\mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) \tag{4.104}$$

The result (4.102) gives the diffusely reflected intensity $I_r^*(0,\mu)$ and the result (4.104) gives the desired emergent intensity $I_r(0,\mu)$ in n^{th} approximation.

4.3.5 The Exact Diffusely Reflected Intensity and the Exact Solution for the Emergent Intensity.

Following Busbridge and Stibbs,[33] we change the variable $\zeta_r\mu$ and $\zeta_s\mu'$ to x and x' respectively [and consequently $\zeta_r\mu_{(r)i}$ and $\zeta_s\mu_{(s)j}$ to x_i and x'_j respectively] to get from the equation (4.24)

$$\sum_{s=1}^{m} \eta_s\omega_s \sum_j \frac{a_j}{8C\left(1+k\zeta_s\mu_{(s)j}\right)} \Big\{ (4+\varpi_2) + \zeta_s^{-2}\Big(4M\varpi_1 - 3(1+M)\varpi_2 + M(1-M)\varpi_1\varpi_2\Big)x_j^2 + \zeta_s^{-4}M\Big(9\varpi_2 - 3(1-M)\varpi_1\varpi_2\Big)x_j^4 \Big\} = 1$$

i.e.

$$\sum_j \frac{a'_j\Psi(x_j)}{1+kx_j} = 1\ , \text{ with } a'_j = \zeta_r a_j \tag{4.105}$$

where, assumption made is

$$\eta_1 > \eta_2 > \cdots > \eta_m \tag{4.106}$$

so that

$$0 \le \zeta_1 < \zeta_2 < \cdots < \zeta_m \le 1 \tag{4.107}$$

$$\Psi\left(x'\right)=\begin{cases}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left[\frac{1}{8C}\left\{(4+\varpi_2)+\zeta_s^{-2}\Big(4M\varpi_1-3\left(1+M\right)\varpi_2\right.\right.\\ \qquad\left.+M\left(1-M\right)\varpi_1\varpi_2\Big)x_j^2+\zeta_s^{-4}M\Big(9\varpi_2\\ \qquad\left.\left.-3\left(1-M\right)\varpi_1\varpi_2\Big)\,x_j^4\right\}\right], & \text{if } 0\le x'\le\zeta_1\\ \sum_{s=r+1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left[\frac{1}{8C}\left\{(4+\varpi_2)+\zeta_s^{-2}\Big(4M\varpi_1-3\left(1+M\right)\varpi_2\right.\right.\\ \qquad\left.+M\left(1-M\right)\varpi_1\varpi_2\Big)x_j^2+\zeta_s^{-4}M\Big(9\varpi_2\\ \qquad\left.\left.-3\left(1-M\right)\varpi_1\varpi_2\Big)\,x_j^4\right\}\right], & \text{if } \zeta_r\le x'\le\zeta_{r+1}\\ 0, & \text{if } \zeta_m\le x'\le 1\end{cases}$$

(4.108)

Then the characteristic function $\Psi\left(x'\right)$ in the equation (4.108) satisfies the necessary condition:

$$\int_0^1\Psi\left(x'\right)dx'\le\frac{1}{2}\qquad(4.109)$$

if the following condition is satisfied

$$\frac{1}{3}\varpi_1\left(1-M\right)+\frac{1}{5}\varpi_2\left(1-M\right)-\frac{1}{15}\varpi_1\varpi_2\left(1-M\right)^2\le 1\qquad(4.110)$$

Now, from the theory of H-function, developed by Chandrasekhar,[45] H-function $H_r\left(\mu\right)$ i.e. $H\left(\zeta_r\mu\right)$, defined by the equation (4.55) which is expressed in terms of the positive roots of the Characteristic Equation (4.24),must satisfies, in the limit of infinite approximation, the non-integral equation:

$$H\left(x\right)=1+xH\left(x\right)\int_0^1\frac{\Psi\left(x'\right)H\left(x\right)}{x+x'}dx'$$

i.e.

$$H_r\left(\mu\right)=1+\zeta_r\mu H_r\left(\mu\right)\int_0^1\frac{\zeta_r\Psi\left(\zeta_r\mu'\right)}{\zeta_r\mu+\zeta_r\mu'}H_s\left(\mu'\right)d\mu'\qquad(4.111)$$

which is bounded in the entire half plane $\mathbb{R}(x) \geq 0$.

So, we shall get, from the equations (4.102) and (4.104), in infinite approximation, the exact diffusely reflected intensity $I_r^*(0, \mu)$ and the exact emergent intensity $I_r(0, \mu)$ expressible in the form

$$I_r^*(0, \mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) - \zeta_r \mu b_1 - b_0 \tag{4.112}$$

and

$$I_r(0, \mu) = G_r \cdot (\mu + \xi_r) \cdot H_r(\mu) \tag{4.113}$$

where ξ_r and G_r are given by the equations (4.101) and (4.103) respectively and $H_r(\mu)$ is the solution of the equation (4.111)

4.4 Discussion

In the phase function (4.6), if we put $\varpi_1 = \varpi$ and $\varpi_2 = 0$, it turns into the **planetary phase function**, if we $\varpi_1 = 0$ and $\varpi_2 = \frac{1}{2}\lambda$; $\lambda = \dfrac{5\varpi_0}{5 - 3\varpi_0}$, ϖ_0 is albedo of single scattering, the phase function will be **Pomraning phase function** and if we put $\varpi_1 = 0$ and $\varpi_2 = \frac{1}{2}$, it will be **Rayleigh Phase Function.**

Now, putting $\varpi_1 = \varpi$ and $\varpi_2 = 0$ in the term G_r and ξ_r, given by the equations (4.103) and (4.101), we shall get the expressions for G_r and ξ_r for the case of planetary phase function.

Now, for $\varpi_1 = \varpi$ and $\varpi_2 = 0$, the equation (4.103) and (4.101) reduce to the forms:

$$G_r = \frac{\begin{array}{l}(V_1^2 - V_2^2)\Big\{4b_1\zeta_r\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right. \\ \left.+\left(a_{(r)1} - a_{(r)2}\right)\right\} - 12b_0\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right) \\ +\frac{4}{3}b_1\varpi_r\varpi N\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\Big\}\end{array}}{\begin{array}{l}\Big\{-4V_1^2\left(3V_2^2 - 1\right)H_r\left(V_1\right)\cdot H_r\left(-V_1\right) - 4V_2^2\left(3V_1^2 - 1\right)\times \\ \times H_r\left(V_2\right)\cdot H_r\left(-V_2\right) + 4\left(3\left(V_2^4 + V_1^4\right) - \left(V_1^2 + V_2^2\right)\right)a_{(r)1}a_{(r)2} \\ +4V_1V_2\left(2 - 3\left(V_1^2 + V_2^2\right)\right)b_{(r)1}b_{(r)2}\Big\}\end{array}}$$

and

$$\xi_r = \frac{\begin{array}{l}\Big\{b_0\left\{12\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - 3\left(a_{(r)1} - a_{(r)2}\right)\right\} \\ -4b_1\zeta_r\left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + \left(V_1 b_{(r)1}\right.\right. \\ \left.\left.-V_2 b_{(r)2}\right)\right\}\Big\} - \frac{4}{3}b_1\varpi\varpi_r N\left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right)\right. \\ \left.+\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\end{array}}{\begin{array}{l}\Big\{4b_1\zeta_r\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\} \\ +b_0\left\{-12\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} \\ +\frac{4}{3}b_1\varpi_r\varpi N\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\Big\}\end{array}}$$

Now, as in chapter2, we can show that $a_{(r)1} \to a_r'$, $b_{(r)1} \to b_r'$, $a_{(r)2} \to \dfrac{1}{\left\{M\left(1 - \frac{1}{3}\varpi_1(1 - M)\right)\right\}^{1/2}}$ and $b_{(r)2} \to 0$ as we substitute the values $V_1 = i\dfrac{1}{\sqrt{\varpi M}}$ and $\dfrac{1}{V_2} = 0$ for the case of planetary phase function. Substituting these values in

$$G_r = \frac{\begin{array}{l}\left(\frac{V_1^2}{V_2^2}-1\right)\left\{4b_1\zeta_r\left\{3\left(\frac{V_1^2}{V_2^2}a_{(r)2}-a_{(r)1}\right)\right.\right.\\ \left.+\frac{1}{V_2^2}\left(a_{(r)1}-a_{(r)2}\right)\right\}-12\frac{1}{V_2^2}b_0\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\\ \left.+\frac{4}{3}b_1\varpi_r\varpi N\left\{3\left(\frac{V_1^2}{V_2^2}a_{(r)2}-a_{(r)1}\right)+\frac{1}{V_2^2}\left(a_{(r)1}-a_{(r)2}\right)\right\}\right\}\end{array}}{\begin{array}{l}\left\{-4\frac{V_1^2}{V_2^2}\left(3-\frac{1}{V_2^2}\right)H_r(V_1)\cdot H_r(-V_1)-4\left(3\frac{V_1^2}{V_2^2}-\frac{1}{V_2^2}\right)\times\right.\\ \times H_r(V_2)\cdot H_r(-V_2)+4\left(3\left(\frac{V_1^4}{V_2^4}+1\right)-\frac{1}{V_2^2}\left(\frac{V_1^2}{V_2^2}+1\right)\right)a_{(r)1}a_{(r)2}\\ \left.+4\frac{V_1}{V_2}\left(2\frac{1}{V_2^2}-3\left(\frac{V_1^2}{V_2^2}+1\right)\right)b_{(r)1}b_{(r)2}\right\}\end{array}}$$

we get

$$G_r = b_1\left(\zeta_r+\frac{1}{3}\varpi_r\varpi N\right)\left\{M\left(1-\frac{1}{3}\varpi\left(1-M\right)\right)\right\}^{1/2}$$

which is same as the equation (2.89) of the section-2.3 of chapter-2

Again substituting the same values in

$$\xi_r = \frac{\begin{array}{l}\left\{b_0\left\{12\left(\frac{V_1^2}{V_2^2}a_{(r)1}-a_{(r)2}\right)-3\frac{1}{V_2^2}\left(a_{(r)1}-a_{(r)2}\right)\right\}\right.\\ -4b_1\zeta_r\left\{3V_1\left(\frac{V_1}{V_2}b_{(r)2}-b_{(r)1}\right)+\frac{1}{V_2^2}\left(V_1b_{(r)1}\right.\right.\\ \left.\left.\left.-V_2b_{(r)2}\right)\right\}\right\}-\frac{4}{3}b_1\varpi_1\varpi_rN\left\{3V_1\left(\frac{V_1}{V_2}b_{(r)2}-b_{(r)1}\right)\right.\\ \left.+\frac{1}{V_2^2}\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}\end{array}}{\begin{array}{l}\left\{4b_1\zeta_r\left\{3\left(\frac{V_1^2}{V_2^2}a_{(r)2}-a_{(r)1}\right)+\frac{1}{V_2^2}\left(a_{(r)1}-a_{(r)2}\right)\right\}\right.\\ +b_0\left\{-12\frac{1}{V_2^2}\left(V_1b_{(r)1}-V_2b_{(r)2}\right)\right\}\\ \left.+\frac{4}{3}b_1\varpi_r\varpi N\left\{3\left(\frac{V_1^2}{V_2^2}a_{(r)2}-a_{(r)1}\right)+\frac{1}{V_2^2}\left(a_{(r)1}-a_{(r)2}\right)\right\}\right\}\end{array}}$$

we get

$$\xi_r = \frac{-12b_0a_{(r)2}+12b_1\zeta_rV_1b_{(r)1}+4b_1\varpi_1\varpi_rNV_1b_{(r)1}}{-12b_1\zeta_r3a_{(r)1}-4b_1\varpi_r\varpi_1Na_{(r)1}}$$

i.e.

$$\xi_r = \frac{b_0}{b_1\left(\zeta_r + \frac{1}{3}\varpi_r\varpi_1 N\right) a_r' \left\{M\left(1 - \frac{1}{3}\varpi\left(1 - M\right)\right)\right\}^{1/2}} - i\frac{1}{\sqrt{\varpi M}}\frac{b_r'}{a_r'}$$

i.e.

$$\xi_r = \frac{b_0}{a_r' G_r} - i\frac{1}{\sqrt{\varpi M}}\frac{b_r'}{a_r'}$$

which is same as the equation (2.88) of the section-2.3 of chapter-2.

Thus the emergent intensity and diffusely reflected intensity for anisotropically scattering medium with planetary phase function can be derived as a particular case from the results for the case of anisotropically scattering medium with three term scattering indicatrix with $\varpi_1 = \varpi$ and $\varpi_2 = 0$.

Again putting $\varpi_1 = 0$ and $\varpi_2 = \frac{1}{2}\lambda$ in the equations (4.103) and (4.101), we get

$$G_r = \frac{\begin{array}{l}\left(V_1^2 - V_2^2\right)\left\{b_1\zeta_r\left(8+\lambda\right)\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right.\right. \\ \left. + \left(a_{(r)1} - a_{(r)2}\right)\right\} + b_0\left\{9\lambda V_1 V_2\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right. \\ \left.\left. -3\left(8+\lambda\right)\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\right\}\end{array}}{\begin{array}{l}\left\{V_1^2\left(3V_2^2 - 1\right)\left\{3\lambda V_2^2 - \left(8+\lambda\right)\right\} H_r\left(V_1\right)\cdot H_r\left(-V_1\right)\right. \\ +V_2^2\left(3V_1^2 - 1\right)\left\{3\lambda V_1^2 - \left(8+\lambda\right)\right\} H_r\left(V_2\right)\cdot H_r\left(-V_2\right) \\ +\left\{\left(8+\lambda\right)\left(3\left(V_1^4 + V_2^4\right) - \left(V_1^2 + V_2^2\right)\right) - 3\lambda V_1^2 V_2^2\left(3\left(V_1^2\right.\right.\right. \\ \left.\left.\left.+V_2^2\right) - 2\right)\right\} a_{(r)1} a_{(r)2} + \left\{\left(8+\lambda\right) V_1 V_2\left(2 - 3\left(V_1^2 + V_2^2\right)\right)\right. \\ \left.\left. +3\lambda V_1 V_2\left(6V_1^2 V_2^2 - \left(V_1^2 + V_2^2\right)\right)\right\} b_{(r)1} b_{(r)2}\right\}\end{array}}$$

and

$$\xi_r = \frac{\begin{bmatrix} b_0 \left\{3\left(8+\lambda\right)\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - 9\lambda V_1^2 V_2^2 \left(a_{(r)1}\right.\right. \\ \left.\left. -a_{(r)2}\right)\right\} - b_1 \zeta_r \left(8+\lambda\right)\left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right)\right. \\ \left. + \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} \end{bmatrix}}{\begin{bmatrix} b_0 \left\{9\lambda V_1 V_2 \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3\left(8+\lambda\right)\left(V_1 b_{(r)1}\right.\right. \\ \left.\left. -V_2 b_{(r)2}\right)\right\} + b_1 \zeta_r \left(8+\lambda\right)\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right. \\ \left. + \left(a_{(r)1} - a_{(r)2}\right)\right\} \end{bmatrix}}$$

i.e.

$$G_r = \frac{\begin{gathered} \left(V_1^2 - V_2^2\right)\left\{b_1 \zeta_r \left(8+\lambda\right)\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right)\right.\right. \\ \left. + \left(a_{(r)1} - a_{(r)2}\right)\right\} + b_0 \left\{9\lambda V_1 V_2 \left(V_2 b_{(r)1} - V_1 b_{(r)2}\right)\right. \\ \left.\left. -3\left(8+\lambda\right)\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\right\} \end{gathered}}{\begin{gathered} \left\{\left(8+\lambda\right)\left\{\left(\frac{V_2^2}{M\epsilon_{rs}^2} + V_1^2\right) - \frac{2\left(4+\lambda\right)}{3M\epsilon_{rs}^2}\right\} H_r\left(V_1\right)\cdot H_r\left(-V_1\right)\right. \\ + \left(8+\lambda\right)\left\{\left(\frac{V_1^2}{M\epsilon_{rs}^2} + V_2^2\right) - \frac{2\left(4+\lambda\right)}{3M\epsilon_{rs}^2}\right\} H_r\left(V_2\right)\cdot H_r\left(-V_2\right) \\ \left. - \frac{16\left(8+\lambda\right)}{3M\lambda\epsilon_{rs}^2} a_{(r)1} a_{(r)2} + 8V_1V_2\left(\frac{1+M\epsilon_{rs}^2}{\epsilon_{rs}^2}\right) b_{(r)1} b_{(r)2}\right\} \end{gathered}}$$

and

$$\xi_r = \frac{\left[b_0\left\{3\left(8+\lambda\right)\left(V_1^2 a_{(r)1} - V_2^2 a_{(r)2}\right) - 9\lambda V_1^2 V_2^2\left(a_{(r)1} - a_{(r)2}\right)\right\} - b_1\zeta_r\left(8+\lambda\right)\left\{3V_1V_2\left(V_1 b_{(r)2} - V_2 b_{(r)1}\right) + \left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\}\right]}{\left[b_0\left\{9\lambda V_1 V_2\left(V_2 b_{(r)1} - V_1 b_{(r)2}\right) - 3\left(8+\lambda\right)\left(V_1 b_{(r)1} - V_2 b_{(r)2}\right)\right\} + b_1\zeta_r\left(8+\lambda\right)\left\{3\left(V_1^2 a_{(r)2} - V_2^2 a_{(r)1}\right) + \left(a_{(r)1} - a_{(r)2}\right)\right\}\right]}$$

which are the same as the equations (3.188) and (3.186) of Section-3.2.2.1 of Chapter-3

Again we have shown, in the section-3.2.4 of Chapter-3, that G_r and ξ_r, given by the equations (3.91) and (3.89) of section-3.1.3.3 of Chapter-3 for Rayleigh phase function can be derived from the equations (3.188) and (3.186) of Section-3.2.2.1 of Chapter-3 by substituting $\lambda = 1$.

So, the emergent intensity and diffusely reflected intensity for the case of anisotropically scattering medium with three term scattering indicatrix yield the same for anisotropically scattering medium with Pomraning phase function as a particular case with $\varpi_1 = 0$ and $\varpi_2 = \frac{1}{2}\lambda$ and hence those for anisotropically scattering medium with Rayleigh phase function as a particular case with $\varpi_1 = 0$ and $\varpi_2 = \frac{1}{2}$.

Chapter 5

Approximation of H-functions and Residual Intensities for the Multiplet Lines in Radiative Transfer

5.1 Introduction

H-function plays an important role in the solution of a radiative transfer. Sometimes it becomes very complicated to determine its value even for a simple form of a radiative transfer equation. Its determination for the case of interlocked multiplet lines will no doubt be a very hard job. The H-functions involved in the discrete ordinate solution of a radiative transfer equation for the case of coherent scattering (pages xxxvii - xliii) and for interlocking problems (chapters: 2, 3 and 4) in an atmosphere of anisotropically scattering medium are not only the functions of real argument μ, but also the imaginary argument μ. Formula for determining the H-functions with complex argument μ is also developed by Chandrasekhar,[45] Kourganoff and Busbridge,[129] Busbridge and Stibbs.[33] Viik[214] studied the H-functions for complex albedo ω of single scattering. Other argument of the H-function — the angular variable μ is kept real and allowed to vary in the region $0 \leq \mu \leq \infty$. Here we have attempted to derive approximate expression for H-functions considering the case of the interlocked multiplets in an atmosphere which is isotropically scattering in which the the variable μ of the H-function remains always real.

Many authors tried from time to time to give some suitable approximation of the H-function for different atmospheres so that the labour in computing its value is minimized. Abu-Shumays[6,7]Harris,[82] Karanjai,[102,103,104] Karanjai and Sen,[115],[67] Holubec and McConnell[85] are some of them. Abu-Shumays[7] gave three approximated forms of H-functions and compared the arithmetic mean of last two of those three approximate forms with the values of H-function given by Chandrasekhar[45] and Carlstedt and Mullikin[35] and showed that his values were remarkably accurate to the approximation of H-function obtained by them.

In section 5.2.1, we propose the approximate form of H-function involved in the solution of a radiative transfer equation of interlocked multiplet lines in isotropically scattering atmosphere for minimizing the labour involved in the numerical calculation for determining its the values at non-zero μ :

$$H\left(x\right) = \frac{a_0 + a_1 x + a_2 x^2}{1 + kx}, x = \zeta_r \mu$$

where $a_s's$ are functions of albedo ω, so that, for $a_0 = A$ or 1, $a_1 = B$ or A' and $a_2 = 0$ or $B\ '$, it reduces to the two approximate forms of Abu-Shumays[7] needed for the purpose. Likewise in section 5.2.2 for H-functions for multiplet lines without interlocking, we take the same approximate form, but the variable x taken in this case is $x = \mu$.

5.2 H-function for Multiplet Lines

5.2.1 H-function for Multiplet Lines with Interlocking

H-function involved in the solution of a radiative transfer equation of interlocked multiplet lines in isotropically scattering medium, as given by Busbridge and Stibbs,[33] is given by

$$H\left(x\right)=1+xH\left(x\right)\int_{0}^{1}\frac{\Psi'\left(x\right)H\left(x'\right)}{x+x'}dx'\ ;\ \ x=\zeta_{r}\mu \tag{5.1}$$

where $\Psi\left(x'\right)$ is given by

$$\Psi\left(x'\right)=\begin{cases}\frac{1}{2C}\sum\limits_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s},0\leq x'\leq\zeta_1\\ \frac{1}{2C}\sum\limits_{s=r+1}^{m}\frac{\eta_s\omega_s}{\zeta_s},\zeta_r\leq x'\leq\zeta_{r+1}\\ 0,\zeta_k\leq x'\leq 1\end{cases} \tag{5.2}$$

in which C is $C=\sum\limits_{s=1}^{m}\eta_s$ so that

$$\int_{0}^{1}\Psi\left(x'\right)dx=\frac{1}{2}\left(1-M\right) \tag{5.3}$$

writing $M=\dfrac{\sum\limits_{s=1}^{m}\eta_s\left(1-\omega_s\right)}{\sum\limits_{s=1}^{m}\eta_s}$

We observe that $\Psi\left(x'\right)$ satisfies the condition

$$\int_{0}^{1}\Psi\left(x'\right)dx\leq\frac{1}{2} \tag{5.4}$$

provided that $M \geq 0$

Now, the equation (5.2) is equivalent to

$$H(x) = 1 + \frac{1}{2C} x H(x) \sum_{s=1}^{m} \eta_s \omega_s \zeta_s^{-1} \int_0^{\zeta_s} \frac{H(x')}{x + x'} dx' \; ; \quad x = \zeta_r \mu \quad (5.5)$$

and the above form of H-function can also be written, by rearranging its terms, in the form :

$$\frac{1}{H(x)} = 1 - \frac{1}{2C} x \sum_{s=1}^{m} \eta_s \omega_s \zeta_s^{-1} \int_0^{\zeta_s} \frac{H(x')}{x + x'} dx' \quad (5.6)$$

Some of the properties which a H-function satisfies are

1.

$$\int_0^1 H(x) \psi(x) \, dx = 1 - \left[1 - 2 \int_0^1 \Psi(x) \, dx \right]^{\frac{1}{2}} \quad (5.7)$$

2.

$$\int_0^1 \frac{\psi(x) H(x)}{1 - kx} = 1; \quad (5.8)$$

where k is determined by

$$\frac{1}{2kC} \cdot \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \cdot \ln \left(\frac{1 + k\zeta_s}{1 - k\zeta_s} \right) = 1 \quad (5.9)$$

We shall find a_s's by using the properties (1 and 2).

Now, by using the definition (5.2) and the relation(5.3) derived from the definition, we can rewrite the (5.7) and (5.8) as

$$\frac{1}{2C} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \int_0^{\zeta_s} H(x') \, dx' = 1 - \sqrt{M} \quad (5.10)$$

and

$$\frac{1}{2C}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\int_0^{\zeta_s}\frac{H\left(x'\right)}{1-kx'}dx'=1 \tag{5.11}$$

5.2.1.1 An Approximate Form of H-function:

First Approximation Form:

Keeping Abu-Shumays'[7] approximation of H-function in mind, we approximate H-function involved in the solution of the r^{th}interlocked multiplet line for non-zero μ as

$$H\left(x\right)=\frac{a_0+a_1x+a_2x^2}{1+kx}\quad ;\ x=\zeta_r\mu\neq 0 \tag{5.12}$$

where a_s's are the functions of albedo to be determined from the properties which a H-function satisfies and k is a root of the transcendental equation (5.9)

Second Approximation Form:

Now, from (5.10), by using the equation (5.12), we can construct a linear equation involving a_0, a_1 and a_2 as

$$A_0a_0+A_1a_1+A_2a_2=c_0 \tag{5.13}$$

where

$$A_0 = \frac{1}{k}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(1+k\zeta_s\right) \tag{5.14a}$$

$$A_1 = \frac{1}{k}\sum_{s=1}^{m}\eta_s\omega_s - \frac{1}{k^2}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(1+k\zeta_s\right) \tag{5.14b}$$

$$A_2 = \frac{1}{2k}\sum_{s=1}^{m}\eta_s\omega_s\zeta_s - \frac{1}{k^2}\sum_{s=1}^{m}\eta_s\omega_s + \frac{1}{k^3}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(1+k\zeta_s\right) \tag{5.14c}$$

$$c_0 = 2C\left(1-\sqrt{M}\right) \tag{5.14d}$$

Again, the equation (5.11) and (5.12) can be used to constitute the second linear equation involving a_0, a_1 and a_2 as

$$A_0'a_0 - A_1'a_1 + A_2'a_2 = c_0' \tag{5.15}$$

where

$$A_0' = \frac{1}{2k}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(\frac{1+k\zeta_s}{1-k\zeta_s}\right) \tag{5.16a}$$

$$A_1' = \frac{1}{2k^2}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(1-k^2\zeta_s^2\right) \tag{5.16b}$$

$$A_2' = \frac{1}{2k^3}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(\frac{1+k\zeta_s}{1-k\zeta_s}\right) - \frac{1}{k^2}\sum_{s=1}^{m}\eta_s\omega_s \tag{5.16c}$$

$$c_0' = 2C \tag{5.16d}$$

Now, from the equation (5.6), we get the second approximation of H-function as:

$$\frac{1}{H_r(\mu)} = 1 - \frac{\zeta_r\mu}{2C\left(1-k\zeta_r\mu\right)}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\left[\left\{\left(a_0 - a_1\zeta_r\mu + a_2\zeta_r^2\mu^2\right)\right\}\ln\left(\frac{\zeta_r\mu+\zeta_s}{\zeta_r\mu}\right) - \left(a_0 - \frac{a_1}{k} + \frac{a_2}{k^2}\right)\ln\left(1+k\zeta_s\right) + a_2\zeta_s\left(\frac{1}{k} - \zeta_r\mu\right)\right] \tag{5.17}$$

Thus we get the approximate form of H-function suggested by Abu-Shumays[7] for getting its value at non-zero μ, with a little error lying within the limit from 0.001% to 0.005%, as

$$H_{(AS)}(x) = \frac{1}{2}\left\{H_{(AS-I)}(x) + H_{(AS-II)}(x)\right\} \quad ; \; x = \zeta_r\mu \tag{5.18}$$

where $H_{(AS-I)}(x)$ is obtained from the equations (5.1) and (5.17) by putting $a_0 = A$, $a_1 = B$ and $a_2 = 0$ and $H_{(AS-II)}(x)$ by putting $a_0 = 1$, $a_1 = A'$ and $a_2 = B'$. So, their first approximation forms are as

$$H_{(AS-I)}(x) = \frac{A + Bx}{1 + kx} \tag{5.19}$$

and

$$H_{(AS-II)}(x) = \frac{1 + A'x + B'\mu^2}{1 + kx} \tag{5.20}$$

and second approximation forms are as

$$\frac{1}{H_{(AS-I)}(\zeta_r\mu)} = 1 - \frac{\zeta_r\mu}{2C(1 - k\zeta_r\mu)} \sum_{s=1}^{m} \frac{\eta_s\omega_s}{\zeta_s}\left[\left(A - B\zeta_r\mu\right) \times \times \ln\left(\frac{\zeta_r\mu + \zeta_s}{\zeta_r\mu}\right) - \left(A - \frac{B}{k}\right)\ln(1 + k\zeta_s)\right] \tag{5.21}$$

and

$$\frac{1}{H_{(AS-II)}(\zeta_r\mu)} = 1 - \frac{\zeta_r\mu}{2C(1 - k\zeta_r\mu)} \sum_{s=1}^{m} \frac{\eta_s\omega_s}{\zeta_s}\left[\left\{(1 - A'\zeta_r\mu + B'\zeta_r^2\mu^2)\right\}\ln\left(\frac{\zeta_r\mu + \zeta_s}{\zeta_r\mu}\right) - \left(1 - \frac{A'}{k} + \frac{B'}{k^2}\right)\ln(1 + k\zeta_s) + B'\zeta_s\left(\frac{1}{k} - \zeta_r\mu\right)\right] \tag{5.22}$$

in which A, B, A' and B' as

$$A = \frac{A_1 c'_0 + c_0 A'_1}{A_0 A'_1 + A_1 A'_0} \tag{5.23a}$$

$$B = \frac{c_0 A'_0 - A_0 c'_0}{A_0 A'_1 + A_1 A'_0} \tag{5.23b}$$

$$A' = \frac{A_2 A'_0 - A_0 A'_2 - A_2 c'_0 + c_0 A'_2}{A_1 A'_2 + A_2 A'_1} \tag{5.23c}$$

and $$B' = \frac{A_1 c'_0 - A_1 A'_0 - A_0 A'_1 + c_0 A'_1}{A_1 A'_2 + A_2 A'_1} \tag{5.23d}$$

with

$$A_0 = \frac{1}{k} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \ln(1 + k\zeta_s) \tag{5.24a}$$

$$A_1 = \frac{1}{k} \sum_{s=1}^{m} \eta_s \omega_s - \frac{1}{k^2} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \ln(1 + k\zeta_s) \tag{5.24b}$$

$$A_2 = \frac{1}{2k} \sum_{s=1}^{m} \eta_s \omega_s \zeta_s - \frac{1}{k^2} \sum_{s=1}^{m} \eta_s \omega_s + \frac{1}{k^3} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \ln(1 + k\zeta_s) \tag{5.24c}$$

$$c_0 = 2C\left(1 - \sqrt{M}\right) \tag{5.24d}$$

$$A'_0 = \frac{1}{2k} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \ln\left(\frac{1 + k\zeta_s}{1 - k\zeta_s}\right) \tag{5.24e}$$

$$A'_1 = \frac{1}{2k^2} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \ln(1 - k^2 \zeta_s^2) \tag{5.24f}$$

$$A'_2 = \frac{1}{2k^3} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \ln\left(\frac{1 + k\zeta_s}{1 - k\zeta_s}\right) - \frac{1}{k^2} \sum_{s=1}^{m} \eta_s \omega_s \tag{5.24g}$$

$$c'_0 = 2C \tag{5.24h}$$

5.2.2 H-function for Multiplet Lines without Interlocking

Busbridge and Stibbs[33] obtained the residual intensities for multiplet lines without interlocking which are nothing but the lines in isotropically coherent scattering and showed the effect of interlocking on them. Of course, they didn't form any table showing the values of the H-function of the multiplet lines for this case. In this section, we shall calculate the value of the H-function for multiplet lines without interlocking (i.e. for lines in coherent scattering) with the help of Abu-Shumays'[7] approximate form of H-function and compare the results with value of the H-function for multiplet lines with interlocking.

We have the characteristic function $\psi(\mu)$ of the H-function (pages xxxvii - xliii) for the solution of a radiative transfer equation of anisotropic coherent scattering with planetary phase function is

$$\psi(\mu) = \rho + \rho_1\mu^2$$

where $\rho = \frac{1}{2}\omega$ and $\rho_1 = \frac{1}{2}\varpi\,(1-\omega)$ in which ω is $\omega = \dfrac{\eta\,(1-\epsilon)}{1+\eta}$

The characteristic function $\psi(\mu)$ of the H-function $H\,(\mu)$ for coherent isotropically scattering medium can be obtained as a particular case of the above by putting $\varpi = 0$. So, in that case the characteristic function $\psi(\mu)$ is given by

$$\psi(\mu) = \frac{1}{2}\omega \tag{5.25}$$

For a H-function the characteristic function satisfies the relation

$$\int_0^1 \psi(\mu)d\mu \leq \frac{1}{2} \tag{5.26}$$

So, we must have

$$\omega \leq 1 \tag{5.27}$$

Abu-Shumays'[7] approximate form of H-function for multiplet lines without interlocking is given by

$$H(\mu) = \frac{1}{2}\left\{H_{(AS-I)}(\mu) + H_{(AS-II)}(\mu)\right\}$$

in which $H_{(AS-I)}(\mu)$ and $H_{(AS-II)}(\mu)$, in first approximate forms, are

$$H_{(AS-I)}(\mu) = \frac{A + B\mu}{1 + k\mu} \tag{5.28}$$

and

$$H_{(AS-II)}(\mu) = \frac{1 + A'\mu + B'\mu^2}{1 + k\mu} \tag{5.29}$$

and in the second approximation forms, they are

$$H_{(AS-I)}(\mu) = \left[1 - \frac{\omega\mu}{2(1 - k\mu)}\left\{(A - B\mu)\ln\left(\frac{\mu + 1}{\mu}\right) - \left(A - \frac{B}{k}\right)\ln(1 + k)\right\}\right]^{-1} \tag{5.30}$$

and

$$H_{(AS-II)}(\mu) = \left[1 - \frac{\omega\mu}{2(1 - k\mu)}\left\{\left(1 - A'\mu + B'\mu^2\right)\ln\left(\frac{\mu + 1}{\mu}\right) - \left(1 - \frac{A'}{k} + \frac{B'}{k^2}\right)\ln(1 + k) + B'\left(\frac{1}{k} - \mu\right)\right\}\right]^{-1} \tag{5.31}$$

in which k is given by the transcendental equation

$$\frac{\omega}{2k}\ln\frac{1 + k}{1 - k} = 1 \tag{5.32}$$

and the constant terms A, B, A' and B' are given by

$$A = \frac{A_1 c' + cA_1'}{A_0 A_1' + A_1 A_0'} \tag{5.33a}$$

(5.33b)

$$B = \frac{cA_0' - A_0 c'}{A_0 A_1' + A_1 A_0'} \tag{5.33c}$$

$$A' = \frac{A_2 A_0' - A_0 A_2' - A_2 c' + cA_2'}{A_1 A_2' + A_2 A_1'} \tag{5.33d}$$

and

$$B' = \frac{A_1 c' - A_1 A_0' - A_0 A_1' + cA_1'}{A_1 A_2' + A_2 A_1'} \tag{5.33e}$$

with

$$A_0 = \frac{1}{k} \ln(1+k) \tag{5.34a}$$

$$A_1 = \frac{1}{k} - \frac{1}{k^2} \ln(1+k) = \frac{1}{k} - \frac{1}{k} A_0 \tag{5.34b}$$

$$A_2 = \frac{1}{2k} - \frac{1}{k^2} + \frac{1}{k^3} \ln(1+k) = \frac{1}{2k} - \frac{1}{k} A_1 \tag{5.34c}$$

$$c = \frac{2 - 2\sqrt{1-\omega}}{\omega} \tag{5.34d}$$

$$A_0' = \frac{1}{2k} \ln\left(\frac{1+k}{1-k}\right) \tag{5.34e}$$

$$A_1' = \frac{1}{2k^2} \ln\left(1-k^2\right) \tag{5.34f}$$

$$A_2' = \frac{1}{2k^2} \ln\left(\frac{1+k}{1-k}\right) - \frac{1}{k^2} = \frac{1}{k^2} A_0' - \frac{1}{k^2} \tag{5.34g}$$

$$c' = \frac{2}{\omega} \tag{5.34h}$$

5.3 Doublet Lines:

5.3.1 H-functions for Doublet Lines with Interlocking:

A few calculations had been done only for interlocked doublets and triplets. Busbridge and Stibbs[33] calculated for the first time the numerical values of H-functions for doublets only to use the results in determining the residual intensity of the doublet lines. They approximated the first line of the doublets as follows:

$$\begin{aligned} H(n_1\mu) &= H(1-\alpha_1\lambda_1-\alpha_2\lambda_2-k_1,\mu)\;; && (5.35)\\ \alpha_r &= \frac{\eta_r}{\eta_1+\eta_2}\;;\; r=1,2 && (5.36)\\ \lambda_r &= \frac{1+\epsilon\eta_r}{1+\eta_r} && (5.37)\\ n_r &= \frac{1}{1+\eta_r}\;;\; r=1,2 && (5.38)\\ k_1 &= \frac{1}{2}\alpha_s(1-\lambda_s)\left(1-\frac{n_1}{n_2}\right) && (5.39) \end{aligned}$$

and wrote down the value of $H(n_1\mu)$ directly from the table of $H(\varpi,\mu)$ prepared by Chandrasekhar[45] for $0 \le \mu \le 1$ or by interpolating linearly between two successive values from the table when necessary. After that to find the values of $H(n_2\mu)$, they used the relation

$$H(n_2\mu) = H\left(n_1\cdot\frac{n_2}{n_1}\mu\right) \tag{5.40}$$

which shows that the values of $H(n_2\mu)$ in the interval become not only equivalent to a value of $H(n_1\mu)$ in the interval $0 \le \mu \le 1$, but also becomes sometimes equivalent to its value in the interval $1 \le \mu \le \frac{n_2}{n_1}$. To have those values, they used an extrapolation formula of the form

$$H(n_1\mu) = (\alpha_1\lambda_1+\alpha_2\lambda_2)^{-\frac{1}{2}} - (c_1\mu+c_2)^{-1} \tag{5.41}$$

in which the values of c_1 and c_2 are obtained from the two values of $H(n_1\mu)$ for $\mu = 0.95$ and $\mu = 1$. They did not calculate the values for triplets or higher multiplets.

Karanjai[103] first attempted to calculate H-functional values and residual intensities for higher multiplates. He calculated H-functions numerically for interlocked doublets and triplets by using his approximation of H-function

$$H(\varpi_0, \mu) = 1 + \frac{\alpha\mu}{1 - \varpi_0 + 2\mu\sqrt{1-\varpi_0}} \tag{5.42}$$

where α is a function of ϖ_0 and is given by

$$\alpha = \varpi_0(1-\varpi_0)\exp\left[\varpi_0\left(1 + 2\left|\varpi_0 - \frac{1}{2}\right|\right)\right] \tag{5.43}$$

and compared his results for H-functions of doublets with those given by Busbridge and Stibbs.[33] He used only one case of the three cases of Busbridge and Stibbs.[33] Comparison of the values of H-functions of Karanjai[103] with the results of Busbridge and Stibbs,[33] are mentioned in the Table-5.1 (prepared by Karanjai) and the comparison is made more clear by drawing graph in Fig. 5.1.

μ	$\eta_1=1$ and $\eta_2=0.5$			
	Busbridge and Stibbs 1954		Karanjai 1968	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000	1.0000	1.0000
0.1	1.0598	1.0724	1.06016	1.07269
0.2	1.0935	1.1108	1.09353	1.11049
0.4	1.1382	1.1591	1.13703	1.15717
0.6	1.1679	1.1896	1.17139	1.18620
0.8	1.1896	1.2111	1.18620	1.20634
1.0	1.2064	1.2271	1.20189	1.22121

Table-5.1

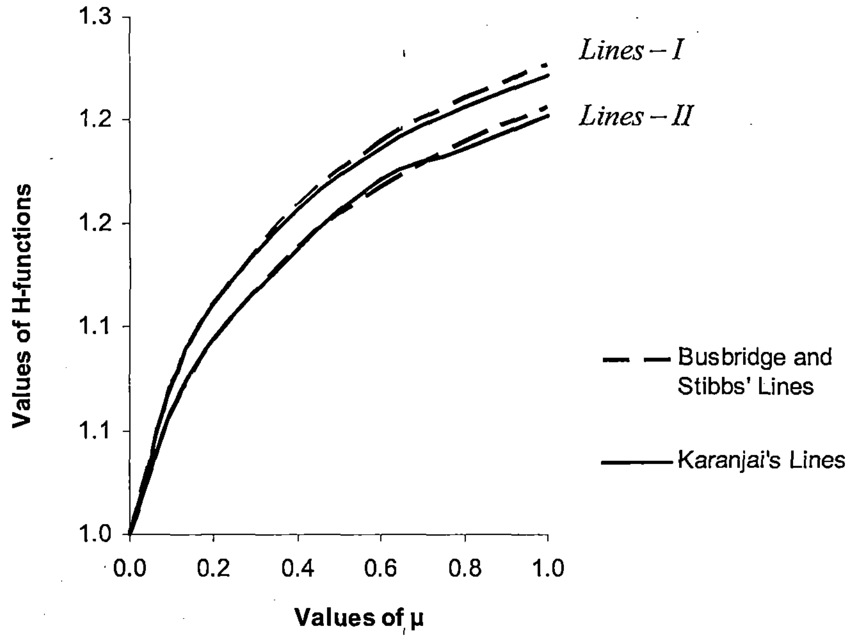


Fig 5.1 (η_1 = 1 and η_2 = 1/2)

Siewert and Özişik[190] wrote the solution of interlocked doublets deriving the expression for emergent intensity of interlocked multiplets lines, but they did not show any calculation in their paper, though they reported in their paper that some numerical calculation for H-function of interest had been made and would be available upon request.

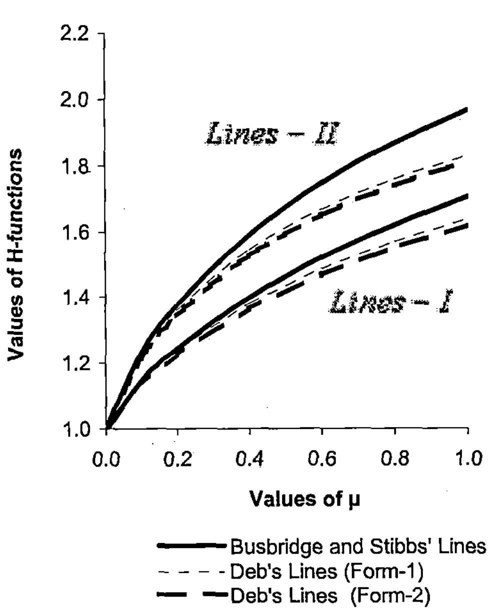
Deb[65] calculated H-functions for doublets and triplets by using its four approximate forms given by Karanjai and Sen[115,116] which are given below:

1. $H(\omega, \mu) = 1 + a\mu + b\mu^2 + c\mu^3$
2. $H(\omega, \mu) = 1 + \dfrac{a\mu + b\mu^2}{A + 2\mu}$ in which $A = \sqrt{1-\omega}$
3. $H(\omega, \mu) = 1 + \dfrac{a\mu + b\mu^2 + c\mu^3}{A + 2\mu}$ in which A is same as (2)
4. $H(\omega, \mu) = 1 + \dfrac{a\mu + b\mu^2 + c\mu^3}{1 + k\mu}$ in which k is a root of the transcendental equation $\dfrac{\omega}{2k} = \log\left[\dfrac{1+k}{1-k}\right]$

In the above four approximate forms of H-function, a, b and c are the functions of albedo ω. He used these approximations to find the values of $H\ (n_1\mu)$ and $H\ (n_2\mu)$, following the Busbridge and Stibbs' procedure, by taking ω as $\omega = 1 - \alpha_1\lambda_1 - \alpha_2\lambda_2 - k_1$ and determined corresponding residual intensities for each case chosen by Busbridge and Stibbs.[33] Only difference is that he used the second approximation forms, given by Karanjai and Sen[115,116] to find the values of $H\ (n_1\mu)$ in lieu of using interpolation and extrapolation. He also compared his results by the results of Busbridge and Stibbs.[33] For ready reference these are tabulated in Table 5.2 (a), 5.2(b) and 5.2(c) (prepared by Deb[65]) and comparisons are made more clear by drawing their graphs in the fig.5.2 (a), 5.2(b) and 5.2(c).

μ	η_1=10 and η_2=5									
	Busbridge and Stibbs (1954)		Deb(1996) (Form-1)		Deb(1996) (Form-2)		Deb(1996) (Form-3)		Deb(1996) (Form-4)	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.1440	1.2278	1.1443	1.2249	1.1370	1.2142	1.1443	1.2249	1.1447	1.2253
0.2	1.2429	1.3745	1.2392	1.3602	1.2279	1.3452	1.2391	1.3600	1.2396	1.3605
0.4	1.3975	1.5893	1.3807	1.5444	1.3652	1.5261	1.3806	1.5441	1.3811	1.5446
0.6	1.5197	1.7462	1.4866	1.6686	1.4690	1.6497	1.4863	1.6683	1.4868	1.6687
0.8	1.6211	1.8684	1.5703	1.7592	1.5518	1.7407	1.5700	1.7589	1.5705	1.7961
1.0	1.7075	1.9670	1.6388	1.8286	1.6199	1.8108	1.6385	1.8282	1.6389	1.8286

Table-5.2 (a)



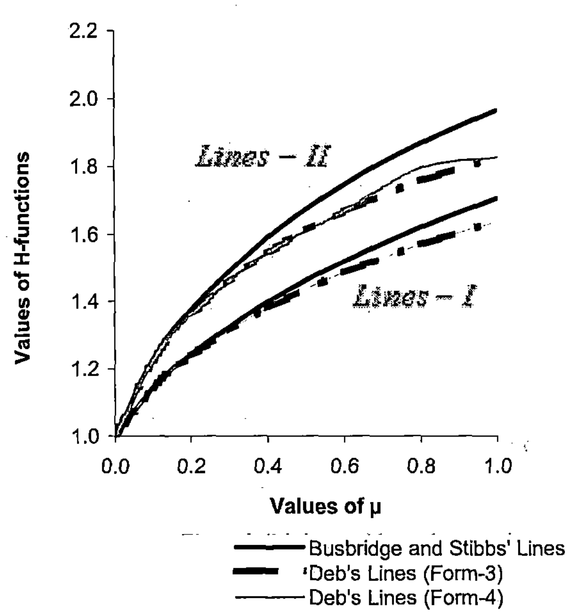


Fig-5.2 (a)

μ	$\eta_1=4$ and $\eta_2=2$									
	Busbridge and Stibbs (1954)		Deb(1996) (Form-1)		Deb(1996) (Form-2)		Deb(1996) (Form-3)		Deb(1996) (Form-4)	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.1140	1.1647	1.1153	1.1651	1.1104	1.1586	1.1154	1.1652	1.1155	1.1653
0.2	1.1869	1.2620	1.1867	1.2585	1.1795	1.2496	1.1868	1.2586	1.1869	1.2587
0.4	1.2936	1.3944	1.2881	1.3802	1.2786	1.3697	1.2882	1.3801	1.2883	1.3802
0.6	1.3722	1.4844	1.3603	1.4590	1.3499	1.4484	1.3602	1.4589	1.3604	1.4590
0.8	1.4341	1.5509	1.4153	1.5150	1.4047	1.5049	1.4552	1.5149	1.4153	1.5150
1.0	1.4844	1.6020	1.4590	1.5571	1.4484	1.5475	1.4589	1.5570	1.4590	1.5570

Table-5.2 (b)

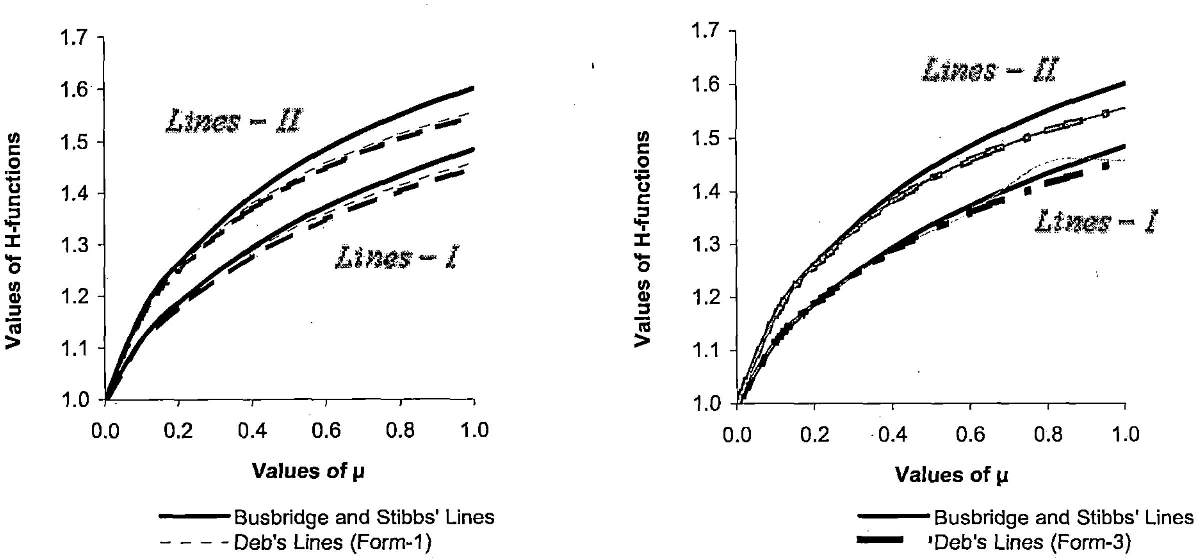
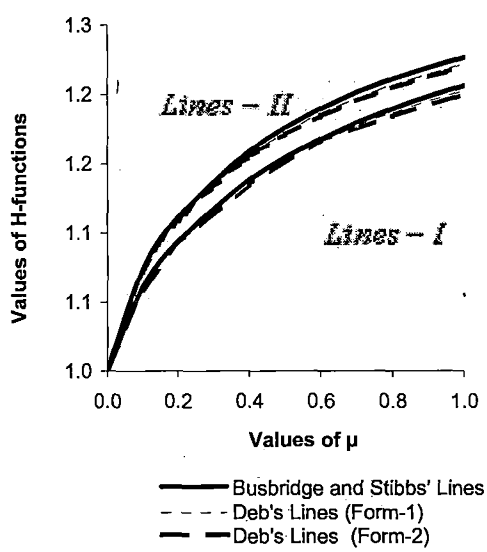


Fig-5.2 (b)

μ	$\eta_1=1$ and $\eta_2=1/2$									
	Busbridge and Stibbs (1954)		Deb(1996) (Form-1)		Deb(1996) (Form-2)		Deb(1996) (Form-3)		Deb(1996) (Form-4)	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0598	1.0724	1.0602	1.0728	1.0589	1.0712	1.0603	1.0729	1.0603	1.0729
0.2	1.0935	1.1108	1.0937	1.1107	1.0918	1.1086	1.0938	1.1108	1.0938	1.1108
0.4	1.1382	1.1591	1.1374	1.1577	1.1351	1.1553	1.1375	1.1577	1.1373	1.1577
0.6	1.1679	1.1896	1.1661	1.1869	1.1673	1.1845	1.1662	1.1869	1.1662	1.1869
0.8	1.1896	1.2111	1.1869	1.2071	1.1845	1.2049	1.1869	1.2072	1.1869	1.2072
1.0	1.2064	1.2271	1.2027	1.2221	1.2004	1.2200	1.2027	1.2221	1.2027	1.2221

Table-5.2 (c)



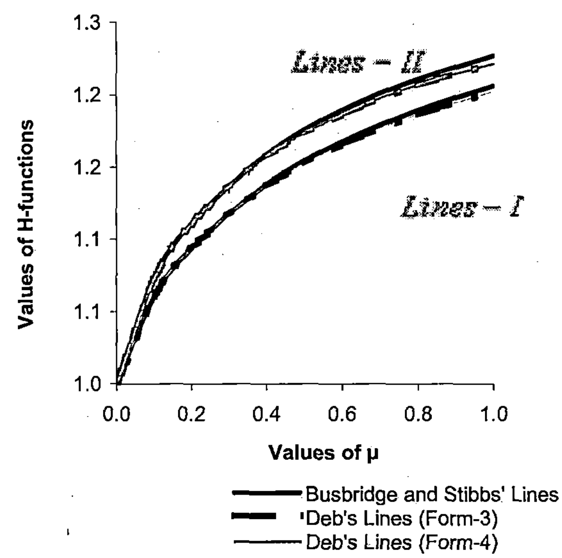


Fig-5.2 (c)

Abu-Shumays[7] used two forms (5.19) and (5.20) to evaluate the numerical values of H-function involved in a basic radiative transfer for isotropically scattering medium and showed that accuracy of the form is four places of decimal in most cases. We have intended it to use here to have the numerical results of H-function involved in the solution of a radiative transfer equation of interlocked multiplet lines in isotropically scattering atmosphere with an expectation to get more accurate results than the previous. We have neither used the results nor used the second approximation form of H-function for lines of coherent scattering as the other authors did. Here, we develop a new method. We derive the second approximation form considering the effect of interlocking and multiplets as well on the lines and use it to find the numerical values of H-function.

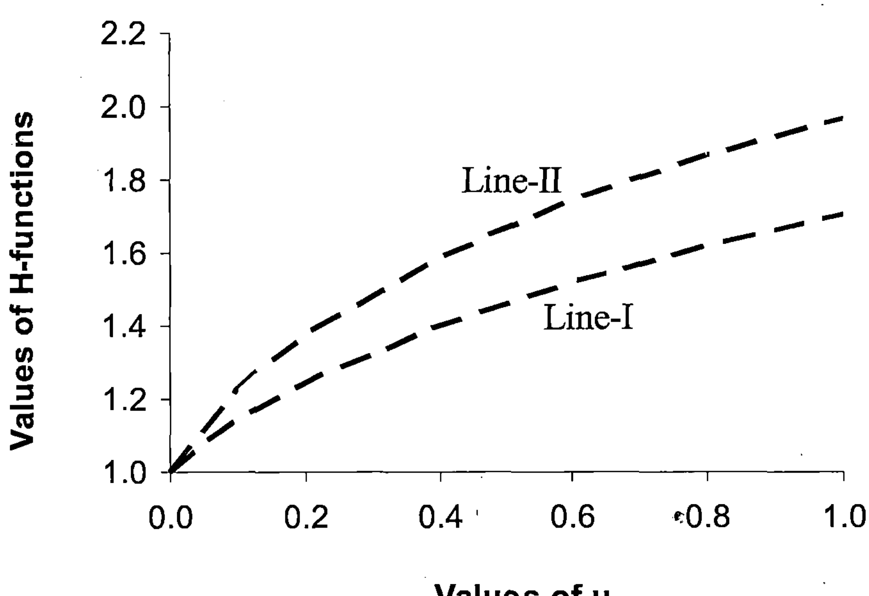
The calculated values of H-function for a set of interlocked doublet lines are given in Table-5.3.

μ	H-functions for Doublets	Case-I	Case-II	Case-III	Case-IV	Case-V
		η_1=16.0 ; η_2=8.0	η_1=10.0 ; η_2=5.0	η_1=6.0 ; η_2=3.0	η_1=4.0 ; η_2=2.0	η_1=1.0 ; η_2=0.5
0.1	$H_1(\mu)$	1.15632	1.14443	1.12874	1.11435	1.05993
	$H_2(\mu)$	1.25528	1.22820	1.19420	1.16498	1.07255
0.2	$H_1(\mu)$	1.26656	1.24330	1.21358	1.18717	1.09365
	$H_2(\mu)$	1.42606	1.37485	1.31319	1.26226	1.11096
0.3	$H_1(\mu)$	1.36049	1.32587	1.28265	1.24517	1.11856
	$H_2(\mu)$	1.56591	1.49174	1.40519	1.33571	1.13826
0.4	$H_1(\mu)$	1.44382	1.39788	1.34163	1.29380	1.13826
	$H_2(\mu)$	1.68550	1.58952	1.48030	1.39459	1.15919
0.5	$H_1(\mu)$	1.51923	1.46206	1.39324	1.33571	1.15442
	$H_2(\mu)$	1.79005	1.67340	1.54343	1.44334	1.17594
0.6	$H_1(\mu)$	1.58830	1.52003	1.43910	1.37244	1.16800
	$H_2(\mu)$	1.88275	1.74656	1.59754	1.48461	1.18971
0.7	$H_1(\mu)$	1.65207	1.57289	1.48030	1.40505	1.17962
	$H_2(\mu)$	1.96580	1.81114	1.64458	1.52010	1.20128
0.8	$H_1(\mu)$	1.71131	1.62144	1.51762	1.43426	1.18971
	$H_2(\mu)$	2.04077	1.86869	1.68595	1.55101	1.21115
0.9	$H_1(\mu)$	1.76661	1.66626	1.55166	1.46064	1.19856
	$H_2(\mu)$	2.10891	1.92037	1.72265	1.57822	1.21969
1.0	$H_1(\mu)$	1.81843	1.70784	1.58287	1.48461	1.20640
	$H_2(\mu)$	2.17117	1.96709	1.75548	1.60237	1.22716

Tables 5.3(a)– 5.3(c) and Figs. 5.3(a)– 5.3(i) are given below for the comparative study of our results with the results of Busbridge and Stibbs.[33]

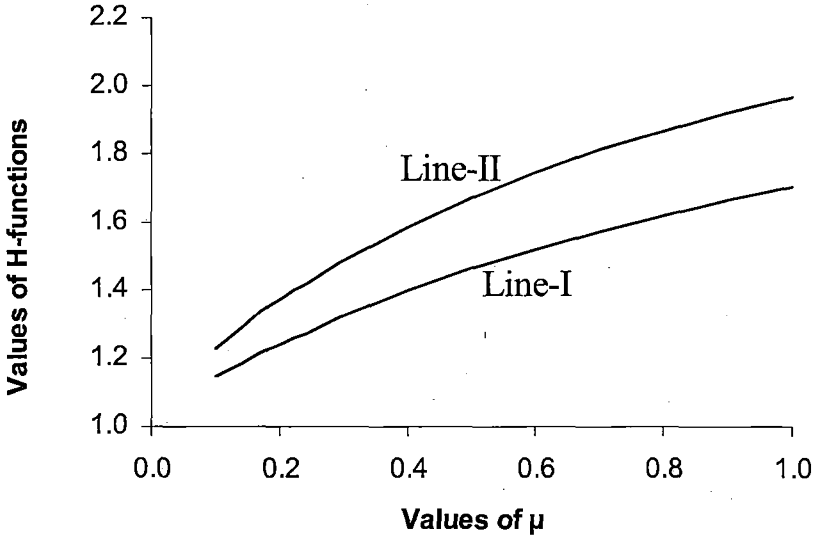
μ	$\eta_1 = 10$ and		$\eta_2 = 5$	
	Busbridge and Stibbs (1954)		Present Results	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000		
0.1	1.1440	1.2278	1.14443	1.22820
0.2	1.2429	1.3745	1.24330	1.37485
0.4	1.3975	1.5893	1.39788	1.58952
0.6	1.5197	1.7462	1.52003	1.74656
0.8	1.6211	1.8684	1.62144	1.86869
1.0	1.7075	1.9670	1.70784	1.96709

Table-5.3(a)



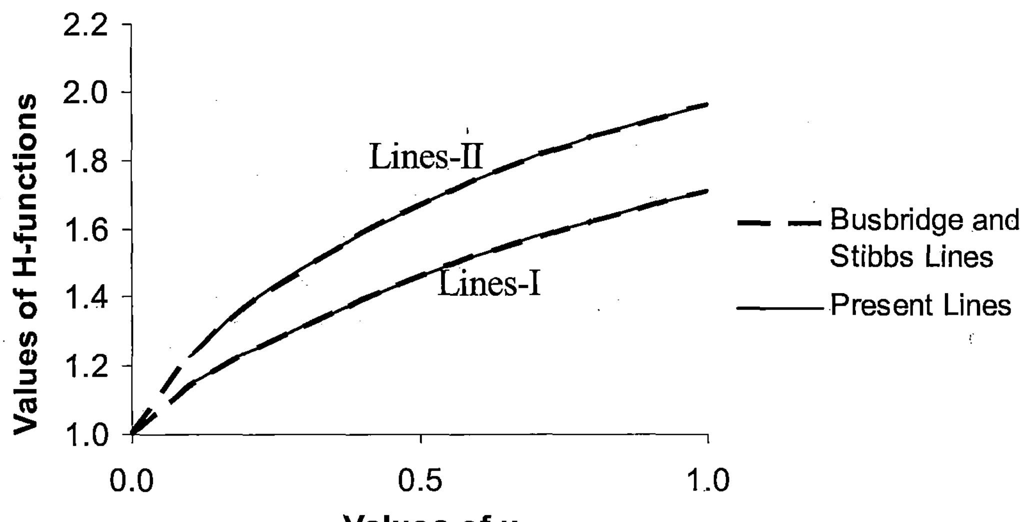
Busbridge and Stibbs Lines (Case-I)

Fig. 5.3 (a)



Present Lines (Case-I)

Fig. 5.3 (b)

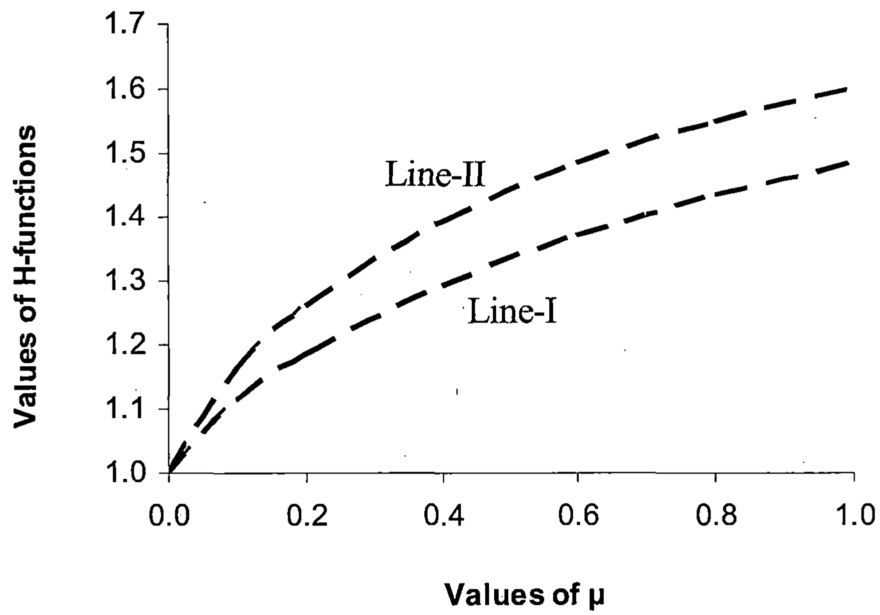


Both Sets of Lines together (Case-I)

Fig. 5.3 (c)

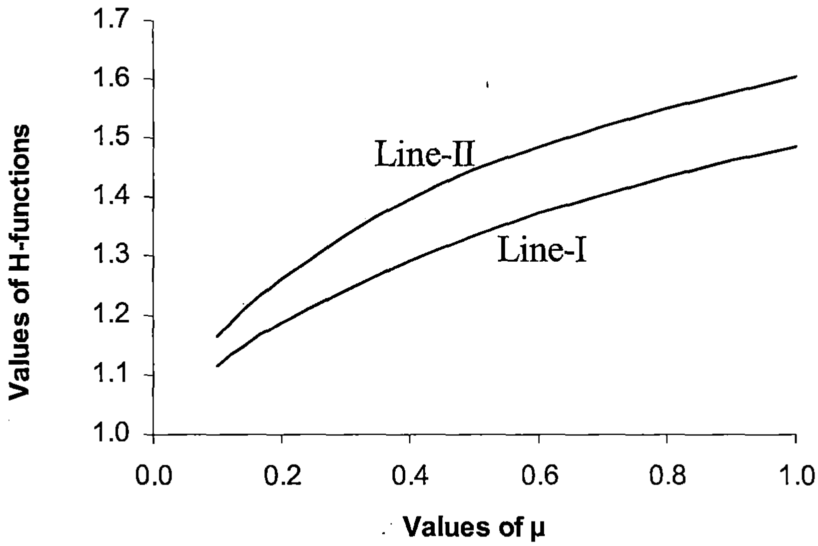
μ	$\eta_1=4$ and $\eta_2=2$			
	Busbridge and Stibbs (1954)		Present Results	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000		
0.1	1.1140	1.1647	1.11435	1.16498
0.2	1.1869	1.2620	1.18717	1.26226
0.4	1.2936	1.3944	1.29380	1.39459
0.6	1.3722	1.4844	1.37244	1.48461
0.8	1.4341	1.5509	1.43426	1.55101
1.0	1.4844	1.6020	1.48461	1.60237

Table-5.3(b)



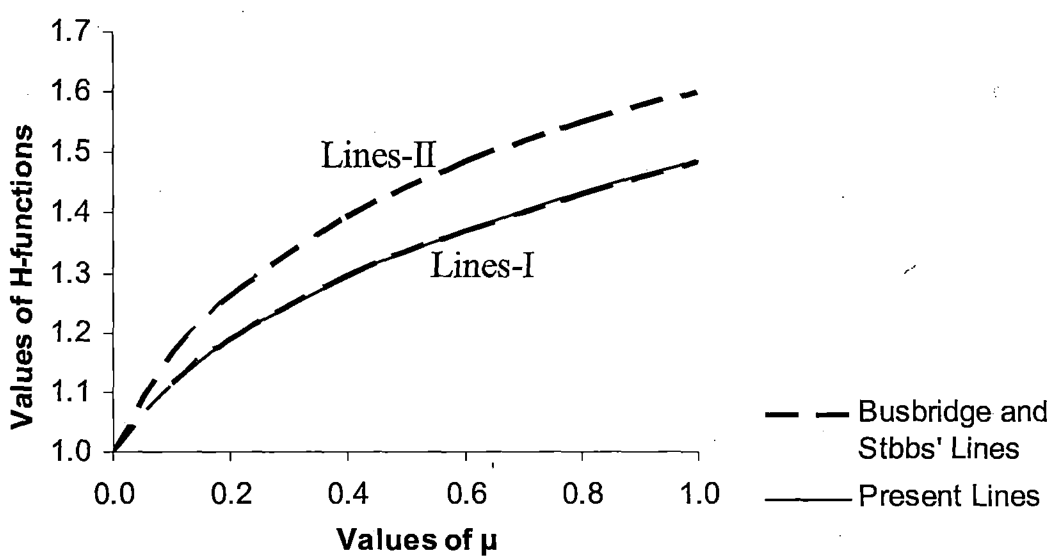
Busbridge and Stibbs Lines (Case-II)

Fig. 5.3 (d)



Present Lines (Case-II)

Fig. 5.3 (e)

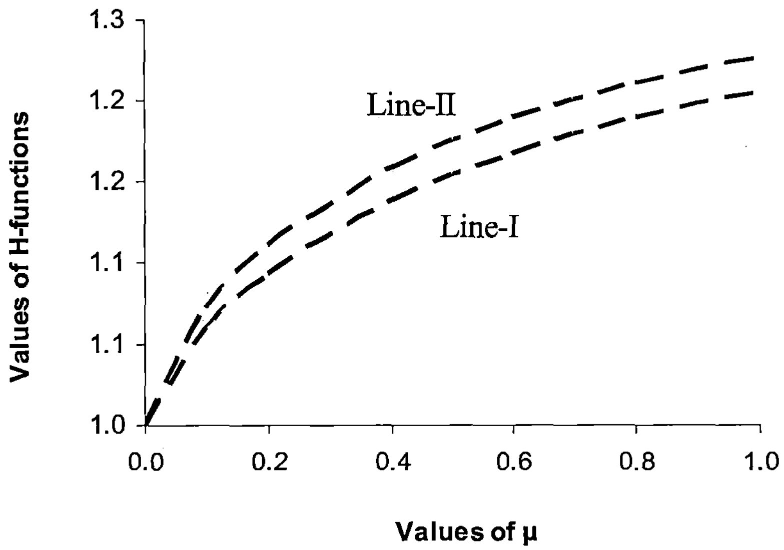


Both Sets of Lines together (Case-II)

Fig. 5.3 (f)

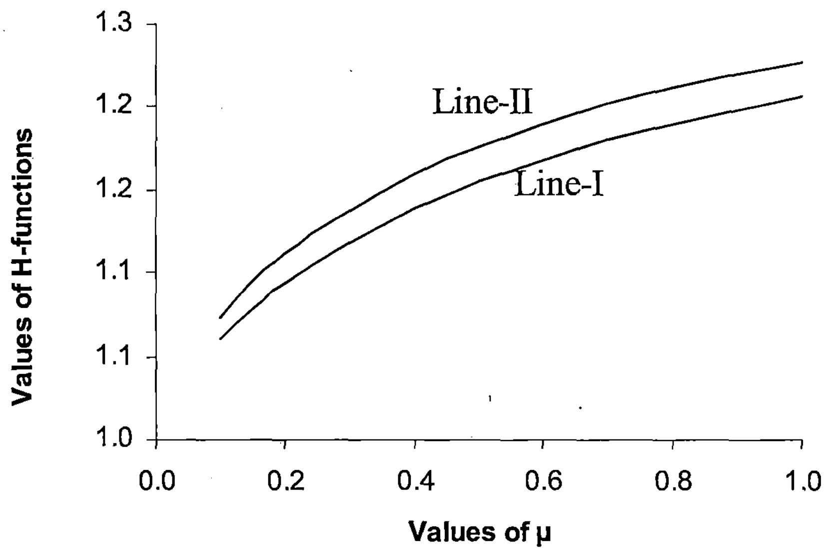
μ	$\eta_1=1$ and $\eta_2=1/2$			
	Busbridge and Stibbs (1954)		Present Results	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.0	1.0000	1.0000		
0.1	1.0598	1.0724	1.05993	1.07255
0.2	1.0935	1.1108	1.09365	1.11096
0.4	1.1382	1.1591	1.13826	1.15919
0.6	1.1679	1.1896	1.16800	1.18971
0.8	1.1896	1.2111	1.18971	1.21115
1.0	1.2064	1.2271	1.20640	1.22716

Table-5.3(c)



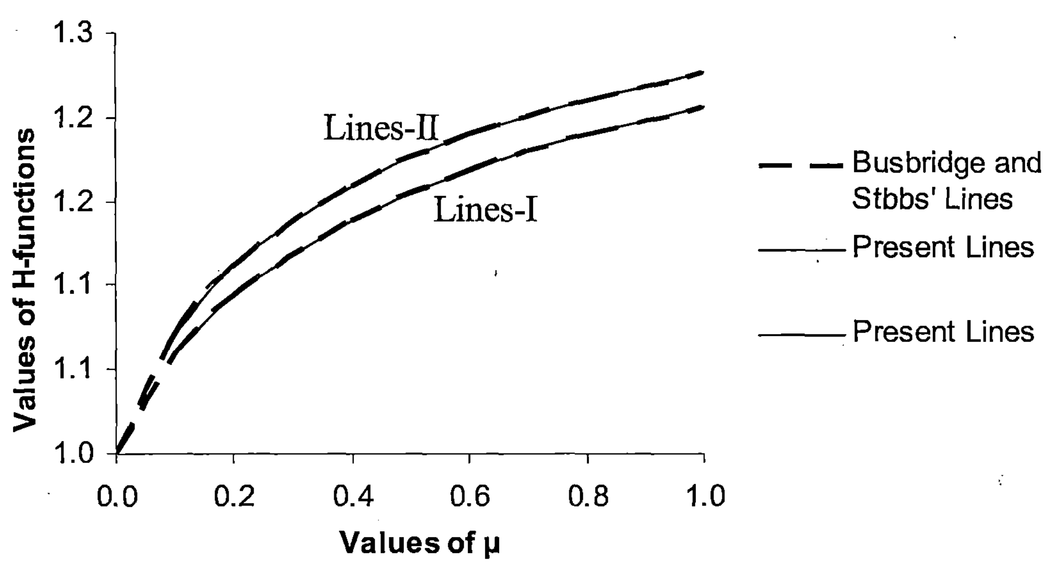
Busbridge and Stibbs Lines (Case-III)

Fig. 5.3 (g)



Present Lines (Case-III)

Fig. 5.3 (h)



Both Sets of Lines together (Case-III)

Fig. 5.3 (i)

Conclusions-I:

Abu-Shumays[7] was surprised to notice that his results for H-functions are much closure to the values obtained by Breen and Chandrasekhar[45] considering H-functions as the solution of the exact integral equation which they satisfy. It was found that the agreement is better than 0.005% for albedo equal to 1 and better than 0.001% for the albedos less than equal to 0.5. So, we may expect that the values of H-functions obtained in Table 5.3 by using Abu-Shumays' [7] approximation for interlocked doublet lines are much more closure to its exact values. We have compared our present results with the results of Busbridge and Stibbs[33] and found them to be almost coincident. The graphs of our present results (as shown figures 5.3(c), 5.3(f), 5.3(i)) in the three cases adopted by Busbridge and Stibbs[33] coincide with the graphs of their results. For comparative study of the work of Karanjai[103] and Deb[65] with the work of Busbridge and Stibbs,[33] the tables 5.2, 5.2 (a), 5.2(b) and 5.2(c) and the graphs 5.2, 5.2 (a), 5.2(b) and 5.2(c)are furnished. As our results are almost the same as those of Busbridge and Stibbs,[33] the comparison of our present results with those of Karanjai[103] and Deb[65] will be nothing but the repetition of the same work.

5.3.2 H-functions for Doublet Lines without Interlocking:

The following table 5.4 shows the value of the H-function for multiplet lines without interlocking.

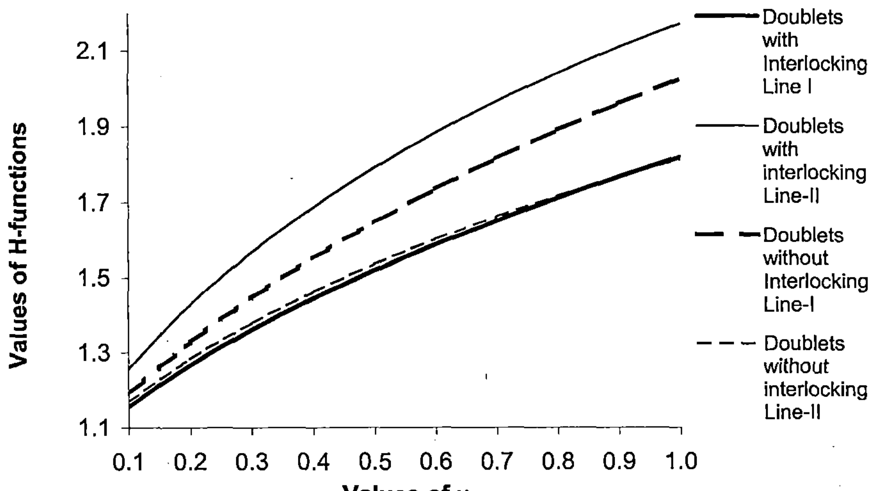
μ	H-functions for Doublets without Interlocking	Case-I	Case-II	Case-III	Case-IV	Case-V
		η_1=16.0 ; η_2=8.0	η_1=10.0 ; η_2=5.0	η_1=6.0 ; η_2=3.0	η_1=4.0 ; η_2=2.0	η_1=1.0 ; η_2=0.5
0.1	$H_1(\mu)$	1.19068	1.17596	1.15660	1.13890	1.07241
	$H_2(\mu)$	1.16793	1.14887	1.12536	1.10551	1.04491
0.2	$H_1(\mu)$	1.32793	1.29870	1.26153	1.22868	1.11348
	$H_2(\mu)$	1.28313	1.24707	1.20422	1.16933	1.06908
0.3	$H_1(\mu)$	1.44601	1.40207	1.34756	1.30060	1.14390
	$H_2(\mu)$	1.37904	1.32675	1.26632	1.21840	1.08646
0.4	$H_1(\mu)$	1.55137	1.49263	1.42128	1.36108	1.16797
	$H_2(\mu)$	1.46229	1.39446	1.31784	1.25835	1.09992
0.5	$H_1(\mu)$	1.64700	1.57352	1.48588	1.41324	1.18773
	$H_2(\mu)$	1.53604	1.45337	1.36178	1.29191	1.11079
0.6	$H_1(\mu)$	1.73469	1.64663	1.54329	1.45896	1.20434
	$H_2(\mu)$	1.60222	1.50539	1.39992	1.32066	1.11979
0.7	$H_1(\mu)$	1.81569	1.71328	1.59483	1.49950	1.21855
	$H_2(\mu)$	1.66215	1.55185	1.43346	1.34566	1.12741
0.8	$H_1(\mu)$	1.89089	1.77441	1.64145	1.53579	1.23087
	$H_2(\mu)$	1.71679	1.59367	1.46324	1.36765	1.13395
0.9	$H_1(\mu)$	1.96100	1.83078	1.68391	1.56850	1.24168
	$H_2(\mu)$	1.76690	1.63157	1.48992	1.38717	1.13964
1.0	$H_1(\mu)$	2.02658	1.88297	1.72277	1.59817	1.25125
	$H_2(\mu)$	1.81306	1.66613	1.51398	1.40463	1.14464

Table-5.4

5.3.3 Comparison of H-functions for Doublet Lines with and without Interlocking:

The values computed in the table 5.4 for H-function of multiplet lines without interlocking are compared with table 5.3 calculated for multiplet lines with interlocking in the tables 5.4(a), 5.4(b), 5.4(c), 5.4(d) and 5.4(e). Comparisons are also shown by drawing their graphs in the figures 5.4(a), 5.4(b), 5.4(c), 5.4(d) and 5.4(e).

μ	η_1=16 and η_2=8			
	Doblets with interlocking		Doblets without interlocking	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.1	1.15632	1.25528	1.19068	1.16793
0.2	1.26656	1.42606	1.32793	1.28313
0.3	1.36049	1.56591	1.44601	1.37904
0.4	1.44382	1.68550	1.55137	1.46229
0.5	1.51923	1.79005	1.64700	1.53604
0.6	1.58830	1.88275	1.73469	1.60222
0.7	1.65207	1.96580	1.81569	1.66215
0.8	1.71131	2.04077	1.89089	1.71679
0.9	1.76661	2.10891	1.96100	1.76690
1.0	1.81843	2.17117	2.02658	1.81306

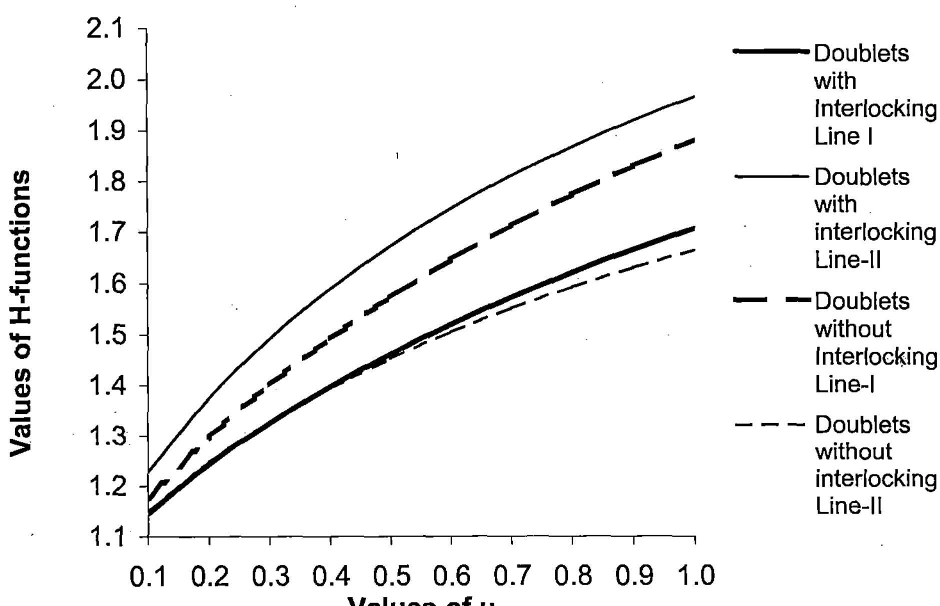


Interlocked Doublets verses Non-interlocked Doublets (Case-I)

Fig-5.4 (a)

μ	η_1=10 and η_2=5			
	Doblets with interlocking		Doblets without interlocking	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.1	1.14443	1.22820	1.17596	1.14887
0.2	1.24330	1.37485	1.29870	1.24707
0.3	1.32587	1.49174	1.40207	1.32675
0.4	1.39788	1.58952	1.49263	1.39446
0.5	1.46206	1.67340	1.57352	1.45337
0.6	1.52003	1.74656	1.64663	1.50539
0.7	1.57289	1.81114	1.71328	1.55185
0.8	1.62144	1.86869	1.77441	1.59367
0.9	1.66626	1.92037	1.83078	1.63157
1.0	1.70784	1.96709	1.88297	1.66613

Table-5.4 (b)

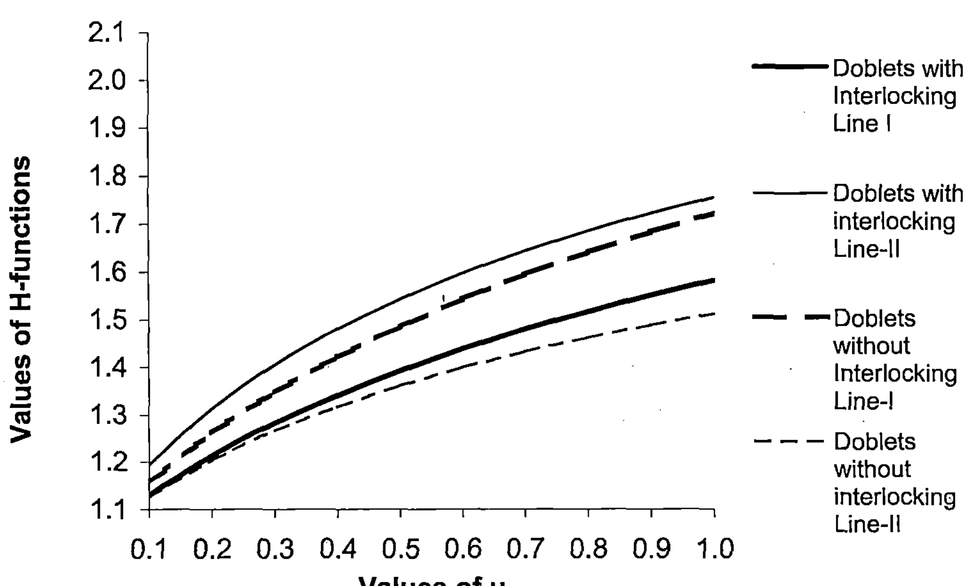


Interlocked Doublets verses Non-interlocked Doublets (Case-II)

Fig-5.4 (b)

μ	η_1=6 and η_2=3			
	Doblets with interlocking		Doblets without interlocking	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.1	1.12874	1.19420	1.15660	1.12536
0.2	1.21358	1.31319	1.26153	1.20422
0.3	1.28265	1.40519	1.34756	1.26632
0.4	1.34163	1.48030	1.42128	1.31784
0.5	1.39324	1.54343	1.48588	1.36178
0.6	1.43910	1.59754	1.54329	1.39992
0.7	1.48030	1.64458	1.59483	1.43346
0.8	1.51762	1.68595	1.64145	1.46324
0.9	1.55166	1.72265	1.68391	1.48992
1.0	1.58287	1.75548	1.72277	1.51398

Table-5.4 (c)

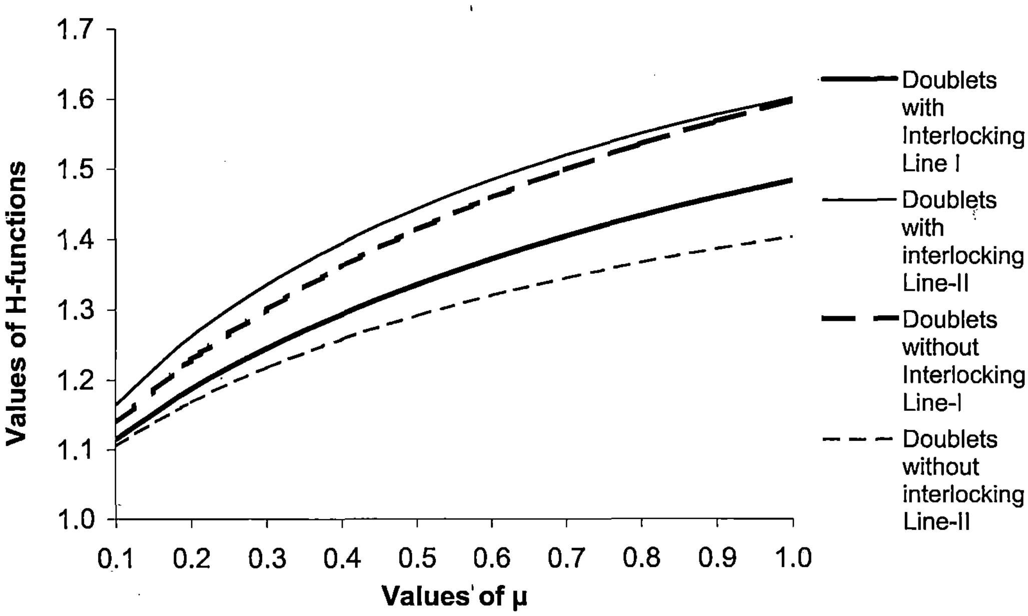


Interlocked Doblets verses Non-interlocked Doublets (Case-III)

Fig-5.4 (c)

μ	η_1=4 and η_2=2			
	Doblets with interlocking		Doblets without interlocking	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.1	1.11435	1.16498	1.13890	1.10551
0.2	1.18717	1.26226	1.22868	1.16933
0.3	1.24517	1.33571	1.30060	1.21840
0.4	1.29380	1.39459	1.36108	1.25835
0.5	1.33571	1.44334	1.41324	1.29191
0.6	1.37244	1.48461	1.45896	1.32066
0.7	1.40505	1.52010	1.49950	1.34566
0.8	1.43426	1.55101	1.53579	1.36765
0.9	1.46064	1.57822	1.56850	1.38717
1.0	1.48461	1.60237	1.59817	1.40463

Table-5.4 (d)

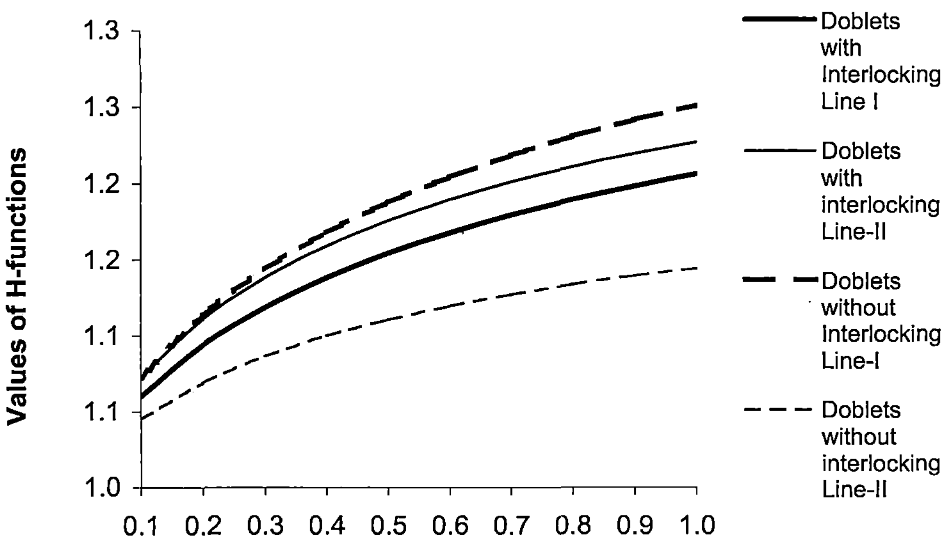


Interlocked Doublets verses Non-interlocked Doublets (Case-IV)

Fig-5.4 (d)

μ	$\eta_1=1$ and $\eta_2=1/2$			
	Doblets with interlocking		Doblets without interlocking	
	$H_1(\mu)$	$H_2(\mu)$	$H_1(\mu)$	$H_2(\mu)$
0.1	1.05993	1.07255	1.07241	1.04491
0.2	1.09365	1.11096	1.11348	1.06908
0.3	1.11856	1.13826	1.14390	1.08646
0.4	1.13826	1.15919	1.16797	1.09992
0.5	1.15442	1.17594	1.18773	1.11079
0.6	1.16800	1.18971	1.20434	1.11979
0.7	1.17962	1.20128	1.21855	1.12741
0.8	1.18971	1.21115	1.23087	1.13395
0.9	1.19856	1.21969	1.24168	1.13964
1.0	1.20640	1.22716	1.25125	1.14464

Table-5.4 (e)



Interlocked Doblets verses Non-interlocked Doublets (Case-V)

Fig-5.4 (e)

Conclusions-II:

The effect of interlocking decrease the value of $H_1(\mu)$ and increases the value of $H_2(\mu)$ for all values of μ i.e. the effect of interlocking is to increase the value of $H_1(\mu)$ and to decrease the value of $H_2(\mu)$ for all values of μ.

5.3.4 Residual Intensities for Interlocked Multiplet Lines

The residual intensity $r_r(\mu)$ of the r^{th} interlocked line is given by

$$r_r(\mu) = \frac{100 I_r(0,\mu)}{I^{(cont)}(0,\mu)} \tag{5.44}$$

where $I^{(cont)}(0,\mu)$ is given by

$$I^{(cont)}(0,\mu) = b_0 + b_1\mu \tag{5.45}$$

Busbridge and Stibbs[33] calculated $r_1(\mu)$ and $r_2(\mu)$ for the following three cases

(I): $\eta_1 = 10, \eta_2 = 5$

(II): $\eta_1 = 4, \eta_2 = 2$

(III): $\eta_1 = 1, \eta_2 = 1/2$

for a region of spectrum where $b_1 = \frac{3}{2}b_0$

But, we have obtained the emergent intensity $I_r(0,\mu)$ for r^{th} interlocked line for isotropically scattering medium in equation (2.104) of section (2.3.3) of chapter-2 as

$$I_r(0,\mu) = \sqrt{M} H_r(\mu) \left\{ b_0 + b_1\zeta_r\mu + b_1\zeta_r \left(\sum_{\alpha=1}^{n} \frac{1}{\zeta_r k_{(s)\alpha}} - \sum_{i=1}^{n} \mu_{(r)i} \right) \right\}$$

i.e. for the region of spectrum where $b_1 = \frac{3}{2}b_0$,

$$r_r(\mu) = \frac{100\sqrt{M}H_r(\mu)\left\{2 + 3\zeta_r\mu + 3\zeta_r\left(\sum\limits_{\alpha=1}^{n}\frac{1}{\zeta_r k_{(r)\alpha}} - \sum\limits_{j=1}^{n}\mu_{(r)j}\right)\right\}}{(2+3\mu)} \tag{5.46}$$

But from the theory of Chandrasekhar,[45] it can be shown that

$$\lim_{n \to \infty}\left(\sum_{\alpha=1}^{n}\frac{1}{k_\alpha\zeta_r} - \sum_{\alpha=1}^{n}\mu_{(r)i}\right) = \frac{1}{2C\sqrt{M}}\sum_{r=1}^{m}\eta_r\omega_r\alpha_{(r)1} \tag{5.47}$$

where

$$\alpha_{(r)1} = \int_0^1 \mu H_r(\mu)\,d\mu \tag{5.48}$$

$$\therefore\ r_r(\mu) = \frac{100H_r(\mu)}{2C(2+3\mu)}\left\{2C\sqrt{M}(2+3\zeta_r\mu) + 3\zeta_r\sum_{r=1}^{m}\eta_r\omega_r\alpha_{(r)1}\right\} \tag{5.49}$$

where $\alpha_1(r)$ is the first moment of H-function of r^{th} line and is given by

$$\begin{aligned}\alpha_{1(r)} &= \int_0^1 \mu' H(x')\,d\mu'\ ;\ x' = \zeta_s\mu\\ &= a_0\left(\frac{1}{\zeta_s k} - \frac{1}{\zeta_s^2k^2}\ln(1+k\zeta_s)\right) + a_1\left(\frac{1}{2k} - \frac{1}{\zeta_s k^2} + \frac{1}{\zeta_s^2k^3}\times\right.\\ &\left.\times\ln(1+k\zeta_s)\right) + a_2\left(\frac{\zeta_s}{3k} - \frac{1}{2k^2} + \frac{1}{\zeta_s k^3} - \frac{1}{\zeta_s^2k^4}\ln(1+k\zeta_s)\right)\end{aligned} \tag{5.50}$$

so that the first moments of H-functions $\alpha_{1(r)(AS-I)}$ and $\alpha_{1(r)(AS-II)}$ will be respectively

$$\alpha_{1(s)(AS-I)} = A\left(\frac{1}{\zeta_s k} - \frac{1}{\zeta_s^2k^2}\ln(1+k\zeta_s)\right) + B\left(\frac{1}{2k} - \frac{1}{\zeta_s k^2} + \frac{1}{\zeta_s^2k^3}\ln(1+k\zeta_s)\right) \tag{5.51}$$

and

$$\alpha_{1(r)(AS-II)} = \left(\frac{1}{\zeta_r k} - \frac{1}{\zeta_r^2 k^2}\ln(1+k\zeta_r)\right) + A'\left(\frac{1}{2k} - \frac{\zeta_r}{\zeta_r k^2} + \frac{1}{\zeta_r^2 k^3}\times\right.$$
$$\left.\times\ln(1+k\zeta_r)\right) + B'\left(\frac{\zeta_r}{3k} - \frac{1}{2k^2} + \frac{1}{\zeta_r k^3} - \frac{1}{\zeta_r^2 k^4}\ln(1+k\zeta_r)\right) \quad (5.52)$$

We have obtained the residual intensities for interlocked doublet lines in the following table by using our approximated form derived above for three different cases which are used by Eddington and Busbridge and Stibbs[33] also:

μ ↧	Residual Intensities for Doublets	η_1=10 and η_2=5			η_1=4 and η_2=2			η_1=1 and η_2=1/2		
		Eddington	Busbridge & Stibbs	Present	Eddington	Busbridge & Stibbs	Present	Eddington	Busbridge & Stibbs	Present
0.1	$r_1(\mu)$	40.38	39.09	35.27	58.43	56.88	50.85	84.21	82.58	77.07
	$r_2(\mu)$	46.97	46.39	43.26	66.50	65.74	57.68	89.69	88.80	80.39
0.2	$r_1(\mu)$	38.64	38.16	34.34	55.77	55.17	49.27	81.06	80.24	75.05
	$r_2(\mu)$	46.27	46.35	43.74	64.82	64.76	57.68	87.67	87.28	80.03
0.4	$r_1(\mu)$	35.74	35.86	32.18	51.42	51.49	46.03	76.21	75.97	71.42
	$r_2(\mu)$	44.31	44.73	42.78	61.53	61.81	56.09	84.47	84.39	78.65
0.6	$r_1(\mu)$	33.40	33.70	30.21	48.04	48.28	43.26	72.66	72.60	68.59
	$r_2(\mu)$	42.26	42.70	41.15	58.65	58.95	54.15	82.07	82.07	77.31
0.8	$r_1(\mu)$	31.46	31.82	28.52	45.32	45.60	40.98	69.96	69.96	66.38
	$r_2(\mu)$	40.36	40.75	39.47	56.22	56.50	52.34	80.23	80.26	76.17
1.0	$r_1(\mu)$	29.84	30.18	27.07	43.09	43.37	39.09	67.83	67.86	64.63
	$r_2(\mu)$	38.65	39.00	37.91	54.18	54.41	50.75	78.77	78.80	75.24

Table-5.5

Conclusions-III:

The errors in results those we have found in comparison with Eddington's results are in the range 1.9% — 13.2% and with Busbridge and Stibbs' results are in the range 2.8% — 12.3%.

Appendix I

H-functions for Interlocked Multiplets Lines

The equation of transfer of Chandrasekhar,[45] for isotropically scattering media, is

$$\mu\frac{dI(\tau,\mu)}{d\tau}=I(\tau,\mu)-\frac{1}{2}\int_{1}^{-1}I(\tau,\mu')\,d\mu' \tag{I.1}$$

The equation of transfer of Busbridge and Stibbs,[33] for rth interlocked multiplet line, is

$$\begin{aligned}\zeta_r\mu\frac{dI_r(\tau,\mu)}{d\tau} &= I_r(\tau,\mu)-\lambda_r B_\nu(T)\\ &-\frac{1}{2}(1-\lambda_r)\frac{\alpha_r}{\eta_r}\sum_{s=1}^{m}\eta_s\int_{1}^{-1}I_s(\tau,\mu')\,d\mu',\\ &(r=1,\cdots\cdots,m)\end{aligned} \tag{I.2}$$

in which

$$\zeta_r = \frac{1}{1+\eta_r}$$
$$\lambda_r = \frac{1+\epsilon\eta_r}{1+\eta_r} = 1-\omega_r$$

and

$$\alpha_r = \eta_r / \sum_{s=1}^{m} \eta_s \tag{I.3}$$

so that

$$\sum_{r=1}^{m} \alpha_r = 1 \tag{I.4}$$

The solution of the equation (I.1), by Chandrasekhar,[45] involves a H-function $H(\mu)$ which satisfies the non-linear integral equation :

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\Psi(\mu')\ H(\mu)}{\mu+\mu'} d\mu' \tag{I.5}$$

and the solution of the equation (I.2), by Busbridge and Stibbs,[33] involves a H-function $H_r(\mu)$ i.e. $H(\zeta_r\mu)$ satisfying the non-linear integral equation :

$$H(x) = 1 + xH(x) \int_0^1 \frac{\Psi(x')\,H(x)}{x+x'} dx'$$

i.e.

$$H(x) = 1 + \frac{1}{2} xH(x) \sum_{s=1}^{m} \frac{\alpha_s}{\zeta_s} (1-\lambda_s) \int_0^{\zeta_s} \frac{H(x')}{x+x'} dx'$$

i.e.

$$H(\zeta_r\mu) = 1 + \frac{1}{2}\zeta_r\mu H(\zeta_r\mu) \sum_{s=1}^{m} \alpha_s (1-\lambda_s) \int_0^1 \frac{H(\zeta_s\mu')}{\zeta_r\mu+\zeta_s\mu'} d\mu' \tag{I.6}$$

where assumption made is that

$$\eta_1 > \eta_2 > \cdots\cdots > \eta_m \tag{I.7}$$

so that

$$0 \le \zeta_1 < \zeta_2 < \cdots\cdots \zeta_m \le 1 \tag{I.8}$$

The function $\Psi(x')$ is given by

$$\Psi(x') = \begin{cases} \frac{1}{2}\sum_{s=1}^{m} \alpha_s (1-\lambda_s)/\zeta_s, & if\ 0 \le x' \le \zeta_1 \\ \frac{1}{2}\sum_{s=r+1}^{m} \alpha_s (1-\lambda_s)/\zeta_s, & if\ \zeta_s \le x' \le \zeta_{s+1} \\ 0, & if\ \zeta_s \le x' \le 1 \end{cases} \tag{I.9}$$

Chandrasekhar [42] established two theorems regarding H-functions in his paper that

Theorem(I): The solution for the emergent (or the reflected) radiation obtained in the n^{th} approximation of the integro-differential equation of the radiative transfer or neutron transport involves H-functions of the form:

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1} (\mu + \mu_i)}{\prod_{\alpha=1} (1 + k_\alpha \mu)} \tag{I.10}$$

where the μ_i's are the positive zeros of the Legendre polynomial $P_{2n}(\mu)$ and k_α's are the roots of the characteristic equation

$$1 = 2\sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{1 - k^2 \mu_j^2} \tag{I.11}$$

(Ψ an even polynomial in μ and $\int_0^1 \Psi(\mu)\, d\mu \le \frac{1}{2}$)

then, in the limit of infinite approximation, the H-functions become solutions of the functional equations of a certain standard form, namely,

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\Psi(\acute{\mu})\ H(\mu)}{\mu + \acute{\mu}} d\acute{\mu}$$

Theorem(II): The solution of the functional equation:

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\Psi(\acute{\mu})\ H(\mu)}{\mu + \acute{\mu}} d\acute{\mu}$$

where $\Psi(\acute{\mu})$ is an even polynomial satisfying the condition:

$$\int_0^1 \Psi(\mu)\, d\mu \leq \frac{1}{2}$$

is the limit function

$$\lim_{x \longrightarrow \infty} \frac{1}{\mu_1 \cdots \mu_n} \cdot \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha} (1 + k_\alpha \mu)}$$

By the theorems(I) and (II), we can conclude that the solution of the functional equations (I.6) is the limit of the function :

$$H(x) = \frac{1}{x_1 \cdots x_n} \frac{\prod_{i=1}^{n} (x + x_i)}{\prod_{\alpha=1} (1 + k_\alpha x)} \tag{I.12}$$

as n tends to infinity, where the x_i's are the positive zeros of the Legendre polynomial $P_{2n}(x)$ and k_α's are the roots of the characteristic equation :

$$1 = 2\sum_{j=1}^{n} \frac{\acute{a}_j \Psi(x_j)}{1 - k^2 x_j^2} \quad \acute{x} = \zeta_s \acute{\mu} \text{ and } \acute{a}_j = \zeta_s a_j \tag{I.13}$$

The function $\Psi\left(x'\right)$ satisfies the condition:

$$\int_0^1 \Psi\left(x'\right) dx' \leq \frac{1}{2}$$

as proved by Busbridge and Stibbs.[33]

Replacing x by $\zeta_r \mu$ or, equivalently x_i by $\zeta_r \mu_{(r)i}$ in the equation (I.12), we get

$$H\left(\zeta_r \mu\right) = \frac{1}{\zeta_r \mu_{(r)1} \cdots \zeta_r \mu_{(r)n}} \frac{\prod\limits_{i=1}^{n}\left(\zeta_r \mu + \zeta_r \mu_{(r)i}\right)}{\prod\limits_{\alpha=1}\left(1 + k_\alpha \zeta_r \mu\right)}$$

i.e.

$$H\left(\zeta_r \mu\right) = \frac{1}{\mu_{(r)1} \cdots \mu_{(r)n}} \frac{\prod\limits_{i}\left(\mu + \mu_{(r)i}\right)}{\prod\limits_{\alpha}\left(1 + k_\alpha \mu\right)} \tag{I.14}$$

which is used by Karanjai and Barman[107] and the followers.

The function $\Psi\left(x_j\right)$, in the equation (I.13), will take the form:

$$1 = 2\sum_{j=1}^{n} \frac{\zeta_s a_j}{1 - k^2 \zeta_s^2 \mu_{(s)j}^2} \frac{1}{2} \sum_{s=1}^{m} \alpha_s\left(1 - \lambda_s\right)/\zeta_s$$

i.e.

$$1 = \sum_{s=1}^{m} \alpha_s\left(1 - \lambda_s\right)/\zeta_s \sum_{j=1}^{n} \frac{\zeta_s a_j}{1 - k^2 \zeta_s^2 \mu_{(s)j}^2}$$

i.e

$$1 = \frac{1}{\sum\limits_{s=1}^{n} \eta_s} \left[\sum_{s=1}^{m} \eta_s \omega_s \sum_{j=1}^{n} \frac{a_j}{1 - k^2 \zeta_s^2 \mu_{(s)j}^2}\right]$$

which is the characteristic equation (17), constructed by Karanjai and Barman.[107] The above form of characteristic equation can also be put in the form:

$$1 = 2\sum_{s=1}^{m} \eta_s \omega_s \sum_{j=1}^{n} \frac{a_j}{C\left(1 + k \zeta_s \mu_{(s)j}\right)} \tag{I.15}$$

in which $C = \sum_{s=1}^{m} \eta_s$

Now, we take the characteristic equation:

$$1 = 2\sum_{j=1}^{n} \frac{a_j' \Psi\left(x_j\right)}{1 - k^2 x_j^2} \quad x' = \zeta_s \mu' \text{ and } a_j' = \zeta_s a_j \tag{I.16}$$

and the characteristic function $\Psi\left(x'\right)$ in the form:

$$\Psi\left(x'\right) = \begin{cases} \frac{1}{2C} \cdot \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s^{2p+1}} \cdot \left(\xi_p x'^{2p} + \xi_{p-1} x'^{2p-2} + \cdots + \xi_1 x'^2 + \xi_0\right), & \text{if } 0 \le x' \le \zeta_1 \\ \frac{1}{2C} \cdot \sum_{s=r+1}^{m} \frac{\eta_s \omega_s}{\zeta_s^{2p+1}} \cdot \left(\xi_p x'^{2p} + \xi_{p-1} x'^{2p-2} + \cdots + \xi_1 x'^2 + \xi_0\right), & \text{if } \zeta_r \le x' \le \zeta_{r+1} \\ 0, & \text{if } \zeta_m \le x' \le 1 \end{cases} \tag{I.17}$$

We consider the characteristic function

$$\Psi\left(x'\right) = \begin{cases} \frac{1}{2C} \cdot \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s^{2p+1}} \cdot x'^{2p}, & \text{if } 0 \le x' \le \zeta_1 \\ \frac{1}{2C} \cdot \sum_{s=r+1}^{m} \frac{\eta_s \omega_s}{\zeta_s^{2p+1}} \cdot x'^{2p}, & \text{if } \zeta_r \le x' \le \zeta_{r+1} \\ 0, & \text{if } \zeta_m \le x' \le 1 \end{cases} \tag{I.18}$$

Assumption made is

$$\eta_1 > \eta_2 > \cdots > \eta_m$$

so that

$$0 \le \zeta_1 < \zeta_2 < \cdots < \zeta_m \le 1$$

Then

$$\int_0^1 \Psi(x')\,dx' = \frac{1-M}{2(2p+1)}$$

So, for the equation (I.17)

$$\int_0^1 \Psi(x')\,dx' = \frac{1}{2}(1-M)\left(\xi_0 + \frac{\xi_1}{3} + \cdots + \frac{\xi_{p-2}}{(2p-1)} + \frac{\xi_p}{(2p+1)}\right) \tag{I.19}$$

which will satisfy the condition:

$$\int_0^1 \Psi(x')\,dx' \le \frac{1}{2}$$

provided that

$$(1-M)\left(\xi_0 + \frac{\xi_1}{3} + \cdots + \frac{\xi_{p-2}}{(2p-1)} + \frac{\xi_p}{(2p+1)}\right) \le 1$$

Using the definition (I.18) in the equation (I.16), we get

$$1 = 2\sum_{j=1}^{n} \frac{a'_j}{1-k^2x_j^2} \cdot \frac{1}{2C} \cdot \sum_{s=1}^{m} \frac{\eta_s\omega_s}{\zeta_s^{p+1}} \cdot x'^p \ ; \ \ x' = \zeta_s\mu' \text{ and } a'_j = \zeta_s a_j$$

i.e.

$$2C = \sum_{s=1}^{m} \eta_s\omega_s \sum_{j=1}^{n} \frac{a_j\mu_{(r)i}^p}{1+k\zeta_s\mu_{(r)i}}$$

i.e.

$$2C = D_p(k\zeta_s)$$

if we take the following definition

$$D_\ell(x) = \sum_{s=1}^{m} \eta_s\omega_s \sum_j \frac{a_j\mu_{(s)j}^\ell}{1+\mu_{(s)j}\,x} \tag{I.20}$$

From this we can conclude that replacement of x from the definition (I.20) by $k\zeta_s$ is equivalent to the replacement of x from the equation (I.13) by $\zeta_s\mu$ and conversely.

Theorem(III): The solution for the emergent(or the reflected) radiation(obtained in the n^{th} approximation of the integro-differential equation of the radiative transfer or neutron transport involves H-functions of the form:

$$H_r(\mu) = \frac{1}{\mu_{(r)1}\cdots\mu_{(r)n}} \frac{\prod_{i=1} \left(\mu + \mu_{(r)i}\right)}{\prod_{\alpha=1} \left(1 + k_\alpha \mu\right)} \tag{I.21}$$

where the $\mu_{(r)i}$'s are the positive zeros of the Legendre polynomial $P_{2n}(\mu)$ and k_α's are the roots of the characteristic equation

$$1 = 2\sum_{j=1}^{n} \frac{a_j' \Psi(x_j)}{1 - k_\alpha^2 x_j^2} \quad ; \; x' = \zeta_s \mu' \text{ and } a_j' = \zeta_s a_j \tag{I.22}$$

$$\left[\Psi \text{ is an even polynomial in } x' \text{ and } \int_0^1 \Psi(\mu)\, d\mu \leq \frac{1}{2}\right]$$

then, in the limit of infinite approximation, the H-functions become solutions of the functional equations of a certain standard form, namely,

$$H(x) = 1 + xH(x) \int_0^1 \frac{\Psi(x')\; H(x)}{x + x'} dx'$$

Some Properties of H-functions:

For the H-functions involved in the interlocked multiplets lines, the characteristic function Ψ has the form defined by (I.18). With this characteristic function the integrals $\int_0^1 \Psi(x')\,dx'$ and $\int_0^1 \Psi(x')\,x'^2 dx'$ becomes

$$\int_0^1 \Psi(x')\,dx' = \frac{1}{2(2p+1)}\cdot(1-M)$$

and

$$\int_0^1 \Psi(x')\,x'^2 dx' = \frac{1}{2C(2p+3)}\cdot\sum_{s=1}^{m}\eta_s\omega_s\zeta_s^2$$

Property: (I)

$$\int_0^1 H(x')\,\psi(x')\,dx' = 1-\left[1-2\int_0^1 \Psi(x')\,dx'\right]^{\frac{1}{2}}$$

i.e.

$$\frac{1}{2C}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s^{2p+1}}\int_0^{\zeta_s} H(x')\,x'^{2p}dx' = 1-\left(1-\frac{1-M}{2p+1}\right)^{1/2}$$

i.e.

$$\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\int_0^{1} H(\zeta_s\mu')\,\mu'^{2p}d\mu' = 1-\left(1-\frac{1-M}{2p+1}\right)^{1/2} \qquad \text{(I.23)}$$

Property: (II)

$$\int_0^1 \frac{\psi(x')\,H(x')}{1-kx'} = 1;$$

i.e.

$$\frac{1}{2C}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s^{2p+1}}\int_0^{\zeta_s}\frac{H(x')\;x'^{2p}}{1-kx'}\,dx' = 1$$

i.e.

$$\frac{1}{2C}\sum_{s=1}^{m}\eta_s\omega_s\int_0^1 \frac{H(\zeta_s\mu')\ \mu'^{2p}}{1-k\zeta_s\mu'}\,d\mu' = 1$$

where k is determined by

$$\frac{1}{2kC}\cdot\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\cdot\ln\left(\frac{1+k\zeta_s}{1-k\zeta_s}\right) = 1$$

Property: (III)

$$\left[1 - 2\int_0^1 \Psi(x')\,dx'\right]^{\frac{1}{2}}\int_0^1 H(x')\,\psi(x')\,x'^2dx'$$

$$+\frac{1}{2}\left[\int_0^1 H(x')\,\psi(x')\,x'dx'\right]^2 = \int_0^1 \Psi(x')\,x'^2dx'$$

i.e.

$$\left(1-\frac{1-M}{2p+1}\right)^{1/2}\frac{1}{2C}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s^{2p+1}}\int_0^{\zeta_s} H(x')\,x'^{2p+2}dx'$$

$$+\frac{1}{2}\left[\frac{1}{2C}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s^{2p+1}}\int_0^{\zeta_s} H(x')\,x'^{2p+1}dx'\right]^2 = \frac{1}{2C\,(2p+3)}\cdot\sum_{s=1}^{m}\eta_s\omega_s\zeta_s^2$$

i.e.

$$\left(1-\frac{1-M}{2p+1}\right)^{1/2}\sum_{s=1}^{m}\eta_s\omega_s\zeta_s^2\int_0^1 H(\zeta_s\mu')\,\mu'^{2p+2}d\mu'$$

$$+\frac{1}{4C}\left[\sum_{s=1}^{m}\eta_s\omega_s\zeta_s\int_0^1 H(\zeta_s\mu')\,\mu'^{2p+1}d\mu'\right]^2 = \frac{1}{2p+3}\cdot\sum_{s=1}^{m}\eta_s\omega_s\zeta_s^2 \tag{I.24}$$

Property: (IV)

$$H(x)\,H(-x) = \frac{1}{T(x)} \tag{I.25}$$

in which

$$T(x) = 1 - 2x^2 \int_{-1}^{+1} \frac{\Psi(x')\,dx'}{x^2 - x'^2}$$

i.e.

$$T(x) = 1 - \frac{1}{C} x^2 \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s^{2p+1}} \int_0^{\zeta_s} \frac{x'^{2p} dx'}{x^2 - x'^2}$$

i.e.

$$T(x) = 1 - \frac{1}{C} x^2 \sum_{s=1}^{m} \eta_s \omega_s \int_0^{1} \frac{\mu'^{2p} d\mu'}{x^2 - \zeta_s^2 \mu'^2} \tag{I.26}$$

The above properties of H-functions can be used easily to write the properties of H-functions involved in an absorption lines in an isotropically scattering medium or an anisotropically scattering media by replacing the characteristic function μ'^{2p} by 1 (for the case of isotropically scattering medium), $\xi_1 \mu'^2 + \xi_0$ (for the case of anisptropically scattering medium with planetary phase function) and $\xi_2 \mu'^4 + \xi_1 \mu'^2 + \xi_0$ (for the case of anisptropically scattering medium with Rayleigh phase function or Pomraning phase function or three term scattering indicatrix etc.)

For isotropically scattering medium the properties will be

Property: (I)

$$\frac{1}{2C} \sum_{s=1}^{m} \frac{\eta_s \omega_s}{\zeta_s} \int_0^{\zeta_s} H(x')\, dx' = 1 - \sqrt{M} \tag{I.27}$$

Property: (II)

$$\frac{1}{2C}\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\int_0^{\zeta_s}\frac{H(x')}{1-kx'}dx' = 1$$

where k is determined by

$$\frac{1}{2kC}\cdot\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\cdot\ln\left(\frac{1+k\zeta_s}{1-k\zeta_s}\right) = 1$$

Property: (III)

$$\sqrt{M}\sum_{s=1}^{m}\eta_s\omega_s\zeta_s^2\int_0^1 H(\zeta_s\mu')\,\mu'^2 d\mu'$$

$$+\frac{1}{4C}\left[\sum_{s=1}^{m}\eta_s\omega_s\zeta_s\int_0^1 H(\zeta_s\mu')\,\mu' d\mu'\right]^2 = \frac{1}{3}\cdot\sum_{s=1}^{m}\eta_s\omega_s\zeta_s^2 \qquad (I.28)$$

Property: (IV)

$$H(x)\,H(-x) = \frac{1}{T(x)} \qquad (I.29)$$

in which

$$\begin{aligned} T(x) &= 1-\frac{1}{C}x^2\sum_{s=1}^{m}\eta_s\omega_s\int_0^1\frac{d\mu'}{x^2-\zeta_s^2\mu'^2} \\ &= 1-\frac{1}{2C}x\sum_{s=1}^{m}\frac{\eta_s\omega_s}{\zeta_s}\ln\left(\frac{x+\zeta_s}{x-\zeta_s}\right) \end{aligned} \qquad (I.30)$$

Appendix II

Characteristic Equation Generating Functions

We have already defined $D_\ell(x)$ as

$$D_\ell(x) = \sum_{s=1}^{m} \eta_s \omega_s \sum_j \frac{a_j \mu_{(s)j}^\ell}{1 + \mu_{(s)j}\, x}$$

which yields

$$D_\ell(x) = \frac{1}{x}\psi_{\ell-1} - \frac{1}{x} D_{\ell-1}(x),$$

where

$$\psi_\ell = \sum_{s=1}^{m} \eta_s \omega_s \sum_j a_j \mu_{(s)j}^\ell$$

The above recurrence relation will ultimately produce

$$\begin{aligned} D_\ell(x) = \frac{\psi_{\ell-1}}{x} - \frac{\psi_{\ell-2}}{x^2} + \frac{\psi_{\ell-3}}{x^3} - \cdots + (-1)^{\ell-2} \frac{\psi_1}{x^{\ell-1}} \\ + (-1)^{\ell-1} \frac{1}{x^\ell}\left(\psi_0 - D_0(x)\right) \end{aligned} \tag{II.1}$$

where the function ψ_ℓ is defined as

$$\psi_\ell = \sum_{s=1}^{m} \eta_s \omega_s \sum_j a_j \mu_{(s)j}^{\ell} \tag{II.2}$$

The relation (II.1) is same as the equation (49) as obtained by Karanjai and Barman.[107]

Applying the relations (II.1) and (II.2) and using the notation $\mathcal{D}_\ell$ for $D_\ell\,(x)$, we get

$$\begin{aligned}
\mathcal{D}_1 &= D_1\,(x) = \frac{1}{x}\,(\psi_0 - \mathcal{D}_0) && \text{(II.3a)}\\
\mathcal{D}_2 &= D_2\,(x) = \frac{\psi_1}{x} - \frac{1}{x^2}\,(\psi_0 - \mathcal{D}_0) && \text{(II.3b)}\\
\mathcal{D}_3 &= D_3\,(x) = \frac{\psi_2}{x} - \frac{\psi_1}{x^2} + \frac{1}{x^3}\,(\psi_0 - \mathcal{D}_0) && \text{(II.3c)}\\
\mathcal{D}_4 &= D_4\,(x) = \frac{\psi_3}{x} - \frac{\psi_2}{x^2} + \frac{\psi_1}{x^3} - \frac{1}{x^4}\,(\psi_0 - \mathcal{D}_0) && \text{(II.3d)}
\end{aligned}$$

But,

$$\begin{aligned}
\psi_0 &= 2\sum_{s=1}^{m} \eta_s \omega_s && \text{(II.4a)}\\
\psi_1 &= 0 && \text{(II.4b)}\\
\psi_2 &= \frac{1}{3}\psi_0 && \text{(II.4c)}\\
\psi_3 &= 0 && \text{(II.4d)}\\
\psi_4 &= \frac{1}{5}\psi_0 && \text{(II.4e)}
\end{aligned}$$

So, from the equations (II.3a-II.3d), we get, respectively the following equations:

$$\mathcal{D}_1 = \frac{1}{x}(\psi_0 - \mathcal{D}_0) \tag{II.5a}$$

$$\mathcal{D}_2 = -\frac{1}{x^2}(\psi_0 - \mathcal{D}_0) \tag{II.5b}$$

$$\mathcal{D}_3 = \frac{\psi_2}{x} + \frac{1}{x^3}(\psi_0 - \mathcal{D}_0) \tag{II.5c}$$

$$\mathcal{D}_4 = -\frac{\psi_2}{x^2} - \frac{1}{x^4}(\psi_0 - \mathcal{D}_0) \tag{II.5d}$$

Now, from the equations (II.5a) and (II.5b), we get

$$\mathcal{D}_1 = -x \cdot \mathcal{D}_2 \tag{II.6}$$

Again, from the equations (II.5c) and (II.5d), we get

$$\mathcal{D}_3 = -x\mathcal{D}_4 \tag{II.7}$$

and from the equation (II.5c), using the equation (II.5b), we get

$$x\mathcal{D}_3 = \psi_2 - \mathcal{D}_2 \tag{II.8}$$

From the equations (II.6) and (II.7), we get

$$\mathcal{D}_1\mathcal{D}_4 = \mathcal{D}_2\mathcal{D}_3 \tag{II.9}$$

Multiplying the both sides of the equation (II.5d) by $\mathcal{D}_0$, we get

$$\begin{aligned}\mathcal{D}_0\mathcal{D}_4 - \mathcal{D}_1\mathcal{D}_3 = {} & \frac{\psi_2(\psi_0 - \mathcal{D}_0)}{x^2} + \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2 + \psi_0\mathcal{D}_4 \\ & -\frac{\psi_2}{x^2}(\psi_0 - \mathcal{D}_0) - \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2\end{aligned} \tag{II.10}$$

and using the equation (II.5d), we get

$$\mathcal{D}_0\mathcal{D}_4 = \frac{\psi_2(\psi_0 - \mathcal{D}_0)}{x^2} + \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2 + \psi_0\mathcal{D}_4 \tag{II.11}$$

Therefore, the equations (II.11) and (II.10) gives

$$\mathcal{D}_0\mathcal{D}_4 - \mathcal{D}_1\mathcal{D}_3 = \psi_0\mathcal{D}_4 \tag{II.12}$$

From the equations (II.5c) and (II.5d), we get

$$\mathcal{D}_3\mathcal{D}_4 = -\frac{\psi_2^2}{x^3} - \frac{2\psi_2}{x^5}(\psi_0 - \mathcal{D}_0) - \frac{1}{x^7}(\psi_0 - \mathcal{D}_0)^2 \tag{II.13}$$

Multiplying the equation (II.5a) by the equation (II.8), we get

$$\mathcal{D}_0\mathcal{D}_3 - \mathcal{D}_1\mathcal{D}_2 = \psi_0\mathcal{D}_3 - \psi_2\mathcal{D}_1 \tag{II.14}$$

Now, from the equations (II.5a) and (II.6), we obtain,

$$\mathcal{D}_0\mathcal{D}_2 - \mathcal{D}_1^2 = \psi_0\mathcal{D}_2 \tag{II.15}$$

Again, multiplying the equations (II.5c) and (II.7), we get

$$\mathcal{D}_2\mathcal{D}_4 - \mathcal{D}_3^2 = \psi_2\mathcal{D}_4 \tag{II.16}$$

From the equations (II.5a) and (II.5c), we get

$$\mathcal{D}_1\mathcal{D}_3 = \frac{1}{x}(\psi_0 - \mathcal{D}_0)\left\{\frac{\psi_2}{x} + \frac{1}{x^3}(\psi_0 - \mathcal{D}_0)\right\}$$

i.e.

$$\mathcal{D}_1\mathcal{D}_3 = \frac{\psi_2}{x^2}(\psi_0 - \mathcal{D}_0) + \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2 \tag{II.17}$$

and from the equations (II.5b), we get

$$\mathcal{D}_2^2 = \left\{-\frac{1}{x^2}(\psi_0 - \mathcal{D}_0)\right\}^2 = \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2 \tag{II.18}$$

So, from the equations (II.17) and (II.18), we get

$$\mathcal{D}_2^2 - \mathcal{D}_1\mathcal{D}_3 = \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2 - \frac{\psi_2}{x^2}(\psi_0 - \mathcal{D}_0) - \frac{1}{x^4}(\psi_0 - \mathcal{D}_0)^2$$

i.e.

$$\mathcal{D}_2^2 - \mathcal{D}_1\mathcal{D}_3 = -\frac{\psi_2}{x^2}(\psi_0 - \mathcal{D}_0)$$

i.e.

$$\mathcal{D}_2^2 - \mathcal{D}_1\mathcal{D}_3 = \psi_2 \left\{ -\frac{1}{x^2} (\psi_0 - \mathcal{D}_0) \right\}$$

Therefore , using the equation (II.5b), we get

$$\mathcal{D}_2^2 - \mathcal{D}_1\mathcal{D}_3 = \psi_2\mathcal{D}_2 \tag{II.19}$$

Thus, we have obtained, as in the equations (II.9,II.12, II.14 , II.16, II.19, and II.19), the following relations :

$$\mathcal{D}_1\mathcal{D}_4 = \mathcal{D}_2\mathcal{D}_3 \tag{II.20a}$$
$$\mathcal{D}_0\mathcal{D}_4 - \mathcal{D}_1\mathcal{D}_3 = \psi_0\mathcal{D}_4 \tag{II.20b}$$
$$\mathcal{D}_0\mathcal{D}_3 - \mathcal{D}_1\mathcal{D}_2 = \psi_0\mathcal{D}_3 - \psi_2\mathcal{D}_1 \tag{II.20c}$$
$$\mathcal{D}_0\mathcal{D}_2 - \mathcal{D}_1^2 = \psi_0\mathcal{D}_2 \tag{II.20d}$$
$$\mathcal{D}_2\mathcal{D}_4 - \mathcal{D}_3^2 = \psi_2\mathcal{D}_4 \tag{II.20e}$$
$$\mathcal{D}_4\mathcal{D}_6 - \mathcal{D}_5^2 = \psi_4\mathcal{D}_6 \tag{II.20f}$$
$$\mathcal{D}_2^2 - \mathcal{D}_1\mathcal{D}_3 = \psi_2\mathcal{D}_2 \tag{II.20g}$$

Again, by virtue of the relations (II.4a, II.4c, II.4c), the above seven relations can also be reduced further.

Appendix III

An Important Identity

$$\lim_{n \to \infty} \left(\sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} - \sum_{\alpha=1}^{n} \mu_{(r)i} \right) = \frac{1}{2C\sqrt{M}} \sum_{r=1}^{m} \eta_r \omega_r \alpha_{(r)1}$$

We construct the function

$$s_r(\mu) = \sum_{\alpha=1}^{n} \frac{l_\alpha}{1 - k_{(r)\alpha}\zeta_r \mu} + 1 \tag{III.1}$$

where $k'_\alpha s$ $(\alpha = 1, 2, \cdots, n)$ are the positive roots of the characteristic equation and l_α $'s$ $(\alpha = 1, 2, \cdots, n)$ are the n constants to be determined from the condition

$$s_r\left(\mu_{(r)i}\right) = 0\ ;\ i = 1, 2, \cdots, n \tag{III.2}$$

Now,

$$\begin{aligned} s_r(\mu) R_r(\mu) &= \prod_{\alpha=1}^{n} (1 - k_\alpha \zeta_r \mu) \sum_{\alpha=1}^{n} \frac{l_\alpha}{1 - k_\alpha \zeta_r \mu} \\ &\quad + \prod_{\alpha=1}^{n} (1 - k_\alpha \zeta_r \mu) \end{aligned}$$

i.e.

$$s_r(\mu) R_r(\mu) = (-1)^n k_1 \zeta_r \cdot k_2 \zeta_r \cdot \ \cdots \ k_n \zeta_r \cdot P_r(\mu)$$

i.e.

$$s_r(\mu) = (-1)^n k_1 \zeta_r \cdot k_2 \zeta_r \cdot \ \cdots \ k_n \zeta_r \cdot \frac{P_r(\mu)}{R_r(\mu)}$$

i.e.

$$s_r(\mu) = \sqrt{M} H_r(-\mu) \tag{III.3}$$

Again

$$\begin{aligned} & (1 - k_\alpha \zeta_r \mu) \, s_r(\mu) \\ = \ & (1 - k_\alpha \zeta_r \mu) \left(\sum_{\alpha=1}^{n} \frac{l_\alpha}{1 - k_\alpha \zeta_r \mu} + 1 \right) \\ \rightarrow \ & l_\alpha \ \text{as} \ \ \mu \rightarrow (k_\alpha \zeta_r)^{-1} \end{aligned}$$

$$\therefore \ (1 - k_\alpha \zeta_r \mu) \, s_r(\mu) \rightarrow l_\alpha \ \text{as} \ \ \mu \rightarrow (k_\alpha \zeta_r)^{-1}$$

So,.

$$l_\alpha = \lim_{\mu \to (k_\alpha \zeta_r)^{-1}} (1 - k_\alpha \zeta_r \mu)\, s_r(\mu)$$

i.e.

$$\frac{l_\alpha}{k_\alpha \zeta_r} = (-1)^n k_1\zeta_r \cdot k_2\zeta_r \cdot \cdots k_n\zeta_r \cdot \frac{1}{k_\alpha \zeta_r} \frac{P_r\left(\frac{1}{k_\alpha\zeta_r}\right)}{R_{(r)\alpha}\left(\frac{1}{k_\alpha\zeta_r}\right)}$$

i.e.

$$\sum_{\alpha=1}^{n} \frac{l_\alpha}{k_\alpha \zeta_r} = (-1)^n k_1\zeta_r . k_2\zeta_r \ldots k_n\zeta_r f_r(0) \qquad \text{(III.4)}$$

where

$$f_r(x) = \sum_{\alpha=1}^{n} \frac{P_r(1/k_\alpha\zeta_r)}{R_{(r)\alpha}(1/k_\alpha\zeta_r)} . \frac{1}{k_\alpha\zeta_r} R_{(r)\alpha}(x) \qquad \text{(III.5)}$$

Now, $f_r(x)$ is a dègree of polynomial $(n-1)$ in x which takes the values $\frac{1}{k_\alpha\zeta_r} P_r(1/k_\alpha\zeta_r)$ for $x = \frac{1}{k_\alpha\zeta_r}$, $\alpha = 1, 2, \cdots, n$.

So,

$$xP_r(x) - f_r(x) = 0 \text{ for } x = \frac{1}{k_\alpha\zeta_r}, \ \alpha = 1, 2, \cdots, n \qquad \text{(III.6)}$$

This helps us to conclude that the polynomial on the left hand side of the above equation must divide the polynomial $R_r(x)$. Hence, we get the following relation:

$$f_r(x) = xP_r(x) - (A_r x + B_r)\, R_r(x) \qquad \text{(III.7)}$$

Putting $x = 0$, we get

$$-f_r(0) = B_r R_r(0) \tag{III.8}$$

and comparing the co-efficient of x^{n+1} and x^n, we get

$$1 + (-)^{n+1} A_r \cdot k_1\zeta_r \cdot k_2\zeta_r \cdot \cdots k_n\zeta_r = 0 \tag{III.9a}$$

$$-\sum_{i=1}^{n} \mu_{(r)i} + (-1)^n k_1\zeta_r \cdot k_2\zeta_r \cdot \cdots k_n\zeta_r A_r \sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} + (-1)^{n-1} k_1\zeta_r \cdot k_2\zeta_r \cdot \cdots k_n\zeta_r B_r = 0 \tag{III.9b}$$

giving

$$B_r = \frac{(-1)^{n-1}}{k_1\zeta_r \cdot k_2\zeta_r \cdot \cdots k_n\zeta_r} \left(\sum_{i=1}^{n} \mu_{(r)i} - \sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} \right) \tag{III.10}$$

So,

$$f_r(0) = (-1)^{n+1} \frac{1}{\sqrt{M}} \left(\sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} - \sum_{i=1}^{n} \mu_{(r)i} \right) \mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n} \tag{III.11}$$

Therefore

$$\sum_{\alpha=1}^{n} \frac{l_\alpha}{k_\alpha \zeta_r} = \sum_{i=1}^{n} \mu_{(r)i} - \sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} \tag{III.12}$$

But $\frac{l_\alpha}{k_\alpha \zeta_r}$ can also be expressed in terms of $H_r(\mu)$. We consider

$$\sum_{i=1}^{n} a_i \mu_{(r)i} s\left(-\mu_{(r)i}\right)$$

Since $s\left(\mu_{(r)i}\right) = 0 \quad (i = 1, 2, \cdots, n)$, we can extend the summation for the negative values of i also. Hence

$$\sum_{i=1}^{n} a_i \mu_{(r)i} s\left(-\mu_{(r)i}\right) = \sum_{i=-n}^{+n} a_i \mu_{(r)i} s\left(-\mu_{(r)i}\right) \tag{III.13}$$

Now substituting the expressions for $s\left(-\mu_{(r)i}\right)$ from the equation (III.1) in the above relation we obtain

$$\sum_{i=1}^{n} a_i \mu_{(r)i} s\left(-\mu_{(r)i}\right) = \sum_{i=-n}^{+n} a_i \mu_{(r)i} \left(\sum_{\alpha=1}^{n} \frac{l_\alpha}{1 + k_\alpha \zeta_r \mu} + 1\right)$$

Then using the equations (III.3) and (III.12), we get

$$\sum_{i=1}^{n} a_i \mu_{(r)i} s\left(-\mu_{(r)i}\right) = \left(\sum_{i=-n}^{+n} a_i \mu_{(r)i} \sum_{\alpha=1}^{n} \frac{l_\alpha}{1 + k_\alpha \zeta_r \mu} + \sum_{i=-n}^{+n} a_i \mu_{(r)i}\right)$$

i.e.

$$\sqrt{M} \sum_{i=1}^{n} a_i \mu_{(r)i} H_r\left(\mu_{(r)i}\right) = \sum_{\alpha=1}^{n} l_\alpha \sum_{i=-n}^{+n} \frac{a_i \mu_{(r)i}}{1 + k_\alpha \zeta_r \mu}$$

i.e.

$$\sqrt{M} \sum_{i=1}^{n} a_i \mu_{(r)i} H_r\left(\mu_{(r)i}\right) = \sum_{\alpha=1}^{n} \frac{l_\alpha}{k_\alpha \zeta_r} \sum_{i=-n}^{+n} \frac{a_i k_\alpha \zeta_r \mu_{(r)i}}{1 + k_\alpha \zeta_r \mu}$$

i.e.

$$\sum_{r=1}^{m}\eta_r\omega_r\sqrt{M}\sum_{i=1}^{n}a_i\mu_{(r)i}H_r\left(\mu_{(r)i}\right)=\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=-n}^{+n}\frac{l_\alpha}{k_\alpha\zeta_r}\sum_{\alpha=1}^{n}\frac{a_ik_\alpha\zeta_r\mu_{(r)i}}{1+k_\alpha\zeta_r\mu}$$

i.e.

$$\sum_{i=-n}^{+n}\frac{l_\alpha}{k_\alpha\zeta_r}\sum_{r=1}^{m}\eta_r\omega_r\sum_{\alpha=1}^{n}\frac{a_ik_\alpha\zeta_r\mu_{(r)i}}{1+k_\alpha\zeta_r\mu}=\sqrt{M}\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=1}^{n}a_i\mu_{(r)i}H_r\left(\mu_{(r)i}\right)$$

i.e.

$$\sum_{\alpha=1}^{n}\frac{l_\alpha}{k_\alpha\zeta_r}\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=-n}^{+n}a_i\left(1-\frac{1}{1+k_\alpha\zeta_r\mu}\right)$$
$$=\sqrt{M}\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=1}^{n}a_i\mu_{(r)i}H_r\left(\mu_{(r)i}\right)$$

i.e.

$$\sum_{\alpha=1}^{n}\frac{l_\alpha}{k_\alpha\zeta_r}\left(\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=-n}^{+n}a_i-\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=-n}^{+n}\frac{a_i}{1+k_\alpha\zeta_r\mu}\right)$$
$$=\sqrt{M}\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=1}^{n}a_i\mu_{(r)i}H_r\left(\mu_{(r)i}\right)$$

i.e.

$$\sum_{\alpha=1}^{n}\frac{l_\alpha}{k_\alpha\zeta_r}\left(\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=-n}^{+n}a_i-\sum_{r=1}^{m}\eta_r\omega_r\sum_{i=-n}^{+n}\frac{a_i}{1+k_\alpha\zeta_r\mu}\right)$$
$$=\sqrt{M}\sum_{i=1}^{n}a_i\mu_{(r)i}H_r\left(\mu_{(r)i}\right)$$

i.e.

$$\sum_{\alpha=1}^{n} \frac{l_\alpha}{k_\alpha \zeta_r} (\psi_0 - D_0) = \sqrt{M} \sum_{r=1}^{m} \eta_r \omega_r \sum_{i=1}^{n} a_i \mu_{(r)i} H_r \left(\mu_{(r)i}\right)$$

i.e.

$$-\sum_{\alpha=1}^{n} \frac{l_\alpha}{k_\alpha \zeta_r} \left(\frac{2C - \psi_0}{2C}\right) = \frac{\sqrt{M}}{2C} \sum_{r=1}^{m} \eta_r \omega_r \sum_{i=1}^{n} a_i \mu_{(r)i} H_r \left(\mu_{(r)i}\right)$$

i.e.

$$-M \sum_{\alpha=1}^{n} \frac{l_\alpha}{k_\alpha \zeta_r} = \frac{\sqrt{M}}{2C} \sum_{r=1}^{m} \eta_r \omega_r \sum_{i=1}^{n} a_i \mu_{(r)i} H_r \left(\mu_{(r)i}\right)$$

i.e.

$$\sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} - \sum_{\alpha=1}^{n} \mu_{(r)i} = \frac{\sqrt{M}}{2CM} \sum_{r=1}^{m} \eta_r \omega_r \sum_{i=1}^{n} a_i \mu_{(r)i} H_r \left(\mu_{(r)i}\right)$$

Now, allowing $n \to \infty$, $H_r(\mu)$ becomes the solution of the equation (I.6) of Appendix- I which is bounded in the half plane $\mathbb{R} \geq 0$ and therefore, we get

$$\lim_{n \to \infty} \left(\sum_{\alpha=1}^{n} \frac{1}{k_\alpha \zeta_r} - \sum_{\alpha=1}^{n} \mu_{(r)i} \right) = \frac{1}{2C\sqrt{M}} \sum_{r=1}^{m} \eta_r \omega_r \alpha_{(r)1}$$

where

$$\alpha_{(r)1} = \int_0^1 \mu H_r(\mu)\, d\mu \tag{III.14}$$

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INTERLOCKING PROBLEM IN ANISOTROPICALLY SCATTERING MEDIA (Planetary Phase Function)

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The equation of transfer for the interlocked multiplets with planck-function as a linear function of optical depth in anisotropically scattering media with planetary phase functions is solved by the method of discrete-ordinates an attempt has also been made for getting the exact solution by infinite approximation.

INTRODUCTION : Woolley and Stibbs [1] applied the theory of absorption lines by coherent scattering to the case of interlocking without redistribution to deduce the equation of transfer for interlocked triplets in the Milne-Eddigton model and they solved the problem by Eddington approximation method.

The same problem with the linear Planck-function has been solved by (i) Busbridge and Stibbs [2] by the method of principal of invariance governing the law of diffuse reflection with a silght modification; (ii) Dasgupta and Karanjai [5] by applying Sobolev's probabilistic method; (iii) Dasgupta [4] by Laplace Transform and Wiener Hopf technique, (iv) Karanjai and Barman [7] by using the extension of the method of discrete-ordinates.

Karanjai and Karanjai [8] and Deb, Biswas and Karanjai [6] solved the same problem with non-linear Planck-function.

Here, we have solved, the equation of transfer for interlocked multiplets in anisotropically scattering media. The phase function considered here is Planetary phase function.

FORMULATION OF THE PROBLEM : The equation transfer and the boundary condition:

We take the equaiton of transfer for r^{th} interlocked line in the form :

$$\mu \frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r) I_r(\tau,\mu) - (1+\varepsilon\eta_r) B_\nu(T) - \frac{1}{2}(1-\varepsilon)\alpha_r \sum_{p=1}^{k} \eta_p \int_{-1}^{+1} p(\mu,\mu') I_p(\tau,\mu')\, d\mu' \quad \ldots(1)$$

where $\alpha_r = \eta_r \Big/ \sum_{p=1}^{k} \eta_p$, $r = 1, 2, \ldots, k$ so that $\sum_{r=1}^{k} \alpha_r = 1$

the Planck-function $B_\nu(T)$, consider in this case, is of the form :

$$B_\nu(T) = B(\tau) = b_0 + b_1 \tau$$

b_0 and b_1, being positive constants and the (azimuth independent) Planetary phase function $p(\mu, \mu')$, taken here, is given by $p(\mu,\mu') = 1 + W_1 \mu \mu'$

In the above equation, τ denotes the optical depth and $\eta_1 = K_1/K$. K_1 denotes the absorption co-efficient for the r^{th} interlocked line and K denotes the continuous absorption which is

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supposed to be constant for each line. ε, the co-efficient, which is introduced to allow for thermal emission associated with the line-absorption, and $B_v(T)$, the Planck-function, are considered to be constant for each line.

So, the Eq. (1) becomes

$$\mu \frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)\, I_r(\tau,\mu) - (1+\varepsilon\eta_r)\, B_v(T)$$

$$-\frac{1}{2}(1-\varepsilon)\alpha_r \sum_{p=1}^{k} \eta_p \int_{-1}^{+1} (1+W_l\mu\mu')\, I_p(\tau,\mu')\, d\mu' \qquad \ldots(2)$$

The boundary conditions for solving the Eq. (2) are

$$I_r(0,-\mu) = 0, \quad (0<\mu\le 1) \qquad \ldots(3)$$

and $$I_r(\tau,\mu)\, e^{-\tau/\mu} \to 0 \qquad \text{as } \tau\to\infty$$

i.e., $I_r(\tau,\mu)$ is atmost linear in τ as τ tends to infinity. ...(4)

SOLUTION OF THE EQUATION : We observe that if we assume, like Busbridge and Stibbs [1], that one of the solution of the Eq. (2) to be

$$I_r(\tau,\mu) = b_0 + b_1\left(\tau + \frac{\mu}{1+\eta_r}\right) + I_r^*(\tau,\mu) \qquad \ldots(5)$$

which consists of two parts, the first part being the solution for an infinitely unbounded atmosphere as τ tends to infinity and the second part $I_r^*(\tau,\mu)$ being the departure of the asymptotic solution from the value $I_r(\tau,\mu)$ as we approach the boundary $\tau=0$

Now, writing

$$\zeta_r = \frac{1}{1+\eta_r} \quad \text{and} \quad \omega_r = \frac{(1-\varepsilon)\,\eta_r}{1+\eta_r}$$

and using the Eq. (5), the Eq. (2) can be reduced to the form:

$$\zeta_r\mu \frac{dI_r^*(\tau,\mu)}{d\tau} = I_r^*(\tau,\mu) - \frac{1}{2}\omega_r \cdot \frac{1}{\sum_{p=1}^{k}\eta_p}\left\{\sum_{p=1}^{k}\eta_p \int (1+W\mu\mu')\, I_p^*(\tau,\mu')\, d\mu'\right.$$

$$\left. -\frac{1}{3}\omega_r b_l W_l \frac{1}{\sum_{p=1}^{k}\eta_p}\left\{\sum_{p=1}^{k}\left(\frac{\eta_p}{1+\eta_p}\right)\right\}\mu \qquad \ldots(6)$$

and the boundary conditions can be transformed to:

$$I_r^*(0,-\mu) = b_1\zeta_r\mu - b_0, \text{ where } 0<\mu\le 1 \qquad \ldots(7)$$

and

$I_r^*(\tau,\mu)$ is atmost linear in τ as τ tends to infinity. ...(8)

In the n^{th} approximation, we replace the integro-differential Eq. (6) by the system of $2n$ linear differential equations :

$$\zeta_r \mu_{(r)_i} \frac{dI^*_{(r)}}{dr} = I^*_{(r)_i} - \frac{1}{2}\omega_r \frac{1}{\sum_{p=1}^{k} \eta_p} \sum_{p=1}^{k} \eta_p \sum_j (1 + W_{\mu_{(r)_i}} \mu_{(p)_j}) I^*_{(p)_j} a_j$$

$$-\frac{1}{3}\omega_r b_1 W \left\{ \frac{\sum_{p=1}^{k} \left(\frac{\eta_p}{1+\eta_p} \right)}{\sum_{p=1}^{k} \eta_p} \right\} \mu_{(r)_i} \qquad \ldots(9)$$

where the symbol $I^*_{(r)_i}$, for brevity, is used for $I_r^*(\tau, \mu_{(r)_i})$, $\mu_{(r)_i}$'s ($i = 1, 2, \ldots, n$; assuming that $\mu_{(r)i} = -\mu_{(r)i}$) are the zeros of the Legendre polynomial $P_{2n}(\mu)$ which are independent on the lines of interlocking and α_j's ($j = 1, 2, \ldots, n$; having the property that $\alpha_{-j} = \alpha_j$) are the corresponding Gaussian Weights. However, it is to be noted that there is no term with $j = 0$. For the simplicity we have used here $I^*_{(r)_i}$ for $I_r^*(\tau, \mu)$ in the Eq. (9).

Now, we try to get the solution of the equation :

$$\zeta_r \mu_{(r)_i} = \frac{dI^*_{(r)_i}}{d\tau} = I^*_{(r)_i} - \frac{1}{2}\omega_r \frac{1}{\sum_{p=1}^{k} \eta_p} \sum_{p=1}^{k} \eta_p \sum_j (1 + W\mu_{(r)_i} \mu_{(p)_j}) I^*_{(p)_j} a_j \qquad \ldots(10)$$

The system of Eq. (10) admits integrals of the form :

$$I^*_{(r)_i} = g_{(r)_i} e^{-k\tau}, \quad i = \pm 1, \pm 2, \ldots, \pm n \qquad \ldots(11)$$

then using the Eq. (11) in Eq. (10), we get

$$g_{(r)_i}(1 + \zeta_r k \mu_{(r)_i}) = \frac{\omega_r}{2C} \sum_{p=1}^{k} \eta_k \sum_j (1 + W \mu_{(r)_i} \mu_{(p)_j}) g_{(p)_j} a_j \qquad \ldots(12)$$

Hence,

$$g_{(r)_i} = \omega_r \frac{\sigma \mu_{(r)_i} + \rho}{1 + \zeta_r k \mu_{(r)_i}} \qquad \ldots(13)$$

where σ and ρ are constants which are independent of $\mu_{(r)_i}$.

Now, defining

$$D_m(x) = \sum_{p=1}^{k} \eta_p \omega_p \sum_j \frac{a_j \mu^m_{(r)_j}}{1 + \mu_{(r)_j} x} \qquad \ldots(14)$$

and, for simplification, replacing $D_m(\zeta_p k)$ by D_m, we can write the equation, obtained by using the Eq. (14) in the Eq. (13), in the compact form:

$$\sigma \mu_{(r)_i} + \rho = \frac{1}{2C} \{\sigma D_1(\zeta_p k) + \rho D_0 (\zeta_p k)\} + W \mu_{(r)_i} \{\sigma D_2 (\zeta_p k) + \rho D_1 (\zeta_p k)\} \ldots(15)$$

where $C = \sum_{p=1}^{k} \eta_p$

The Eq. (15), as it is true for all $\mu_{(r)_i}$, will produce the following two relations:

$$(2C - W D_2)\sigma - W \rho D_1 = 0 \quad \text{and} \quad \sigma D_1 + (D_0 - 2C) \rho = 0 \qquad \ldots(16)$$

which, on elimination of σ and ρ, give

$$2C - W D_0 D_2 - 4C^2 + 2W C D_2 + W D_1^2 = 0 \qquad \ldots(17)$$

which, by using the result (48) and (49) of Karanjai and Barman [7], becomes equivalent to :

$$\overset{k}{\underset{p=1}{\sigma}} \eta_p \omega_p \sum_j \frac{a_j}{2C(1+\zeta_r k \mu_{(p)_j})} (1 + M W \mu_{(p)_j}^2) = 1 \qquad \ldots(18)$$

where

$$M = \left(\sum_{p=1}^{k} \eta_p (1 - \omega_p) \right) \Big/ \sum_{p=1}^{k} \eta_p = \frac{2C - \psi_0}{2C} \quad \text{and} \quad \psi_m = \sum_{p=1}^{k} \eta_p \omega_p \sum_j a_j \mu_{(p)_j}^m$$

The Eq. (18) is the **characteristic equation** which is an equation in K of order $2n$ and it will give $2n$ distinct non-zero roots which occur in pair as $\pm k_{(r)_\alpha}$, $(\alpha = 1, 2, \ldots, n)$, if $\omega_r < 1$.

Now, from two equations in (16), we can construct the relation: $\sigma = -\dfrac{M W}{\zeta_p K} \rho$

which, on using in the Eq. (13) and using the result in the Eq. (11), we get $2n$ independent integrals, corresponding to $2n$ distinct non-zero roots of the characteristic Eq. (18) which occur in pair as $\pm k_{(r)\alpha}$, $(\alpha = 1, 2, \ldots, n)$, of the form

$$I^*_{(r)_i} = \omega_r \rho \frac{\zeta_p k_{(p)_\alpha} \mp M W \mu_{(r)_i}}{(1 \pm \zeta_p k \mu_{(r)_i}) \zeta_p k_{(p)_\alpha}} . e^{\mp k_{(r)} \alpha^\tau}, \alpha = 1, 2, \ldots, n; i = \pm 1, \ldots, \pm n \qquad \ldots(19)$$

To get the complete solution of the Eq. (9), we require a particular integral which can be obtained as follows :

$$I^*_{(r)_i} = \omega_r A h_{(r)_i} \mu_{(r)_i}, \qquad i = \pm 1, \ldots \pm n \qquad \ldots(20)$$

where
$$-A = \frac{1}{3} b_1 \omega W \left\{ \frac{\sum_{p=1}^{k} \left(\frac{\eta_p}{1+\eta_p} \right)}{\sum_{p=1}^{k} \eta_p} \right\}$$

Now, using the Eq. (20) in the Eq. (9), we get the relation :

$$h_{(r)_i} \mu_{(r)_i} = \frac{1}{2C} \left\{ \sum_{p=1}^{k} \eta_p \omega_p \sum_j (1 + W \mu_{(r)_i} \mu_{(p)_j}) h_{(p)_j} \mu_{(p)_j} \right\} - \mu_{(r)_i} \qquad \ldots(21)$$

which is of the form :

$$h_{(r)_i} = \gamma - 1 + \frac{\delta}{\mu_{(r)_i}} \qquad \ldots(22)$$

where γ and δ are constants which are independent of $\mu_{(r)_i}$. Putting this expression for $h_{(r)_i}$ in the Eq. (21) we shall get a relation which is valid for all $\mu_{(r)_i}$ producing two equations which give γ and δ. Using the Eq. (22) with these γ and δ in Eq. (20), we get the required particular integral as :

$$I^*_{(r)_i} = \frac{1}{3} \omega_r b_1 W N \mu_{(r)_i} \qquad \ldots(23)$$

where
$$N = \left\{ \sum_{p=1}^{k} \left(\frac{\eta_p}{1+\eta_p} \right) \right\} \Big/ \left\{ \sum_{p=1}^{k} \eta_p \left(1 - \frac{1}{3} \omega_p W \right) \right\}$$

The general solution (23) of the Eq. (10) together with the particular integral (23) will constitute the complete solution of the Eq. (9).

According to Chandrasekhar [3], the solution of the Eq. (9), satisfying the boundary conditions (14) can be put in the form :

$$I^{*}_{(r)_i} = \frac{1}{3}\,\omega_r\, b_1 \left\{ \sum_{\alpha=1}^{n} \frac{(\zeta_p\, k_{(p)_\alpha} - M\,W\,\mu_{(r)_i})}{(1+\zeta_p\, k_{(p)_\alpha}\mu_{(r)_i})\,\zeta_p\, k_{(p)_\alpha}} \cdot L_{(r)_\alpha} e^{-k_{(p)}\,\alpha^{\tau}} + W\,N\,\mu_{(r)_i} \right\} \qquad \ldots(24)$$

where $K_{(p)_\alpha}$'s ($\alpha = 1, 2, \ldots, n$) are the positive roots of the characteristic Eq. (27) and $L_{(r)_\alpha}$'s are the constants of the integration to be determined by the boundary condtions (7)

i.e.,

$$I^{*}_{(r)_{-i}} = b_1\,\zeta_r\,\mu_{(r)_i} - b_0, \quad \text{where} \quad 0 < \mu_{(r)_i} \le 1 \qquad \ldots(25)$$

THE ELEMINTATION OF THE CONSTANTS AND THE EXPRESSION FOR THE LAW OF DIFFUSE REFLECTION IN CLOSED FORM : From the Eq. (25) we can write

$$I_r^{*}(\tau, \mu) = \frac{1}{3}\,\omega_r\, b_1 \left\{ \sum_{\alpha=1}^{n} \frac{(\zeta_p\, k_{(p)_\alpha} - M\,W\,\mu)}{(1+\zeta_p\, k_{(p)_\alpha}\,\mu)\,\zeta_p\, k_{(p)_\alpha}} \cdot L_{(r)_\alpha} e^{-k_{(p)}\,\alpha^{\tau}} + W\,N\,\mu \right\} \qquad \ldots(26)$$

Now we define

$$S_r(\mu) = \sum_{\alpha=1}^{n} \frac{(\zeta_p\, k_{(p)_\alpha} - MW\mu)\, L_{(r)_\alpha}}{(1+\zeta_p\, k_{(p)_\alpha}\,\mu)\zeta_p\, k_{(p)_\alpha}} - WN\mu - \frac{3\zeta_r\,\mu}{\omega_r} + \frac{3b_0}{\omega_r\, b_1} \qquad \ldots(27)$$

Then the boundary conditions (25) are expressible in the form :

$$S_r(\mu_{(r)_i}) = 0 \qquad \ldots(28)$$

Again, we can express $I_r^{*}(0, \mu)$ in terms of $S_r(\mu)$ as follows :

$$I_r^{*}(0, \mu) = \frac{1}{3}\,\omega_r\, b_1 \left\{ S_r(\mu) - \frac{3\zeta_1\,\mu}{\omega_r} + \frac{3b_0}{\omega_r\, b_1} \right\} \qquad \ldots(29)$$

Now, we define two polynomials :

$$P_r(\mu) = \prod_{i=1}^{n} (\mu - \mu_{(r)_i}) \quad \text{and} \quad R_r(\mu) = \prod_{\alpha=1}^{n} (1 - \zeta_r k_{(p)_\alpha}\,\mu)$$

Then

$$S_r(\mu) = (-1)^{n+1}\left(NW + \frac{3\zeta_1}{\omega_r}\right) \zeta_r k_{(p)_1}\, \zeta_r k_{(p)_2}, \ldots, \zeta_r k_{(p)_n} \frac{P_r(\mu)}{R_r(\mu)} (\mu - \lambda_r) \qquad \ldots(30)$$

Moreover, we observe that

$$L_{(r)_\alpha} = \lim_{\mu \to (\zeta_r k_{(p)_\alpha})^{-1}} \left\{ \frac{1}{T_\alpha} (1 - \zeta_r\, k_{(p)_\alpha}\mu)\, S_r(\mu) \right\} \qquad \ldots(31)$$

provided that

$$T_\alpha = 1 + \frac{MW}{\zeta_r\,\zeta_p\, k^2_{(p)_\alpha}}$$

Therefore, using the Eq. (30) in the Eq. (31), we get

$$L_{(r)_\alpha} = (-1)^{n+1} \cdot \frac{1}{T_\alpha} \cdot \left(NW + \frac{3\zeta_r}{\omega_r}\right) \zeta_r\, k_{(p)_1}\, \zeta_r\, k_{(p)_2}, \ldots, \zeta_r\, k_{(p)_n} \frac{P_r\left(\dfrac{1}{\zeta_r\, k_{(p)_\alpha}}\right)}{R_{(r)_\alpha}\left(\dfrac{1}{\zeta_r k_{(p)_\alpha}}\right)} \times \left(\frac{1}{\zeta_r\, k_{(p)_\alpha}} - \lambda_r\right) \qquad \ldots(32)$$

where

$$R_{(r)_\alpha}(x) = \prod_{\beta(\neq\alpha)=1}^{n} (1 - \zeta_r k_{(p)_\beta} x), \qquad 1 \le \alpha \le n \quad \ldots(33)$$

Summing up both sides of the Eq. (32) over α, we get

$$\sum_{\alpha=1}^{n} L_{(r)_\alpha} = (-1)^{n+1}\left(NW + \frac{3\zeta_r}{\omega_r}\right)\zeta_r k_{(p)_1}\, \zeta_r k_{(p)_2}, \ldots, \zeta_r k_{(p)_n} f_r(0) \quad \ldots(34)$$

where

$$f_r(x) = \sigma_{\alpha=1}^{n} \frac{1}{T_\alpha} \frac{P_r\left(\frac{1}{\zeta_r k_{(p)_\alpha}}\right)}{R_{(r)_\alpha}\left(\frac{1}{\zeta_r k_{(p)_\alpha}}\right)}\left(\frac{1}{\zeta_r k_{(p)_\alpha}} - \lambda_r\right) R_{(r)_\alpha}(x) \quad \ldots(35)$$

Now, we observe that the polynomial $f_r(x)$ is of degree $(n-1)$ in x which takes the values :

$$\frac{1}{T_\alpha} P_r\left(\frac{1}{\zeta_r k_{(p)_\alpha}}\right)\left(\frac{1}{\zeta_r k_{(p)_\alpha}} - \lambda_r\right) \quad \text{for} \quad x = \frac{1}{\zeta_r k_{(p)_\alpha}}, \alpha = 1, 2, \ldots, n$$

So,

$$(1 + MWx^2) f_r(x) - P_r(x)(x - \lambda_r) = 0$$

for

$$x = \frac{1}{\zeta_r k_{(p)_\alpha}}, \alpha = 1, 2, \ldots, n \quad \ldots(36)$$

This helps us to conclude that polynomial on the left hand side of the Eq. (36) must be divisible by $R_r(x)$. Hence, we get the following relation :

$$(1 + M\, Wx^2) f_r(x) - P_r(x)(x - \lambda_r) = R_r(x)(A_r x + B_r) \quad \ldots(37)$$

where A_r and B_r are constants.

Now, assuming that $W \neq 0$, putting as $x = i/\sqrt{(MWt)}$ and $x = -i/\sqrt{(MWt)}$ in the Eq. (37) we can derive the values of A_r and B_r as ;

$$A_r = (-1)^{n+1}(a_r' - i\lambda_r \sqrt{MW} b_r')\, \mu_{(r)_1}, \ldots, \mu_{(r)_n} \quad \ldots(38)$$

and

$$B_r = (-1)^n \left(\lambda_r a_r' + i\frac{1}{\sqrt{MW}} b_r'\right) \mu_{r_1}, \ldots, \mu_{(r)_n} \quad \ldots(39)$$

where

$$a_r' = \frac{1}{2}\left\{H_r\left(+i\frac{1}{\sqrt{MW}}\right) + H_r\left(-i\frac{1}{\sqrt{MW}}\right)\right\} \quad \ldots(40)$$

$$b_r' = \frac{1}{2}\left\{H_r\left(+i\frac{1}{\sqrt{MW}}\right) - H_r\left(-i\frac{1}{\sqrt{MW}}\right)\right\} \quad \ldots(41)$$

in which the function $H_r(\mu)$ is defined as :

$$H_r(\mu) = \frac{1}{\mu_{(r)_1}\mu_{(r)_2}\cdots\mu_{(r)_n}} \cdot \frac{\prod_{i=1}^{n}(\mu + \mu_{(r)_i})}{\prod_{\alpha=1}^{n}(1 + \zeta_r k_{(p)_\alpha}\mu)} \quad \ldots(42)$$

Now, putting $x = 0$ in the Eq. (37), using the Eq. (39), we get

$$f_r(0) = (-1)^{n+1}\left\{(1 - a_r')\lambda_r - i\frac{1}{\sqrt{MW}} b_r'\right\} \mu_{(r)_1} \cdots \mu_{(r)_n} \quad \ldots(43)$$

So, using the Eq. (43) in the Eq. (34), we get

$$\sum_{\alpha=1}^{n} L_{(r)_\alpha} = \left(NW + \frac{3\zeta_r}{\omega_r}\right)\left\{(1-\alpha_r')\,\lambda_r - i\frac{1}{\sqrt{MW}}\,b_r'\right\}\zeta_r\,k_{(p)_1}\,\zeta_r\,k_{(p)_2},\ldots,\zeta_r\,k_{(p)_n}\times\mu_{(r)_1},\ldots,\mu_{r(n)} \qquad \ldots(44)$$

But the roots of the characteristic Eq. (18) obey the relation :

$$\zeta_r\,k_{(p)_1}\,\zeta_r\,k_{(p)_2},\ldots,\zeta_r\,k_{(p)_n}\,\mu_{(r)_1}\,\mu_{(r)_2},\ldots,\mu_{(r)_n} = \left\{M\left(1-\frac{1}{3}W_1\,(1-M)\right)\right\}^{1/2} \qquad \ldots(45)$$

Applying the relation (45) in the Eq. (44), we, therefore, obtain :

$$\sum_{\alpha=1}^{n} L_{(r)_\alpha} = \left(NW + \frac{3\zeta_r}{\omega_r}\right)\left\{(1-a_r')\,\lambda_r - i\frac{1}{\sqrt{MW}}\,b_r'\right\}\left\{M\left(1-\frac{1}{3}W_1\,(1-M)\right)\right\}^{1/2} \qquad \ldots(46)$$

Again, from the Eq. (30) and (45), we shall get

$$S_r(\mu) = \left(NW + \frac{3\zeta_r}{\omega_r}\right)\left\{M\left(1-\frac{1}{3}W_1(1-M)\right)\right\}^{1/2}(\mu-\lambda_r)\,H_r(-\mu) \qquad \ldots(47)$$

Now, by putting $\mu = 0$ in the Eq. (30) and (47), we can construct the equation ;

$$\sum_{\alpha=1}^{n} L_{(r)_\alpha} = \lambda_r\left(NW + \frac{3\zeta_r}{\omega_r}\right)\left\{M\left(1-\frac{1}{3}W_1(1-M)\right)\right\}^{1/2} - \frac{3b_0}{\omega_r\,b_1} \qquad \ldots(48)$$

Comparing the Eq. (46) and (48), we get

$$\lambda_r = \frac{b_0}{a_r'G_r} - i\frac{1}{\sqrt{MW}}\frac{b_r'}{a_r'} \qquad \ldots(49)$$

where

$$G_r = b_1\left(\zeta_r + \frac{1}{3}\,\omega_r\,WN\right)\left\{M\left(1-\frac{1}{3}W_1\,(1-M)\right)\right\}^{1/2} \qquad \ldots(50)$$

Now, from the Eq. (29), by using the Eq. (47), we can write

$$I_r^*(0,\mu) = G_r\,(\mu+\lambda_r)\,H_r\,(\mu) - b_1\zeta_r\mu - b_0 \qquad \ldots(51)$$

where λ_r and G_r are given by the Eqs. (49) and (50) respectively.

Now, from the Eq. (5), we get

$$I_r\,(0,\mu) = G_r\,(\mu+\lambda_r)\,H_r\,(\mu) \qquad \ldots(52)$$

The Eq. (51) will give diffusely reflected intensity $I_r^*(0,\mu)$ and the Eq. (52) will give the emergent intensity $I_r(0,\mu)$ in the n^{th} approximation.

THE EXACT DIFFUSELY REFLECTED INTENSITY AND THE EXACT SOLUTION FOR THE EMERGENT INTENSITY : Folloing Busbridge and Stibbs [2], we change the variables $\zeta_r\,\mu$ and $\zeta_p\,\mu'$ to x and x' respectively [consequently $\zeta_r\,\mu_{(r)_i}$ and $\zeta_p\,\mu_{(p)_j}$ to x_i and x_j respectively] to get, from the Eq. (18), that

$$2\sum_j \frac{a_j'\,\psi\,(x_j)}{1-k^2x_j^2} = 1 \qquad \text{with } a_j' = \zeta_r\,a_j \qquad \ldots(53)$$

where assuming that

$$\eta_1 > \eta_2 > \ldots > \eta_k \qquad \ldots(54)$$

so that

$$0 \le \zeta_1 < \zeta_2 < \ldots < \zeta_k \le 1 \qquad \ldots(55)$$

the characteristic function $\psi(x')$ is defined as

$$\psi(x') = \begin{cases} \sum_{p=1}^{k} \frac{\eta_p \omega_p}{\zeta_p} \left\{ \frac{1}{2C} \left(\frac{1}{\zeta_r} . MWx_j'^2 \right) \right\}, & \text{if } 0 \le x' < \zeta_1 \\ \sum_{p=r+1}^{k} \frac{\eta_p \omega_p}{\zeta_p} \left\{ \frac{1}{2C} \left(1 + \frac{1}{\zeta_r} . MWx_j'^2 \right) \right\}, & \text{if } \zeta_r < x' < \zeta_{r+1} \\ 0, & \text{if } \zeta_k \le x' \le 1 \end{cases} \qquad \ldots(56)$$

Then,

$$\int_0^1 \psi(x')\, dx' < \frac{1}{2} \qquad \ldots(57)$$

Therefore, following the theory of H-function, developed by Chandrasekhar [3], we shall be able to show, in the present case, that $H(x)$, where $x = \zeta_r \mu$ or equivalently $H_r(\mu)$, given by the Eq. (42) satisfies, in the limit of infinite approximation, the non-integral equation :

$$H(x) = 1 + xH(x) \int_0^1 \frac{\psi(x')}{x + x'} H(x')\, dx', \; x' = \zeta_r \mu'$$

i.e.,

$$H_r(\mu) = 1 + \zeta_r \mu\, H_r(\mu) \int_0^1 \frac{\zeta_p \psi(\zeta_p \mu')}{\zeta_r \mu + \zeta_p \mu'} H_p(\mu)\, d\mu' \qquad \ldots(58)$$

which is bounded in the entire half plane $R(x) \ge 0$.

The characteristic function $\psi(x')$ in the Eq. (58) satisfies the necessary condition :

$$\int_0^1 \psi(x')\, dx' \le \frac{1}{2} \qquad \ldots(59)$$

Now, we allow n to tend to infinity for both the Eqs. (51) and (52) to get the exact diffusely reflected intensity $I_r^*(0, \mu)$ and the exact emergent intensity $I_r(0, \mu)$, given by :

$$I_r^*(0, \mu) = G_r(\mu + \lambda_r) H_r(\mu) - b_1 \zeta_r \mu - b_0 \qquad \ldots(60)$$

and

$$I_r(0, \mu) = G_r(\mu + \lambda_r) H_r(\mu) \qquad \ldots(61)$$

where λ_r and G_r are given by the eqs. (49) and (50) and the H-function $H_r(\mu)$ is the solution of the Eq. (58).

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Anisotropic coherent scattering with planetary phase function

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Abstract

The diffusely reflected intensity and emergent intensity for the case of coherent anisotropic scattering are derived in the nth approximation from a radiative transfer equation with linear Planck function and planetary phase function. This is then made exact allowing n tending to infinity.

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Keywords: Discrete ordinate method; Planetary phase function; Linear Planck function; Characteristic equation; Emergent intensity

1. Introduction

Woolley and Stibbs (1953) and Chandrasekhar (1960) gave the equation of the radiative transfer for the case of coherent scattering. Woolley and Stibbs (1953) solved it by Eddington method and Strömgren method, Chandrasekhar (1960) solved it with linear form of Planck function by the discrete ordinate method, Busbridge (1953) by using principle of invariance.

The same equation of radiative transfer was solved by Karanjai and Deb (1992) with an exponential form of Planck function by discrete ordinate method and again by Karanjai and Deb (1991) using Eddington's approximation method. Ghosh and Karanjai (2004,2006) solved the equation of transfer for coherent anisotropic scattering with different phase functions by double interval spherical harmonic method.

Here, we have solved the problem for the case of coherent anisotropic scattering by Chandrasekhar's discrete ordinate method where scattering occur in accordance with planetary phase function.

2. Equation of transfer and its boundary conditions

2.1. Equation of transfer

The equation of transfer suitable for this problem is

$$\mu\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau}=(1+\eta)I(\tau,\mu)-\frac{(1-\varepsilon)\eta}{2}\int_{-1}^{+1}p(\mu,\mu')\times I(\tau,\mu')\,\mathrm{d}\mu'-(1+\varepsilon\eta)B_v(T), \tag{1}$$

in which all the symbols are brought from Woolley and Stibbs (1953).

Here, we have taken the phase function $p(\mu,\mu')$, given by

$$p(\mu,\mu')=1+\varpi\mu\mu' \tag{2}$$

and a linear form of Planck function $B_v(T)$, given by

$$B_v(T)=b_0+b_1\tau. \tag{3}$$

With these two, Eq. (1) is expressible as

$$\zeta\mu\cdot\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau}=I(\tau,\mu)-\frac{1}{2}\,\omega\int_{-1}^{+1}(1+\varpi\mu\mu')\times I(\tau,\mu')\,\mathrm{d}\mu'-(1-\omega)(b_0+b_1\tau), \tag{4}$$

where ζ and ω are

$$\zeta=\frac{1}{1+\eta} \tag{5}$$

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and

$$\omega = \frac{(1-\varepsilon)\eta}{(1+\eta)}. \tag{6}$$

Now changing the variable τ to t as follows:

$$\tau = \zeta t \tag{7}$$

Eq. (4) can be reduced to the form:

$$\mu\frac{\mathrm{d}\Phi(t,\mu)}{\mathrm{d}\tau} = \Phi(t,\mu) - \frac{1}{2}\,\omega\int_{-1}^{+1}(1+\varpi\mu\mu') \times \Phi(t,\mu')\,\mathrm{d}\mu' - (1-\omega)(b_0 + b_1\zeta t), \tag{8}$$

where

$$\Phi(t,\mu) = I(\zeta t,\mu). \tag{9}$$

In nth approximation,

$$\mu_i\cdot\frac{\mathrm{d}\Phi_i}{\mathrm{d}\tau} = \Phi_i - \frac{1}{2}\,\omega\sum_j(1+\varpi\mu_i\mu_j)\Phi_j a_j - (1-\omega)(b_0+b_1\zeta t), \quad \Phi_j = \Phi(t,\mu_j). \tag{10}$$

2.2. *Reduction of Eq. (10) to a standard form*

Now, let

$$\Phi(t,\mu) = b_0 + \frac{b_1}{1+\eta}(t+\mu) + \Phi^*(t,\mu)$$

i.e.

$$\Phi_i = b_0 + b_1\zeta t + b_1\mu_i\zeta + \Phi_i^* \tag{11}$$

be a solution of Eq. (10).
Then

$$\mu_i\frac{\mathrm{d}\Phi_i^*}{\mathrm{d}t} = \Phi_i^* - \frac{1}{2}\,\omega\sum_j(1+\varpi\mu_i\mu_j)\Phi_j^* a_j - \frac{1}{3}\,\omega\varpi b_1\zeta\mu_i. \tag{12}$$

2.3. *Boundary conditions*

The equation of transfer (12) is to be solved subject to the boundary conditions:

(i) $I(0,\mu) = 0$ for $0 \leqslant \mu < 1$,
(ii) $I(\tau,\mu)\mathrm{e}^{-\tau} \to 0$ as $\tau \to \infty$

so that

(i) $\Phi(0,\mu) = 0$ for $0 \leqslant \mu < 1$,
(ii) $\Phi(t,\mu)\mathrm{e}^{-t} \to 0$ as $t \to \infty$

each of which can be split into $2n$ parts which by the use of Eq. (11) as follows:

$$\Phi_i^* = b_1\mu_{-i}\zeta - b_0 \quad \text{for } -1 < \mu_i \leqslant 0 \tag{13}$$

and

Φ_i^* is almost linear in t as t tends to ∞. (14)

3. Solution of the equation

Now, we shall try to get the solution of the part

$$\mu_i\frac{\mathrm{d}\Phi_i^*}{\mathrm{d}t} = \Phi_i^* - \frac{1}{2}\,\omega\sum_j(1+\varpi\mu_i\mu_j)\Phi_j^* a_j \tag{15}$$

of Eq. (12).
Let

$$\Phi_i^* = g_i\mathrm{e}^{-kt} \tag{16}$$

be a solution of Eq. (15).
Then

$$g_i(1+k\mu_i) = \frac{1}{2}\,\omega\sum_j(1+\varpi\mu_i\mu_j)g_j a_j \tag{17}$$

i.e.

$$g_i = \omega\frac{\rho+\rho_1\mu_i}{1+k\mu_i}. \tag{18}$$

3.1. *Determination of the characteristic equation*

Using Eq. (18) in Eq. (17), we get

$$\rho+\rho_1\mu_i = \tfrac{1}{2}\omega\{(\rho D_0+\rho_1 D_1)+\mu_i(\rho\varpi D_1+\rho_1\varpi D_2)\},$$

which gives

$$(2-\omega D_0)\rho + (-\omega D_1)\rho_1 = 0, \tag{19a}$$

$$\varpi\omega D_1\rho + (\varpi\omega D_2 - 2)\rho_1 = 0, \tag{19b}$$

where

$$D_\ell = D_\ell(k)$$

and

$$D_\ell = \sum_j\frac{a_j\mu_j^\ell}{1+k\mu_j}. \tag{20}$$

Eliminating ρ and ρ_1 between Eqs. (19a) and (19b), we get

$$4 - 2\omega D_0 - 2\varpi\omega D_2 + \omega^2\varpi(D_2D_0 - D_1^2) = 0.$$

Now the use of relation (A.9) of Appendix A will give

$$\omega D_0 + \varpi\omega(1-\omega)D_2 = 2 \tag{21}$$

i.e.

$$\omega\sum_j\frac{a_j}{2(1+k\mu_j)}\{1+\varpi(1-\omega)\mu_j^2\} = 1, \tag{22}$$

which is known as the *characteristic equation* and an equation in k of order $2n$. This equation will give $2n$ non-zero roots of the form $\pm k_\alpha$, $\alpha = 1,\ldots,n$.

3.2. Solution of Eq. (15)

Now, using Eq. (A.7) of Appendix A in Eq. (19a), we get

$$\rho_1 = \frac{k(2-\omega D_0)}{\omega(2-D_0)}\rho. \tag{23}$$

Now, using relation (A.8) of Appendix A, we get, from the form (21) of the characteristic equation, that

$$D_0 = \frac{2k^2 + 2\varpi\omega(1-\omega)}{k^2\omega + \varpi\omega(1-\omega)}. \tag{24}$$

Therefore,

$$2 - D_0 = -\frac{2k^2(1-\omega)}{k^2\omega + \varpi\omega(1-\omega)}$$

and

$$2 - \omega D_0 = \frac{2\varpi\omega(1-\omega)^2}{k^2\omega + \varpi\omega(1-\omega)}.$$

So, from Eq. (23)

$$\rho_1 = -\frac{\varpi(1-\omega)}{k}\rho. \tag{25}$$

So, Eq. (18) will take the form

$$g_i = \omega\frac{\rho - (\varpi(1-\omega)/k)\rho\mu_i}{1+k\mu_i}$$

i.e.

$$g_i = \omega\rho\frac{k - \varpi(1-\omega)\mu_i}{k(1+k\mu_i)}.$$

So, Eq. (15) admits $2n$ independent integrals of the form:

$$\Phi_i^* = \omega\rho\frac{k_\alpha \mp \varpi(1-\omega)\mu_i}{k_\alpha(1 \pm k_\alpha\mu_i)}e^{\mp k_\alpha\tau},$$

$$\alpha = \pm 1, \pm 2, \ldots, \pm n. \tag{26}$$

3.3. Particular integral of Eq. (12)

To get the complete solution of Eq. (12), we require a particular integral of the form. To obtain this we put

$$\Phi_i^* = \tfrac{1}{3}\omega\varpi b_1 h_i \zeta\mu_i. \tag{27}$$

Then

$$\frac{d\Phi_i^*}{dt} = 0.$$

So, Eq. (12) becomes

$$h_i = \left(\frac{1}{2}\,\omega\sum_j a_j h_j \mu_j\right)\frac{1}{\mu_i} + \left(\frac{1}{2}\,\omega\sum_j a_j\varpi\mu_j^2 h_j + 1\right) \tag{28}$$

i.e.

$$h_i = \frac{\sigma_1}{\mu_i} + \sigma, \tag{29}$$

where σ and σ_1 are the constants independent of μ_i.

Using Eq. (29) in Eq. (28), we get

$$\sigma_1 + \sigma\mu_i = \left(\frac{1}{2}\,\omega\sum_j a_j\right)(\sigma_1 + \sigma\mu_j) + \mu_i\left(\frac{1}{2}\,\omega\sum_j a_j\varpi\mu_j(\sigma_1 + \sigma\mu_j) + 1\right),$$

which gives

$$(\omega - 1)\sigma_1 = 0, \quad (\tfrac{1}{3}\omega\varpi - 1)\sigma + 1 = 0$$

i.e.

$$\sigma_1 = 0, \quad \sigma = \frac{1}{1 - \frac{1}{3}\varpi\omega}.$$

So, from Eq. (29), we get

$$h_i = \frac{1}{1 - \frac{1}{3}\varpi\omega}.$$

Therefore, the particular integral of Eq. (12)

$$I_i^* = b_1\frac{\frac{1}{3}\varpi\omega}{1 - \frac{1}{3}\varpi\omega}\zeta\mu_i. \tag{30}$$

3.4. Complete solution of Eq. (12)

The complete solution of Eq. (12) can be written from the integrals (26) and the particular integrals (30).

Following Chandrasekhar (1960), we can write the complete solution of Eq. (12) as

$$\Phi_i^* = \frac{1}{3}\,b_1\omega\left\{\sum_{\alpha=1}^{n}\frac{k_\alpha - \varpi(1-\omega)\mu_i}{k_\alpha(1+k_\alpha\mu_i)}L_\alpha e^{-k_\alpha t} + \varpi\frac{\zeta\mu_i}{1 - \frac{1}{3}\varpi\omega}\right\}, \quad i = \pm 1, \pm 2, \ldots, \pm n, \tag{31}$$

where k_α's ($\alpha = 1, 2, \ldots, n$) are the positive roots of the characteristic equation (22) and L_α's are the constants of integration to be determined by the boundary conditions (13).

Writing

$$M = 1 - \omega \tag{32}$$

and

$$N = \frac{\zeta}{1 - \frac{1}{3}\varpi\omega} \tag{33}$$

Eq. (31) can be put in a shorter form:

$$\Phi_i^* = \frac{1}{3}b_1\omega\left\{\sum_{\alpha=1}^{n}\frac{(k_\alpha - \varpi M\mu_i)L_\alpha e^{-k_\alpha t}}{k_\alpha(1+k_\alpha\mu_i)} + \varpi N\mu_i\right\},$$

$$i = \pm 1, \pm 2, \ldots, \pm n. \tag{34}$$

3.5. Diffusely reflected intensity and emergent intensity in closed form

From Eq. (34), we can write

$$\Phi^*(t,\mu) = \frac{1}{3}b_1\omega\left\{\sum_{\alpha=1}^{n}\frac{(k_\alpha - \varpi M\mu)L_\alpha e^{-k_\alpha t}}{k_\alpha(1+k_\alpha\mu)} + \varpi N\mu\right\}. \tag{35}$$

Now, we define

$$S(\mu) = \sum_{\alpha=1}^{n}\frac{(k_\alpha + \varpi M\mu)L_\alpha}{k_\alpha(1-k_\alpha\mu)} - \varpi N\mu - \frac{3\zeta\mu}{\omega} + \frac{3b_0}{\omega b_1}. \tag{36}$$

Now, using definition (36) of $S(\mu)$, we get, from the set of boundary conditions (13), we get

$$S(\mu_i) = 0 \quad \text{for } 0 \leqslant \mu_i \leqslant 1 \tag{37}$$

and from Eq. (35), applying relations (36) and (9), we get

$$I^*(0,\mu) = \tfrac{1}{3}b_1\omega S(-\mu) - b_1\zeta\mu - b_0. \tag{38}$$

Now, we define two new functions $P(\mu)$ and $R(\mu)$ as

$$P(\mu) = \prod_{i=1}^{n}(\mu - \mu_i) \tag{39}$$

and

$$R(\mu) = \prod_{\alpha=1}^{n}(1 - k_\alpha\mu). \tag{40}$$

Then, by virtue of relation (37), we can conclude that the polynomial $S(\mu)R(\mu)$ has n zeros $\mu_i; i = 1,\ldots,n$ and as it is a polynomial of degree $n+1$, it has, therefore, one more zero which is different from these n zeros, say λ, in which the co-efficient of μ^{n+1} is $(-1)^{n+1}(\varpi N + 3\zeta/\omega)k_1\cdots k_n$.

Clearly, the polynomial $(\mu-\lambda)P(\mu)$ has also the same zeros as the polynomial $S(\mu)R(\mu)$ has, but its co-efficient of μ^{n+1} is 1.

Therefore,

$$S(\mu) = (-1)^{n+1}\left(\varpi N + \frac{3\zeta}{\omega}\right)k_1\cdots k_n(\mu-\lambda)\frac{P(\mu)}{R(\mu)}, \tag{41}$$

where λ is a constant.

Again, taking form (36) of $S(\mu)$, we observe that

$$\lim_{\mu\to(k_\alpha)^{-1}}\frac{1}{T_\alpha}(1-k_\alpha\mu)S(\mu) = \frac{1}{T_\alpha}\left(1+\frac{\varpi M}{k_\alpha^2}\right)L_\alpha.$$

Therefore,

$$L_\alpha = \lim_{\mu\to(k_\alpha)^{-1}}\frac{1}{T_\alpha}(1-k_\alpha\mu)S_r(\mu) \tag{42}$$

provided that

$$T_\alpha = 1 + \frac{\varpi M}{k_\alpha^2}. \tag{43}$$

Again, from Eq. (42), using Eq. (41),

$$L_\alpha = (-1)^{n+1}\frac{1}{T_\alpha}\left(\varpi N + \frac{3\zeta}{\omega}\right)k_1\cdots k_n \times \left(\frac{1}{k_\alpha} - \lambda\right)\frac{P(1/k_\alpha)}{R_\alpha(1/k_\alpha)}, \tag{44}$$

where R_α is given by

$$R_\alpha(\mu) = \prod_{\beta(\neq\alpha)=1}^{n}(1 - k_\beta\mu). \tag{45}$$

Summing over α to both sides of Eq. (44)

$$\sum_{\alpha=1}^{n}L_\alpha = (-1)^{n+1}\left(\varpi N + \frac{3\zeta}{\omega}\right)k_1\cdots k_n f(0), \tag{46}$$

where

$$f(x) = \sum_{\alpha=1}^{n}\frac{1}{T_\alpha}\left(\frac{1}{k_\alpha} - \lambda\right)\frac{P(1/k_\alpha)}{R_\alpha(1/k_\alpha)}R_\alpha(x).$$

Now, we see that $f(x)$ is a polynomial of degree $n-1$ in x. It takes the values

$$\frac{1}{T_\alpha}\left(\frac{1}{k_\alpha} - \lambda\right)P\left(\frac{1}{k_\alpha}\right) \quad \text{for } x = \frac{1}{k_\alpha},\ \alpha = 1,\ldots,n.$$

So,

$$(1+\varpi Mx^2)f(x) - (x-\lambda)P(x) = 0 \quad \text{for } x = \frac{1}{k_\alpha},\ \alpha = 1,\ldots,n. \tag{47}$$

Therefore,

$$(1+\varpi Mx^2)f(x) - (x-\lambda)P(x) = (Ax+B)R(x). \tag{48}$$

Now, defining another function $H(\mu)$ as

$$H(\mu) = \frac{1}{\mu_1\cdots\mu_n}\cdot\frac{\prod_{i=1}^{n}(\mu+\mu_i)}{\prod_{\alpha=1}^{n}(1+k_\alpha\zeta\mu)}, \tag{49}$$

we put $x = +\mathrm{i}/\sqrt{\varpi M}$ and $x = -\mathrm{i}/\sqrt{\varpi M}$, with the assumption that $\varpi \neq 0$, to get

$$\frac{\mathrm{i}A}{\sqrt{\varpi M}} + B = \left(\lambda - \frac{\mathrm{i}}{\sqrt{\varpi M}}\right)(-1)^n\mu_1\cdots\mu_n \times H\left(-\frac{\mathrm{i}}{\sqrt{\varpi M}}\right), \tag{50a}$$

$$-\frac{\mathrm{i}A}{\sqrt{\varpi M}} + B = \left(\lambda + \frac{\mathrm{i}}{\sqrt{\varpi M}}\right)(-1)^n\mu_1\cdots\mu_n \times H\left(+\frac{i}{\sqrt{\varpi M}}\right). \tag{50b}$$

Adding and subtracting the two relations (50a) and (50b) and writing

$$a = \frac{1}{2}\left\{H\left(+\frac{\mathrm{i}}{\sqrt{\varpi M}}\right) + H\left(-\frac{\mathrm{i}}{\sqrt{\varpi M}}\right)\right\} \tag{51a}$$

and

$$b = \frac{1}{2}\left\{H\left(+\frac{\mathrm{i}}{\sqrt{\varpi M}}\right) - H\left(-\frac{\mathrm{i}}{\sqrt{\varpi M}}\right)\right\} \tag{51b}$$

we can express A and B as

$$A=(-1)^{n+1}\mu_1\cdots\mu_n(a-\mathrm{i}\lambda\sqrt{\varpi M}\cdot b) \tag{52a}$$

and

$$B=(-1)^n\mu_1\cdots\mu_n\left(\lambda\cdot a+\frac{\mathrm{i}}{\sqrt{\varpi M}}\cdot b\right). \tag{52b}$$

Now, putting $x=0$ in Eq. (48), we get

$$f(0)=(-1)^{n+1}\lambda\cdot\mu_1\cdots\mu_n+B.$$

Therefore, using Eq. (52b)

$$f(0)=(-1)^n\left\{\frac{ib}{\sqrt{\varpi M}}-\lambda(1-a)\right\}\cdot\mu_1\cdots\mu_n, \tag{53}$$

which on using the equation in Eq. (46), yields

$$\sum_{\alpha=1}^{n}L_\alpha=\left(\varpi N+\frac{3\zeta}{\omega}\right)\left\{\lambda(1-a)-\frac{ib}{\sqrt{\varpi M}}\right\}\times k_1\cdots k_n\cdot\mu_1\cdots\mu_n.$$

But

$$k_1\cdots k_n\cdot\mu_1\cdots\mu_n=\left\{M\left(1-\tfrac{1}{3}\varpi(1-M)\right)\right\}^{1/2},$$

which is Eq. (B.5) from Appendix A.

Therefore,

$$\sum_{\alpha=1}^{n}L_\alpha=\left(\varpi N+\frac{3\zeta}{\omega}\right)\left\{\lambda(1-a)-\frac{ib}{\sqrt{\varpi M}}\right\}\times\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}. \tag{54}$$

Again from Eq. (41), using Eq. (B.5) from Appendix A, we get

$$S(\mu)=-\left(\varpi N+\frac{3\zeta}{\omega}\right)\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}\times(\mu-\lambda)H(-\mu). \tag{55}$$

So, putting $\mu=0$, we get

$$S(0)=\lambda\left(\varpi N+\frac{3\zeta}{\omega}\right)\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}. \tag{56}$$

Again, putting $\mu=0$ in Eq. (36), we get

$$S(0)=\sum_{\alpha=1}^{n}L_\alpha+\frac{3\,b_0}{\omega b_1}.$$

Therefore, using Eq. (54), we get

$$S(0)=\left(\varpi N+\frac{3\zeta}{\omega}\right)\left\{\lambda(1-a)-\frac{ib}{\sqrt{\varpi M}}\right\}\times\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}+\frac{3b_0}{\omega b_1}. \tag{57}$$

So, comparing Eqs. (56) and (57) and simplifying we get

$$\lambda=\frac{b_0}{aG}-\frac{\mathrm{i}}{\sqrt{\varpi M}}\cdot\frac{b}{a}, \tag{58}$$

where

$$G=b_1\left(\zeta+\tfrac{1}{3}\varpi\omega N\right)\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}. \tag{59}$$

So, from Eq. (55),

$$S(\mu)=\left(\varpi N+\frac{3\zeta}{\omega}\right)\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}\times\left(\frac{b_0}{aG}-\frac{\mathrm{i}}{\sqrt{\varpi M}}\cdot\frac{b}{a}-\mu\right)H(-\mu),$$

where G is given by Eq. (58).

So, from Eq. (38),

$$I^*(0,\mu)=b_1\left(\zeta+\frac{1}{3}\omega\varpi N\right)\left\{M\left(1-\frac{1}{3}\varpi(1-M)\right)\right\}^{1/2}\times\left(\frac{b_0}{aG}-\frac{\mathrm{i}}{\sqrt{\varpi M}}\cdot\frac{b}{a}+\mu\right)H(\mu)-b_1\zeta\mu-b_0$$

i.e.

$$I^*(0,\mu)=G(\mu+\lambda)H(\mu)-b_1\zeta\mu-b_0, \tag{60}$$

where λ and G are given, respectively, by Eqs. (58) and (59).

Again from Eq. (11), using Eq. (60), we get

$$I(0,\mu)=G(\mu+\lambda)H(\mu), \tag{61}$$

which is the desired solution in nth approximation.

3.6. *The exact diffusely reflected intensity and the exact emergent intensity*

The *characteristic equation* (22) can be written as

$$\sum_j\frac{a_j}{(1+k\mu_j)}\frac{1}{2}\,\omega\{1+\varpi(1-\omega)\mu_j^2\}=1$$

i.e.

$$2\sum_j\frac{a_j\Psi(\mu_j)}{1-k^2\mu_j^2}=1, \tag{62}$$

where

$$\Psi(\mu)=\tfrac{1}{2}\omega\{1+\varpi(1-\omega)\mu^2\}. \tag{63}$$

Then

$$\int_0^1\Psi(\mu')\,\mathrm{d}\mu'=\int_0^1\frac{1}{2}\,\omega\{1+\varpi(1-\omega)\mu'^2\}\,\mathrm{d}\mu'<\frac{1}{2}\quad\text{if we choose }\omega<1\text{ and since }0<\varpi<1.$$

Therefore, following the theory of H-function, developed by Chandrasekhar (1960), we can show, in the present case, that $H(\mu)$, given by Eq. (49) satisfies, in the limit of infinite approximation, the non-integral equation:

$$H(\mu)=1+\mu H(\mu)\int_0^1\frac{\Psi(\mu')H(\mu')}{\mu+\mu'}\,\mathrm{d}\mu', \tag{64}$$

which is bounded in the entire half plane $\mathbb{R}(\mu)\geqslant 0$. The characteristic function (63) satisfies the necessary condition:

$$\int_0^1 \Psi(\mu')\,d\mu' \leqslant \frac{1}{2}. \tag{65}$$

Allowing $n \to \infty$, we shall get, the exact diffusely reflected intensity from Eq. (60) as

$$I^*(0,\mu) = G(\mu+\lambda)H(\mu) - b_1\mu - b_0 \tag{66}$$

and the exact emergent intensity from Eq. (61) as

$$I(0,\mu) = G(\mu+\lambda)H(\mu), \tag{67}$$

where λ and G are given, in infinite approximation, by Eqs. (58) and (59), respectively, and $H(\mu)$ is given by Eq. (64).

4. Discussion

Now, as $\varpi \to 0$, we observe that

$$a \to \frac{1}{M^{1/2}} \quad \text{and} \quad i\frac{1}{\sqrt{\varpi M}}b \to \frac{1}{\sqrt{M}}\left(\sum_{i=1}^{n}\mu_i - \sum_{\alpha=1}^{n}\frac{1}{k_\alpha}\right)$$

so that

$$I(0,\mu) \to \sqrt{M}\left[b_1\zeta\left\{\mu - \left(\sum_{i=1}^{n}\mu_i - \sum_{\alpha=1}^{n}\frac{1}{k_\alpha}\right)\right\} + b_0\right]H_r(\mu)$$

i.e., in the limiting position as ϖ approaches zero,

$$I(0,\mu) = \sqrt{M}H(\mu)\left\{b_0 + b_1\zeta\mu + b_1\zeta\left(\sum_{\alpha=1}^{n}\frac{1}{k_\alpha} - \sum_{i=1}^{n}\mu_i\right)\right\}, \tag{68}$$

which is the solution for the case of coherent isotropic scattering.

Like emergent intensity, we can show that the diffusely reflected intensity for isotropically scattering media can also be derived from the anisotropically scattering media.

The solution obtained by Chandrasekhar (1960) from the equation of the form:

$$\mu\frac{dI_v}{dt_v} = I_v - \frac{1}{2}(1-\lambda_v)\int_{-1}^{+1} I_v(t_v,\mu')\,d\mu' - \lambda_v B_v(t_v), \tag{69}$$

where

$$\lambda_v = \frac{k_v + \sigma_v\varepsilon_v}{k_v+\sigma_v} = \frac{1+\varepsilon_v\eta_v}{1+\eta_v}\left(\eta_v = \frac{\sigma_v}{k_v}\right) \quad \text{and}$$

$$B_v = B_v^{(0)} + B_v^{(1)}\tau_v$$

is

$$I(0,\mu) = \frac{\lambda^{1/2}B^{(1)}}{1+\varepsilon\eta}H(\mu)\left(\mu + \frac{1+\varepsilon\eta}{\lambda}\cdot\frac{B^{(0)}}{B^{(1)}} + \sum_{\alpha=1}^{n}\frac{1}{k_\alpha} - \sum_{i=1}^{n}\mu_i\right).$$

Busbridge (1953)'s result with $\varepsilon = 0$, obtained by using the principle of invariance, is

$$I_v(0,\mu) = (b_v^{(0)} + \lambda_v b_v^{(1)}\mu)\lambda_v^{1/2}H(\mu) + \tfrac{1}{2}(1-\lambda_v)\lambda_v b_v^{(1)}\alpha_1 H(\mu).$$

The exact diffusely reflected intensity $I^*(0,\mu)$ and exact emergent intensity $I(0,\mu)$ for isotropically scattering media can also be derived from the anisotropically scattering media associated with the planetary phase function by allowing ϖ of the phase function (2) to tend to zero. The characteristic function $\Psi(\mu')$ of the H-function $H(\mu)$ for isotropically scattering media is to be taken from Eq. (63) by putting $\varpi = 0$ directly which is given by

$$\Psi(\mu') = \tfrac{1}{2}\omega \quad \text{if } 0 \leqslant \mu' \leqslant 1. \tag{70}$$

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Appendix A. Relation among $D_\ell(k)$'s ; $\ell = 0, 1, 2$

We have defined in Eq. (20)

$$D_\ell = \sum_j \frac{a_j\mu_j^\ell}{1+k\mu_j}$$

$$\Rightarrow D_\ell(x) = \frac{1}{x}\left(\sum_j a_j\mu_j^{\ell-1} - D_{\ell-1}(x)\right)$$

$$\Rightarrow D_\ell(x) = \frac{1}{x}\left(\frac{2}{\ell}\varepsilon_{\ell,\text{odd}} - D_{\ell-1}(x)\right), \tag{A.1}$$

where

$$\varepsilon_{\ell,\text{odd}} = \begin{cases} 1 & \text{if } \ell \text{ is odd}, \\ 0 & \text{if } \ell \text{ is even}. \end{cases} \tag{A.2}$$

But Eq. (A.1) gives

$$D_{2j-1}(x) = \frac{1}{x}\left(\frac{2}{2j-1} - D_{\ell-1}(x)\right), \tag{A.3}$$

$$D_{2j}(x) = -\frac{1}{x}D_{2j-1}(x). \tag{A.4}$$

From the two relations (A.3) and (A.4), we readily deduce that

$$D_{2j-1}(x) = \frac{2}{(2j-1)x} + \frac{2}{(2j-1)x^3} + \cdots + \frac{2}{3x^{2j-3}} + \frac{1}{x^{2j-1}}\{2 - D_0(x)\}(j = 1, \ldots, 2n), \tag{A.5}$$

$$D_{2j}(x) = -\frac{2}{(2j-1)x^2} - \frac{2}{(2j-1)x^4} - \cdots - \frac{2}{3x^{2j-2}} - \frac{1}{x^{2j}}\{2 - D_0(x)\}(j = 1, \ldots, 2n), \tag{A.6}$$

which are Eqs. (24) and (25) of Chandrasekhar (1960, Chapter-III, p.73)

Now from Eq. (A.5), putting $j = 1$, we get

$$D_1(x) = \frac{1}{x}\{2 - D_0(x)\}.$$

Now from Eq. (A.6), putting $j = 1$, we get

$$D_2(x) = -\frac{1}{x^2}\{2 - D_0(x)\}.$$

Replacing x by k in the above equations and using the notation D_ℓ for $D_\ell(k)$, we get

$$D_1(k) = \frac{1}{k}(2 - D_0) \tag{A.7}$$

and

$$D_2 = -\frac{1}{k^2}(2 - D_0) \tag{A.8}$$

giving

$$D_0D_2 - D_1^2 = 2D_2. \tag{A.9}$$

Appendix B. Relation involving characteristic roots k_α's and μ_i's

Let p_{2j} be the co-efficients of μ^{2j} of the Legendre polynomial $P_{2n}(\mu)$.

Then,

$$P_{2n}(\mu) = \sum_{j=1}^{n} p_{2j}\mu^{2j}. \tag{B.1}$$

Now, we have

$$\sum_{j=1}^{n} p_{2j}D_{2j}(k) = \sum_{j} \frac{a_j}{1+\mu_j k}\left(\sum_{j=1}^{n} p_{2j}\mu^{2j}\right) = 0.$$

Since μ_i's are zeros of the Legendre polynomial $P_{2n}(\mu)$.

Therefore,

$$\sum_{j=1}^{n} p_{2j}D_{2j}(k) = 0. \tag{B.2}$$

Now Eq. (B.2), using Eqs. (A.6) and (24), gives

$$t^n\{1 - \omega - \tfrac{1}{3}\varpi\omega(1-\omega)\}p_{2n} + \cdots + p_0 = 0, \quad t = \frac{1}{k^2}.$$

Therefore,

$$k_1^2 \cdot k_2^2 \cdots k_n^2 = (-1)^n(1-\omega)\left(1 - \frac{1}{3}\varpi\omega\right)\frac{p_{2n}}{p_0}. \tag{B.3}$$

Again, μ_i's are zeros of the Legendre polynomial $P_{2n}(\mu)$ and so,

$$\mu_1^2 \cdot \mu_2^2 \cdots \mu_n^2 = (-1)^n \frac{p_0}{p_{2n}}. \tag{B.4}$$

Multiplying Eqs. (B.3) and (B.4), we get

$$k_1 \cdot k_2 \cdots k_n \cdot \mu_1 \cdot \mu_2 \cdots \mu_n = \left\{M\left(1 - \tfrac{1}{3}\varpi(1-M)\right)\right\}^{1/2}. \tag{B.5}$$

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