Chapter 5

Solution of the equation of transfer with Carlstedt and Mullikin's phase function

5.1 The equations and boundary conditions :

The equation of transfer for the plane- parallel scattering atmosphere with axial symmetry is given by

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{5.1}$$

where, $p(\mu,\mu')$ is the phase function which gives the measure (probability) through which a pencil of radiation is scattered from (μ,ϕ) direction to (μ',ϕ') direction and defined by

$$p(\mu,\mu') = \frac{1}{2\pi}\int_{0}^{2\pi} P(\mu,\Phi;\mu',\Phi')d\Phi' \tag{5.2}$$

$I(\tau,\mu)$ is the specific intensity of radiation at an optical depth τ, given by $\tau = \int_z^\infty k\rho dz$. k being the coefficient of scattering and ρ , the density. $\mu = cos\theta$,where θ is the angle made by intensity with the outward drown normal. The equation of transfer (5.1)is to be solved subject to the boundary conditions,

(a) Absence of incident radiation from outside at the free surface $\tau = 0$.

$$I(0,\mu) \equiv 0 \quad for -1 \leq \mu \leq 0 \tag{5.3}$$

(b) The convergence of the intensity as $\tau \to \infty$

$$I(\tau, \mu)e^{-\tau} \quad \to 0 \;\; as \;\; \tau \to \infty \tag{5.4}$$

We represent $I(\tau, \mu)$ by two different expansions $I_+(\tau, \mu)$ and $I_-(\tau, \mu)$ for μ in the intervals (0,1) and (-1,0)respectively in the form

$$I_+(\tau, \mu) = A\tau + \sum_{l=0}^{l_0}(2l + 1)I_l^+(\tau)\mu P_l(2l - 1) \;\; for \;\; 0 \leq \mu \leq 1 \tag{5.5}$$

$$I_-(\tau, \mu) = A\tau + \sum_{l=0}^{l_0}(2l + 1)I_l^-(\tau)\mu P_l(2l + 1) \;\; for \;\; -1 \leq \mu \leq 0 \tag{5.6}$$

as series in Legendre Polynomials. Here A is a constant independent of τ and μ. Taking that $P(\mu, \mu')$ is representable by

$$P(\mu, \mu') = \sum_{k=0}^{\infty} \omega_k P_k(\mu) P_k(\mu') \tag{5.7}$$

using (5.5),(5.6),(5.7) and the recurrence formulae

$$\mu P_l(2\mu \pm 1) = \frac{1}{2l+1}\left[\frac{l+1}{2}P_{l+1}(2\mu \pm 1) \mp \frac{2l+1}{2}P_l(2\mu \pm 1) + \frac{l}{2}P_{l+1}(2\mu \pm 1)\right] \tag{5.8}$$

the equation of transfer takes the form

$$\mu\frac{dI_+(\tau, \mu)}{d\tau} = I_+(\tau, \mu) - \frac{1}{2}\sum_{k=0} \omega_k P_k(\mu)[...] \tag{5.9}$$

$$\mu\frac{dI_-(\tau, \mu)}{d\tau} = I_-(\tau, \mu) - \frac{1}{2}\sum_{k=0} \omega_k P_k(\mu)[...] \tag{5.10}$$

where

$$\begin{aligned}
[...] &= \left[A\tau \int_{-1}^{+1} p_k(\mu')d\mu' + \sum_{l=0} \frac{I_l^+(\tau)}{2}\{...\}_1 + \sum_{l=0} \frac{I_l^-(\tau)}{2}\{...\}_2\right] \\
\{...\}_1 &= (l + 1)J_{l+1,k}^+ + (2l + 1)J_{l,k}^+ + lJ_{l-1,k}^+ \\
\{...\}_2 &= (l + 1)J_{l+1,k}^- + (2l + 1)J_{l,k}^- + lJ_{l-1,k}^- \\
J_{l,k}^+ &= \int_0^1 P_k(\mu)P_l(2\mu - 1)d\mu \\
J_{l,k}^- &= \int_{-1}^0 P_k(\mu)P_l(2\mu + 1)d\mu
\end{aligned}$$

Multiplying equation (5.9) and (5.10) by $P_l(2\mu - 1)$ and $P_l(2\mu + 1)$ respectively, integrating over μ in their respective ranges and using the recurrence formulae (5.8) we have the following equations.

$$A\int_0^1 \mu P_l(2\mu - 1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2 - l}{2l-1}I_{l-2}^{+'} + 2lI_{l-1}^{+'} + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)}I_l^{+'} + 2(l+1)I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3}I_{l+2}^{+'}\right] + A\tau\delta_{0l}$$
$$= \frac{1}{2(2l+1)}[lI_{l-1}^{+} + (2l+1)I_l^{+} + (l+1)I_{l+1}^{+}] - \frac{1}{2}\sum_{k=0}\omega_k J_{l,k}^{+}[\cdots] \quad (5.11)$$

and

$$A\int_{-1}^0 \mu P_l(2\mu + 1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2 - l}{2l-1}I_{l-2}^{-'} + 2lI_{l-1}^{-'} + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)}I_l^{-'} - 2(l+1)I_{l+1}^{-'} + \frac{l^2 + 3l + 2}{2l+3}I_{l+2}^{-'}\right] + A\tau\delta_{0l}$$
$$= \frac{1}{2(2l+1)}[lI_{l-1}^{-} + (2l+1)I_l^{+} + (l+1)I_{l+1}^{-}] - \frac{1}{2}\sum_{k=0}\omega_k J_{l,k}^{-}[\cdots] \quad (5.12)$$

where I_l' is derivative of I_l with respect to the optical thickness τ and δ_{0l} is the Kronecker delta. Now,

$$\sum_{k=0}\omega_k J_{l,k}^{+}[...] = \delta_{0,l}\left\{2A\tau + \frac{1}{2}(I_0^{+} - I_0^{-} + I_1^{+} - I_1^{-})\right\} + \sum_{k=1}\omega_k J_{l,k}^{+}[...] \quad (5.13)$$

and

$$\sum_{k=0}\omega_k J_{l,k}^{-}[...] = \delta_{0,l}\left\{2A\tau + \frac{1}{2}(I_0^{+} - I_0^{-} + I_1^{+} - I_1^{-})\right\} + \sum_{k=1}\omega_k J_{l,k}^{-}[...] \quad (5.14)$$

The above differential equations (5.13) and (5.14) are to be solved in the desired approximation with the boundary conditions stated in (5.3)and (5.4).

5.2 First approximate solution in case of scattering with Carlstedt and Mullikin's phase function

We now demonstrate the method with particular reference to the Carlstedt and Mullikins phase function given by

$$\begin{aligned} P(\mu,\mu') &= 1+b_0P_4(\mu) \\ &= 1+\frac{3b_0}{8}+b_0(\frac{35}{8}\mu^4-\frac{15}{4}\mu^2) \end{aligned} \tag{5.15}$$

where $1 \le b_0 \le 2$.

In the case of scattering with Carlstedt and Mullikin's phase function the equation of transfer (5.1) takes the form

$$\mu\frac{dI_+(\tau,\mu)}{d\tau} = I_+(\tau,\mu)-[1+b_0P_4(\mu)]\left[A\tau+\frac{I_0^+-I_0^-+I_1^++I_1^-}{4}\right] \tag{5.16}$$

$$\mu\frac{dI_-(\tau,\mu)}{d\tau} = I_-(\tau,\mu)-[1+b_0P_4(\mu)]\left[A\tau+\frac{I_0^+-I_0^-+I_1^++I_1^-}{4}\right] \tag{5.17}$$

Multiplying (5.16) by $P_l(2\mu-1)$ and (5.17) by $P_l(2\mu+1)$ and integrating over respective ranges of μ we obtain

$$\begin{aligned} &\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{l-2}^{+'}+2lI_{l-1}^{+'}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{+'}+2(l+1)I_{l+1}^{+'}\right. \\ &\left.+\frac{l^2+3l+2}{2l+3}l_{l+2}^{+'}\right]+A\int_0^1\mu P_l(2\mu-1)d\mu = \frac{1}{2(2l+1)}[lI_{l-1}^{+}+(2l+1)I_l^{+} \\ &+(l+1)I_{l+1}^{+}]+A\tau-A\tau\int_0^1[1+b_0P_4(\mu)]P_l(2\mu-1)d\mu-\frac{1}{4}[I_0^+-I_0^-+I_1^+ \\ &+I_1^-]\int_0^1[1+b_0p_4(\mu)]P_l(2\mu-1)d\mu \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} &\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{l-2}^{-'}-2lI_{l-1}^{-'}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{-'}-2(l+1)I_{l+1}^{-'}\right. \\ &\left.+\frac{l^2+3l+2}{2l+3}l_{l+2}^{-'}\right]+A\int_{-1}^0\mu P_l(2\mu+1)d\mu = \frac{1}{2(2l+1)}[lI_{l-1}^{-}-(2l+1)I_l^{-} \\ &+(l+1)I_{l+1}^{-}]+A\tau-A\tau\int_{-1}^0[1+b_0P_4(\mu)]P_l(2\mu+1)d\mu-\frac{1}{4}[I_0^+-I_0^-+I_1^+ \\ &+I_1^-]\int_{-1}^0[1+b_0p_4(\mu)]P_l(2\mu-1)d\mu \end{aligned} \tag{5.19}$$

For $l_0 = 1$ we get the following set of differential equations from (5.18) and (5.19) by separation of equations for $l = 0$ and $l = 1$.

$$\left.\begin{array}{l}
(\frac{4}{3}I_0^{+'} + 2I_1^{+'}) - (I_0^+ + I_0^- + I_1^+ - I_1^-) = -2A \\[2ex]
(\frac{4}{3}I_0^{-'} + 2I_1^{-'}) + (I_0^+ + I_0^- + I_1^+ - I_1^-) = 2A \\[2ex]
(2I_0^{+'} + \frac{24}{5}I_1^{+'}) - [(2 + \frac{b_0}{8})I_0^+ - \frac{b_0}{8}I_0^- + (6 + \frac{b_0}{8})I_1^+ + \frac{b_0}{8}I_1^-] \\
= -2A + A\tau(12 + \frac{b_0}{2}) \\[2ex]
(-2I_0^{-'} + \frac{24}{5}I_1^{-'}) + [\frac{b_0}{8}I_0^+ - (2 + \frac{b_0}{8})I_0^- + \frac{b_0}{8}I_1^+ + (6 + \frac{b_0}{8})I_1^-] \\
= -2A + A\tau(12 - \frac{b_0}{2})
\end{array}\right\} \quad (5.20)$$

where

$$\left.\begin{array}{l}
I_+(\tau, \mu) = A\tau + I_0^+(\tau)\mu + 3I_1^+(\tau)\mu P_1(2\mu - 1), 0 \le \mu \le 1 \\[2ex]
I_-(\tau, \mu) = A\tau + I_0^-(\tau)\mu + 3I_1^-(\tau)\mu P_1(2\mu + 1), -1 \le \mu \le 0
\end{array}\right\} \quad (5.21)$$

We take the trial solution as

$$\left.\begin{array}{l}
I_l^+(\tau) = A(g_{l,\alpha}e^{-k\tau} + g_{l,\beta}) \\[2ex]
I_l^-(\tau) = A(h_{l,\alpha}e^{-k\tau} + h_{l,\beta})
\end{array}\right\} \quad (5.22)$$

where $g_{l,\alpha}, g_{l,\beta}, h_{l,\alpha}, h_{l,\beta}$ are constants to be determined.

When substituted in (5.20) and the coefficient of $e^{k\tau}$ and constant term are equated, we obtain

$$\left.\begin{array}{l}
(1 + \frac{4k}{3})g_{0,\alpha} + (1 + 2k)g_{1,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \\[2ex]
(2 + \frac{b_0}{8} + 2k)g_{0,\alpha} + (6 + \frac{b_0}{8} + \frac{24k}{5})g_{1,\alpha} - \frac{b_0}{8}h_{0,\alpha} + \frac{b_0}{8}h_{1,\alpha} = 0 \\[2ex]
g_{0,\beta} + g_{1,\beta} + (1 - \frac{4k}{3})h_{0,\beta} - (1 - 2k)h_{1,\beta} = 0 \\[2ex]
\frac{b_0}{8}g_{0,\alpha} + \frac{b_0}{8}g_{1,\alpha} + (2k - 2 - \frac{b_0}{8})h_{0,\alpha} + (6 + \frac{b_0}{8} - \frac{24k}{5})h_{1,\alpha} = 0
\end{array}\right\} \quad (5.23)$$

and

$$\left.\begin{array}{l} g_{0,\beta}+g_{1,\beta}+h_{0,\beta}-h_{1,\beta}=2 \\ (2+\frac{b_0}{8})g_{0,beta}+(6+\frac{b_0}{8})g_{1,\beta}-\frac{b_0}{8}h_{0,\beta}+\frac{b_0}{8}h_{1,\beta}=2 \\ \frac{b_0}{8}g_{0,\beta}+\frac{b_0}{8}g_{1,\beta}-(2+\frac{b_0}{8})h_{0,\beta}+(6+\frac{b_0}{8})h_{1,\beta}=-2 \end{array}\right\} \quad (5.24)$$

The values of k which will make the above equations(5.23) consistent could be obtained by evaluating the determinantal equation

$$\Delta(k)=\begin{vmatrix} 1+\frac{4k}{3} & 1+2k & 1 & -1 \\ 2+\frac{b_0}{8}+2k & 6+\frac{b_0}{8}+\frac{24k}{5} & -\frac{b_0}{8} & \frac{b_0}{8} \\ 1 & 1 & 1-\frac{4k}{3} & -1+2k \\ \frac{b_0}{8} & \frac{b_0}{8} & 2k-2-\frac{b_0}{8} & 6+\frac{b_0}{8}-\frac{24k}{5} \end{vmatrix}=0 \quad (5.25)$$

and $\Delta(k)=0$ yields $k=0,0\pm\sqrt{\frac{10}{3}-\frac{5b_0}{8}}$.
Since $1\leq b_0\leq 2$, therefore we obtain the following table for different values of b_0 in the interval [1,2].

Table-1

b_0	k
1	$0,0,\pm1.7480174$
1.25	$0,0,\pm1,7280368$
1.5	$0,0,\pm1.7078251$
1.75	$0,0,\pm1.6873714$
2	$0,0,\pm1.6666667$

To satisfy the boundary condition (5.4)

$$I_l^{+}e^{-\tau}\longrightarrow 0 \;\; as \;\; \tau\longrightarrow\infty$$
$$I_l^{+}e^{-\tau}\longrightarrow 0 \;\; as \;\; \tau\longrightarrow\infty$$

we retain only the positive roots for different value of b_0

Using the boundary condition (5.3) at the free surface, we obtain

$$\left.\begin{array}{l} h_{0,\alpha} + h_{0,\beta} = 0 \\ h_{1,\alpha} + h_{1,\beta} = 0 \end{array}\right\} \tag{5.26}$$

From (5.23), (5.24) and (5.26) we obtain the following table of values of $g_{l,\alpha}, g_{l,\beta}, h_{l,\alpha}, h_{l,\alpha}$ for different values of b_0

Table-II

b_0	1	1.25	1.5	1.75	2
k	1.7480174	1.7280368	1.7078251	1.6873714	1.666667
$g_{0,\alpha}$	-0.255707	-0.428826	-0.457904	-0.488844	-0.521729
$g_{1,\alpha}$	0.045844	0.173505	0.187147	0.201757	0.217387
$h_{0,\alpha}$	1.563439	1.593994	1.668301	1.745652	1.826052
$h_{1,\alpha}$	0.921867	0.950283	0.994078	1.039629	1.086936
$g_{0,\beta}$	3.563148	3.593996	3.668308	3.745659	3.826053
$g_{1,\beta}$	-0.922734	-0.950276	-0.994074	-1.039618	-1.086943
$h_{0,\beta}$	-1.563138	-1.593996	-1.668304	-1.745658	-1.826051
$h_{1,\beta}$	-0.922708	-0.950276	-0.994074	-1.039618	-1.086943

Since

$$\left.\begin{array}{l} I_+(\tau,\mu) = A\tau + \mu I_0^+(\tau) + \mu P_1(2\mu-1)I_1^+(\tau) \\ I_-(\tau,\mu) = A\tau + \mu I_0^-(\tau) + \mu P_1(2\mu-1)I_1^-(\tau) \end{array}\right\} \tag{5.27}$$

$$\left.\begin{array}{l} F = 2[\int_0^1 I_+(\tau,\mu)\mu d\mu + \int_{-1}^0 I_(\tau,\mu)\mu d\mu] \\ = \frac{2}{3}[(I_0^+ + I_0^-) + \frac{3}{2}(I_1^+ - I_1^-)] \quad \cdots (a) \\ = \frac{1}{2}[\frac{4}{3}(I_0^+ + I_0^-) + 2(I_1^+ - I_1^-)] \quad \cdots (b) \end{array}\right\} \tag{5.28}$$

From first two equation of (5.20) right hand side of (5.28b) turns out to a constant.

Substituting the values of $I_l^+, I_l^-(l=0,1)$ in (5.28a) for any values of $b_0(1 \le b_0 \le 2)$ we obtain $3F = A[4 + \ \textit{function of} \ \tau]$

with this two result $A = \frac{3}{4}F$.

Here the source function is given by

$$\begin{aligned} J(\tau,\mu) &= \frac{1}{2}\int_{-1}^{1} p(\mu,\mu,)I(\tau,\mu,)d\mu, \\ &= [1+b_0P_4(\mu)]\left[A\tau + \frac{I_0^+ - I_0^- + I_1^+ + I_1^-}{4}\right] \end{aligned} \quad (5.29)$$

The law of darkening is obtained using the relation

$$\begin{aligned} I(0,\mu) &= \int_0^\infty J(\tau,\mu)e^{-\frac{\tau}{\mu}}\frac{d\tau}{\mu} \\ &= \int_0^\infty [1+b_0P_4(\mu)]\left[A\tau + \frac{I_0^+ - I_0^- + I_1^+ + I_1^-}{4}\right] e^{-\frac{\tau}{\mu}}\frac{d\tau}{\mu} \end{aligned} \quad (5.30)$$

for different values of b_0 in the respective range.

5.3 Second approximation

Seperating the equations for $l = 0, 1, 2$ we get the following set of equations for $l_0 = 2$ from (5.18) and (5.19).

$$\left.\begin{aligned}
&\tfrac{4}{3}I_0^{+'} + 2I_1^{+'} + \tfrac{2}{3}I_2^{+'} - (I_0^+ + I_0^- + I_1^+ - I_1^-) = -2A \\
&\tfrac{4}{3}I_0^{-'} - 2I_1^{-'} + \tfrac{2}{3}I_2^{-'} + (I_0^+ + I_0^- + I_1^+ - I_1^-) = 2A \\
&(2I_0^{+'} + \tfrac{24}{5}I_1^{+'} + 4I_2^{+'}) - \left[(2+\tfrac{b_0}{8})I_0^+ - \tfrac{b_0}{8}I_0^- + (6+\tfrac{b_0}{8})I_1^+ + \tfrac{b_0}{8}I_1^- + 4I_2^+\right] \\
&= -2A + (\tfrac{b_0}{2}+12)A\tau \\
&(-2I_0^{-'} + \tfrac{24}{5}I_1^{-'} - 4I_2^{-'}) + \left[\tfrac{b_0}{8}I_0^+ - (2+\tfrac{b_0}{8})I_0^- + \tfrac{b_0}{8}I_1^+ + (6+\tfrac{b_0}{8})I_1^- - 4I_2^+\right] \\
&= -2A + (12-\tfrac{b_0}{2})A\tau \\
&(\tfrac{1}{3}I_0^{+'} + 2I_1^{+'} + \tfrac{80}{21}I_2^{+'}) + \left[\tfrac{5b_0}{16}I_0^+ - \tfrac{5b_0}{16}I_0^-(2-\tfrac{5b_0}{16})I_1^+ + \tfrac{5b_0}{16}I_1^- - 5I_2^+\right] \\
&= (10-\tfrac{5b_0}{4})A\tau \\
&(\tfrac{1}{3}I_0^{-'} - 2I_1^{-'} + \tfrac{80}{21}I_2^{-'}) + \left[\tfrac{5b_0}{16}I_0^+ - \tfrac{5b_0}{16}I_0^+\tfrac{5b_0}{16}I_1^+ - (2-\tfrac{5b_0}{16})I_1^- + 5I_2^-\right] \\
&= (10-\tfrac{5b_0}{4})A\tau
\end{aligned}\right\} \quad (5.31)$$

where

$$\left.\begin{aligned} I_+(\tau,\mu) &= A\tau + I_0^+(\tau)\mu + 3I_1^+(\tau)\mu P_1(2\mu-1) + 5I_2^+(\tau)\mu P_2(2\mu-1), \\ &\qquad 0 \le \mu \le 1 \\ I_-(\tau,\mu) &= A\tau + I_0^-(\tau)\mu + 3I_1^-(\tau)\mu P_1(2\mu+1) + 5I_2^-(\tau)\mu P_2(2\mu+1), \\ &\qquad -1 \le \mu \le 0 \end{aligned}\right\} \quad (5.32)$$

As previous we take the trial solution as

$$\left.\begin{aligned} I_l^+(\tau) &= A(g_{l,\alpha}e^{-k\tau} + g_{l,\beta}) \\ I_l^-(\tau) &= A(h_{l,\alpha}e^{-k\tau} + h_{l,\beta}) \end{aligned}\right\} \quad (5.33)$$

where $g_{l,\alpha}, g_{l,\beta}, h_{l,\alpha}, h_{l,\beta}$ are constants to be determined.

When substituted in (5.31) and the coefficient of $e^{k\tau}$ and constant term are equated, we obtain

$$\left.\begin{aligned} &(1+\tfrac{4k}{3})g_{0,\alpha} + (1+2k)g_{1,\alpha} + \tfrac{2k}{3}g_{2,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \\ &g_{0,\alpha} + g_{1,\alpha} + (1-\tfrac{4k}{3})h_{0,\alpha} + (1-2k)h_{1,\alpha} - \tfrac{2k}{3}h_{2,\alpha} = 0 \\ &(2+\tfrac{b_0}{8}+2k)g_{0,\alpha} + (6+\tfrac{b_0}{8}+\tfrac{24k}{5})g_{1,\alpha} + (4+4k)g_{2,\alpha} - \tfrac{b_0}{8}h_{0,\alpha} + \tfrac{b_0}{8}h_{1,\alpha} = 0 \\ &\tfrac{b_0}{8}g_{0,\alpha} + \tfrac{b_0}{8}g_{1,\alpha} + (2k-2-\tfrac{b_0}{8})h_{0,\alpha} + (6+\tfrac{b_0}{8}-\tfrac{24k}{5})h_{1,\alpha} + (-4+4k)h_{2,\alpha} = 0 \\ &(\tfrac{k}{3}-\tfrac{5b_0}{16})g_{0,\alpha} + (2+2k-\tfrac{5b_0}{16})g_{1,\alpha} + 5(1+\tfrac{16k}{21})g_{2,\alpha} + \tfrac{5b_0}{16}h_{0,\alpha} - \tfrac{5b_0}{16}h_{1,\alpha} = 0 \\ &\tfrac{5b_0}{16}g_{0,\alpha} + \tfrac{5b_0}{16}g_{1,\alpha} - (\tfrac{k}{3}+\tfrac{5b_0}{16})h_{0,\alpha} + (-2+\tfrac{5b_0}{16}+2k)h_{1,\alpha} + 5(1-\tfrac{16k}{21})h_{2,\alpha} = 0 \end{aligned}\right\} \quad (5.34)$$

and

$$\left.\begin{aligned} &g_{0,\beta} + g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = 2 \\ &(2+\tfrac{b_0}{8})g_{0,\beta} + (6+\tfrac{b_0}{8})g_{1,\beta} + 4g_{2,\beta} - \tfrac{b_0}{8}h_{0,\beta} + \tfrac{b_0}{8}h_{1,\beta} = 2 \\ &\tfrac{b_0}{8}g_{0,\beta} + \tfrac{b_0}{8}g_{1,\beta} - (2+\tfrac{b_0}{8})h_{0,beta} + (6+\tfrac{b_0}{8})h_{1,\beta} - 4h_{2,\beta} = -2 \\ &\tfrac{5b_0}{16}g_{0,\beta} - (2-\tfrac{5b_0}{16})g_{1,\beta} - 5g_{2,\beta} - \tfrac{5b_)}{16}h_{0,\beta} + \tfrac{5b_0}{16}h_{1,\beta} = 0 \\ &\tfrac{5b_)}{16}g_{0,\beta} + \tfrac{5b_)}{16}g_{1,\beta} - \tfrac{5b_0}{16}h_{0,\beta} - (2-\tfrac{5b_0}{16})h_{1,\beta} + 5h_{2,\beta} = 0 \end{aligned}\right\} \quad (5.35)$$

The above equations (5.34) have a non trivial solution if the determinant of the coefficient of the constant $g_{l,\alpha}$ and $h_{l,\alpha}$ is zero, that is

$$\Delta(k) = \begin{vmatrix} 1+\frac{4k}{3} & 1+2k & \frac{2k}{3} & 1 & -1 & 0 \\ 2+\frac{b_0}{8}+2k & 6+\frac{b_0}{8}+\frac{24k}{5} & 4+4k & -\frac{b_0}{8} & \frac{b_0}{8} & 0 \\ \frac{k}{3}-\frac{5b_0}{16} & 2+2k-\frac{5b_0}{16} & 5+\frac{80k}{21} & \frac{5b_0}{16} & -\frac{5b_0}{16} & 0 \\ 1 & 1 & 0 & 1-\frac{4k}{3} & -1+2k & -\frac{2k}{3} \\ \frac{b_0}{8} & \frac{b_0}{8} & 0 & 2k-2-\frac{b_0}{8} & 6+\frac{b_0}{8}-\frac{24k}{5} & -4+4k \\ \frac{5b_0}{16} & \frac{5b_0}{16} & -\frac{k}{3}-\frac{5b_0}{16} & 0 & -2+2k+\frac{5b_0}{16} & 5-\frac{80k}{21} \end{vmatrix} = 0 \tag{5.36}$$

The positive values of k are tabulated for different values of b_0 with width 0.25 in the interval $(1, 2)$. in table (III).

Since we have to satisfy the boundary condition

$$I_l^+ e^{-\tau} \to 0 \ as \ \tau \to \infty$$
$$I_l^+ e^{-\tau} \to 0 \ as \ \tau \to \infty$$

Using the boundary condition at the free surface (5.3), we obtain

$$\left.\begin{array}{l} h_{0,\alpha}+h_{0,\beta}=0 \\ h_{1,\alpha}+h_{1,\beta}=0 \\ h_{2,\alpha}+h_{2,\beta}=0 \end{array}\right\} \tag{5.37}$$

From (5.34), (5.35) and (5.37) we obtain the following values of $g_{l,\alpha}; g_{l,\beta}; h_{l,\alpha}; h_{l,\beta}$, $(l = 0, 1, 2)$ for different values of b_0 in $(1, 2)$.

Table-III

b_0	1	1.25	1.5	1.75	2
k	7.1571949	7.31399696	7.47262994	7.63608251	7.80247327
$g_{0,\alpha}$	0.018905	0.010358	0.007184	0.005409	0.004454
$g_{1,\alpha}$	-0.011916	-0.006625	-0.004711	-0.003523	-0.002901
$g_{2,\alpha}$	0.005134	0.002891	0.002131	0.001559	0.001294
$h_{0,\alpha}$	-0.041092	-0.018229	-0.010028	-0.006814	-0.005149
$h_{1,\alpha}$	0.007817	0.003923	0.002439	0.001822	0.001481
$h_{2,\alpha}$	0.021044	0.009719	0.5584	0.003934	0.003061
$g_{0,\beta}$	-79.666336	-33.620243	-19.978571	-14.352570	-11.298055
$g_{1,\beta}$	44.382442	18.866713	11.604844	8.606177	6.975809
$g_{2,\beta}$	-21.426832	-9.425854	-5.898335	-4.439305	-3.644617
$h_{0,\beta}$	3.135266	1.413850	0.787310	0.542781	0.417019
$h_{1,\beta}$	-0.442634	-0.362486	-0.310719	-0.284025	-0.266008
$h_{2,\beta}$	0.157310	0.153026	0.151598	0.150753	0.152040

since

$$\left.\begin{array}{l} I_+(\tau,\mu) = A\tau + I_0^+(\tau)\mu + 3I_1^+(\tau)\mu P_1(2\mu-1) + 5I_2^+(\tau)\mu P_2(2\mu-1), \\ \qquad\qquad 0 \le \mu \le 1 \\ I_-(\tau,\mu) = A\tau + I_0^-(\tau)\mu + 3I_1^-(\tau)\mu P_1(2\mu+1) + 5I_2^-(\tau)\mu P_2(2\mu+1), \\ \qquad\qquad -1 \le \mu \le 0 \end{array}\right\} \tag{5.38}$$

$$\left.\begin{array}{l} F = 2[\int_0^1 I_+(\tau,\mu)\mu d\mu + \int_{-1}^0 I_(\tau,\mu)\mu d\mu] \\ \quad = \frac{2}{3}[(I_0^+ + I_0^-) + \frac{3}{2}(I_1^+ - I_1^-) + \frac{1}{3}(I_2^+ + I_2^-)] \quad \cdots (a) \\ \quad = \frac{1}{2}[\frac{4}{3}(I_0^+ + I_0^-) + 2(I_1^+ - I_1^-) + \frac{2}{3}(I_2^+ + I_2^-)] \quad \cdots (b) \end{array}\right\} \tag{5.39}$$

From first two equation of (5.31) right hand side of (5.39b) turns out to a constant.

Substituting the values of $I_l^+, I_l^- (l = 0,1)$ in (5.39a) for any one tabulated values of $b_0 (1 \le b_0 \le 2)$ we also obtain $3F = A[4 + function of \tau]$

with this two result $A = \frac{3}{4}F$.

Source function and law of darkening are also same in first approximation which are given in (5.29) and (5.30).