Chapter 4

Solution of radiative transfer problem in thin atmosphere with double interval spherical harmonic method

4.1 The Equation and the Boundary Conditions

We consider that one free surface of a plane parallel finite medium, containing forwarded scattering particles suspended in a gas is irradiated by an isotropic radiation field I_0. The gas is neither absorbing nor emitting and is not bounded by opaque or reflecting walls.
The relevant transfer equation for this model is given by

$$\mu \frac{dI(\tau,\mu)}{d\tau} + I(\tau,\mu) = \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{4.1}$$

where $I(\tau,\mu)$ is the intensity, $p(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\mu,\Phi;\mu',\Phi')$ is the phase function giving the measure of the probability of a ray in the direction (μ',ϕ') being scattered into the direction (μ,ϕ) and $\tau = \int_x^\infty k\rho dz$ is the optical depth, k being the absorption coefficient, ρ the density , μ the cosine of the polar angle measured from the direction of increasing τ. Here we take the general phase function expanded in Legendre polynomials as follows:-

$$p(\mu,\mu') = \sum_k w_k P_k(\mu)P_k(\mu') \tag{4.2}$$

The transfer equation (4.1) is to be solved subject to the following boundary conditions:

$$I(0,\mu) = I_0(say)\ \ 0 \leq \mu \leq 1 \tag{4.3}$$

$$I(\tau,\mu) = 0,\ \ -1 \leq \mu \leq 0 \tag{4.4}$$

4.2 Method of Solution

If we follow Wan et al[61], we consider

$$I_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1)\ ,\ 0 \leq \mu \leq 1 \tag{4.5}$$

$$I_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1)\ ,\ -1 \leq \mu \leq 0 \tag{4.6}$$

as a series of legendre polynomials. Here $\phi(\tau)$ is a function of τ only, the nature of which depends on the extent of the medium and boundary conditions.

The transfer equation (4.1) can be written as

$$\mu\frac{dI_+(\tau,\mu)}{d\tau} + I_+(\tau,\mu) = \frac{1}{2}\int_0^1 p(\mu,\mu')I_+(\tau,\mu')d\mu' + \frac{1}{2}\int_{-1}^0 p(\mu,\mu')I_-(\tau,\mu')d\mu' \tag{4.7}$$

$$\mu\frac{dI_-(\tau,\mu)}{d\tau} + I_-(\tau,\mu) = \frac{1}{2}\int_0^1 p(\mu,\mu')I_+(\tau,\mu')d\mu' + \frac{1}{2}\int_{-1}^0 p(\mu,\mu')I_-(\tau,\mu')d\mu' \tag{4.8}$$

where we have assumed that the linearly anisotropic scattering phase function $p(\mu,\mu')$ is given by equation (4.2).

we also use the recurrence relation relation

$$\mu P_l(2\mu\pm1) = \frac{1}{(2l+1)}\left[\frac{l+1}{2}P_{l+1}(2\mu\pm1) \mp \frac{2l+1}{2}P_l(2\mu\pm1) + \frac{l}{2}P_{l-1}(2\mu\pm1)\right] \tag{4.9}$$

Now using phase function (4.2) and the recurrence relation (4.9) the equation of transfer (4.7) and (4.8) reduces to

$$\mu\frac{dI_+(\tau,\mu)}{d\tau} + I_+(\tau,\mu) \ = \ \frac{1}{2}\sum_{k=0} w_k P_k(\mu)[...] \tag{4.10}$$

and

$$\mu \frac{dI_-(\tau,\mu)}{d\tau} + I_-(\tau,\mu) = \frac{1}{2}\sum_{k=0} w_k P_k(\mu)[...] \qquad (4.11)$$

where

$$[...] = \phi(\tau)\int_1^{-1} p_k \mu' d\mu' + \sum_{l=0}^{l_0} \frac{I_l^+(\tau)}{2}\{...\}_1 + \sum_{l=0}^{l_0} \frac{I_l^+(\tau)}{2}\{...\}_2 \qquad (4.12)$$

$$\{...\}_1 = (l+1)P^+_{l+1,k} + (2l+1)P^+_{l,k} + lP^+_{l-1,k} \qquad (4.13)$$

$$\{...\}_2 = (l+1)P^-_{l+1,k} + (2l+1)P^-_{l,k} + lP^-_{l-1,k} \qquad (4.14)$$

$$P^+_{l,k} = \int_0^1 p_k(\mu)p_l(2\mu-1)d\mu \qquad (4.15)$$

$$P^-_{l,k} = \int_{-1}^0 p_k(\mu)p_l(2\mu+1)d\mu \qquad (4.16)$$

we now multiply both sides of the equation (4.10) and (4.11) by $p_l(2\mu-1)$ and $p_l(2\mu+1)$ respectively and integrate over μ in their respective intervals to get

$$\left.\begin{array}{l} \phi'(\tau)\int_0^1 \mu p_l(2\mu-1)d\mu + \phi(\tau)\int_0^1 \mu p_l(2\mu-1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I^{+'}_{l-2} + 2lI^{+'}_{l-1}\right. \\ \left. + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{+'} + 2(l+1)I^{+'}_{l+1} + \frac{l^2+3l+2}{2l+3}I^{+'}_{l+2}\right] \\ + \frac{1}{2(2l+1)}[lI^+_{l-1} + (2l+1)I^+_l + (l+1)I^+_{l+1}] = \frac{1}{2}\sum_{k=0}^{\infty} w_k P^+_{l,k}[...] \end{array}\right\} \qquad (4.17)$$

and

$$\left.\begin{array}{l} \phi'(\tau)\int_{-1}^0 \mu p_l(2\mu+1)d\mu + \phi(\tau)\int_{-1}^0 \mu p_l(2\mu+1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I^{-'}_{l-2} + 2lI^{-'}_{l-1}\right. \\ \left. + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{-'} + 2(l+1)I^{-'}_{l+1} + \frac{l^2+3l+2}{2l+3}I^{-'}_{l+2}\right] \\ + \frac{1}{2(2l+1)}[lI^-_{l-1} - (2l+1)I^-_l + (l+1)I^-_{l+1}] = \frac{1}{2}\sum_{k=0}^{\infty} w_k P^-_{l,k}[...] \end{array}\right\} \qquad (4.18)$$

we now use the relation (4.15) and (4.16) to reduce the R.H.S of (4.17) as follows

$$\sum_{k=0}^{\infty} W_k P^+_{l,k}[...] = \delta_{0l}[2\phi(\tau) + \frac{1}{2}(I_0^+ + I_1^+ - I_0^- + I_1^-)] + \sum_{k=1}^{\infty} w_k P^+_{l,k}[...] \qquad (4.19)$$

Similarly, R.H.S of (4.18) reduces to

$$\sum_{k=0}^{\infty} W_k P^-_{l,k}[...] = \delta_{0l}[2\phi(\tau) + \frac{1}{2}(I_0^+ + I_1^+ - I_0^- + I_1^-)] + \sum_{k=1}^{\infty} w_k P^-_{l,k}[...] \qquad (4.20)$$

Where δ_{0l} = kronecker delta function

We now separate the equation (4.17) and (4.18) for $l = 0$ and $l = 1$ from the rest of the equations as

$$\left.\begin{array}{l}
\underline{for\ l = o} \\
\left(\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'}\right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2\phi'(\tau) + 2\sum_{k=1}^{\infty} w_k P_{0,k}^+[...] \\
\\
\underline{for\ l = 1} \\
\left(2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} + \frac{6}{5}I_3^{+'}\right) + (I_0^+ + 3I_1^+ + 2I_2^+) = -2\phi'(\tau) \\
+6\sum_{k=1}^{\infty} w_k P_{1,k}^+[...] \\
\\
\underline{for\ l \neq 0, 1} \\
\frac{1}{2}\left[\frac{l^2-l}{2l-1}I_{l-2}^{+'} + 2lI_{l-1}^{+'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{+'} + 2(l+1)I_{l+1}^{+'} + \frac{l^2+3l+2}{2l+3}I_{l+2}^{+'}\right] \\
-[lI_{l-1}^+ + (2l+1)I_l^+ + (l+1)I_{l+1}^+] = (2l+1)\sum_{k=1}^{\infty} w_k P_{l,k}^+[...]
\end{array}\right\} \quad (4.21)$$

and

$$\left.\begin{array}{l}
\underline{for\ l = o} \\
\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'}\right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2\phi'(\tau) + 2\sum_{k=1}^{\infty} w_k P_{0,k}^-[...] \\
\\
\underline{for\ l = 1} \\
-\left(2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} + \frac{6}{5}I_3^{-'}\right) + 2(I_0^- - 3I_1^- + 2I_2^-) = -2\phi'(\tau) \\
+6\sum_{k=1}^{\infty} w_k P_{1,k}^-[...] \\
\\
\underline{for\ l \neq 0, 1} \\
\frac{1}{2}\left[\frac{l^2-l}{2l-1}I_{l-2}^{-'} - 2lI_{l-1}^{-'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{-'} - 2(l+1)I_{l+1}^{-'} + \frac{l^2+3l+2}{2l+3}I_{l+2}^{-'}\right] \\
+[lI_{l-1}^- - (2l+1)I_l^- + (l+1)I_{l+1}^-] = (2l+1)\sum_{k=1}^{\infty} w_k P_{l,k}^-[...]
\end{array}\right\} \quad (4.22)$$

The above differential equations are to be solved in the desired approximation with the boundary conditions stated in equations (4.3) and (4.4). It is assumed that when we are working in the l-th approximation,

$I_{l+1}^+ = I_{l+1}^- = 0$

4.3 First approximate solution for scattering with Rayleigh phase function

Here we demonstrate the method with the particular case of scattering with Rayleigh's phase function which given by

$$\begin{aligned} p(\mu, \mu') &= \frac{3}{4}(1 + cos^2\theta), \ \textit{where } \theta \textit{ ,the angle of scattering} \\ &= 1 + \frac{1}{2}P_2(\mu)P_2(\mu') \end{aligned} \tag{4.23}$$

Relation (4.23) can thus be obtained from the general relation (4.2) by putting $w_0 = 1; w_1 = 0; w_2 = \frac{1}{2}; w_n = 0$ for $n > 2$. First approximate solution of the transfer equation with Rayleigh's phase function has been discussed and obtained (Karanjai & Biswas) already. Here we find that putting $w_0 = 1; w_1 = 0; w_2 = \frac{1}{2}; w_n = 0$ for $n > 2$ in equation (4.21) and (4.22) we get the same set of equations (17a)-(17d) obtained by Karanjai and Biswas[30] and thus the first approximate solution for Rayleigh's phase function obtained from this solution with general phase function is same as obtained by Karanjai & Biswas[30].

4.4 Second approximate solution for scattering with Rayleigh phase function

We name the solution the second approximation when $l_0 = 2$ and in this case it is assumed that $I_3^+ = I_3^- = 0$. So as in section 4 in the second approximation we obtain the following differential equations

$$\left.\begin{array}{l}
\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2\phi'(\tau) \\
\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2\phi'(\tau) \\
2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} + \frac{61}{32}I_0^+ + \frac{909}{160}I_1^+ + \frac{119}{32}I_2^+ + \frac{3}{32}I_0^- - \frac{51}{160}I_1^- + \frac{9}{32}I_2^- \\
= -2\phi'(\tau) \\
-2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} + \frac{3}{32}I_0^+ + \frac{51}{160}I_1^+ + \frac{9}{32}I_2^+ + \frac{61}{32}I_0^- - \frac{909}{160}I_1^- + \frac{119}{32}I_2^- \\
= -2\phi'(\tau) \\
\frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} - \frac{1}{64}I_0^+ + \frac{623}{320}I_1^+ + \frac{317}{64}I_2^+ + \frac{1}{64}I_0^- - \frac{17}{320}I_1^- + \frac{3}{64}I_2^- = 0 \\
\frac{1}{3}I_0^{-'} - 2I_1^{-'} + \frac{80}{21}I_2^{-'} - \frac{1}{64}I_0^+ - \frac{17}{320}I_1^+ - \frac{3}{64}I_2^+ + \frac{1}{64}I_0^- + \frac{623}{320}I_1^- - \frac{317}{64}I_2^- = 0
\end{array}\right\} \tag{4.24}$$

where

$$\left.\begin{array}{l}
I_+(\tau,\mu) = \phi(\tau) + I_0^+(\tau)\mu + 3I_1^+(\tau)\mu P_1(2\mu-1) + 5I_2^+(\tau)\mu P_2(2\mu-1), \\
\qquad\qquad 0 \le \mu \le 1 \\
I_-(\tau,\mu) = \phi(\tau) + I_0^-(\tau)\mu + 3I_1^-(\tau)\mu P_1(2\mu+1) + 5I_2^-(\tau)\mu P_2(2\mu+1), \\
\qquad\qquad -1 \le \mu \le 0
\end{array}\right\} \tag{4.25}$$

We now take the trial solution as

$$I_l^+ = g_{l,\alpha}e^{-k\tau} + g_{l,\beta}e^{-\tau} + g_{l,\gamma}e^{\tau} \tag{4.26}$$

$$I_l^- = h_{l,\alpha}e^{-k\tau} + h_{l,\beta}e^{-\tau} + h_{l,\gamma}e^{\tau} \tag{4.27}$$

where $k \neq 1$.

For thin atmosphere following Wan et al[61] we have

$$\phi(\tau) = Ae^{-\tau} + Be^{\tau} \tag{4.28}$$

$$A = \frac{e^{2\tau_0}}{e^{2\tau_0} - 1} \tag{4.29}$$

$$B = \frac{1}{1 - e^{2\tau_0}} \tag{4.30}$$

We now substitute (4.26), (4.27) and (4.28) in the set of equations (4.24) and equating the coefficients of $e^{-k\tau}$ from both sides, we obtain

$$\left.\begin{array}{l}
(1-\frac{4k}{3})g_{0,\alpha}+(1-2k)g_{1,\alpha}-\frac{2k}{3}g_{2,\alpha}+h_{0,\alpha}-h_{1,\alpha} \\
g_{0,\alpha}+g_{1,\alpha}+(1+\frac{4k}{3})h_{0,\alpha}-(1+2k)h_{1,\alpha}+\frac{2k}{3}h_{2,\alpha}=0 \\
(2k-\frac{61}{32})g_{0,\alpha}+(\frac{24k}{5}-\frac{909}{160})g_{1,\alpha}+(4k-\frac{119}{32})g_{2,\alpha}-\frac{3}{32}h_{0,\alpha}+\frac{51}{160}h_{1,\alpha}-\frac{9}{32}h_{2,\alpha}=0 \\
\frac{3}{32}g_{0,\alpha}+\frac{51}{160}g_{1,\alpha}+\frac{9}{32}g_{2,\alpha}-(2k+\frac{61}{32})h_{0,\alpha}-(\frac{24k}{5}+\frac{909}{160})h_{1,\alpha}+(4k+\frac{119}{32})h_{2,\alpha}=0 \\
(\frac{k}{3}+\frac{1}{64})g_{0,\alpha}+(2k-\frac{623}{320})g_{1,\alpha}+(\frac{80k}{21}-\frac{317}{64})g_{2,\alpha}-\frac{1}{64}h_{0,\alpha}+\frac{17}{320}h_{1,\alpha}-\frac{3}{64}h_{2,\alpha}=0 \\
\frac{1}{64}g_{0,\alpha}+\frac{17}{320}g_{1,\alpha}+\frac{3}{64}g_{2,\alpha}+(\frac{k}{3}-\frac{1}{64})h_{0,\alpha}-(2k+\frac{623}{320})h_{1,\alpha}+(\frac{80k}{21}+\frac{317}{64})h_{2,\alpha}=0
\end{array}\right\} \tag{4.31}$$

The above set of equations has non-trivial solution, if

$$\Delta(k)=0 \tag{4.32}$$

where $\Delta(k)=$

$$\begin{vmatrix}
(1-\frac{4k}{3}) & (1-2k) & -\frac{2k}{3} & 1 & -1 & 0 \\
1 & 1 & 0 & (1+\frac{4k}{3}) & -(1+2k) & \frac{2k}{3} \\
(2k-\frac{61}{32}) & (\frac{24k}{5}-\frac{909}{160}) & (4k-\frac{119}{32}) & -\frac{3}{32} & \frac{51}{160} & -\frac{9}{32} \\
\frac{3}{32} & \frac{51}{160} & \frac{9}{32} & (2k+\frac{61}{32}) & -(\frac{24k}{5}+\frac{909}{160}) & (4k+\frac{119}{32}) \\
(\frac{k}{3}+\frac{1}{64}) & (2k-\frac{623}{320}) & (\frac{80k}{21}-\frac{317}{64}) & -\frac{1}{64} & \frac{17}{320} & -\frac{3}{64} \\
\frac{1}{64} & \frac{17}{320} & -\frac{3}{64} & (\frac{k}{3}-\frac{1}{64}) & -(2k+\frac{623}{320}) & (\frac{80k}{21}+\frac{317}{64})
\end{vmatrix} \tag{4.33}$$

and $\Delta(k)=0$ yields
$k=0,\ 1.35634131,\ -1.83136563,\ -1.10538893$
We take $k=1.35634131$
Again substituting (4.26), (4.27) and (4.28) in the set of equations (4.24)

and equating the coefficients of $e^{-\tau}$ from the both sides, we obtain

$$\left.\begin{array}{l}
\frac{1}{3}g_{0,\beta} + g_{1,\beta} + \frac{2}{3}g_{2,\beta} - h_{0,\beta} + h_{1,\beta} = 2A \\
g_{0,\beta} + g_{1,\beta} + \frac{7}{3}h_{0,\beta} - 3h_{1,\beta} + \frac{2}{3}h_{2,\beta} = 2A \\
3g_{0,\beta} - \frac{141}{5}g_{1,\beta} + 9g_{2,\beta} - 3h_{0,\beta} + \frac{51}{5}h_{1,\beta} + 9h_{2,\beta} = -64A \\
3g_{0,\beta} - \frac{51}{5}g_{1,\beta} + 9g_{2,\beta} + 125h_{0,\beta} - \frac{1677}{5}h_{1,\beta} + 247h_{2,\beta} = 64A \\
\frac{67}{3}g_{0,\beta} + \frac{17}{5}g_{1,\beta} - \frac{1537}{21}g_{2,\beta} - h_{0,\beta} + \frac{17}{5}h_{1,\beta} - 3h_{2,\beta} = 0 \\
g_{0,\beta} + \frac{17}{5}g_{1,\beta} + 3g_{2,\beta} + \frac{61}{3}h_{0,\beta} - \frac{1263}{5}h_{1,\beta} + \frac{11777}{21}h_{2,\beta} = 0
\end{array}\right\} \quad (4.34)$$

solving (4.34) we obtain

$$\left.\begin{array}{l}
g_{0,\beta} = -10.763859A \\
g_{1,\beta} = -2.439640A \\
g_{2,\beta} = -3.452224A \\
h_{0,\beta} = -15.060650A \\
h_{1,\beta} = -8.418612 \\
h_{2,\beta} = -3.193409
\end{array}\right\} \quad (4.35)$$

and the constant A is given by (4.29).

$$\left.\begin{array}{l}
\frac{7}{3}g_{0,\gamma} + 3g_{1,\gamma} + \frac{3}{3}g_{2,\gamma} + h_{0,\gamma} - h_{1,\gamma} = 2B \\
g_{0,\gamma} + g_{1,\gamma} - \frac{1}{3}h_{0,\gamma} + h_{1,\gamma} - \frac{2}{3}h_{2,\gamma} = -2B \\
125g_{0,\gamma} + \frac{1677}{5}g_{1,\gamma} + 247g_{2,\gamma} + h_{0,\gamma} - \frac{51}{5}h_{1,\gamma} + 9h_{2,\gamma} = -64B \\
3g_{0,\gamma} + \frac{51}{5}g_{1,\gamma} + g_{2,\gamma} - 3h_{0,\gamma} - \frac{141}{5}h_{1,\gamma} - 9h_{2,\gamma} = -64B \\
\frac{61}{3}g_{0,\gamma} + \frac{1263}{5}g_{1,\gamma} + \frac{11777}{21}g_{2,\gamma} + h_{0,\gamma} - \frac{76}{3}h_{0,\gamma} - \frac{17}{5}h_{1,\gamma} + 3h_{2,\gamma} = 0 \\
g_{0,\gamma} + \frac{17}{5}g_{1,\gamma} + 3g_{2,\gamma} - \frac{67}{3}h_{0,\gamma} + \frac{17}{5}h_{1,\gamma} + \frac{1537}{21}h_{2,\gamma} = 0
\end{array}\right\} \quad (4.36)$$

$$\left.\begin{aligned} g_{0,\gamma} &= 1.136095B \\ g_{1,\gamma} &= -0.955960B \\ g_{2,\gamma} &= 0.376548B \\ h_{0,\gamma} &= 0.000001B \\ h_{1,\gamma} &= -2.023598 \\ h_{2,\gamma} &= 0.107666 \end{aligned}\right\} \tag{4.37}$$

and the constant B is given by (4.30).
Applying boundary condition on (4.26) & (4.27) we have

$$\left.\begin{aligned} g_{l,\alpha} + g_{l,\beta} + h_{l,\gamma} &= 0 \\ h_{l,\alpha} + h_{l,\beta} + h_{l,\gamma} &= 0 \end{aligned} \quad for\ all\ l\ at\ \tau = 0\right\} \tag{4.38}$$

Thus

$$\left.\begin{aligned} g_{0,\alpha} &= 10.763859A - 1.136095B \\ g_{1,\alpha} &= 2.439640A + 0.955960B \\ g_{2,\alpha} &= 3.452224A - 0.376548B \\ h_{0,\alpha} &= 15.060650A - 0.000001B \\ h_{1,\alpha} &= 10.44221 \\ h_{2,\alpha} &= 3.085743 \end{aligned}\right\} \tag{4.39}$$

where A and B are given by (4.29) and (4.30) and the complete solution can thus be obtained.

4.5 First approximate solution in the case of scattering with the phase function $p(\mu, \mu') = 1 + w_1 p_1(\mu) p_1(\mu') + w_2 p_2(\mu) p_2(\mu')$

In this section we compare this phase function with the general phase function (4.2), we obtain $w_0 = 1$ and $w_k = 0$ for $k > 2$. Using these values the differential equations (4.21) and (4.22) reduces to

$$\left.\begin{array}{l}
\frac{4}{3}I_0^{+'} + 2I_1^{+'} + (I_0^+ + I_1^+ + I_0^- - I_1^-) - w_1[\frac{1}{3}(I_0^+ + I_0^-) + \frac{1}{2}(I_1^+ - I_1^-)] = -2\phi'(\tau) \\
\frac{4}{3}I_0^{-'} - 2I_1^{-'} - (I_0^+ + I_1^+ + I_0^- - I_1^-) + w_1[\frac{1}{3}(I_0^+ + I_0^-) + \frac{1}{2}(I_1^+ - I_1^-)] = 2\phi'(\tau) \\
2I_0^{+'} + \frac{24}{5}I_1^{+'} + 6(\frac{1}{3}I_0^+ + I_1^+) - w_1[\frac{1}{3}(I_0^+ + I_0^-) + \frac{1}{2}(I_1^+ - I_1^-)] \\
\qquad -\frac{3w_2}{16}[(I_0^+ - I_0^-) + \frac{17}{5}(I_1^+ + I_1^-)] = -2\phi'(\tau) \\
-2I_0^{-'} + \frac{24}{5}I_1^{-'} + 6(\frac{1}{3}I_0^- - I_1^-) - w_1[\frac{1}{3}(I_0^+ + I_0^-) + \frac{1}{2}(I_1^+ - I_1^-)] \\
\qquad +\frac{3w_2}{16}[(I_0^+ - I_0^-) + \frac{17}{5}(I_1^+ + I_1^-)] = -2\phi'(\tau)
\end{array}\right\} \tag{4.40}$$

The above set of differential equations is to be solved under the same boundary conditions (3) and (4) which in the present case can be written as

$$\phi(0) = 1 \ and \ \phi(\tau_0) = 0 \tag{4.41}$$

$$I_0^+(0) = 0 = I_1^+(0) \tag{4.42}$$

$$I_0^-(\tau_0) = 0 = I_1^-(\tau_0) \tag{4.43}$$

The boundary condition (4.43) is essentially for thin atmosphere. Here, $\phi(\tau)$ is the μ- independent term in the representation of intensity and according to Wan et al [61] $\phi(\tau)$ can be chosen as

$$\phi(\tau) = Ae^{-\tau} + Be^{\tau} \tag{4.44}$$

for thin atmosphere. Here A and B are constants, the values of which in the case of thin atmosphere can be obtained from equation (4.41) and (4.44) as

$$A = \frac{e^{2\tau_0}}{e^{2\tau_0} - 1} \tag{4.45}$$

$$B = \frac{1}{1 - e^{2\tau_0}} \tag{4.46}$$

According to Wan et al[60], we take the trial solution for the set of equations (4.40) as

$$I_l^+(\tau) = g_{l,\alpha}e^{-k\tau} + g_{l,\beta}e^{-\tau} + g_{l,\gamma}e^{\tau} \tag{4.47}$$

$$I_l^-(\tau) = h_{l,\alpha}e^{-k\tau} + h_{l,\beta}e^{-\tau} + h_{l,\gamma}e^{\tau} \tag{4.48}$$

We now substitute (4.44).(4.47) & (4.48) in the set of equations (4.40) and then equating the coefficients of $e^{-k\tau}$ from both sides we have

$$\left.\begin{array}{l}
(1-\frac{4k}{3}-\frac{w_1}{3})g_{0,\alpha}+(1-2k-\frac{w_1}{2})g_{1,\alpha}+(1-\frac{w_1}{3})h_{0,\alpha}+(\frac{w_1}{2}-1)h_{1,\alpha}=0\\
(\frac{w_1}{3}-1)g_{0,\alpha}+(\frac{w_1}{2}-1)g_{1,\alpha}+(\frac{w_1}{3}-\frac{4k}{3}-1)h_{0,\alpha}+(1+2k-\frac{w_1}{2})h_{1,\alpha}=0\\
(2-2k-\frac{w_1}{3}-\frac{3w_2}{16})g_{1,\alpha}+(6-\frac{24k}{5}-\frac{w_1}{2}-\frac{51w_2}{80})g_{1,\alpha}-(\frac{w_1}{3}-\frac{3W_2}{16})h_{0,\alpha}+\\
(\frac{w_1}{2}-\frac{51w_2}{80})h_{1,\alpha}=0\\
(\frac{3w_2}{16}-\frac{w_1}{3})g_{0,\alpha}+(\frac{51w_2}{80}-\frac{w_1}{2})g_{1,\alpha}+(2+2k-\frac{w_1}{3}-\frac{3w_2}{16})h_{0,\alpha}+(\frac{w_1}{2}+\frac{51w_2}{80}\\
-\frac{24k}{5}-6)h_{1,\alpha}=0
\end{array}\right\} \tag{4.49}$$

For the above set of equations to have non-trivial solution

$$\Delta(k) = 0 \tag{4.50}$$

where $\Delta(k) =$

$$\begin{vmatrix}
(1-\frac{4k}{3}-\frac{w_1}{3}) & (1-2k-\frac{w_1}{2}) & (1-\frac{w_1}{3}) & (\frac{w_1}{2}-1)\\
(\frac{w_1}{3}-1) & (\frac{w_1}{2}-1) & (\frac{w_1}{3}-\frac{4k}{3}-1) & (1+2k-\frac{w_1}{2})\\
(2-2k-\frac{w_1}{3}-\frac{3w_2}{16}) & (6-\frac{24k}{5}-\frac{w_1}{2}-\frac{51w_2}{80}) & -(\frac{w_1}{3}-\frac{3w_2}{16}) & (\frac{w_1}{2}-\frac{51w_2}{80})\\
(\frac{3w_2}{16}-\frac{w_1}{3}) & (\frac{51w_2}{80}-\frac{w_1}{2}) & (2+2k-\frac{w_1}{3}-\frac{3w_2}{16}) & (\frac{w_1}{2}+\frac{51w_2}{80}-\frac{24k}{5}-6)
\end{vmatrix} \tag{4.51}$$

from which k can be determined.

Again, substituting (4.44) , (4.47) & (4.48) in the set of equations (4.40) and then equating the coefficients $e^{-\tau}$ from the both sides, we obtain

$$\left.\begin{array}{l}
\frac{1}{3}(w_1+1)g_{0,\beta}+(\frac{w_1}{2}+1)g_{1,\beta}+(\frac{w_1}{3}-1)h_{0,\beta}+(1-\frac{w_1}{2})h_{1,\beta}=-2A \\
(\frac{w_1}{3}-1)g_{0,\beta}+(\frac{w_1}{2}-1)g_{1,\beta}+\frac{1}{3}(w_1-7)h_{0,\beta}+(3-\frac{w_1}{2})h_{1,\beta}=-2A \\
(\frac{w_1}{3}+\frac{3w_2}{16})g_{0,\beta}+(\frac{w_1}{2}+\frac{51w_2}{80}-\frac{6}{5})g_{1,\beta}+(\frac{w_1}{3}-\frac{3w_2}{16})h_{0,\beta}+(\frac{51w_2}{80}-\frac{w_1}{2})h_{1,\beta} \\
=-2A \\
(\frac{w_1}{3}-\frac{3w_2}{16})g_{0,\beta}+(\frac{w_1}{2}+\frac{51w_2}{80})g_{1,\beta}+(\frac{w_1}{3}+\frac{3w_2}{16}-4)h_{0,\beta}+(\frac{51w_2}{80}-\frac{w_1}{2}-\frac{51w_2}{80})h_{1,\beta} \\
=-2A
\end{array}\right\} \tag{4.52}$$

where A is given by (4.45). From this set of equations the coefficients $(g_{0,\beta}, g_{1,\beta}, h_{0,\beta}, h_{1,\beta})$ can be determined.
Similarly, following the same procedure as above and equating the coefficients of e^{τ} from the both sides, we obtain

$$\left.\begin{array}{l}
\frac{1}{3}(7-w_1)g_{0,\gamma}+(3-w_1)g_{1,\gamma}+(1-\frac{w_1}{3})h_{0,\gamma}+(\frac{w_1}{2}-1)h_{1,\gamma}=-2B \\
(1-\frac{w_1}{3})g_{0,\gamma}+(1-\frac{w_1}{2})g_{1,\gamma}-\frac{1}{3}(w_1+1)h_{0,\gamma}+(\frac{w_1}{2}+1)h_{1,\gamma}=2B \\
(4-\frac{w_1}{3}-\frac{3w_2}{16})g_{0,\gamma}+(\frac{54}{5}-\frac{w_1}{2}-\frac{5w_2}{80})g_{1,\gamma}+(\frac{3w_2}{16}-\frac{w_1}{3})h_{0,\gamma}+(\frac{w_1}{2}-\frac{51w_2}{80})h_{1,\gamma} \\
=-2B \\
(\frac{3w_2}{16}-\frac{w_1}{3})g_{0,\gamma}+(\frac{51w_2}{80}-\frac{w_1}{2})g_{1,\gamma}-(\frac{w_1}{3}+\frac{3w_2}{16})h_{0,\gamma}+(\frac{w_1}{2}+\frac{51w_2}{80}-\frac{6}{5})h_{1,\gamma} \\
=-2B
\end{array}\right\} \tag{4.53}$$

From this set of equations the coefficients $(g_{0,\gamma}, g_{1,\gamma}, h_{0,\gamma}, h_{1,\gamma})$ can be determined.

4.6 Second approximate solution in the case of scattering with the phase function $p(\mu,\mu') = 1 + w_1 p_1(\mu)p_1(\mu') + w_2 p_2(\mu)p_2(\mu')$

Following the same procedure as in section (4.5), we obtain the following differential equations from equations (4.21) and (4.22)

$$\left.\begin{array}{l}
\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} + (\frac{w_1}{3}+1)(I_0^+ + I_)^-) + (\frac{w_2}{2}+1)(I_1^+ - I_1^-) + \frac{w_1}{6}(I_2^+ + I_2^-) \\
= -2\phi'(\tau) \\
\\
\frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'} + (\frac{w_1}{3}+1)(I_0^+ + I_)^-) + (\frac{w_2}{2}+1)(I_1^+ - I_1^-) + \frac{w_1}{6}(I_2^+ + I_2^-) \\
= -2\phi'(\tau) \\
\\
2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} - (\frac{w_1}{3} + \frac{3w_2}{16} - 2)I_0^+ - (\frac{w_1}{2} + \frac{51w_2}{80} - 6)I_1^+ \\
-(\frac{w_1}{6} + \frac{9w_2}{16} - 4)I_2^+ - (\frac{w_1}{3} - \frac{3w_2}{16})I_0^- + (\frac{w_1}{2} - \frac{51w_2}{80})I_1^- - (\frac{w_1}{6} - \frac{9w_2}{16})I_2^- \\
= -2\phi'(\tau) \\
\\
-2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} - (\frac{w_1}{3} - \frac{3w_2}{16})I_0^+ - (\frac{w_1}{2} - \frac{51w_2}{80})I_1^+ - (\frac{w_1}{6} - \frac{9w_2}{16})I_2^+ \\
-(\frac{w_1}{3} - \frac{3w_2}{16} - 2)I_0^- + (\frac{w_1}{2} + \frac{51w_2}{80} - 6)I_1^- - (\frac{w_1}{6} + \frac{9w_2}{16} - 4)I_2^- \\
= -2\phi'(\tau) \\
\\
\frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} + (2I_1^+ + 5I_2^+) - \frac{w_2}{32}[(I_0^+ - I_0^-) + \frac{17}{5}(I_1^+ + I_1^-) \\
+3(I_2^+ - I_2^-)] = 0 \\
\\
\frac{1}{3}I_0^{-'} - 2I_1^{-'} + \frac{80}{21}I_2^{-'} + (2I_1^- + 5I_2^-) - \frac{w_2}{32}[(I_0^+ - I_0^-) + \frac{17}{5}(I_1^+ + I_1^-) \\
+3(I_2^+ - I_2^-)] = 0
\end{array}\right\} \quad (4.54)$$

The above differential equations can be solved by the same procedure as in section (4.5).

4.7 First approximate solution for scattering with Pomraning phase function

Here we demonstrate the method with the particular case of scattering with Pomraning phase function which is given by

$$p(\mu,\mu') = 1 + \frac{\alpha}{2}P_2(\mu)P_2(\mu') \tag{4.55}$$

where $\alpha = \frac{5\lambda}{5-3\lambda}$ and λ,the albedo of single scattering.
In this section we compare this phase function with general phase function (2) and obtained $w_0 = 1; w_1 = 0; w_2 = \frac{\alpha}{2}; w_n = 0\ for\ n > 2$. Using this values the differential equation (4.21), (4.22) reduces to

$$\left.\begin{aligned}
&\tfrac{4}{3}I_0^{+'} + 2I_1^{+'} + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2\phi'(\tau)\\
&\tfrac{4}{3}I_0^{-'} - 2I_1^{-'} - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2\phi'(\tau)\\
&2I_0^{+'} + \tfrac{24}{5}I_1^{+'} + 6(\tfrac{1}{3}I_0^+ + I_1^+) - \tfrac{3\alpha}{32}[(I_0^+ - I_0^-) + \tfrac{17}{5}(I_1^+ + I_1^-)] = -2\phi'(\tau)\\
&-2I_0^{-'} + \tfrac{24}{5}I_1^{-'} + 6(\tfrac{1}{3}I_0^- - I_1^-) + \tfrac{3\alpha}{32}[(I_0^+ - I_0^-) + \tfrac{17}{5}(I_1^+ + I_1^-)] = -2\phi'(\tau)
\end{aligned}\right\} \tag{4.56}$$

where

$$\left.\begin{aligned}
I_+(\tau,\mu) &= \phi(\tau) + I_0^+(\tau)\mu + 3I_1^+(\tau)\mu P_1(2\mu-1), 0 \le \mu \le 1\\
I_-(\tau,\mu) &= \phi(\tau) + I_0^-(\tau)\mu + 3I_1^-(\tau)\mu P_1(2\mu+1), -1 \le \mu \le 0
\end{aligned}\right\} \tag{4.57}$$

The above set of differential equations is to be solved under the same boundary condition (4.3) and (4.4) which in the present case can be written as

$$\phi(0) = 1\ and\ \phi(\tau_0) = 0 \tag{4.58}$$

$$I_0^+(0) = 0 = I_1^+(0) \tag{4.59}$$

$$I_0^-(\tau_0) = 0 = I_1^-(\tau_0) \tag{4.60}$$

The boundary condition (4.28) is essentially for thin atmosphere. Here $\phi(\tau)$ is the μ-independent term in the representation of intensity and following to Wan et al[61] $\phi(\tau)$ can be chosen as

$$\phi(\tau) = Ae^{-\tau} + Be^{\tau} \tag{4.61}$$

for thin atmosphere.
Here A and B are constants, the values of which, in this case of thin atmosphere, can be obtained from equations (4.26) and (4.29) as

$$A = \frac{e^{2\tau_0}}{e^{2\tau_0} - 1} \tag{4.62}$$

$$B = \frac{1}{1 - e^{2\tau_0}} \tag{4.63}$$

Following Wan et al[61] we take the trial solution for the set of equation (4.24) as

$$I_l^+ = g_{l,a}e^{-k\tau} + g_{l,b}e^{-\tau} + g_{l,c}e^{\tau} \tag{4.64}$$

$$I_l^- = h_{l,a}e^{-k\tau} + h_{l,b}e^{-\tau} + h_{l,c}e^{\tau} \tag{4.65}$$

where $k \neq 1$.
We now substitute (4.29),(4.32),(4.33) in the set of equations (4.24) and then equating the coefficients of $e^{-k\tau}$ from the both sides we have

$$\left.\begin{aligned}
&(1-\tfrac{4k}{3})g_{0,a} + (1-2k)g_{1,a} + h_{0,a} - h_{1,a} = 0\\
&-g_{0,a} - g_{1,a} - (1+\tfrac{4k}{3})h_{0,a} + (1+2k)h_{1,a} = 0\\
&(2-2k-\tfrac{3\alpha}{32})g_{0,a} + (6-\tfrac{24k}{5}-\tfrac{51\alpha}{160})g_{1,a} + \tfrac{3\alpha}{32}h_{0,a} - \tfrac{51\alpha}{160}h_{1,a} = 0\\
&\tfrac{3\alpha}{32}g_{0,a} + \tfrac{51\alpha}{160}g_{1,a} + (2+2k-\tfrac{3\alpha}{32})h_{0,a} + (\tfrac{51\alpha}{160}-\tfrac{24k}{5}-6)h_{1,a} = 0
\end{aligned}\right\} \tag{4.66}$$

The above set of equations has non-trivial solution, if

$$\Delta(k) = 0 \tag{4.67}$$

where

$$\Delta(k) = \begin{vmatrix}
(1-\frac{4k}{3}) & (1-2k) & 1 & -1\\
-1 & -1 & (-1-\frac{4k}{3}) & (1+2k)\\
(2-2k-\frac{3\alpha}{32}) & (6-\frac{24k}{5}-\frac{51\alpha}{160}) & \frac{3\alpha}{32} & -\frac{51\alpha}{160}\\
\frac{3\alpha}{32} & \frac{51\alpha}{160} & (2+2k-\frac{3\alpha}{32}) & (\frac{51\alpha}{160}-\frac{24k}{5}-6)
\end{vmatrix} \tag{4.68}$$

and $\Delta(k) = 0$ yields $k = 0, 0, -0.416667\sqrt{19.2 - 2.4\alpha}, 0.416667\sqrt{19.2 - 2.4\alpha}$.

Again substituting (4.29),(4.32),(4.33) in the set of equations (4.24) and then equating the coefficients of $e^{-\tau}$ from the both sides we obtain

$$\left.\begin{array}{l} \frac{1}{3}g_{0,b} + g_{1,b} - h_{0,b} + h_{1,b} = -2A \\ -g_{0,b} - g_{1,b} - \frac{7}{3}h_{0,b} + 3h_{1,b} = -2A \\ \frac{3\alpha}{32}g_{0,b} + (\frac{51\alpha}{160} - \frac{6}{5})g_{1,b} - \frac{3\alpha}{32}h_{0,b} + \frac{51\alpha}{160}h_{1,b} = -2A \\ -\frac{3\alpha}{32}g_{0,b} + \frac{51\alpha}{160}g_{1,b} + (\frac{3\alpha}{32} - 4)h_{0,b} + (\frac{54}{5} - \frac{51\alpha}{160})h_{1,b} = -2A \end{array}\right\} \quad (4.69)$$

where A is given by (4.29).

Solving (4.37) we obtain

$$\left.\begin{array}{l} g_{0,b} = \frac{-182272A+20320A\alpha-969A\alpha^2}{43008-7680\alpha+901\alpha^2} \\ g_{1,b} = \frac{-1280(-168A+17A\alpha)}{3(43008-7680\alpha+901\alpha^2)} \\ h_{0,b} = \frac{141312A-28640A\alpha+2499A\alpha^2}{43008-7680\alpha+901\alpha^2} \\ h_{1,b} = \frac{20(6656A-1920A\alpha+153A\alpha^2)}{343008-7680\alpha+901\alpha^2} \end{array}\right\} \quad (4.70)$$

where α is given by (4.23).

Similarly , following the same procedure as above and equating the coefficients of e^{τ} from both sides, we obtain

$$\left.\begin{array}{l} \frac{7}{3}g_{0,c} + 3g_{1,c} + h_{0,c} - h_{1,c} = -2B \\ g_{0,c} + g_{1,c} - \frac{1}{3}h_{0,c} + h_{1,c} = -2B \\ (4 - \frac{3\alpha}{32})g_{0,c} + (\frac{54}{5} - \frac{51\alpha}{160})g_{1,c} + \frac{3\alpha}{32}h_{0,c} - \frac{51\alpha}{160}h_{1,c} = -2B \\ \frac{3\alpha}{32}g_{0,c} + \frac{51\alpha}{160}g_{1,c} - \frac{3\alpha}{32}h_{0,c} + (\frac{51\alpha}{160} - \frac{6}{5})h_{1,c} = -2B \end{array}\right\} \quad (4.71)$$

Solving (4.39) we obtain

$$\left.\begin{aligned} g_{0,c} &= \frac{-4(-69B+10B\alpha)}{3(-28+5\alpha)} \\ g_{1,c} &= \frac{65(-8B+B\alpha)}{18(-28+5\alpha)} \\ h_{0,c} &= \frac{-356B+45B\alpha}{3(-28+5\alpha)} \\ h_{1,c} &= \frac{5(-168B+17B\alpha)}{18(-28+5\alpha)} \end{aligned}\right\} \tag{4.72}$$

where α and B are given by (4.23) and (4.31).

Applying boundary condition on (4.32) and (4.33) we have

$$\left.\begin{aligned} g_{l,a}+g_{l,b}+g_{l,c} &= 0 \\ h_{l,a}+h_{l,b}+h_{l,c} &= 0 \end{aligned} \quad for\ all\ l\ at\ \tau = 0\right\} \tag{4.73}$$

Thus

$$\left.\begin{aligned} g_{0,a} &= \frac{182272A-20320A\alpha+969A\alpha^2}{43008-7680\alpha+901\alpha^2} + \frac{4(-69B+10B\alpha)}{3(-28+5\alpha)} \\ g_{1,a} &= \frac{1280(-168A+17A\alpha)}{3(43008-7680\alpha+901\alpha^2)} + \frac{65(8B-B\alpha)}{18(-28+5\alpha)} \\ h_{0,a} &= \frac{-141312A+28640A\alpha-2499A\alpha^2}{43008-7680\alpha+901\alpha^2} + \frac{356B-45B\alpha}{3(-28+5\alpha)} \\ h_{1,a} &= \frac{-20(6656A-1920A\alpha+153A\alpha^2)}{343008-7680\alpha+901\alpha^2} + \frac{5(168B-17B\alpha)}{18(-28+5\alpha)} \end{aligned}\right\} \tag{4.74}$$

where A , B and α are given by (4.30) , (4.31) and (4.23) and the complete solution can thus be obtained.

4.8 Second approximate solution for scattering with Pomraning phase function

We name the solution the second approximation when $l_0 = 2$ and in this case it is assumed that $I_3^+ = I_3^- = 0$. So as in section (4.4) in the second approximation we obtain the following differential equations

$$\left.\begin{array}{l}
\frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} + I_0^+ + (\frac{\alpha}{4}+1)I_1^+ + I_0^- - (\frac{\alpha}{4}+1)I_1^- = -2\phi'(\tau) \\
\frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'} - I_0^+ + (\frac{\alpha}{4}-1)I_1^+ - I_0^- + (\frac{\alpha}{4}-1)I_1^- = 2\phi'(\tau) \\
2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} - (\frac{3\alpha}{32}-2)I_0^+ - (\frac{51\alpha}{160}-6)I_1^+ - (\frac{9\alpha}{32}-4)I_2^+ \\
+\frac{3\alpha}{32}I_0^- - \frac{51\alpha}{160}I_1^- + \frac{9\alpha}{32}I_2^- = -2\phi'(\tau) \\
-2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} + \frac{3\alpha}{32}I_0^+ + \frac{51\alpha}{160}I_1^+ + \frac{9\alpha}{32}I_2^+ \\
+(\frac{3\alpha}{32}-2)I_0^- + (\frac{51\alpha}{160}-6)I_1^- - (\frac{9\alpha}{32}-4)I_2^- = -2\phi'(\tau) \\
\frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} + (2I_1^+ + 5I_2^+) - \frac{\alpha}{64}[(I_0^+ - I_0^-) + \frac{17}{5}(I_1^+ + I_1^-) \\
+3(I_2^+ - I_2^-)] = 0 \\
\frac{1}{3}I_0^{-'} - 2I_1^{-'} + \frac{80}{21}I_2^{-'} + (2I_1^- - 5I_2^-) - \frac{\alpha}{64}[(I_0^+ - I_0^-) + \frac{17}{5}(I_1^+ + I_1^-) \\
+3(I_2^+ - I_2^-)] = 0
\end{array}\right\} \quad (4.75)$$

where

$$\left.\begin{array}{l}
I_+(\tau,\mu) = \phi(\tau) + I_0^+(\tau)\mu + 3I_1^+(\tau)\mu P_1(2\mu-1) + 5I_2^+(\tau)\mu P_2(2\mu-1), \\
\qquad\qquad 0 \le \mu \le 1 \\
I_-(\tau,\mu) = \phi(\tau) + I_0^-(\tau)\mu + 3I_1^-(\tau)\mu P_1(2\mu+1) + 5I_2^-(\tau)\mu P_2(2\mu+1), \\
\qquad\qquad -1 \le \mu \le 0
\end{array}\right\} \quad (4.76)$$

We now take the trial solution as

$$I_l^+ = g_{l,a}e^{-k\tau} + g_{l,b}e^{-\tau} + g_{l,c}e^{\tau} \quad (4.77)$$

$$I_l^- = h_{l,a}e^{-k\tau} + h_{l,b}e^{-\tau} + h_{l,c}e^{\tau} \quad (4.78)$$

where $k \neq 1$. For thin atmosphere following Wan et al[61] we have

$$\phi(\tau) = Ae^{-\tau} + Be^{\tau} \quad (4.79)$$

$$A = \frac{e^{2\tau_0}}{e^{2\tau_0}-1} \quad (4.80)$$

$$B = \frac{1}{1-e^{2\tau_0}} \quad (4.81)$$

We now substitute (4.45), (4.46) and (4.47) in the set of equations (4.43) and equating the coefficients of $e^{-k\tau}$ from both sides, we obtain

$$\left.\begin{array}{l}
(1-\frac{4k}{3})g_{0,a}+(1+\frac{\alpha}{4}-2k)g_{1,a}-\frac{2k}{3}g_{2,a}+h_{0,a}-(\frac{\alpha}{4}+1)h_{1,a}=0\\
g_{0,a}+(1-\frac{\alpha}{4})g_{1,a}+(1+\frac{4k}{3})h_{0,a}+(1-\frac{\alpha}{4}-2k)h_{1,a}+\frac{2k}{3}h_{2,a}=0\\
(2-2k-\frac{3\alpha}{32})g_{0,a}+(6-\frac{24k}{5}-\frac{51\alpha}{160})g_{1,a}+(4-4k-\frac{9\alpha}{32})g_{2,a}+\frac{3\alpha}{32}h_{0,a}\\
-\frac{51\alpha}{160}h_{1,a}+\frac{9\alpha}{32}h_{2,a}=0\\
\frac{3\alpha}{32}g_{0,a}+\frac{51\alpha}{160}g_{1,a}+\frac{9\alpha}{32}g_{2,a}+(2+2k-\frac{3\alpha}{32})h_{0,a}-(6+\frac{24k}{5}-\frac{51\alpha}{160})h_{1,a}\\
+(4+4k-\frac{9\alpha}{32})h_{2,a}=0\\
-(\frac{k}{3}+\frac{\alpha}{64})g_{0,a}+(2-2k-\frac{17\alpha}{320})g_{1,a}+(5-\frac{80k}{21}-\frac{3\alpha}{64})g_{2,a}+\frac{\alpha}{64}h_{0,a}\\
-\frac{17\alpha}{320}h_{1,a}+\frac{3\alpha}{64}h_{2,a}=0\\
-\frac{\alpha}{64}g_{0,a}-\frac{17\alpha}{320}g_{1,a}-\frac{3\alpha}{64}g_{2,a}-(\frac{k}{3}-\frac{\alpha}{64})h_{0,a}+(2+2k-\frac{17\alpha}{320})h_{1,a}\\
-(5+\frac{80k}{21}-\frac{3\alpha}{64})h_{2,a}=0
\end{array}\right\} \tag{4.82}$$

The above set of equations has non-trivial solution, if

$$\Delta(k)=0 \tag{4.83}$$

where $\Delta(k)=$

$$\begin{vmatrix}
(1-\frac{4k}{3}) & (1+\frac{\alpha}{4}-2k) & -\frac{2k}{3} & 1 & -(1+\frac{\alpha}{4}) & 0\\
1 & 1-\frac{\alpha}{4} & 0 & (1+\frac{4k}{3}) & (1-\frac{\alpha}{4}-2k) & \frac{2k}{3}\\
(2-2k-\frac{3\alpha}{32}) & (6-\frac{24k}{5}-\frac{51\alpha}{160}) & (4-4k-\frac{9\alpha}{32}) & \frac{3\alpha}{32} & -\frac{51\alpha}{160} & \frac{9\alpha}{32}\\
\frac{3\alpha}{32} & \frac{51\alpha}{160} & \frac{9\alpha}{32} & (2+2k-\frac{3\alpha}{32}) & -(6+\frac{24k}{5}-\frac{51\alpha}{160}) & (4+4k-\frac{9\alpha}{32})\\
-(\frac{k}{3}+\frac{\alpha}{64}) & (2-2k-\frac{17\alpha}{320}) & (5-\frac{80k}{21}-\frac{3\alpha}{64}) & \frac{\alpha}{64} & -\frac{17\alpha}{320} & \frac{3\alpha}{64}\\
-\frac{\alpha}{64} & -\frac{17\alpha}{320} & -\frac{3\alpha}{64} & -(\frac{k}{3}-\frac{\alpha}{64}) & (2+2k-\frac{17\alpha}{320}) & (\frac{3\alpha}{64}-5-\frac{80k}{21})
\end{vmatrix} \tag{4.84}$$

$$\left.\begin{aligned}\Delta(k) &= \tfrac{9216}{1225}k^6 + \tfrac{8448}{245}k^5 + (\tfrac{2964\alpha}{49} - \tfrac{1397\alpha^2}{245} - \tfrac{11360}{49})k^4 + (\tfrac{4377\alpha}{245} - \tfrac{939\alpha^2}{986} - \tfrac{1056}{7})k^3 \\ &+(\tfrac{28184}{49} - \tfrac{48943\alpha}{245} + \tfrac{44783\alpha^2}{1960} - \tfrac{649\alpha^3}{1568})k^2 + (\tfrac{984}{7} - \tfrac{1362\alpha}{35} + \tfrac{87\alpha^2}{35})k \\ &+(\tfrac{5\alpha^3}{4} - \tfrac{47\alpha^2}{2} + 134\alpha - 240)\end{aligned}\right\} \tag{4.85}$$

and values of $\Delta(k) = 0$ are tabulated in the following table for different values of λ .

Table-1

λ	$\alpha = \frac{5}{5-3\lambda}$	k
0.1	1.06383	-7.29372,-1.48824,-0.713354,0.502632,1.38268,3.02667
0.2	1.13636	-7.24949,1.48701,-0.705079,0.493842,1.3786,2.98579
0.3	1.21951	-7.19929,-1.48556,-0.695419,0.483581,1.37372,2.93963
0.4	1.31579	-7.14186,-1.48382,-0.683993,0.471443,1.36777,2.88714
0.5	1.42857	-7.07559,-1.48172,-0.670264,0.456863,1.36037,2.827
0.6	1.5625	-6.99833,-1.47912,-0.65345,0.439024,1.35096,2.75758
0.7	1.72414	-6.90728,-1.47584,-0.632373,0.416703,1.33864,2.67683
0.8	1.92308	-6.79872,-1.4716,-0.605156,0.387986,1.3219,2.58225
0.9	2.17391	-6.66772,-1.46595,-0.568633,0.347932,1.29817,2.47107
1.0	2.5	-6.50812,-1.45818,0.517001,0.296461,1.26264,2.34088

Again substituting (4.45), (4.46) and (4.47) in the set of equations (4.43) and equating the coefficients of $e^{-\tau}$ from the both sides, we obtain

$$\left.\begin{aligned}&-\tfrac{1}{3}g_{0,b} + (\tfrac{\alpha}{4} - 1)g_{1,b} - \tfrac{2}{3}g_{2,b} + h_{0,b} - (\tfrac{\alpha}{4} + 1)h_{1,b} = 2A \\ &-g_{0,b} + (\tfrac{\alpha}{4} - 1)g_{1,b} - \tfrac{7}{3}h_{0,b} + (\tfrac{\alpha}{4} + 1)h_{1,b} - \tfrac{2}{3}h_{2,b} = -2A \\ &-\tfrac{3\alpha}{32}g_{0,b} + (\tfrac{6}{5} - \tfrac{51\alpha}{160})g_{1,b} - \tfrac{9\alpha}{32}g_{2,b} + \tfrac{3\alpha}{32}h_{0,b} - \tfrac{51\alpha}{160}h_{1,b} + \tfrac{9\alpha}{32}h_{2,b} = 2A \\ &\tfrac{3\alpha}{32}g_{0,b} + \tfrac{51\alpha}{160}g_{1,b} + \tfrac{9\alpha}{32}g_{2,b} + (4 - \tfrac{3\alpha}{32})h_{0,b} + (\tfrac{51\alpha}{160} - \tfrac{54}{5})h_{1,b} + (8 - \tfrac{9\alpha}{32})h_{2,b} = 2A \\ &-(\tfrac{1}{3} + \tfrac{\alpha}{64})g_{0,b} - \tfrac{17\alpha}{320}g_{1,b} + (\tfrac{25}{21} - \tfrac{3\alpha}{64}g_{2,b} + \tfrac{\alpha}{64}h_{0,b} - \tfrac{17\alpha}{64}h_{1,b} + \tfrac{3\alpha}{64}h_{2,b} = 0 \\ &-\tfrac{\alpha}{64}g_{0,b} - \tfrac{17\alpha}{64}g_{1,b} - \tfrac{3\alpha}{64}g_{2,b} + (\tfrac{\alpha}{64} - \tfrac{1}{3})h_{0,b} + (4 - \tfrac{17\alpha}{320})h_{1,b} + (\tfrac{3\alpha}{64} - \tfrac{185}{21})h_{2,b} = 0\end{aligned}\right\} \tag{4.86}$$

solving (4.86) we obtain.

$$\left.\begin{aligned}
g_{0,b} &= \frac{(\frac{1768}{3}-\frac{119579\alpha}{525}+\frac{145459\alpha^2}{5880}-\frac{1311\alpha^3}{1568})A}{\Delta_1}\\
g_{1,b} &= \frac{(-\frac{55152}{245}+\frac{17342\alpha}{245}-\frac{437\alpha^2}{98})A}{\Delta_1}\\
g_{2,b} &= \frac{(\frac{12376}{75}-\frac{31109\alpha}{525}+\frac{2027\alpha^2}{420})A}{\Delta_1}\\
h_{0,b} &= \frac{(\frac{-229384}{245}+\frac{227257\alpha}{3675}+\frac{35599\alpha^2}{5880}-\frac{1311\alpha^3}{1568})A}{\Delta_1}\\
h_{1,b} &= \frac{(\frac{-30208}{245}+\frac{6953\alpha}{245})A}{\Delta_1}\\
h_{2,b} &= \frac{(\frac{-23192}{525}+\frac{149\alpha}{15}+\frac{23\alpha^2}{420})A}{\Delta_1}
\end{aligned}\right\} \tag{4.87}$$

where $\Delta_1 = -\frac{1311\alpha^3}{1568} + \frac{9459\alpha^2}{1960} + \frac{1290\alpha}{49} - \frac{165456}{1225}$ and A , α are given by (4.48),(4.23).

Similarly , applying the same procedure as in previous case and equating the coefficients of e^{τ} from both sides we obtain

$$\left.\begin{aligned}
&\tfrac{7}{3}g_{0,c} + (\tfrac{\alpha}{4}+3)g_{1,c} + \tfrac{2}{3}g_{2,c} + h_{0,c} - (\tfrac{\alpha}{4}+1)h_{1,c} = -2B\\
&-g_{0,c} + (\tfrac{\alpha}{4}-1)g_{1,c} + \tfrac{1}{3}h_{0,c} + (\tfrac{\alpha}{4}-3)h_{1,c} + \tfrac{2}{3}h_{2,c} = 2B\\
&(4-\tfrac{3\alpha}{32})g_{0,c} + (\tfrac{54}{5}-\tfrac{51\alpha}{160})g_{1,c} + (8-\tfrac{9\alpha}{32})g_{2,c} + \tfrac{3\alpha}{32}h_{0,c} - \tfrac{51\alpha}{160}h_{1,c} + \tfrac{9\alpha}{32}h_{2,c} = -2B\\
&\tfrac{3\alpha}{32}g_{0,c} + \tfrac{51\alpha}{160}g_{1,c} + \tfrac{9\alpha}{32}g_{2,c} - \tfrac{3\alpha}{32}h_{0,c} + (\tfrac{51\alpha}{160}-\tfrac{6}{5})h_{1,c} - \tfrac{9\alpha}{32}h_{2,c} = -2B\\
&(\tfrac{1}{3}-\tfrac{\alpha}{64})g_{0,c} + (4-\tfrac{17\alpha}{320})g_{1,c} + (\tfrac{185}{21}-\tfrac{3\alpha}{64})g_{2,c} + \tfrac{\alpha}{64}h_{0,c} - \tfrac{17\alpha}{320}h_{1,c} + \tfrac{3\alpha}{64}h_{2,c} = 0\\
&-\tfrac{\alpha}{64}g_{0,c} - \tfrac{17\alpha}{320}g_{1,c} - \tfrac{3\alpha}{64}g_{2,c} + (\tfrac{1}{3}+\tfrac{\alpha}{64})h_{0,c} - \tfrac{17\alpha}{64}h_{1,c} + (\tfrac{3\alpha}{64}-\tfrac{25}{21})h_{2,c} = 0
\end{aligned}\right\} \tag{4.88}$$

solving we obtain

$$
\left.\begin{aligned}
g_{0,c} &= \frac{(\frac{61592}{105}-\frac{400577\alpha}{3675}+\frac{4489\alpha^2}{5880}+\frac{1311\alpha^3}{1568})B}{\Delta_2}\\
g_{1,c} &= \frac{(\frac{-1392}{5}+\frac{16262\alpha}{245}-\frac{437\alpha^2}{98})B}{\Delta_2}\\
g_{2,c} &= \frac{(\frac{7816}{75}-\frac{527\alpha}{21}+\frac{713\alpha^2}{420})B}{\Delta_2}\\
h_{0,c} &= \frac{(-\frac{12056}{21}+\frac{555433\alpha}{3675}-\frac{15413\alpha^2}{840}+\frac{1311\alpha^3}{1568})B}{\Delta_2}\\
h_{1,c} &= \frac{(-\frac{5056}{35}+\frac{496\alpha}{35})B}{\Delta_2}\\
h_{2,c} &= \frac{(\frac{-12056}{75}+\frac{18649\alpha}{525}-\frac{1231\alpha^2}{420})B}{\Delta_2}
\end{aligned}\right\} \quad (4.89)
$$

where the constant B and α are given by (4.49),(4.23) and $\Delta_2 = -\frac{1311\alpha^3}{1568} + \frac{15447\alpha^2}{1960} - \frac{552\alpha}{35} - \frac{15168}{175}$.
Applying boundary condition on trial solution we have

$$
\left.\begin{aligned}
g_{l,a}+g_{l,b}+h_{l,c} &= 0\\
h_{l,a}+h_{l,b}+h_{l,c} &= 0
\end{aligned}\quad for\ all\ l\ at\ \tau = 0\right\} \quad (4.90)
$$

From which the unknowns $g_{l,a}$ and $h_{l,a}$ for $l = 0, 1, 2$ can be easily determined using (4.55) and (4.57) and the complete solution can thus be obtained.