# Chapter 3

# Solution of the equation of radiative transfer for interlocked triplets by double interval spherical harmonic method

## 3.1 Equation of Transfer

The equation of transfer for the r-th line of multiplets in the case of interlocking without redistribution is

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\epsilon\eta_r)(a+b\tau)$$
$$-(1-\epsilon)\alpha_r\sum_{p=1}^{k}\frac{1}{2}\eta_p\int_{-1}^{1}I_p(\tau,\mu')d\mu';\ r=1,2,\ldots\ldots k \tag{3.1}$$

where

$$\alpha_r = \frac{\eta_r}{\eta_1+\eta_2+\ldots\ldots+\eta_k} \tag{3.2}$$

so that

$$\alpha_1+\alpha_2+\ldots\ldots+\alpha_k = 1 \tag{3.3}$$

and $\eta_r$, the ratio of line to the continuuam absorption coefficient for the r-th line is independent of depth but is a function of frequency. $\epsilon$ is the coefficient of thermal emission, is independent of both frequency and depth.

For triplet (3.1) reduces to

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\epsilon\eta_r)(a+b\tau)$$
$$-(1-\epsilon)\alpha_r\frac{1}{2}\left[\sum_{p=1}^{3}\eta_p\int_{-1}^{1} I_p(\tau,\mu')d\mu'\right] \; ; \; r = 1,2,3. \tag{3.4}$$

The above equation of transfer (3.4) is to be solved subject to the boundary conditions:

$$I_r(0,\mu) \equiv 0 \;\; for \;\; -1 \le \mu \le 0\,, \;\; r = 1,2,3. \tag{3.5}$$

and

$$I_r(\tau,\mu)e^{-\tau} \to 0 \;\; as \;\; \tau \to \infty \;, r = 1,2,3. \tag{3.6}$$

We shall seek a solution of equations (4) $I_r(\tau,\mu)$ can be expansions $I_r^+(\tau,\mu)$ and $I_r^-(\tau,\mu)$ for $\mu$ in the interval (0,1) and (-1,0) respectively in the form[65]

$$I_r^+(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_{rl}^+(\tau)\mu P_l(2\mu-1)\,, \; 0 \le \mu \le 1, r = 1,2,3. \tag{3.7}$$

$$I_r^-(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_{rl}^-(\tau)\mu P_l(2\mu+1)\,, \; -1 \le \mu \le 0, r = 1,2,3. \tag{3.8}$$

where A is a constant(independent of $\mu$)to be determined and the recurrence formulae

$$\mu P_l(2\mu\pm 1) = \frac{1}{(2l+1)}\left[\frac{l+1}{2}P_{l+1}(2\mu\pm 1) \mp \frac{2l+1}{2}P_l(2\mu\pm 1) + \frac{l}{2}P_{l-1}(2\mu\pm 1)\right] \tag{3.9}$$

has the advantages due to orthogonality of $P_l(2\mu-1)$ in $(0,1)$ and $P_l(2\mu+1)$ in $(-1,0)$

The equation of transfer(3.4) in the present representation is equivalent to

$$\mu\frac{dI_1(\tau,\mu)}{d\tau} = (1+\eta_1)I_1(\tau,\mu) - (1+\epsilon\eta_1)(a+b\tau)$$
$$-(1-\epsilon)\alpha_1\frac{1}{2}\left[\eta_1\int_{-1}^{1} I_1(\tau,\mu')d\mu' + \eta_2\int_{-1}^{1} I_2(\tau,\mu')d\mu' + \eta_3\int_{-1}^{1} I_3(\tau,\mu')d\mu'\right] \tag{3.10}$$

and

$$\mu\frac{dI_2(\tau,\mu)}{d\tau} = (1+\eta_2)I_2(\tau,\mu) - (1+\epsilon\eta_2)(a+b\tau)$$
$$-(1-\epsilon)\alpha_2\frac{1}{2}\left[\eta_1\int_{-1}^{1} I_1(\tau,\mu')d\mu' + \eta_2\int_{-1}^{1} I_2(\tau,\mu')d\mu' + \eta_3\int_{-1}^{1} I_3(\tau,\mu')d\mu'\right] \tag{3.11}$$

and

$$\mu\frac{dI_3(\tau,\mu)}{d\tau} = (1+\eta_3)I_2(\tau,\mu) - (1+\epsilon\eta_3)(a+b\tau)$$
$$-(1-\epsilon)\alpha_3\frac{1}{2}\left[\eta_1\int_{-1}^{1} I_1(\tau,\mu')d\mu' + \eta_2\int_{-1}^{1} I_2(\tau,\mu')d\mu' + \eta_3\int_{-1}^{1} I_3(\tau,\mu')d\mu'\right] \tag{3.12}$$

also we have

$$\int_{-1}^{1} I_1(\tau,\mu')d\mu' = 2A\tau + \frac{1}{2}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) \tag{3.13}$$
$$\int_{-1}^{1} I_2(\tau,\mu')d\mu' = 2A\tau + \frac{1}{2}(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \tag{3.14}$$
$$\int_{-1}^{1} I_3(\tau,\mu')d\mu' = 2A\tau + \frac{1}{2}(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) \tag{3.15}$$

Since

$$\begin{aligned}
\int_{-1}^{+1} I_r(\tau,\mu')d\mu' &= \int_0^1 I_r^{+}(\tau,\mu')d\mu' + \int_{-1}^{0} I_r^{-}(\tau,\mu')d\mu' \\
&= \int_0^1 \left\{A\tau + \sum_{l=0}^{l_0}(2l+1)I_{rl}^{+}(\tau)\mu' P_l(2\mu'-1)\right\} d\mu' \\
&\quad + \int_{-1}^{0} \left\{A\tau + \sum_{l=0}^{l_0}(2l+1)I_{rl}^{-}(\tau)\mu' P_l(2\mu'+1)\right\} d\mu' \\
&= \int_0^1 \left\{A\tau + I_{r0}^{+}(\tau)\mu' + 3I_{r1}^{+}(\tau)\mu' P_l(2\mu'-1)\right\} d\mu'
\end{aligned}$$

$$+\int_{-1}^{0}\left\{A\tau + I_{r0}^{-}(\tau)\mu' + 3I_{r1}^{-}(\tau)\mu' P_l(2\mu'+1)\right\}d\mu'$$
$$= \; 2A\tau + \frac{1}{2}(I_{r0}^{+} - I_{r0}^{-} + I_{r1}^{+} + I_{r1}^{-}) \qquad (3.16)$$

Note here that if $l_0 \geq 2$, then the contribution of other terms in the summation after integration become zero due to orthogonality of Legendre polynomial which is

$$\int_{-1}^{+1} P_m(\mu)P_m(\mu)d\mu = \begin{cases} 0, \; if \;\; m \neq n \\ \frac{2}{2n+1}, \; if \;\; m = n \end{cases} \qquad (3.17)$$

Let us first consider the equation for $r = 1$

The equation of transfer(3.10) can be written as

$$\mu\frac{dI_1^{+}(\tau,\mu)}{d\tau} = (1+\eta_1)I_1^{+}(\tau,\mu) - (1+\epsilon\eta_1)(a+b\tau) - (1-\epsilon)\eta_1 A\tau$$
$$-\frac{(1-\epsilon)\alpha_1}{4}\Big[\eta_1(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) + \eta_2(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})$$
$$+\eta_3\frac{1}{2}(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-})\Big] \qquad (3.18)$$

*and*

$$\mu\frac{dI_1^{-}(\tau,\mu)}{d\tau} = (1+\eta_1)I_1^{-}(\tau,\mu) - (1+\epsilon\eta_1)(a+b\tau) - (1-\epsilon)\eta_1 A\tau$$
$$-\frac{(1-\epsilon)\alpha_1}{4}\Big[\eta_1(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) + \eta_2(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})$$
$$+\eta_3\frac{1}{2}(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-})\Big] \qquad (3.19)$$

Multiplying equation (3.18) by $P_l(2\mu-1)$ and equation (3.19) by $P_l(2\mu+1)$ respectively and integrating over $\mu$ in their respective ranges and using the recurrence formulae (3.9), we have the following equations:

$$\left.\begin{array}{l}
\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{1l-2}^{+'} + 2lI_{1l-1}^{+'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{+'} + 2(l+1)I_{1l+1}^{+'} + \frac{l^2+3l+2}{2l+3}I_{1l+2}^{+'}\right] \\
+A\int_0^1 \mu P_l(2\mu-1)d\mu = \frac{(1+\eta_1)}{2(2l+1)}[lI_{1l-1}^{+} + (2l+1)I_{1l}^{+} + (l+1)I_{1l+1}^{+}] - [(1+\epsilon\eta_1) \\
(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau]\int_0^1 P_l(2\mu-1)d\mu - \frac{(1-\epsilon)\alpha_1}{4}[\eta_1(I_{10}^{+} - I_{10}^{-} \\
+I_{11}^{+} + I_{11}^{-}) + \eta_2(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) + \eta_3(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-})]
\end{array}\right\}$$
$$(3.20)$$

and

$$\left.\begin{array}{l}\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I^{-'}_{1l-2} - 2lI^{-'}_{1l-1} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{-'}_{1l} - 2(l+1)I^{-'}_{1l+1} + \frac{l^2+3l+2}{2l+3}I^{-'}_{1l+2}\right] \\ +A\int_{-1}^{0}\mu P_l(2\mu+1)d\mu = \frac{(1+\eta_1)}{2(2l+1)}[lI^{-}_{1l-1} - (2l+1)I^{-}_{1l} + (l+1)I^{-}_{1l+1}] - [(1+\epsilon\eta_1) \\ (a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau]\int_{-1}^{0} P_l(2\mu+1)d\mu - \frac{(1-\epsilon)\alpha_1}{4}[\eta_1(I^{+}_{10} - I^{-}_{10} \\ +I^{+}_{11} + I^{-}_{11}) + \eta_2(I^{+}_{20} - I^{-}_{20} + I^{+}_{21} + I^{-}_{21}) + \eta_3(I^{+}_{30} - I^{-}_{30} + I^{+}_{31} + I^{-}_{31})]\end{array}\right\} \tag{3.21}$$

where $I'_l$ are differentials of $I_l$ with respect to the optical thickness $\tau$. Separating the equations for $l = 0$ and $l = 1$ from the equations (3.20) and (3.21) we can write

$$\left.\begin{array}{l}\underline{for\ \ l = o} \\ \left(\frac{4}{3}I^{+'}_{10} + 2I^{+'}_{11} + \frac{2}{3}I^{+'}_{12}\right) - [\{2(1+\eta_1) - \eta_1(1-\epsilon)\alpha_1\}I^{+}_{10} + (1-\epsilon)\alpha_1\eta_1 I^{-}_{10} + \\ \{2(1+\eta_1) - \eta_1(1-\epsilon)\alpha_1\}I^{+}_{11} - (1-\epsilon)\alpha_1\eta_1 I^{-}_{11}] + (1-\epsilon)\alpha_1\eta_2[I^{+}_{20} - I^{-}_{20} \\ +I^{+}_{21} + I^{-}_{21}] + (1-\epsilon)\alpha_1\eta_3[I^{+}_{30} - I^{-}_{30} + I^{+}_{31} + I^{-}_{31}] \\ = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\ \\ \underline{for\ \ l = 1} \\ \left(2I^{+'}_{10} + \frac{24}{5}I^{+'}_{11} + 4I^{+'}_{12} + \frac{6}{5}I^{+'}_{13}\right) - 2(1+\eta_1)(I^{+}_{10} + 3I^{+}_{11} + 2I^{+}_{12}) = -2A \\ \\ \underline{for\ \ l \neq 0, 1} \\ \left[\frac{l^2-l}{2l-1}I^{+'}_{1l-2} + 2lI^{+'}_{1l-1} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{+'}_{1l} + 2(l+1)I^{+'}_{1l+1} + \frac{l^2+3l+2}{2l+3}I^{+'}_{1l+2}\right] \\ -2(1+\eta_1)[lI^{+}_{1l-1} + (2l+1)I^{+}_{1l} + (l+1)I^{+}_{1l+1}] = 0\end{array}\right\} \tag{3.22}$$

and

$$
\left.\begin{array}{l}
\underline{for\ \ l=o} \\
(\frac{4}{3}I_{10}^{-'}+2I_{11}^{-'}+\frac{2}{3}I_{12}^{-'})-[(1-\epsilon)\alpha_1\eta_1 I_{10}^{+}+\{2(1+\eta_1)-\eta_1(1-\epsilon)\alpha_1\}I_{10}^{-} \\
+(1-\epsilon)\alpha_1\eta_1 I_{11}^{+}-\{2(1+\eta_1)-\eta_1(1-\epsilon)\alpha_1\}I_{11}^{-}]+(1-\epsilon)\alpha_1\eta_2[I_{20}^{+}-I_{20}^{-} \\
+I_{21}^{+}+I_{21}^{-}]+(1-\epsilon)\alpha_1\eta_3[I_{30}^{+}-I_{30}^{-}+I_{31}^{+}+I_{31}^{-}] \\
=2A-4[(1+\epsilon\eta_1)(a+b\tau)+(1-\epsilon)\eta_1 A\tau-(1+\eta_1)A\tau] \\
\\
\underline{for\ \ l=1} \\
(-2I_{10}^{-'}+\frac{24}{5}I_{11}^{-'}-4I_{12}^{-'}+\frac{6}{5}I_{13}^{-'})-2(1+\eta_1)(I_{10}^{-}-3I_{11}^{-}+2I_{12}^{-})=-2A \\
\\
\underline{for\ \ l\neq 0,1} \\
\left[\frac{l^2-l}{2l-1}I_{1l-2}^{-'}-2lI_{1l-1}^{-'}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{-'}-2(l+1)I_{1l+1}^{-'}+\frac{l^2+3l+2}{2l+3}I_{1l+2}^{-'}\right] \\
-2(1+\eta_1)[lI_{1l-1}^{-}-(2l+1)I_{1l}^{-}+(l+1)I_{1l+1}^{-}]=0
\end{array}\right\} \tag{3.23}
$$

The equations (3.22) and (3.23) are to be solved subject to the boundary conditions (3.5) and (3.6) which are restated below

$$
I_{il}^{-}(0)\equiv 0 \ \ and \ \ \left.\begin{array}{l} I_{il}^{+}(\tau)\,e^{-\tau}\to 0 \\ I_{il}^{-}(\tau)\,e^{-\tau}\to 0 \end{array}\right\} \ as\ \tau\to\infty\ ,\ i=1,2,3. \tag{3.24}
$$

## 3.2 Solution

It is assumed that at the N-th approximation

$$
I_{i\,N+1}^{+}=I_{i\,N+1}^{-}=0\ ,\ i=1,2,3. \tag{3.25}
$$

We assume a trial solution of the form

$$
\left.\begin{array}{l} I_{rl}^{+}(\tau)=A\left[g_{rl,\alpha}\,e^{-k\tau}+g_{rl,\beta}\right] \\ \\ I_{rl}^{-}(\tau)=A\left[h_{1l,\alpha}\,e^{-k\tau}+h_{rl,\beta}\right] \end{array}\right\}\ ,\ r=1,2,3. \tag{3.26}
$$

where $g_{rl,\alpha}, g_{rl,\beta}, h_{rl,\alpha}, h_{rl,\beta}, r=1,2,3.$ are constants to be determined.

Substituting these in (3.22) and (3.23) and equating the coefficients of $e^{-k\tau}$ and constant term we obtain (3.27) and (3.28).

$$\left.\begin{array}{l}
\underline{for\ \ l=0} \\
\{\frac{4k}{3}+2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)\}g_{10,\alpha}+\{2k+2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)\}g_{11,\alpha} \\
+\frac{2k}{3}g_{12,\alpha}-(1-\epsilon)\alpha_1\eta_2 g_{20,\alpha}-(1-\epsilon)\alpha_1\eta_2 g_{21,\alpha}-(1-\epsilon)\alpha_1\eta_3 g_{30,\alpha} \\
-(1-\epsilon)\alpha_1\eta_3 g_{31,\alpha}+(1-\epsilon)\alpha_1\eta_1 h_{10,\alpha}-(1-\epsilon)\alpha_1\eta_1 h_{11,\alpha}+(1-\epsilon)\alpha_1\eta_2 h_{20,\alpha} \\
-(1-\epsilon)\alpha_1\eta_2\, h_{21,\alpha}+(1-\epsilon)\alpha_1\eta_3 h_{30,\alpha}-(1-\epsilon)\alpha_1\eta_3 h_{31,\alpha}=0 \\
\underline{for\ \ l=1} \\
2(1+\eta_1+k)g_{10,\alpha}+6(1+\eta_1+\frac{4k}{5})g_{11,\alpha}+4(1+\eta_1+k)g_{12,\alpha}+\frac{6k}{5}\,g_{13,\alpha}=0 \\
\underline{for\ \ l\neq 0,1} \\
\frac{l^2-l}{2l-1}g_{1l-2,\alpha}+2l(1+\eta_1+k)g_{1l-1,\alpha}+\{\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}k+2(1+\eta_1)(2l+1)\}\times \\
g_{1l,\alpha}+2(l+1)(1+\eta_1+k)g_{1l+1,\alpha}+\frac{l^2+3l+2}{2l+3}kg_{1l+2,\alpha}=0 \\
\underline{for\ \ l=0} \\
\{2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)-\frac{4k}{3}\}h_{10,\alpha}+\{2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)-2k\}h_{11,\alpha} \\
-\frac{2k}{3}h_{12,\alpha}-(1-\epsilon)\alpha_1\eta_2\, h_{20,\alpha}+(1-\epsilon)\alpha_1\eta_2\, h_{21,\alpha}+(1-\epsilon)\alpha_1\eta_1 g_{10,\alpha} \\
+(1-\epsilon)\alpha_1\eta_1\, g_{11,\alpha}+(1-\epsilon)\alpha_1\eta_2\, g_{20,\alpha}+(1-\epsilon)\alpha_1\eta_2 g_{21,\alpha}+(1-\epsilon)\alpha_1\eta_3 g_{30,\alpha}+ \\
(1-\epsilon)\alpha_1\eta_3 g_{31,\alpha}-(1-\epsilon)\alpha_1\eta_3 h_{30,\alpha}-(1-\epsilon)\alpha_1\eta_3 h_{31,\alpha}=0 \\
\underline{for\ \ l=1} \\
-2(1+\eta_1-k)h_{10,\alpha}+6(1+\eta_1-\frac{4k}{5})h_{11,\alpha}-4(1+\eta_1-k)h_{12,\alpha}-\frac{6k}{5}\,g_{13,\alpha}=0 \\
\underline{for\ \ l\neq 0,1} \\
-\frac{l^2-l}{2l-1}h_{1l-2,\alpha}-2l(1+\eta_1-k)h_{1l-1,\alpha}-\{\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}k-2(1+\eta_1)(2l+1)\}\times \\
h_{1l,\alpha}-2(l+1)(1+\eta_1-k)h_{1l+1,\alpha}-\frac{l^2+3l+2}{2l+3}kh_{1l+2,\alpha}=0
\end{array}\right\} \tag{3.27}$$

and

$$
\left.
\begin{array}{l}
\underline{for\ \ l=0} \\
\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}g_{10,\beta}+\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}g_{11,\beta}- \\
(1-\epsilon)\alpha_1\eta_2 g_{20,\beta}-(1-\epsilon)\alpha_1\eta_2 g_{21,\beta}-(1-\epsilon)\alpha_1\eta_3 g_{30,\beta}-(1-\epsilon)\alpha_1\eta_3 g_{31,\beta} \\
+(1-\epsilon)\alpha_1\eta_1 h_{10,\beta}-(1-\epsilon)\alpha_1\eta_1 h_{11,\beta}+(1-\epsilon)\alpha_1\eta_2 h_{20,\beta}-(1-\epsilon)\alpha_1\eta_2 h_{21,\beta} \\
+(1-\epsilon)\alpha_1\eta_3 h_{30,\beta}-(1-\epsilon)\alpha_1\eta_3 h_{31,\beta}=2+4\frac{a}{A}(1+\epsilon\eta_1) \\
\underline{for\ \ l=1} \\
g_{10,\beta}+3g_{11,\beta}+2g_{12,\beta}=1 \\
\underline{for\ \ l\neq 0,1} \\
lg_{1l-1,\beta}+(2l+1)g_{1l,\beta}+(l+1)g_{1l+1,\beta}=0 \\
\underline{for\ \ l=0} \\
\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}h_{10,\beta}-\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}h_{11,\beta}+ \\
(1-\epsilon)\alpha_1\eta_1 g_{10,\beta}+(1-\epsilon)\alpha_1\eta_1 g_{11,\beta}+(1-\epsilon)\alpha_1\eta_2 h_{20,\beta}+(1-\epsilon)\alpha_1\eta_2 g_{21,\beta} \\
-(1-\epsilon)\alpha_1\eta_2 h_{20,\beta}+(1-\epsilon)\alpha_1\eta_2 h_{21,\beta}+(1-\epsilon)\alpha_1\eta_3 g_{30,\beta}+(1-\epsilon)\alpha_1\eta_3 g_{31,\beta} \\
-(1-\epsilon)\alpha_1\eta_3 h_{30,\beta}+(1-\epsilon)\alpha_1\eta_3 h_{31,\beta}=2-4\frac{a}{A}(1+\epsilon\eta_1) \\
\underline{for\ \ l=1} \\
h_{10,\beta}-3h_{11,\beta}+2h_{12,\beta}=1 \\
\underline{for\ \ l\neq 0,1} \\
lh_{1l-1,\beta}-(2l+1)h_{1l,\beta}+(l+1)h_{1l+1,\beta}=0
\end{array}
\right\}
\tag{3.28}
$$

A similar sets of equations like (3.27) and (3.28) will be obtained considering $r=2,3$.

Solving (3.27) and the similar set of equation obtained when $r = 2,3$

combinedly by the method described by Wilson and Sen[65] we obtain $k = k_m$ ; $m = 12, 16, \ldots$

Using boundary conditions (3.24) we obtain

$$\sum_{r=1}^{n-1} h_{i\,l,\alpha}^{(m)} + h_{i\,l,\beta} = 0\,, \quad i = 1, 2, 3. \tag{3.29}$$

Thus equations (3.27), (3.28) and similar sets of equations (for $r = 2, 3$) and (3.29) are sufficient to determine the unknowns $g_{il,j}, h_{il,j}; i = 1, 2, 3; j = \alpha, \beta\,;\ l = 1, 2, \ldots, n$.

Thus we have

$$\left.\begin{aligned} I_{il}^{+}(\tau) &= A\left[g_{il,\alpha}^{(m)}\, e^{-k\tau} + g_{il,\beta}\right] \\ I_{il}^{-}(\tau) &= A\left[h_{il,\alpha}^{(m)}\, e^{-k\tau} + h_{il,\beta}\right] \end{aligned}\right\} \tag{3.30}$$

where $i = 1, 2, 3;\ \ l = 1, 2, \ldots, n$.

Now we will consider two approximation, viz. $l_0 = 1$ and $l_0 = 2$.

## 3.3 First approximation

### 3.3.1 First approximation when r = 1

We name the solution first approximation when $l_0 = 1$.

In this case we have from equations (3.22) and (3.23)

$$\left.\begin{array}{l}
\frac{4}{3}I_{10}^{+'} + 2I_{11}^{+'} - (\xi_1 I_{10}^{+} + \xi_2 I_{10}^{-} + \xi_1 I_{11}^{+} - \xi_2 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \\
+\xi_4(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) \\
+(1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\
\\
\frac{4}{3}I_{10}^{-'} - 2I_{11}^{-'} + (\xi_2 I_{10}^{+} + \xi_1 I_{10}^{-} + \xi_2 I_{11}^{+} - \xi_1 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \\
+\xi_4(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) \\
+(1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\
\\
2I_{10}^{+'} + \frac{24}{5}I_{11}^{+'} - 2\xi_5(I_{10}^{+} + 3I_{11}^{+}) = -2A \\
\\
-2I_{10}^{-'} + \frac{24}{5}I_{11}^{-'} - 2\xi_5(I_{10}^{-} - 3I_{11}^{-}) = -2A
\end{array}\right\} \quad (3.31)$$

where

$$\left.\begin{array}{l}
I_{1l}^{+}(\tau,\mu) = A\tau + I_{10}^{+}(\tau)\mu + 3I_{11}^{+}(\tau)\mu P_1(2\mu-1)\,, \;\; 0 \le \mu \le 1 \\
I_{1l}^{-}(\tau,\mu) = A\tau + I_{10}^{-}(\tau)\mu + 3I_{11}^{-}(\tau)\mu P_1(2\mu+1)\,, \;\; -1 \le \mu \le 0
\end{array}\right\} \quad (3.32)$$

We now take the trial solution given by (3.26) and substituting these in (3.31) and then equating the coefficient of $e^{k\tau}$ and constant term we obtain

$$\left.\begin{array}{l}
(\xi_1 + \frac{4k}{3})g_{10,\alpha} + (\xi_1 + 2k)g_{11,\alpha} + \xi_2 h_{10,\alpha} - \xi_2 h_{11,\alpha} - \xi_3 g_{20,\alpha} \\
-\xi_3 g_{21,\alpha} + \xi_3 h_{20,\alpha} - \xi_3 h_{21,\alpha} - \xi_4 g_{30,\alpha} - \xi_4 g_{31,\alpha} + \xi_4 h_{30,\alpha} - \xi_4 h_{31,\alpha} = 0 \\
\\
2(\xi_5 + k)g_{10,\alpha} + 6(\xi_5 + \frac{4k}{5})g_{11,\alpha} = 0 \\
\\
\xi_2 g_{10,\alpha} + \xi_2 g_{11,\alpha} + (\xi_1 - \frac{4k}{3})h_{10,\alpha} + (-\xi_1 + 2k)h_{11,\alpha} + \xi_3 g_{20,\alpha} \\
+\xi_3 g_{21,\alpha} - \xi_3 h_{20,\alpha} + \xi_3 h_{21,\alpha} + \xi_4 g_{30,\alpha} + \xi_4 g_{31,\alpha} - \xi_4 h_{30,\alpha} + \xi_4 h_{31,\alpha} = 0 \\
\\
2(-\xi_5 + k)h_{10,\alpha} + 6(\xi_5 - \frac{4k}{5}) = 0
\end{array}\right\} \quad (3.33)$$

and

$$\left.\begin{array}{l}
\xi_1 g_{10,\beta}+\xi_1 g_{11,\beta}+\xi_2 h_{10,\beta}-\xi_2 h_{11,\beta}-\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}\\
+h_{21,\beta})-\xi_4(g_{30,\beta}+g_{31,\beta}-h_{30,\beta}+h_{31,\beta})=2+\frac{4a}{A}(1+\epsilon\eta_1)\\
\\
2\xi_5 g_{10,\beta}+6\xi_5 g_{11,\beta} \qquad\qquad\qquad\qquad\qquad =2\\
\\
\xi_2 g_{10,\beta}+\xi_2 g_{11,\beta}+\xi_1 h_{10,\beta}-\xi_1 h_{11,\beta}+\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}\\
+h_{21,\beta})+\xi_4(g_{30,\beta}+g_{31,\beta}-h_{30,\beta}+h_{31,\beta})=2-\frac{4a}{A}(1+\epsilon\eta_1)\\
\\
2\xi_5 h_{10,\beta}-6\xi_5 h_{11,\beta} \qquad\qquad\qquad =2
\end{array}\right\} \tag{3.34}$$

where we make the abbreviation

$$\left.\begin{array}{l}
\xi_1=2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)\\
\xi_2=(1-\epsilon)\alpha_1\eta_1\\
\xi_3=(1-\epsilon)\alpha_1\eta_2\\
\xi_4=(1-\epsilon)\alpha_1\eta_3\\
\xi_5=(1+\eta_1)
\end{array}\right\} \tag{3.35}$$

### 3.3.2 First approximation when r = 2 .

Similarly considering the rest equations (3.11) and and proceeding in the same manner described in section (3.2) and subsection (3.3.1) and taking $l_0 = 1$,we have, by equating coefficients of $e^{-k\tau}$ and constant term , the following set of equations (3.36) and (3.37)

$$\left.\begin{array}{l}
-\lambda_3 g_{10,\alpha}-\lambda_3 g_{11,\alpha}+\lambda_3 h_{10,\alpha}-\lambda_3 h_{11,\alpha}+(\lambda_1+\frac{4k}{3})g_{20,\alpha}+(\lambda_1+2k)g_{21,\alpha}\\
+\lambda_2 h_{20,\alpha}-\lambda_2 h_{21,\alpha}-\lambda_4 g_{30,\alpha}-\lambda_4 g_{31,\alpha}+\lambda_4 h_{30,\alpha}-\lambda_4 h_{31,\alpha}=0\\
\\
\qquad 2(\lambda_5+k)g_{20,\alpha}+6(\lambda_5+\frac{4k}{3})g_{21,\alpha}=0\\
\\
\lambda_3 g_{10,\alpha}+\lambda_3 g_{11,\alpha}-\lambda_3 h_{10,\alpha}+\lambda_3 h_{11,\alpha}+\lambda_2 g_{20,\alpha}+\lambda_2 g_{21,\alpha}+(\lambda_1-\frac{4k}{3})h_{20,\alpha}\\
+(-\lambda_1+2k)h_{21,\alpha}+\lambda_4 g_{30,\alpha}-\lambda_4 g_{31,\alpha}+\lambda_4 h_{30,\alpha}-\lambda_4 h_{31,\alpha}=0\\
\\
\qquad 2(-\lambda_5+k)h_{20,\alpha}+6(\lambda_5-\frac{4k}{5})h_{21,\alpha}=0
\end{array}\right\} \tag{3.36}$$

and

$$\left.\begin{array}{l} \lambda_1 g_{20,\beta} + \lambda_1 g_{21,\beta} + \lambda_2 h_{20,\beta} - \lambda_2 h_{21,\beta} - \lambda_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} \\ +h_{11,\beta}) - \lambda_4(g_{30,\beta} + g_{31,\beta} - h_{30,\beta} + h_{31,\beta}) = 2 + \frac{4a}{A}(1+\epsilon\eta_2) \\ \\ 2\lambda_5 g_{20,\beta} + 6\lambda_5 g_{21,\beta} \qquad\qquad = 2 \\ \\ \lambda_2 g_{20,\beta} + \lambda_2 g_{21,\beta} + \lambda_1 h_{20,\beta} - \lambda_1 h_{21,\beta} + \lambda_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} \\ +h_{11,\beta}) + \lambda_4(g_{30,\beta} + g_{31,\beta} - h_{30,\beta} + h_{31,\beta}) = 2 - \frac{4a}{A}(1+\epsilon\eta_2) \\ \\ 2\lambda_5 h_{20,\beta} - 6\lambda_5 h_{21,\beta} \qquad\qquad = 2 \end{array}\right\} \tag{3.37}$$

where

$$\left.\begin{array}{l} \lambda_1 = 2(1+\eta_2) - (1-\epsilon)\alpha_2\eta_2 \\ \lambda_2 = (1-\epsilon)\alpha_2\eta_2 \\ \lambda_3 = (1-\epsilon)\alpha_2\eta_1 \\ \lambda_4 = (1-\epsilon)\alpha_2\eta_3 \\ \lambda_5 = (1+\eta_2) \end{array}\right\} \tag{3.38}$$

### 3.3.3 First approximation when r = 3 .

Similarly considering the rest equations (3.12) and and proceeding in the same manner described in section (3.2) and subsection (3.3.1) and (3.3.2) and taking $l_0 = 1$,we have, by equating coefficients of $e^{-k\tau}$ and constant term , the following set of equations (3.39) and (3.40)

$$\left.\begin{array}{l} -\gamma_3 g_{10,\alpha} - \gamma_3 g_{11,\alpha} + \gamma_3 h_{10,\alpha} - \gamma_3 h_{11,\alpha} - \gamma_4 g_{20,\alpha} - \gamma_4 g_{21,\alpha} \\ +\gamma_4 h_{20,\alpha} - \gamma_4 h_{21,\alpha} + (\gamma_1 + \frac{4k}{3})g_{30,\alpha} + (\gamma_1 + 2k)g_{31,\alpha} + \gamma_2 h_{30,\alpha} - \gamma_2 h_{31,\alpha} = 0 \\ \\ \qquad 2(\gamma_5 + k)g_{20,\alpha} + 6(\gamma_5 + \frac{4k}{3})g_{21,\alpha} = 0 \\ \\ \gamma_3 g_{10,\alpha} + \gamma_3 g_{11,\alpha} - \gamma_3 h_{10,\alpha} + \gamma_3 h_{11,\alpha} + \gamma_4 g_{20,\alpha} + \gamma_4 g_{21,\alpha} + (\gamma_1 - \frac{4k}{3})h_{30,\alpha} \\ +(-\gamma_1 + 2k)h_{31,\alpha} - \gamma_4 g_{20,\alpha} + \gamma_4 g_{21,\alpha} + \gamma_2 g_{30,\alpha} + \gamma_2 g_{31,\alpha} = 0 \\ \\ \qquad 2(-\gamma_5 + k)h_{20,\alpha} + 6(\gamma_5 - \frac{4k}{5})h_{21,\alpha} = 0 \end{array}\right\} \tag{3.39}$$

and

$$\left.\begin{array}{l}\gamma_1 g_{30,\beta} + \gamma_1 g_{31,\beta} + \gamma_2 h_{30,\beta} - \gamma_2 h_{31,\beta} - \gamma_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} \\ +h_{11,\beta}) - \gamma_4(g_{20,\beta} + g_{21,\beta} - h_{20,\beta} + h_{21,\beta}) = 2 + \frac{4a}{A}(1+\epsilon\eta_3) \\ \\ 2\gamma_5 g_{20,\beta} + 6\gamma_5 g_{21,\beta} \qquad\qquad\qquad = 2 \\ \\ \gamma_2 g_{30,\beta} + \gamma_2 g_{31,\beta} + \gamma_1 h_{30,\beta} - \gamma_1 h_{31,\beta} + \gamma_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} \\ +h_{11,\beta}) + \gamma_4(g_{20,\beta} + g_{21,\beta} - h_{20,\beta} + h_{21,\beta}) = 2 - \frac{4a}{A}(1+\epsilon\eta_3) \\ \\ 2\gamma_5 h_{20,\beta} - 6\gamma_5 h_{21,\beta} \qquad\qquad\qquad = 2\end{array}\right\} \tag{3.40}$$

where

$$\left.\begin{array}{l}\gamma_1 = 2(1+\eta_2) - (1-\epsilon)\alpha_2\eta_2 \\ \gamma_2 = (1-\epsilon)\alpha_2\eta_2 \\ \gamma_3 = (1-\epsilon)\alpha_2\eta_1 \\ \gamma_4 = (1-\epsilon)\alpha_2\eta_3 \\ \gamma_5 = (1+\eta_2)\end{array}\right\} \tag{3.41}$$

Now the set of equations (3.33),(3.36) and (3.39) has a nontrivial solution if the determinant of the coefficients $g_{il,\alpha}, h_{il,\alpha}$ is zero i.e

$$\Delta(k) = \begin{vmatrix} M_1(k) & M_2(k) \\ \\ M_3(k) & M_4(k) \end{vmatrix} \tag{3.42}$$

where

$$M_1(k) = \begin{pmatrix} (\xi_1 + \frac{4k}{5}) & (\xi_1 + 2k) & \xi_2 & -\xi_2 & -\xi_3 & -\xi_3 \\ \\ 2(\xi_5 + k) & 6(\xi_5 + \frac{4k}{5}) & 0 & 0 & 0 & 0 \\ \\ \xi_2 & \xi_2 & (\xi_1 - \frac{4k}{3}) & (-\xi_1 + 2k) & \xi_3 & \xi_3 \\ \\ 0 & 0 & 2(-\xi_5 + k) & 6(\xi_5 - \frac{4k}{5}) & 0 & 0 \\ \\ -\lambda_3 & -\lambda_3 & \lambda_3 & -\lambda_3 & (\lambda_1 + \frac{4k}{3}) & (\lambda_1 + 2k) \\ \\ 0 & 0 & 0 & 0 & 2(\lambda_5 + k) & 6(\lambda_5 + \frac{4k}{5}) \end{pmatrix} \tag{3.43}$$

$$M_2(k) = \begin{pmatrix} \xi_3 & -\xi_3 & -\xi_4 & -\xi_4 & \xi_4 & -\xi_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\xi_3 & \xi_3 & \xi_4 & \xi_4 & -\xi_4 & \xi_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_3 & -\lambda_3 & -\lambda_4 & -\lambda_4 & \lambda_4 & -\lambda_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.44}$$

$$M_3(k) = \begin{pmatrix} \lambda_3 & \lambda_3 & -\lambda_3 & \lambda_3 & \lambda_2 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma_3 & -\gamma_3 & \gamma_3 & -\gamma_3 & -\gamma_4 & -\gamma_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_3 & \gamma_3 & -\gamma_3 & \gamma_3 & \gamma_4 & \gamma_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.45}$$

$$M_4(k) = \begin{pmatrix} (\lambda_1 - \frac{4k}{5}) & (-\lambda_1 + 2k) & \lambda_4 & \lambda_4 & -\lambda_4 & \lambda_4 \\ 2(-\lambda_5 + k) & 6(\lambda_5 - \frac{4k}{5}) & 0 & 0 & 0 & 0 \\ \gamma_4 & -\gamma_4 & (\gamma_1 + \frac{4k}{3}) & (\gamma_1 + 2k) & \gamma_2 & -\gamma_2 \\ 0 & 0 & 2(\gamma_5 + k) & 6(\gamma_5 + \frac{4k}{5}) & 0 & 0 \\ \gamma_4 & \gamma_4 & \gamma_2 & \gamma_2 & (\gamma_1 - \frac{4k}{3}) & (-\gamma_1 + 2k) \\ 0 & 0 & 0 & 0 & 2(-\gamma_5 + k) & 6(\gamma_5 - \frac{4k}{5}) \end{pmatrix} \tag{3.46}$$

Now we consider the case [9] i.e $\eta_1 = \frac{5}{9}$ , $\eta_2 = \frac{1}{3}$ , $\eta_3 = \frac{1}{9}$ $\epsilon = 0$, so $\alpha_1 = \frac{5}{9}$ , $\alpha_2 = \frac{1}{3}$ , $\alpha_3 = \frac{1}{9}$.
Thus

$$\left.\begin{matrix} \xi_1 = \frac{227}{81} \\ \xi_2 = \frac{25}{81} \\ \xi_3 = \frac{5}{27} \\ \xi_4 = \frac{5}{81} \\ \xi_5 = \frac{14}{9} \end{matrix}\right\} ; \quad \left.\begin{matrix} \lambda_1 = \frac{23}{9} \\ \lambda_2 = \frac{1}{9} \\ \lambda_3 = \frac{5}{27} \\ \lambda_4 = \frac{1}{27} \\ \lambda_5 = \frac{4}{3} \end{matrix}\right\} \ and \ \left.\begin{matrix} \gamma_1 = \frac{179}{81} \\ \gamma_2 = \frac{1}{81} \\ \gamma_3 = \frac{5}{81} \\ \gamma_4 = \frac{1}{27} \\ \gamma_5 = \frac{10}{9} \end{matrix}\right\} \tag{3.47}$$

Therefore $\Delta(k)$ becomes ,

$$\begin{aligned}\Delta(k) \;=\; & \frac{1}{22143375}(243k^2+1312k+392)(486k^2-3024k+784)(27k^2+114k+160)\\ & \times\left(\frac{1108}{125}k^6-\frac{190954}{3375}k^5+\frac{375976}{6075}k^4+\frac{29187296}{164025}k^3-\frac{26784064}{98415}k^2\right.\\ & \left.-\frac{2921600}{19683}k+\frac{43648000}{177146}\right) \end{aligned} \tag{3.48}$$

and $\Delta(k)=0$ yields
$k=-0.317444, -5.081732, 0.271068, 5.951154, -1.578004, -3.755328, 1.167750,$ $1.566356$ , $2.562124$ , $3.631519$.
We take $k=0.271068$
Using (40) in (33) and (36) we obtain in Matrix form

$$\mathbf{A\ X=B} \tag{3.49}$$

where

$$\mathbf{A}=\begin{pmatrix}
\frac{227}{81} & \frac{227}{81} & \frac{25}{81} & -\frac{25}{81} & -\frac{5}{27} & -\frac{5}{27} & \frac{5}{27} & -\frac{5}{27} & -\frac{5}{81} & -\frac{5}{81} & \frac{5}{81} & -\frac{5}{81}\\
\frac{28}{9} & \frac{28}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{25}{81} & \frac{25}{81} & \frac{227}{81} & -\frac{227}{81} & \frac{5}{27} & \frac{5}{27} & -\frac{5}{27} & \frac{5}{27} & \frac{5}{81} & \frac{5}{81} & -\frac{5}{81} & \frac{5}{81}\\
0 & 0 & \frac{28}{9} & -\frac{28}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{5}{27} & -\frac{5}{27} & \frac{5}{27} & -\frac{5}{27} & \frac{23}{9} & \frac{23}{9} & \frac{1}{9} & -\frac{1}{9} & -\frac{1}{27} & -\frac{1}{27} & \frac{1}{27} & -\frac{1}{27}\\
0 & 0 & 0 & 0 & \frac{8}{3} & 8 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{5}{27} & \frac{5}{27} & -\frac{5}{27} & \frac{5}{27} & \frac{1}{9} & -\frac{1}{9} & \frac{23}{9} & -\frac{23}{9} & \frac{1}{27} & \frac{1}{27} & -\frac{1}{27} & \frac{1}{27}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{3} & -8 & 0 & 0 & 0 & 0\\
-\frac{5}{81} & -\frac{5}{81} & \frac{5}{81} & -\frac{5}{81} & -\frac{1}{27} & -\frac{1}{27} & \frac{1}{27} & -\frac{1}{27} & \frac{179}{81} & \frac{179}{81} & \frac{1}{81} & -\frac{1}{81}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{20}{9} & \frac{20}{3} & 0 & 0\\
\frac{5}{81} & \frac{5}{81} & -\frac{5}{81} & \frac{5}{81} & \frac{1}{27} & \frac{1}{27} & -\frac{1}{27} & \frac{1}{27} & \frac{1}{81} & \frac{1}{81} & \frac{179}{81} & -\frac{179}{81}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{20}{9} & -\frac{20}{3}
\end{pmatrix} \tag{3.50}$$

and

$$\mathbf{X}=\begin{pmatrix} g_{10,\beta}\\ g_{11,\beta}\\ h_{10,\beta}\\ h_{11,\beta}\\ g_{20,\beta}\\ g_{21,\beta}\\ h_{20,\beta}\\ h_{21,\beta}\\ g_{30,\beta}\\ g_{31,\beta}\\ h_{30,\beta}\\ h_{31,\beta}\end{pmatrix}, \quad \mathbf{B}=2\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}+4\frac{a}{A}\begin{pmatrix}1\\0\\-1\\0\\1\\0\\-1\\0\\1\\0\\-1\\0\end{pmatrix} \tag{3.51}$$

Inverting (3.49) we obtain

$$\begin{pmatrix} g_{10,\beta}\\ g_{11,\beta}\\ h_{10,\beta}\\ h_{11,\beta}\\ g_{20,\beta}\\ g_{21,\beta}\\ h_{20,\beta}\\ h_{21,\beta}\\ g_{30,\beta}\\ g_{31,\beta}\\ h_{30,\beta}\\ h_{31,\beta}\end{pmatrix}=\begin{pmatrix}0.642857\\0\\0.642857\\0\\0.750001\\0\\0.750001\\0\\0.9\\0\\0.9\\0\end{pmatrix}+\frac{a}{A}\begin{pmatrix}3\\-1\\-2.999999\\-0.999999\\3.000004\\-1.000002\\-3.000003\\-1.000001\\3.000001\\-1\\-3\\-1\end{pmatrix} \tag{3.52}$$

and using the boundary condition given by (3.29) and by (3.52) we get

$$\left.\begin{array}{lll} h_{10,\alpha} &= -0.642857 & +\frac{a}{A}2.999999\\ h_{11,\alpha} &= 0 & +\frac{a}{A}0.999999\\ h_{20,\alpha} &= -0.750001 & +\frac{a}{A}3.000003\\ h_{21,\alpha} &= 0 & +\frac{a}{A}1.000001\\ h_{30,\alpha} &= -0.9 & +\frac{a}{A}3\\ h_{31,\alpha} &= 0 & +\frac{a}{A} \end{array}\right\} \tag{3.53}$$

Now using (3.52),(3.53) and $k = 0.271068$ in the set of equations (3.34),(3.37) and (3.40) we obtain the following system

$$\begin{pmatrix} 3.163893 & 3.344605 & -0.185185 & -0.185185 & -0.061728 & -0.061728 \\ 3.653247 & 10.634460 & 0 & 0 & 0 & 0 \\ 0.308641 & 0.308641 & 0.185185 & 0.185185 & 0.061728 & 0.061728 \\ -0.185185 & -0.185185 & 2.916979 & 3.097691 & -0.037037 & -0.037037 \\ 0 & 0 & 3.208803 & 9.301127 & 0 & 0 \\ 0.185185 & 0.185185 & 0.111111 & 0.111111 & 0.037037 & 0.037037 \\ -0.061728 & -0.061728 & -0.037037 & -0.037037 & 2.5713 & 2.752012 \\ 0 & 0 & 0 & 0 & 2.764358 & 7.967794 \\ 0.061728 & 0.061728 & 0.037037 & 0.037037 & 0.012345 & 0.012345 \end{pmatrix}$$

$$\times \begin{pmatrix} g_{10,\alpha} \\ g_{11,\alpha} \\ g_{20,\alpha} \\ g_{21,\alpha} \\ g_{30,\alpha} \\ g_{31,\alpha} \end{pmatrix} = \begin{pmatrix} 0.392857 \\ 0 \\ 1.374798 \\ 0.235714 \\ 0 \\ 1.493219 \\ 0.039991 \\ 0 \\ 1.596146 \end{pmatrix} + \frac{a}{A} \begin{pmatrix} 1.111111 \\ 0 \\ 4.568974 \\ 0.666666 \\ 0 \\ 4.124534 \\ 0.222222 \\ 0 \\ 3.680085 \end{pmatrix} \tag{3.54}$$

which is the inconsistent system of equations.
Now we have the theorem[47] used in previous chapter (2.4.1) restated below

**Theorem 3.3.1** *Suppose that $A$ is a $m \times n$ matrix whose columns are linearly independent and that $b \in R^m$. Then the vector $x^*$ given $x^* = (A^TA)^{-1}A^Tb$ satisfies $||Ax^* - b|| \leq ||Ax - b||$ for all $x \in R^n$*

Applying the above theorem (3.3.1) in the inconsistent system of equation(3.54) we get

$$\left.\begin{aligned} g_{10,\alpha} &= 0.349579 - \tfrac{a}{A}0.973896 \\ g_{11,\alpha} &= -0.121088 + \tfrac{a}{A}0.337624 \\ g_{20,\alpha} &= 0.249050 - \tfrac{a}{A}0.002823 \\ g_{21,\alpha} &= -0.086489 + \tfrac{a}{A}0.002722 \\ g_{30,\alpha} &= 0.081124 - \tfrac{a}{A}0.021691 \\ g_{31,\alpha} &= -0.028451 + \tfrac{a}{A}0.008464 \end{aligned}\right\} \tag{3.55}$$

### 3.3.4 Determination of a/A

The mean intensity is given by

$$B(\tau) = \frac{1}{2}\int_{-1}^{1} I_r(\tau,\mu')d\mu' = \frac{3}{4}F[\tau + q(\tau)] \tag{3.56}$$

But from equation (3.13) , (3.14) and (3.15) we have

$$\begin{aligned} B(\tau) &= 3A\tau + \frac{1}{4}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-} + I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-} + I_{30}^{+} \\ &\quad -I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) \\ &= 3A\tau + \frac{1}{4}\left[(4.585716 - 1.849133e^{-0.271068})A + a(11.350402e^{-0.271068}\right. \\ &\quad \left.-5.999999)\right] \end{aligned} \tag{3.57}$$

Comparing (3.56) and (3.57)

$$A = \frac{1}{4}F \tag{3.58}$$

Again

$$\begin{aligned} F &= 2\int_{-1}^{1} I_r(\tau,\mu')d\mu' \;,\; r = 1,2,3. \\ &= 12A\tau + [(4.585716 - 1.849133e^{-0.271068})A + a(11.350402e^{-0.271068} \\ &\quad -5.999999)] \end{aligned} \tag{3.59}$$

Thus (3.58) and (3.59) gives

$$\frac{a}{A} = \frac{[4 - 12\tau + (1.849133e^{-0.271068} - 4.585716)]}{(11.350402e^{-0.271068} - 5.999999)} \tag{3.60}$$

## 3.4 Second approximation

We find the solution when $l_o = 2$ and name it second approximation. In this case we have from (3.22) and (3.23)

$$\left.\begin{array}{l}
\frac{4}{3}I_{10}^{+'} + 2I_{11}^{+'} + \frac{2}{3}I_{12}^{+'} - (\xi_1 I_{10}^{+} + \xi_2 I_{10}^{-} + \xi_1 I_{11}^{+} - \xi_2 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \\
+\xi_4(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) \\
+(1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\
\\
\frac{4}{3}I_{10}^{-'} - 2I_{11}^{-'} + \frac{2}{3}I_{12}^{-'} + (\xi_2 I_{10}^{+} + \xi_1 I_{10}^{-} + \xi_2 I_{11}^{+} - \xi_1 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \\
+\xi_4(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) = 2A - 4[(1+\epsilon\eta_1)(a+b\tau) \\
+(1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\
\\
2I_{10}^{+'} + \frac{24}{5}I_{11}^{+'} + 4I_{12}^{+'} - 2\xi_5(I_{10}^{+} + 3I_{11}^{+} + 2I_{12}^{+}) = -2A \\
-2I_{10}^{-'} + \frac{24}{5}I_{11}^{-'} - 4I_{12}^{-'} - 2\xi_5(I_{10}^{-} - 3I_{11}^{-} + 2I_{12}^{-}) = -2A \\
\frac{1}{3}I_{10}^{+'} + 2I_{11}^{+'} + \frac{80}{21}I_{12}^{+'} - \xi_5(2I_{11}^{+} + 5I_{12}^{+}) = 0 \\
\frac{1}{3}I_{10}^{-'} - 2I_{11}^{-'} + \frac{80}{21}I_{12}^{-'} - \xi_5(2I_{11}^{-} - 5I_{12}^{-}) = 0
\end{array}\right\} \quad (3.61)$$

and

$$\left.\begin{array}{l}
\frac{4}{3}I_{20}^{+'} + 2I_{21}^{+'} + \frac{2}{3}I_{22}^{+'} - (\lambda_1 I_{20}^{+} + \lambda_2 I_{20}^{-} + \lambda_1 I_{21}^{+} - \lambda_2 I_{21}^{-}) + \lambda_3(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) \\
+\lambda_4(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) = -2A - 4[(1+\epsilon\eta_2)(a+b\tau) \\
+(1-\epsilon)\eta_2 A\tau - (1+\eta_2)A\tau] \\
\\
\frac{4}{3}I_{20}^{-'} - 2I_{21}^{-'} + \frac{2}{3}I_{22}^{-'} + (\lambda_2 I_{20}^{+} + \lambda_1 I_{20}^{-} + \lambda_2 I_{21}^{+} - \lambda_1 I_{21}^{-}) + \lambda_3(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) \\
+\lambda_4(I_{30}^{+} - I_{30}^{-} + I_{31}^{+} + I_{31}^{-}) = 2A - 4[(1+\epsilon\eta_2)(a+b\tau) \\
+(1-\epsilon)\eta_2 A\tau - (1+\eta_2)A\tau] \\
\\
2I_{20}^{+'} + \frac{24}{5}I_{21}^{+'} + 4I_{22}^{+'} - 2\lambda_5(I_{20}^{+} + 3I_{21}^{+} + 2I_{22}^{+}) = -2A \\
\\
-2I_{20}^{-'} + \frac{24}{5}I_{21}^{-'} - 4I_{22}^{-'} - 2\lambda_5(I_{20}^{-} - 3I_{21}^{-} + 2I_{22}^{-}) = -2A \\
\\
\frac{1}{3}I_{20}^{+'} + 2I_{21}^{+'} + \frac{80}{21}I_{22}^{+'} - \lambda_5(2I_{21}^{+} + 5I_{22}^{+}) = 0 \\
\\
\frac{1}{3}I_{20}^{-'} - 2I_{21}^{-'} + \frac{80}{21}I_{22}^{-'} - \lambda_5(2I_{21}^{-} - 5I_{22}^{-}) = 0
\end{array}\right\} \quad (3.62)$$

and

$$\left.\begin{array}{l}
\frac{4}{3}I_{30}^{+'}+2I_{31}^{+'}+\frac{2}{3}I_{32}^{+'}-(\gamma_1 I_{30}^{+}+\gamma_2 I_{30}^{-}+\gamma_1 I_{31}^{+}-\gamma_2 I_{31}^{-})+\gamma_3(I_{10}^{+}-I_{10}^{-}+I_{11}^{+}+I_{11}^{-})\\
+\gamma_4(I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-})=-2A-4[(1+\epsilon\eta_3)(a+b\tau)\\
+(1-\epsilon)\eta_3 A\tau-(1+\eta_3)A\tau]\\
\\
\frac{4}{3}I_{30}^{-'}-2I_{31}^{-'}+\frac{2}{3}I_{32}^{-'}+(\gamma_2 I_{30}^{+}+\gamma_1 I_{30}^{-}+\gamma_2 I_{31}^{+}-\gamma_1 I_{31}^{-})+\gamma_3(I_{10}^{+}-I_{10}^{-}+I_{11}^{+}+I_{11}^{-})\\
+\gamma_4(I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-})=2A-4[(1+\epsilon\eta_3)(a+b\tau)\\
+(1-\epsilon)\eta_3 A\tau-(1+\eta_1)A\tau]\\
\\
2I_{30}^{+'}+\frac{24}{5}I_{31}^{+'}+4I_{32}^{+'}-2\gamma_5(I_{30}^{+}+3I_{31}^{+}+2I_{32}^{+})=-2A\\
\\
-2I_{30}^{-'}+\frac{24}{5}I_{31}^{-'}-4I_{32}^{-'}-2\gamma_5(I_{30}^{-}-3I_{31}^{-}+2I_{32}^{-})=-2A\\
\\
\frac{1}{3}I_{30}^{+'}+2I_{31}^{+'}+\frac{80}{21}I_{32}^{+'}-\gamma_5(2I_{31}^{+}+5I_{32}^{+})=0\\
\\
\frac{1}{3}I_{30}^{-'}-2I_{31}^{-'}+\frac{80}{21}I_{32}^{-'}-\gamma_5(2I_{31}^{-}-5I_{32}^{-})=0
\end{array}\right\} \tag{3.63}$$

where $\xi_i$ , $\lambda_i$ and $\gamma_i$ , $i = 1, 2, 3, 4, 5.$ are given by (3.35),(3.38) and (3.41).

The above set of equations given by (3.61),(3.62) and (3.63) are obtained (when $l_0 = 2$) by similar process described in section (3.3) considering the equation of transfer for $r = 1$ , 2 and 3 respectively.
and

$$\left.\begin{array}{l}
I_{il}^{+}(\tau,\mu)=A\tau+I_{i0}^{+}(\tau)\mu+3I_{i1}^{+}(\tau)\mu P_1(2\mu-1)+5I_{i2}^{+}(\tau)\mu P_2(2\mu-1),\\
\qquad\qquad\qquad\qquad 0\le\mu\le 1\\
\\
I_{il}^{-}(\tau,\mu)=A\tau+I_{i0}^{-}(\tau)\mu+3I_{i1}^{-}(\tau)\mu P_1(2\mu+1)+5I_{i2}^{-}(\tau)\mu P_2(2\mu+1),\\
\qquad\qquad\qquad\qquad -1\le\mu\le 0\\
i=1,2,3.
\end{array}\right\} \tag{3.64}$$

Now taking the trial solution given by (3.26) and substituting these in (3.61),(3.62)and (3.63)and then equating the coefficient of $e^{-k\tau}$ and constant term we obtain

$$\left.\begin{array}{l}
(\xi_1+\frac{4k}{3})g_{10,\alpha}+(\xi_1+2k)g_{11,\alpha}+\frac{2k}{3}g_{12,\alpha}+\xi_2h_{10,\alpha}-\xi_2h_{11,\alpha}-\xi_3g_{20,\alpha}-\xi_3g_{21,\alpha}\\
+\xi_3h_{20,\alpha}-\xi_3h_{21,\alpha}-\xi_4g_{30,\alpha}-\xi_4g_{31,\alpha}+\xi_4h_{30,\alpha}-\xi_4h_{31,\alpha}=0\\
\\
\xi_2g_{10,\alpha}+\xi_2g_{11,\alpha}+(\xi_1-\frac{4k}{3})h_{10,\alpha}+(-\xi_1+2k)h_{11,\alpha}-\frac{2k}{3}h_{12,\alpha}+\xi_3g_{20,\alpha}+\xi_3g_{21,\alpha}\\
-\xi_3h_{20,\alpha}+\xi_3h_{21,\alpha}+\xi_4g_{30,\alpha}+\xi_4g_{31,\alpha}-\xi_4h_{30,\alpha}+\xi_4h_{31,\alpha}=0\\
\\
2(k+\xi_5)g_{10,\alpha}+6(\xi_5+\frac{4k}{5})g_{11,\alpha}+4(\xi_5+k)g_{12,\alpha}=0\\
2(-\xi_5+k)h_{10,\alpha}+6(\xi_5-\frac{4k}{5})h_{11,\alpha}-4(\xi_5-k)h_{12,\alpha}=0\\
\frac{k}{3}g_{10,\alpha}+2(\xi_5+k)g_{11,\alpha}+5(\xi_5+\frac{16k}{21})g_{21,\alpha}=0\\
\frac{k}{3}h_{10,\alpha}+2(\xi_5-k)h_{11,\alpha}+5(-\xi_5+\frac{16k}{21})h_{12,\alpha}=0\\
\\
-\lambda_3g_{10,\alpha}-\lambda_3g_{11,\alpha}+\lambda_3h_{10,\alpha}-\lambda_3h_{11,\alpha}+(\lambda_1+\frac{4k}{3})g_{20,\alpha}+(\lambda_1+2k)g_{21,\alpha}\\
+\frac{2k}{3}g_{22,\alpha}+\lambda_2h_{20,\alpha}-\lambda_2h_{21,\alpha}-\lambda_4g_{30,\alpha}-\lambda_4g_{31,\alpha}+\lambda_4h_{30,\alpha}-\lambda_4h_{31,\alpha}=0\\
\\
\lambda_3g_{10,\alpha}+\lambda_3g_{11,\alpha}-\lambda_3h_{10,\alpha}+\lambda_3h_{11,\alpha}+\lambda_2g_{20,\alpha}+\lambda_2g_{21,\alpha}+(\lambda_1-\frac{4k}{3})h_{20,\alpha}\\
+(-\lambda_1+2k)h_{21,\alpha}-\frac{2k}{3}h_{22,\alpha}+\lambda_4g_{30,\alpha}+\lambda_4g_{31,\alpha}-\lambda_4h_{30,\alpha}+\lambda_4h_{31,\alpha}=0\\
\\
2(k+\lambda_5)g_{20,\alpha}+6(\lambda_5+\frac{4k}{5})g_{21,\alpha}+4(\lambda_5+k)g_{22,\alpha}=0\\
2(-\lambda_5+k)h_{20,\alpha}+6(\lambda_5-\frac{4k}{5})h_{21,\alpha}-4(\lambda_5-k)h_{22,\alpha}=0\\
\frac{k}{3}g_{20,\alpha}+2(\lambda_5+k)g_{21,\alpha}+5(\lambda_5+\frac{16k}{21})g_{22,\alpha}=0\\
\frac{k}{3}h_{20,\alpha}+2(\lambda_5-k)h_{21,\alpha}+5(-\lambda_5+\frac{16k}{21})h_{22,\alpha}=0\\
\\
-\gamma_3g_{10,\alpha}-\gamma_3g_{11,\alpha}+\gamma_3h_{10,\alpha}-\gamma_3h_{11,\alpha}-\gamma_4g_{20,\alpha}-\gamma_4g_{21,\alpha}+\gamma_4h_{20,\alpha}-\gamma_4h_{21,\alpha}\\
(\gamma_1+\frac{4k}{3})g_{30,\alpha}+(\gamma_1+2k)g_{31,\alpha}+\frac{2k}{3}g_{32,\alpha}+\gamma_2h_{30,\alpha}-\gamma_2h_{31,\alpha}=0\\
\\
\gamma_3g_{10,\alpha}+\gamma_3g_{11,\alpha}-\gamma_3h_{10,\alpha}+\gamma_3h_{11,\alpha}+\gamma_4g_{20,\alpha}+\gamma_4g_{21,\alpha}-\gamma_4h_{20,\alpha}+\gamma_4h_{21,\alpha}\\
\gamma_2g_{30,\alpha}+\gamma_2g_{31,\alpha}+(\gamma_1-\frac{4k}{3})h_{30,\alpha}+(-\gamma_1+2k)h_{31,\alpha}-\frac{2k}{3}h_{32,\alpha}=0\\
\\
2(k+\lambda_5)g_{20,\alpha}+6(\lambda_5+\frac{4k}{5})g_{21,\alpha}+4(\lambda_5+k)g_{22,\alpha}=0\\
2(-\lambda_5+k)h_{20,\alpha}+6(\lambda_5-\frac{4k}{5})h_{21,\alpha}-4(\lambda_5-k)h_{22,\alpha}=0\\
\frac{k}{3}g_{20,\alpha}+2(\lambda_5+k)g_{21,\alpha}+5(\lambda_5+\frac{16k}{21})g_{22,\alpha}=0\\
\frac{k}{3}h_{20,\alpha}+2(\lambda_5-k)h_{21,\alpha}+5(-\lambda_5+\frac{16k}{21})h_{22,\alpha}=0
\end{array}\right\} \quad (3.65)$$

and

$$\left.\begin{array}{l}
\xi_1(g_{10,\beta}+g_{11,\beta})+\xi_2(h_{10,\beta}-h_{11,\beta})-\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}+h_{21,\beta})\\
-\xi_4(g_{30,\beta}+g_{31,\beta}-h_{30,\beta}+h_{31,\beta})=2+4(1+\epsilon\eta_1)\frac{a}{A}\\
\\
\xi_2(g_{10,\beta}+g_{11,\beta})+\xi_1(h_{10,\beta}-h_{11,\beta})+\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}+h_{21,\beta})\\
+\xi_4(g_{30,\beta}+g_{31,\beta}-h_{30,\beta}+h_{31,\beta})=2-4(1+\epsilon\eta_1)\frac{a}{A}\\
\\
2\xi_5 g_{10,\beta}+6\xi_5 g_{11,\beta}+4\xi_5 g_{12,\beta}=2\\
2\xi_5 h_{10,\beta}-6\xi_5 h_{11,\beta}+4\xi_5 h_{12,\beta}=2\\
\\
2\xi_5 g_{11,\beta}+5\xi_5 g_{12,\beta}=0\\
2\xi_5 h_{11,\beta}-5\xi_5 h_{12,\beta}=0\\
\\
-\lambda_3(g_{10,\beta}+g_{11,\beta}-h_{10,\beta}+h_{11,\beta})+\lambda_1(g_{20,\beta}+g_{21,\beta})+\lambda_2(h_{20,\beta}-h_{21,\beta})\\
-\xi_4(g_{30,\beta}+g_{31,\beta}-h_{30,\beta}+h_{31,\beta})=2+4(1+\epsilon\eta_2)\frac{a}{A}\\
\\
\lambda_3(g_{10,\beta}+g_{11,\beta}-h_{10,\beta}+h_{11,\beta})+\lambda_2(g_{20,\beta}+g_{21,\beta})+\lambda_1(h_{20,\beta}-h_{21,\beta})\\
+\xi_4(g_{30,\beta}+g_{31,\beta}-h_{30,\beta}+h_{31,\beta})=2-4(1+\epsilon\eta_2)\frac{a}{A}\\
\\
2\lambda_5 g_{20,\beta}+6\lambda_5 g_{21,\beta}+4\lambda_5 g_{22,\beta}=2\\
2\lambda_5 h_{20,\beta}-6\lambda_5 h_{21,\beta}+4\lambda_5 h_{22,\beta}=2\\
\\
2\lambda_5 g_{21,\beta}+5\lambda_5 g_{22,\beta}=0\\
2\lambda_5 h_{21,\beta}-5\lambda_5 h_{22,\beta}=0\\
\\
-\gamma_3(g_{10,\beta}+g_{11,\beta}-h_{10,\beta}+h_{11,\beta})-\gamma_4(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}+h_{21,\beta})\\
\gamma_1(g_{30,\beta}+g_{31,\beta})+\gamma_2(h_{30,\beta}-h_{31,\beta})=2+4(1+\epsilon\eta_3)\frac{a}{A}\\
\\
\gamma_3(g_{10,\beta}+g_{11,\beta}-h_{10,\beta}+h_{11,\beta})+\gamma_4(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}+h_{21,\beta})\\
\gamma_2(g_{30,\beta}+g_{31,\beta})+\gamma_1(h_{30,\beta}-h_{31,\beta})=2-4(1+\epsilon\eta_3)\frac{a}{A}\\
\\
2\gamma_5 g_{30,\beta}+6\gamma_5 g_{31,\beta}+4\gamma_5 g_{32,\beta}=2\\
2\gamma_5 h_{30,\beta}-6\gamma_5 h_{31,\beta}+4\gamma_5 h_{32,\beta}=2\\
\\
2\gamma_5 g_{31,\beta}+5\gamma_5 g_{32,\beta}=0\\
2\gamma_5 h_{31,\beta}-5\gamma_5 h_{32,\beta}=0
\end{array}\right\} \tag{3.66}$$

where $\xi_i$ , $\lambda_i$ and $\gamma_i$ $i = 1,2,3,4.$ are same as (3.35) , (3.38) and (3.41). The above equations (3.65) have a non trivial solution if the determinant of the coefficient of the constant $g_{il,\alpha}$ and $h_{il,\alpha}$ is zero, that is

$$\Delta(k) = \begin{vmatrix} M_1(k) & M_2(k) & M_3(k) \\ M_4(k) & M_5(k) & M_6(k) \\ M_7(k) & M_8(k) & M_9(k) \end{vmatrix} = 0 \tag{3.67}$$

where

$$M_1(k) = \begin{pmatrix} (\xi_1+\frac{4k}{3}) & (\xi_1+2k) & \frac{2k}{3} & \xi_2 & -\xi_2 & 0 \\ \xi_2 & \xi_2 & 0 & (\xi_1-\frac{4k}{3}) & (-\xi_1+2k) & -\frac{2k}{3} \\ 2(\xi_5+k) & 6(\xi_5+\frac{4k}{5}) & 4(\xi_5+k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\xi_5+k) & 6(\xi_5-\frac{4k}{5}) & 4(\xi_5-k) \\ \frac{k}{3} & 2(\xi_5+k) & 5(\xi_5+\frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\xi_5-k) & 5(-\xi_5+\frac{16k}{21}) \end{pmatrix} \tag{3.68}$$

$$M_2(k) = \begin{pmatrix} -\xi_3 & -\xi_3 & 0 & \xi_3 & -\xi_3 & 0 \\ \xi_3 & \xi_3 & 0 & -\xi_3 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.69}$$

$$M_3(k) = \begin{pmatrix} -\xi_4 & -\xi_4 & 0 & \xi_4 & -\xi_4 & 0 \\ \xi_4 & \xi_4 & 0 & -\xi_4 & \xi_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.70}$$

$$M_4(k) = \begin{pmatrix} -\lambda_3 & -\lambda_3 & 0 & \lambda_3 & -\lambda_3 & 0 \\ \lambda_3 & \lambda_3 & 0 & -\lambda_3 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.71}$$

$$M_5(k) = \begin{pmatrix} (\lambda_1 + \frac{4k}{3}) & (\lambda_1 + 2k) & \frac{2k}{3} & \lambda_2 & -\lambda_2 & 0 \\ \lambda_2 & \lambda_2 & 0 & (\lambda_1 - \frac{4k}{3}) & (-\lambda_1 + 2k) & -\frac{2k}{3} \\ 2(\lambda_5 + k) & 6(\lambda_5 + \frac{4k}{5}) & 4(\lambda_5 + k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\lambda_4 + k) & 6(\lambda_5 - \frac{4k}{5}) & 4(\lambda_5 - k) \\ \frac{k}{3} & 2(\lambda_5 + k) & 5(\lambda_5 + \frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\lambda_5 - k) & 5(-\lambda_5 + \frac{16k}{21}) \end{pmatrix} \tag{3.72}$$

$$M_6(k) = \begin{pmatrix} -\lambda_4 & -\lambda_4 & 0 & \lambda_4 & -\lambda_4 & 0 \\ \lambda_4 & \lambda_4 & 0 & -\lambda_4 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.73}$$

$$M_7(k) = \begin{pmatrix} -\gamma_3 & -\gamma_3 & 0 & \gamma_3 & -\gamma_3 & 0 \\ \gamma_3 & \gamma_3 & 0 & -\gamma_3 & \gamma_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.74}$$

$$M_8(k) = \begin{pmatrix} -\gamma_4 & -\gamma_4 & 0 & \gamma_4 & -\gamma_4 & 0 \\ \gamma_4 & \gamma_4 & 0 & -\gamma_4 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.75}$$

$$M_9(k) = \begin{pmatrix} (\gamma_1 + \frac{4k}{3}) & (\gamma_1 + 2k) & \frac{2k}{3} & \gamma_2 & -\gamma_2 & 0 \\ \gamma_2 & \gamma_2 & 0 & (\gamma_1 - \frac{4k}{3}) & (-\gamma_1 + 2k) & -\frac{2k}{3} \\ 2(\gamma_5 + k) & 6(\gamma_5 + \frac{4k}{5}) & 4(\gamma_5 + k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\gamma_4 + k) & 6(\gamma_5 - \frac{4k}{5}) & 4(\gamma_5 - k) \\ \frac{k}{3} & 2(\gamma_5 + k) & 5(\gamma_5 + \frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\gamma_5 - k) & 5(-\gamma_5 + \frac{16k}{21}) \end{pmatrix} \tag{3.76}$$

Now we consider the case[26] as previous i.e
$\eta_1 = \frac{5}{9}$ , $\eta_2 = \frac{1}{3}$ , $\eta_3 = \frac{1}{9}$ , $\epsilon = 0$ and $\xi_i$ , $\lambda_i$ and $\gamma_i$ are given by (3.47).
Thus $\Delta(k)$ becomes

$$\left.\begin{aligned} \Delta(k) = 512 \ &\times \left(-\tfrac{1152}{1225}k^6 + \tfrac{17216}{315}k^4 - \tfrac{8693440}{19683}k^2 + \tfrac{434561792}{531441}\right) \\ &\times \left(-\tfrac{1152}{1225}k^6 - \tfrac{22}{189}k^5 + \tfrac{93788}{2205}k^4 + \tfrac{2672}{8505}k^3 - \tfrac{3122816}{11907}k^2 + \tfrac{90112}{243}\right) \\ &\times \left(-\tfrac{1152}{1225}k^6 + \tfrac{13376}{441}k^4 - \tfrac{128552000}{964467}k^2 + \tfrac{71200000}{531441}\right) \end{aligned}\right\} \tag{3.77}$$

Now $\Delta(k) = 0$ yields $k =$ -6.99079 , -2.55251 , -1.65252 , 1.65252 , 2.55251 , 6.99079 , -6.2506 , -2.21585 , -1.44456 , 1.4461 , 2.23031 , 6.11083 , -5.21942 , -1.87851 , -1.21736 , 1.21736 , 1.87851 , 5.21942.
We take $k = 1.21736$.
Again using the values of $\xi_i$ , $\lambda_i$ and $\gamma_i$ given by (3.47) in (3.66) we obtain the following matrix form

$$\mathbf{A}\ \mathbf{X} = \mathbf{B} \tag{3.78}$$

where

$$
\mathbf{A}=\begin{pmatrix}
\frac{227}{81} & \frac{227}{81} & 0 & \frac{25}{81} & -\frac{25}{81} & 0 & -\frac{5}{27} & -\frac{5}{27} & 0 & \frac{5}{27} & -\frac{5}{27} & 0 & -\frac{5}{81} & -\frac{5}{81} & 0 & \frac{5}{81} & -\frac{5}{81} & 0 \\
\frac{5}{81} & \frac{5}{81} & 0 & \frac{227}{81} & -\frac{227}{81} & 0 & \frac{5}{27} & \frac{5}{27} & 0 & -\frac{5}{27} & \frac{5}{27} & 0 & \frac{5}{81} & \frac{5}{81} & 0 & -\frac{5}{81} & \frac{5}{81} & 0 \\
\frac{28}{9} & \frac{28}{3} & \frac{56}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{28}{9} & -\frac{28}{3} & \frac{56}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{28}{9} & \frac{70}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{28}{9} & \frac{70}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5}{27} & -\frac{5}{27} & 0 & \frac{5}{27} & -\frac{5}{27} & 0 & \frac{23}{9} & \frac{23}{9} & 0 & \frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{27} & -\frac{1}{27} & 0 & \frac{1}{27} & -\frac{1}{27} & 0 \\
\frac{5}{27} & \frac{5}{27} & 0 & -\frac{5}{27} & \frac{5}{27} & 0 & \frac{1}{9} & -\frac{1}{9} & 0 & \frac{23}{9} & -\frac{23}{9} & 0 & \frac{1}{27} & \frac{1}{27} & 0 & -\frac{1}{27} & \frac{1}{27} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{3} & 8 & \frac{16}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{3} & -8 & \frac{16}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{3} & \frac{20}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{3} & -\frac{20}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5}{81} & -\frac{5}{81} & 0 & \frac{5}{81} & -\frac{5}{81} & 0 & -\frac{1}{27} & -\frac{1}{27} & 0 & \frac{1}{27} & -\frac{1}{27} & 0 & \frac{179}{81} & \frac{179}{81} & 0 & \frac{1}{81} & -\frac{1}{81} & 0 \\
\frac{5}{81} & \frac{5}{81} & 0 & -\frac{5}{81} & \frac{5}{81} & 0 & \frac{1}{27} & \frac{1}{27} & 0 & -\frac{1}{27} & \frac{1}{27} & 0 & \frac{1}{81} & \frac{1}{81} & 0 & \frac{179}{81} & -\frac{179}{81} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{20}{9} & \frac{20}{3} & \frac{40}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{20}{9} & -\frac{20}{3} & \frac{40}{9} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{20}{9} & \frac{50}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{20}{9} & -\frac{50}{9}
\end{pmatrix}
\tag{3.79}
$$

and

$$\mathbf{X} = \begin{pmatrix} g_{10,\beta} \\ g_{11,\beta} \\ g_{12,\beta} \\ h_{10,\beta} \\ h_{11,\beta} \\ h_{12,\beta} \\ g_{20,\beta} \\ g_{21,\beta} \\ g_{22,\beta} \\ h_{20,\beta} \\ h_{21,\beta} \\ h_{22,\beta} \\ g_{30,\beta} \\ g_{31,\beta} \\ g_{32,\beta} \\ h_{30,\beta} \\ h_{31,\beta} \\ h_{32,\beta} \end{pmatrix}, \quad \mathbf{B} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4\frac{a}{A} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.80}$$

Inverting we obtain

$$\left.\begin{aligned}
g_{10,\beta} &= 0.629879 + \tfrac{a}{A}3.62629 \\
g_{11,\beta} &= 0.005899 - \tfrac{a}{A}1.64831 \\
g_{12,\beta} &= -0.002359 + \tfrac{a}{A}0.659326 \\
h_{10,\beta} &= 0.748343 - \tfrac{a}{A}3.33849 \\
h_{11,\beta} &= 0.047947 - \tfrac{a}{A}1.5175 \\
h_{12,\beta} &= 0.019179 - \tfrac{a}{A}0.606998 \\
g_{20,\beta} &= 0.740915 + \tfrac{a}{A}3.6584 \\
g_{21,\beta} &= 0.004129 - \tfrac{a}{A}1.65382 \\
g_{22,\beta} &= -0.001651 + \tfrac{a}{A}0.661528 \\
h_{20,\beta} &= 0.759085 - \tfrac{a}{A}3.6384 \\
h_{21,\beta} &= 0.004129 - \tfrac{a}{A}1.65382 \\
h_{22,\beta} &= 0.001657 - \tfrac{a}{A}0.661528 \\
g_{30,\beta} &= 0.896366 + \tfrac{a}{A}3.65536 \\
g_{31,\beta} &= 0.001657 - \tfrac{a}{A}1.66153 \\
g_{32,\beta} &= -0.000660 + \tfrac{a}{A}0.664661 \\
h_{30,\beta} &= 0.903634 - \tfrac{a}{A}3.65536 \\
h_{31,\beta} &= 0.001651 - \tfrac{a}{A}1.66153 \\
h_{32,\beta} &= 0.00066 - \tfrac{a}{A}0.664611
\end{aligned}\right\} \tag{3.81}$$

Using boundary condition given by (3.29) and by (3.81) we get

$$\left.\begin{array}{l} h_{10,\alpha} = -0.748343 + \frac{a}{A}3.33849 \\ h_{11,\alpha} = -0.047947 + \frac{a}{A}1.5175 \\ h_{12,\alpha} = -0.019179 + \frac{a}{A}0.606998 \\ h_{20,\alpha} = -0.759085 + \frac{a}{A}3.6384 \\ h_{21,\alpha} = -0.004129 + \frac{a}{A}1.65382 \\ h_{22,\alpha} = -0.001657 + \frac{a}{A}0.661528 \\ h_{30,\alpha} = -0.903634 + \frac{a}{A}3.65536 \\ h_{31,\alpha} = -0.001651 + \frac{a}{A}1.66153 \\ h_{32,\alpha} = -0.00066 + \frac{a}{A}0.664611 \end{array}\right\} \tag{3.82}$$

Now using (3.81) , (3.82) and $k = 1.21736$ in (3.65) we have the following system

$$\begin{pmatrix} 3.61404 & 5.23718 & 0.80906 & -\frac{5}{27} & -\frac{5}{27} & 0 & -\frac{5}{81} & -\frac{5}{81} & 0 \\ \frac{25}{81} & \frac{25}{81} & 0 & \frac{5}{27} & \frac{5}{27} & 0 & \frac{5}{81} & \frac{5}{81} & 0 \\ 5.54583 & 15.17666 & 11.09166 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.40578 & 5.54583 & 12.41533 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{27} & -\frac{5}{27} & 0 & 4.1787 & 4.99027 & 0.80906 & -\frac{1}{27} & -\frac{1}{27} & 0 \\ \frac{5}{27} & \frac{5}{27} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{27} & \frac{1}{27} & 0 \\ 0 & 0 & 0 & 5.10138 & 13.84332 & 10.20277 & 0 & 0 & 0 \\ -\frac{5}{81} & -\frac{5}{81} & 0 & -\frac{1}{27} & -\frac{1}{27} & 0 & 3.83302 & 4.64459 & 0.80906 \\ \frac{5}{81} & \frac{5}{81} & 0 & \frac{1}{27} & \frac{1}{27} & 0 & \frac{1}{81} & \frac{1}{81} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.65694 & 12.50999 & 9.31388 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.40578 & 4.65694 & 10.1931 \end{pmatrix}$$

$$\times \begin{pmatrix} g_{10,\alpha} \\ g_{11,\alpha} \\ g_{12,\alpha} \\ g_{20,\alpha} \\ g_{21,\alpha} \\ g_{22,\alpha} \\ g_{30,\alpha} \\ g_{31,\alpha} \\ g_{32,\alpha} \end{pmatrix} = \begin{pmatrix} 0.411655 \\ 0.652325 \\ 0 \\ 0 \\ 0.246993 \\ 0.545514 \\ 0 \\ 0 \\ 0.082331 \\ 0.458828 \\ 0 \\ 0 \end{pmatrix} + \frac{a}{A}\begin{pmatrix} -1.052624 \\ -2,395881 \\ 0 \\ 0 \\ -0.631574 \\ 3.066827 \\ 0 \\ 0 \\ -0.210524 \\ -1.793002 \\ 0 \\ 0 \end{pmatrix} \tag{3.83}$$

which is the inconsistent system of equations. Now applying the theorem

(3.3.1) on the above inconsistent system of equations (3.83) we obtain

$$\begin{pmatrix} g_{10,\alpha} \\ g_{11,\alpha} \\ g_{12,\alpha} \\ g_{20,\alpha} \\ g_{21,\alpha} \\ g_{22,\alpha} \\ g_{30,\alpha} \\ g_{31,\alpha} \\ g_{32,\alpha} \end{pmatrix} = \begin{pmatrix} 0.527225 \\ -0.272536 \\ 0.106247 \\ 0.198501 \\ -0.102481 \\ 0.039235 \\ 0.07826 \\ -0.041 \\ 0.015677 \end{pmatrix} + \frac{a}{A} \begin{pmatrix} -1.07711 \\ 0.549648 \\ -0.211484 \\ -0.433245 \\ 0.222544 \\ -0.084954 \\ -0.16805 \\ 0.087502 \\ -0.033327 \end{pmatrix} \tag{3.84}$$

As in first approximation (Section 3.3) we find $A = \frac{1}{4}F$ and hence

$$\begin{aligned} B(\tau) &= 3A\tau + \frac{1}{4}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-} + I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-} + I_{30}^{+} - I_{30}^{-} \\ &\quad + I_{31}^{+} + I_{31}^{-}) \\ &= 3A\tau + \frac{1}{4}\left[(-2.07685362A + 14.6463899a)e^{1.21736\tau}\right. \\ &\quad \left. + (4.743637A - 2.83173a)\right] \\ &= \frac{1}{4}F(3\tau - 0.519213405e^{1.21736\tau} + 1.18590925) \\ &\quad + a(3.661597475e^{1.21736\tau} - 0.7079325) \end{aligned}$$

(3.85)