Chapter 2

Solution of the equation of radiative transfer for interlocked doublets by double interval spherical harmonic method

2.1 Equation of Transfer

The equation of transfer for the r-th line of multiplets in the case of interlocking without redistribution is

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\epsilon\eta_r)(a+b\tau)$$
$$-(1-\epsilon)\alpha_r\sum_{p=1}^{k}\frac{1}{2}\eta_p\int_{-1}^{1}I_p(\tau,\mu')d\mu';$$
$$r = 1, 2, \ldots\ldots k \tag{2.1}$$

where

$$\alpha_r = \frac{\eta_r}{\eta_1+\eta_2+\ldots\ldots+\eta_k} \tag{2.2}$$

so that

$$\alpha_1+\alpha_2+\ldots\ldots+\alpha_k = 1 \tag{2.3}$$

and η_r, the ratio of line to the continuuam absorption coefficient for the r-th line is independent of depth but is a function of frequency. ϵ is the coefficient of thermal emission, is independent of both frequency and depth.

For doublet (2.1) reduces to

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\epsilon\eta_r)(a+b\tau)$$
$$-(1-\epsilon)\alpha_r\frac{1}{2}\left[\sum_{p=1}^{2}\eta_p\int_{-1}^{1}I_p(\tau,\mu')d\mu'\right];r=1,2. \quad (2.4)$$

The above equation of transfer (2.4) is to be solved subject to the boundary conditions:

$$I_r(0,\mu)\equiv 0 \;\; for \;\; -1\leq\mu\leq 0\,,\;\; r=1,2 \quad (2.5)$$
$$I_r(\tau,\mu)e^{-\tau}\rightarrow 0 \;\; as \;\; \tau\rightarrow\infty\;,r=1,2 \quad (2.6)$$

We shall seek a solution of equations (2.4) $I_r(\tau,\mu)$ can be expansions $I_r^+(\tau,\mu)$ and $I_r^-(\tau,\mu)$ for μ in the interval (0,1) and (-1,0) respectively in the form[65]

$$I_r^+(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_{rl}^+(\tau)\mu P_l(2\mu-1)\;,\;0\leq\mu\leq 1, r=1,2. (2.7)$$
$$I_r^-(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_{rl}^-(\tau)\mu P_l(2\mu+1)\;,\;-1\leq\mu\leq 0, r=1,2. (2.8)$$

where A is a constant(independent of μ)to be determined and the recurrence formulae

$$\mu P_l(2\mu\pm1) = \frac{1}{(2l+1)}\left[\frac{l+1}{2}P_{l+1}(2\mu\pm1)\mp\frac{2l+1}{2}P_l(2\mu\pm1)+\frac{l}{2}P_{l-1}(2\mu\pm1)\right] \quad (2.9)$$

has the advantages due to orthogonality of $P_l(2\mu-1)$ in $(0,1)$ and $P_l(2\mu+1)$ in $(-1,0)$

The equation of transfer (2.4) in the present representation is equivalent to

$$\mu\frac{dI_1(\tau,\mu)}{d\tau} = (1+\eta_1)I_1(\tau,\mu) - (1+\epsilon\eta_1)(a+b\tau)$$
$$-(1-\epsilon)\alpha_1\frac{1}{2}\left[\eta_1\int_{-1}^{1}I_1(\tau,\mu')d\mu' + \eta_2\int_{-1}^{1}I_2(\tau,\mu')d\mu'\right] \quad (2.10)$$

and

$$\mu\frac{dI_2(\tau,\mu)}{d\tau} = (1+\eta_2)I_2(\tau,\mu) - (1+\epsilon\eta_2)(a+b\tau)$$
$$-(1-\epsilon)\alpha_2\frac{1}{2}\left[\eta_1\int_{-1}^{1}I_1(\tau,\mu')d\mu' + \eta_2\int_{-1}^{1}I_2(\tau,\mu')d\mu'\right] \quad (2.11)$$

also we have

$$\int_{-1}^{1}I_1(\tau,\mu')d\mu' = 2A\tau + \frac{1}{2}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) \quad (2.12)$$
$$\int_{-1}^{1}I_2(\tau,\mu')d\mu' = 2A\tau + \frac{1}{2}(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \quad (2.13)$$

Let us first consider the equation for $r = 1$

The equation of transfer (2.10) can be written as

$$\mu\frac{dI_1^{+}(\tau,\mu)}{d\tau} = (1+\eta_1)I_1^{+}(\tau,\mu) - (1+\epsilon\eta_1)(a+b\tau) - (1-\epsilon)\eta_1 A\tau$$
$$-\frac{(1-\epsilon)\alpha_1}{4}\left[\eta_1(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) + \eta_2(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})\right] \quad (2.14)$$

and

$$\mu\frac{dI_1^{-}(\tau,\mu)}{d\tau} = (1+\eta_1)I_1^{-}(\tau,\mu) - (1+\epsilon\eta_1)(a+b\tau) - (1-\epsilon)\eta_1 A\tau$$
$$-\frac{(1-\epsilon)\alpha_1}{4}\left[\eta_1(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) + \eta_2(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})\right] \quad (2.15)$$

Multiplying equation (2.14) by $P_l(2\mu-1)$ and equation (2.15) by $P_l(2\mu+1)$ respectively and integrating over μ in their respective ranges and using the recurrence formulae (2.9), we have the following equations:

$$\left.\begin{aligned}
&\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{1l-2}^{+'} + 2lI_{1l-1}^{+'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{+'} + 2(l+1)I_{1l+1}^{+'} + \frac{l^2+3l+2}{2l+3}I_{1l+2}^{+'}\right] \\
&+A\textstyle\int_0^1 \mu P_l(2\mu-1)d\mu = \frac{(1+\eta_1)}{2(2l+1)}[lI_{1l-1}^{+} + (2l+1)I_{1l}^{+} + (l+1)I_{1l+1}^{+}] - [(1+\epsilon\eta_1) \\
&(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau]\textstyle\int_0^1 P_l(2\mu-1)d\mu - \frac{(1-\epsilon)\alpha_1}{4}[\eta_1(I_{10}^{+} - I_{10}^{-} \\
&+I_{11}^{+} + I_{11}^{-}) + \eta_2(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})]
\end{aligned}\right\} \quad (2.16)$$

and

$$\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I^{-'}_{1l-2} - 2lI^{-'}_{1l-1} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{-'}_{1l} - 2(l+1)I^{-'}_{1l+1} + \frac{l^2+3l+2}{2l+3}I^{-'}_{1l+2}\right]$$
$$+A\int_{-1}^{0}\mu P_l(2\mu+1)d\mu = \frac{(1+\eta_1)}{2(2l+1)}[lI^{-}_{1l-1} - (2l+1)I^{-}_{1l} + (l+1)I^{-}_{1l+1}] - [(1+\epsilon\eta_1)$$
$$(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau]\int_{-1}^{0}P_l(2\mu+1)d\mu - \frac{(1-\epsilon)\alpha_1}{4}[\eta_1(I^{+}_{10} - I^{-}_{10}$$
$$+I^{+}_{11} + I^{-}_{11}) + \eta_2(I^{+}_{20} - I^{-}_{20} + I^{+}_{21} + I^{-}_{21})] \qquad (2.17)$$

where I'_l are differentials of I_l with respect to the optical thickness τ. Separating the equations for $l = 0$ and $l = 1$ from the equations (2.16) and (2.17) we can write

<u>*for* $l = o$</u>

$$\left(\frac{4}{3}I^{+'}_{10} + 2I^{+'}_{11} + \frac{2}{3}I^{+'}_{12}\right) - [\{2(1+\eta_1) - \eta_1(1-\epsilon)\alpha_1\}I^{+}_{10} + (1-\epsilon)\alpha_1\eta_1 I^{-}_{10}+$$
$$\{2(1+\eta_1) - \eta_1(1-\epsilon)\alpha_1\}I^{+}_{11} - (1-\epsilon)\alpha_1\eta_1 I^{-}_{11}] + (1-\epsilon)\alpha_1\eta_2[I^{+}_{20} - I^{-}_{20}$$
$$+I^{+}_{21} + I^{-}_{21}] = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau]$$

<u>*for* $l = 1$</u>

$$\left(2I^{+'}_{10} + \frac{24}{5}I^{+'}_{11} + 4I^{+'}_{12} + \frac{6}{5}I^{+'}_{13}\right) - 2(1+\eta_1)(I^{+}_{10} + 3I^{+}_{11} + 2I^{+}_{12}) = -2A$$

<u>*for* $l \neq 0,1$</u>

$$\left[\frac{l^2-l}{2l-1}I^{+'}_{1l-2} + 2lI^{+'}_{1l-1} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{+'}_{1l} + 2(l+1)I^{+'}_{1l+1} + \frac{l^2+3l+2}{2l+3}I^{+'}_{1l+2}\right]$$
$$-2(1+\eta_1)[lI^{+}_{1l-1} + (2l+1)I^{+}_{1l} + (l+1)I^{+}_{1l+1}] = 0 \qquad (2.18)$$

and

$$\left.\begin{array}{l}
\underline{for\ l=o} \\
\left(\frac{4}{3}I_{10}^{-'}+2I_{11}^{-'}+\frac{2}{3}I_{12}^{-'}\right)-[(1-\epsilon)\alpha_1\eta_1 I_{10}^{+}+\{2(1+\eta_1)-\eta_1(1-\epsilon)\alpha_1\}I_{10}^{-} \\
+(1-\epsilon)\alpha_1\eta_1 I_{11}^{+}-\{2(1+\eta_1)-\eta_1(1-\epsilon)\alpha_1\}I_{11}^{-}]+(1-\epsilon)\alpha_1\eta_2[I_{20}^{+}-I_{20}^{-} \\
+I_{21}^{+}+I_{21}^{-}]=2A-4[(1+\epsilon\eta_1)(a+b\tau)+(1-\epsilon)\eta_1 A\tau-(1+\eta_1)A\tau] \\
\\
\underline{for\ l=1} \\
\left(-2I_{10}^{-'}+\frac{24}{5}I_{11}^{-'}-4I_{12}^{-'}+\frac{6}{5}I_{13}^{-'}\right)-2(1+\eta_1)(I_{10}^{-}-3I_{11}^{-}+2I_{12}^{-})=-2A \\
\\
\underline{for\ l\neq 0,1} \\
\left[\frac{l^2-l}{2l-1}I_{1l-2}^{-'}-2lI_{1l-1}^{-'}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{-'}-2(l+1)I_{1l+1}^{-'}+\frac{l^2+3l+2}{2l+3}I_{1l+2}^{-'}\right] \\
-2(1+\eta_1)[lI_{1l-1}^{-}-(2l+1)I_{1l}^{-}+(l+1)I_{1l+1}^{-}]=0
\end{array}\right\} \tag{2.19}$$

The equations (2.18) and (2.19) are to be solved subject to the boundary conditions (2.5) and (2.6) which are restated below

$$I_{il}^{-}(0)\equiv 0\ \ and\ \ \left.\begin{array}{l} I_{il}^{+}(\tau)\,e^{-\tau}\rightarrow 0 \\ I_{il}^{-}(\tau)\,e^{-\tau}\rightarrow 0 \end{array}\right\}\ as\ \tau\rightarrow\infty,\ \ i=1,2. \tag{2.20}$$

2.2 Solution

It is assumed that at the N-th approximation

$$I_{i\,N+1}^{+}=I_{i\,N+1}^{-}=0\ ,\ \ i=1,2. \tag{2.21}$$

We assume a trial solution of the form

$$\left.\begin{array}{l} I_{il}^{+}(\tau)=A\left[g_{il,\alpha}\,e^{-k\tau}+g_{il,\beta}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad ,i=1,2. \\ I_{il}^{-}(\tau)=A\left[h_{il,\alpha}\,e^{-k\tau}+h_{il,\beta}\right] \end{array}\right\} \tag{2.22}$$

where $g_{1l,\alpha}, g_{1l,\beta}, g_{2l,\alpha}, g_{2l,\beta}, h_{1l,\alpha}, h_{1l,\beta}, h_{2l,\alpha}, h_{2l,\beta}$ are constants to be determined.

Substituting these in (2.18) and (2.19) and equating the coefficients of $e^{-k\tau}$ and constant term we obtain (2.23) and (2.24) which are as follows.

$$\left.\begin{array}{l}
\underline{for\ l=0} \\
\{\frac{4k}{3}+2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)\}g_{10,\alpha}+\{2k+2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)\}g_{11,\alpha} \\
+\frac{2k}{3}g_{12,\alpha}-(1-\epsilon)\alpha_1\eta_2\, g_{20,\alpha}-(1-\epsilon)\alpha_1\eta_2\, g_{21,\alpha}+(1-\epsilon)\alpha_1\eta_1 h_{10,\alpha} \\
-(1-\epsilon)\alpha_1\eta_1\, h_{11,\alpha}+(1-\epsilon)\alpha_1\eta_2\, h_{20,\alpha}-(1-\epsilon)\alpha_1\eta_2\, h_{21,\alpha}=0 \\
\underline{for\ l=1} \\
2(1+\eta_1+k)g_{10,\alpha}+6(1+\eta_1+\frac{4k}{5})g_{11,\alpha}+4(1+\eta_1+k)g_{12,\alpha}+\frac{6k}{5}\, g_{13,\alpha}=0 \\
\underline{for\ l\neq 0,1} \\
\frac{l^2-l}{2l-1}g_{1l-2,\alpha}+2l(1+\eta_1+k)g_{1l-1,\alpha}+\{\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}k+2(1+\eta_1)(2l+1)\}\times \\
g_{1l,\alpha}+2(l+1)(1+\eta_1+k)g_{1l+1,\alpha}+\frac{l^2+3l+2}{2l+3}kg_{1l+2,\alpha}=0 \\
\underline{for\ l=0} \\
\{2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)-\frac{4k}{3}\}h_{10,\alpha}+\{2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon)-2k\}h_{11,\alpha} \\
-\frac{2k}{3}h_{12,\alpha}-(1-\epsilon)\alpha_1\eta_2\, h_{20,\alpha}+(1-\epsilon)\alpha_1\eta_2\, h_{21,\alpha}+(1-\epsilon)\alpha_1\eta_1 g_{10,\alpha} \\
+(1-\epsilon)\alpha_1\eta_1\, g_{11,\alpha}+(1-\epsilon)\alpha_1\eta_2\, g_{20,\alpha}+(1-\epsilon)\alpha_1\eta_2\, g_{21,\alpha}=0 \\
\underline{for\ l=1} \\
-2(1+\eta_1-k)h_{10,\alpha}+6(1+\eta_1-\frac{4k}{5})h_{11,\alpha}-4(1+\eta_1-k)h_{12,\alpha}-\frac{6k}{5}\, g_{13,\alpha}=0 \\
\underline{for\ l\neq 0,1} \\
-\frac{l^2-l}{2l-1}h_{1l-2,\alpha}-2l(1+\eta_1-k)h_{1l-1,\alpha}-\{\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}k-2(1+\eta_1)(2l+1)\}\times \\
h_{1l,\alpha}-2(l+1)(1+\eta_1-k)h_{1l+1,\alpha}-\frac{l^2+3l+2}{2l+3}kh_{1l+2,\alpha}=0
\end{array}\right\} \quad (2.23)$$

and

$$\left.\begin{array}{l}
\underline{for\ \ l=0} \\
\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}g_{10,\beta}+\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}g_{11,\beta}- \\
(1-\epsilon)\alpha_1\eta_2 g_{20,\beta}-(1-\epsilon)\alpha_1\eta_2 g_{21,\beta}+(1-\epsilon)\alpha_1\eta_1 h_{10,\beta}-(1-\epsilon)\alpha_1\eta_1 \\
\times h_{11,\beta}-(1-\epsilon)\alpha_1\eta_2 h_{20,\beta}-(1-\epsilon)\alpha_1\eta_2 h_{21,\beta}=2+4\frac{a}{A}(1+\epsilon\eta_1) \\
\underline{for\ \ l=1} \\
g_{10,\beta}+3g_{11,\beta}+2g_{12,\beta}=1 \\
\underline{for\ \ l\neq 0,1} \\
l g_{1l-1,\beta}+(2l+1)g_{1l,\beta}+(l+1)g_{1l+1,\beta}=0 \\
\underline{for\ \ l=0} \\
\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}h_{10,\beta}-\{2(1+\eta_1)-(1-\epsilon)\alpha_1\eta_1\}h_{11,\beta}+ \\
(1-\epsilon)\alpha_1\eta_1 g_{10,\beta}+(1-\epsilon)\alpha_1\eta_1 g_{11,\beta}+(1-\epsilon)\alpha_1\eta_2 h_{20,\beta}+(1-\epsilon)\alpha_1\eta_2 \\
\times g_{21,\beta}-(1-\epsilon)\alpha_1\eta_2 h_{20,\beta}+(1-\epsilon)\alpha_1\eta_2 h_{21,\beta}=2-4\frac{a}{A}(1+\epsilon\eta_1) \\
\underline{for\ \ l=1} \\
h_{10,\beta}-3h_{11,\beta}+2h_{12,\beta}=1 \\
\underline{for\ \ l\neq 0,1} \\
l h_{1l-1,\beta}-(2l+1)h_{1l,\beta}+(l+1)h_{1l+1,\beta}=0
\end{array}\right\} \quad (2.24)$$

A similar set of equations like (2.23) and (2.24) will be obtained considering $r = 2$.

Solving (2.23) and the similar set of equation obtained when $r = 2$ combinedly by the method described by Wilson and Sen[65] we obtain $k = k_1, k_2, \ldots k_r$; $r = 8, 12, \ldots$

Using boundary conditions (2.20) we obtain

$$\sum_{r=1}^{n-1} h_{i\,l,\alpha}^{(r)} + h_{i\,l,\beta} = 0\,, \quad i = 1,2. \tag{2.25}$$

Thus equations (2.23), (2.24) and similar set of equations (for $r = 2$) and (2.25) are sufficient to determine the unknowns $g_{il,j}, h_{il,j}$; $i = 1,2,$; $j = \alpha,\beta$; $l = 1,2,\ldots,n$.

Thus we have

$$\left.\begin{aligned} I_{il}^{+}(\tau) &= A\left[g_{il,\alpha}^{(r)}\, e^{-k\tau} + g_{il,\beta}\right] \\ I_{il}^{-}(\tau) &= A\left[h_{il,\alpha}^{(r)}\, e^{-k\tau} + h_{il,\beta}\right] \end{aligned}\right\} \tag{2.26}$$

where $i = 1,2;\ l = 1,2,\ldots,n$.

Now we will consider two approximation, viz. $l_0 = 1$ and $l_0 = 2$.

2.3 First approximation

2.3.1 First approximation when r = 1

We name the solution first approximation when $l_0 = 1$.

In this case we have from equations (2.18) and (2.19)

$$\left.\begin{aligned} &\tfrac{4}{3}I_{10}^{+'} + 2I_{11}^{+'} - (\xi_1 I_{10}^{+} + \xi_2 I_{10}^{-} + \xi_1 I_{11}^{+} - \xi_2 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} \\ &+I_{21}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\ &\tfrac{4}{3}I_{10}^{-'} - 2I_{11}^{-'} + (\xi_2 I_{10}^{+} + \xi_1 I_{10}^{-} + \xi_2 I_{11}^{+} - \xi_1 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} \\ &+I_{21}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\ &2I_{10}^{+'} + \tfrac{24}{5}I_{11}^{+'} - 2(1+\eta_1)(I_{10}^{+} + 3I_{11}^{+}) = -2A \\ &-2I_{10}^{-'} + \tfrac{24}{5}I_{11}^{-'} - 2(1+\eta_1)(I_{10}^{-} - 3I_{11}^{-}) = -2A \end{aligned}\right\} \tag{2.27}$$

where

$$\left.\begin{aligned} I_{1l}^{+}(\tau,\mu) &= A\tau + I_{10}^{+}(\tau)\mu + 3I_{11}^{+}(\tau)\mu P_1(2\mu-1)\,, \quad 0\le\mu\le 1 \\ I_{1l}^{-}(\tau,\mu) &= A\tau + I_{10}^{-}(\tau)\mu + 3I_{11}^{-}(\tau)\mu P_1(2\mu+1)\,, \quad -1\le\mu\le 0 \end{aligned}\right\} \quad (2.28)$$

We now take the trial solution given by (2.22) and substituting these in (2.27) and then equating the coefficient of $e^{k\tau}$ and constant term we obtain

$$\left.\begin{aligned} &(\xi_1+\tfrac{4k}{3})g_{10,\alpha}+(\xi_1+2k)g_{11,\alpha}+\xi_2 h_{10,\alpha}-\xi_2 h_{11,\alpha}-\xi_3 g_{20,\alpha} \\ &-\xi_3 g_{21,\alpha}+\xi_3 h_{20,\alpha}-\xi_3 h_{21,\alpha} = 0 \\ &2(\xi_4+k)g_{10,\alpha}+6(\xi_4+\tfrac{4k}{5})g_{11,\alpha} = 0 \\ &\xi_2 g_{10,\alpha}+\xi_2 g_{11,\alpha}+(\xi_1-\tfrac{4k}{3})h_{10,\alpha}+(-\xi_1+2k)h_{11,\alpha}+\xi_3 g_{20,\alpha} \\ &+\xi_3 g_{21,\alpha}-\xi_3 h_{20,\alpha}+\xi_3 h_{21,\alpha} = 0 \\ &2(-\xi_4+k)h_{10,\alpha}+6(\xi_4-\tfrac{4k}{5}) = 0 \end{aligned}\right\} \quad (2.29)$$

and

$$\left.\begin{aligned} &\xi_1 g_{10,\beta}+\xi_1 g_{11,\beta}+\xi_2 h_{10,\beta}-\xi_2 h_{11,\beta}-\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta} \\ &+h_{21,\beta}) = 2+\tfrac{4a}{A}(1+\epsilon\eta_1) \\ &2\xi_4 g_{10,\beta}+6\xi_4 g_{11,\beta} = 2 \\ &\xi_2 g_{10,\beta}+\xi_2 g_{11,\beta}+\xi_1 h_{10,\beta}-\xi_1 h_{11,\beta}+\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta} \\ &+h_{21,\beta} = 2-\tfrac{4a}{A}(1+\epsilon\eta_1) \\ &2\xi_4 h_{10,\beta}-6\xi_4 h_{11,\beta} = 2 \end{aligned}\right\} \quad (2.30)$$

where we make the abbreviation

$$\left.\begin{aligned} \xi_1 &= 2(1+\eta_1)-\alpha_1\eta_1(1-\epsilon) \\ \xi_2 &= (1-\epsilon)\alpha_1\eta_1 \\ \xi_2 &= (1-\epsilon)\alpha_2\eta_2 \\ \xi_4 &= (1+\eta_1) \end{aligned}\right\} \quad (2.31)$$

2.3.2 First approximation when r = 2 .

Similarly considering the rest equation (2.11) and proceeding in the same manner described in section 2.2 and subsection 2.3.1 and taking $l_0 = 1$,we have, by equating coefficients of $e^{-k\tau}$ and constant term , the following set of equations (2.32) and (2.33)

$$\left.\begin{array}{r} -\lambda_3 g_{10,\alpha} - \lambda_3 g_{11,\alpha} + \lambda_3 h_{10,\alpha} - \lambda_3 h_{11,\alpha} + (\lambda_1 + \frac{4k}{3}) g_{20,\alpha} + \\ (\lambda_1 + 2k) g_{21,\alpha} + \lambda_2 h_{20,\alpha} - \lambda_2 h_{21,\alpha} = 0 \\ 2(\lambda_4 + k) g_{20,\alpha} + 6(\lambda_4 + \frac{4k}{3}) g_{21,\alpha} = 0 \\ \lambda_3 g_{10,\alpha} + \lambda_3 g_{11,\alpha} - \lambda_3 h_{10,\alpha} + \lambda_3 h_{11,\alpha} + \lambda_2 g_{20,\alpha} + \lambda_2 g_{21,\alpha} + \\ (\lambda_1 - \frac{4k}{3}) h_{20,\alpha} + (-\lambda_1 + 2k) h_{21,\alpha} = 0 \\ 2(-\lambda_4 + k) h_{20,\alpha} + 6(\lambda_4 - \frac{4k}{5}) h_{21,\alpha} = 0 \end{array}\right\} \tag{2.32}$$

and

$$\left.\begin{array}{l} \lambda_1 g_{20,\beta} + \lambda_1 g_{21,\beta} + \lambda_2 h_{20,\beta} - \lambda_2 h_{21,\beta} - \lambda_3 (g_{10,\beta} + g_{11,\beta} - h_{10,\beta} \\ + h_{11,\beta}) = 2 + \frac{4a}{A}(1 + \epsilon\eta_2) \\ 2\lambda_4 g_{20,\beta} + 6\lambda_4 g_{21,\beta} = 2 \\ \lambda_2 g_{20,\beta} + \lambda_2 g_{21,\beta} + \lambda_1 h_{20,\beta} - \lambda_1 h_{21,\beta} + \lambda_3 (g_{10,\beta} + g_{11,\beta} - h_{10,\beta} \\ + h_{11,\beta}) = 2 - \frac{4a}{A}(1 + \epsilon\eta_2) \\ 2\lambda_4 h_{20,\beta} - 6\lambda_4 h_{21,\beta} = 2 \end{array}\right\} \tag{2.33}$$

where

$$\left.\begin{array}{l} \lambda_1 = 2(1 + \eta_2) - (1 - \epsilon)\alpha_2\eta_2 \\ \lambda_2 = (1 - \epsilon)\alpha_2\eta_2 \\ \lambda_3 = (1 - \epsilon)\alpha_2\eta_1 \\ \lambda_4 = (1 + \eta_2) \end{array}\right\} \tag{2.34}$$

Now the set of equations (2.29) and (2.32) has a nontrivial solution if

$$\Delta(k) = 0 \tag{2.35}$$

where $\Delta(k) =$

$$\begin{vmatrix} \xi_1+\frac{4k}{3} & \xi_1+2k & \xi_2 & -\xi_2 & -\xi_3 & -\xi_3 & \xi_3 & -\xi_3 \\ 2\xi_4+2k & 6\xi_4+\frac{24k}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_2 & \xi_2 & \xi_1-\frac{4k}{3} & -\xi_1+2k & \xi_3 & \xi_3 & -\xi_3 & \xi_3 \\ 0 & 0 & -2\xi_4+2k & 6\xi_4-\frac{24k}{5} & 0 & 0 & 0 & 0 \\ -\lambda_3 & -\lambda_3 & \lambda_3 & -\lambda_3 & \lambda_1+\frac{4k}{3} & \lambda_1+2k & \lambda_2 & -\lambda_2 \\ 0 & 0 & 0 & 0 & 2\lambda_4+2k & 6\lambda_4+\frac{8k}{3} & 0 & 0 \\ \lambda_3 & \lambda_3 & -\lambda_3 & \lambda_3 & \lambda_2 & \lambda_2 & \lambda_1-\frac{4k}{3} & -\lambda_1+2k \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda_4+2k & 6\lambda_4-\frac{24k}{5} \end{vmatrix} \tag{2.36}$$

Now we consider the case[26] i.e $\eta_1 = 1$, $\eta_2 = \frac{1}{2}$, $\epsilon = 0$, so $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{1}{3}$.

Thus

$$\left.\begin{aligned} \xi_1 &= \tfrac{10}{3} \\ \xi_2 &= \tfrac{2}{3} \\ \xi_3 &= \tfrac{1}{3} \\ \xi_4 &= 2 \end{aligned}\right\} \; and \; \left.\begin{aligned} \lambda_1 &= \tfrac{17}{6} \\ \lambda_2 &= \tfrac{1}{6} \\ \lambda_3 &= \tfrac{1}{3} \\ \lambda_4 &= \tfrac{3}{2} \end{aligned}\right\} \tag{2.37}$$

Therefore (2.36) becomes ,$\Delta(k) =$

$$\begin{vmatrix} \frac{10}{3}+\frac{4k}{3} & \frac{10}{3}+2k & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 4+2k & 12+\frac{24k}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{10}{3}-\frac{4k}{3} & -\frac{10}{3}+2k & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -4+2k & 12-\frac{24k}{5} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{17}{6}+\frac{4k}{3} & \frac{17}{6}+2k & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 3+2k & 9+\frac{8k}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{17}{6}-\frac{4k}{3} & -\frac{17}{6}+2k \\ 0 & 0 & 0 & 0 & 0 & 0 & -3+2k & 9-\frac{24k}{5} \end{vmatrix} \tag{2.38}$$

and $\Delta(k) = 0$ yields
$k = \pm 2.36701\,, \pm 5.63299\,, -1.12729\,, -2.14644\,, 1.63853\,, 4.67409$
We take $k = 2.36701$

Using (2.37) in (2.30) and (2.33) we obtain in Matrix form

$$\mathbf{A\ X = B} \tag{2.39}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{10}{3} & \frac{10}{3} & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 4 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{10}{3} & -\frac{10}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 4 & -12 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{17}{6} & \frac{17}{6} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 3 & 9 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{17}{6} & -\frac{17}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -9 \end{pmatrix} \tag{2.40}$$

and

$$\mathbf{X} = \begin{pmatrix} g_{10,\beta} \\ g_{11,\beta} \\ h_{10,\beta} \\ h_{11,\beta} \\ g_{20,\beta} \\ g_{21,\beta} \\ h_{20,\beta} \\ h_{21,\beta} \end{pmatrix}, \quad \mathbf{B} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 4\frac{a}{A}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.41}$$

Using the boundary conditions we obtain

$$\left.\begin{array}{l} h_{1l,\alpha} + h_{1l,\beta} = 0 \\ \qquad\qquad\qquad (\,for\ \ all\ \ l) \\ h_{2l,\alpha} + h_{2l,\beta} = 0 \end{array}\right\} \tag{2.42}$$

Now from (2.29), (2.32) , (2.39) and (2.42) we obtain

$$\left.\begin{array}{l} g_{10,\alpha} = 0.171344 + \frac{a}{A}0.235598 \\ g_{11,\alpha} = -0.064059 - \frac{a}{A}0.088081 \\ g_{20,\alpha} = 2.159601 + \frac{a}{A}2.933029 \\ g_{21,\alpha} = -1.707505 - \frac{a}{A}2.011397 \\ g_{10,\beta} = 0.5 + \frac{a}{A}0.75 \\ g_{11,\beta} = \frac{a}{A} \\ g_{20,\beta} = 0.333333 + \frac{a}{A}2 \\ g_{21,\beta} = \frac{a}{A}0.666668 \\ h_{10,\alpha} = -0.5 - \frac{a}{A}0.75 \\ h_{11,\alpha} = -\frac{a}{A}0.5 \\ h_{20,\alpha} = -0.333333 - \frac{a}{A}2 \\ h_{21,\alpha} = -\frac{a}{A}0.666668 \\ h_{10,\beta} = 0.5 + \frac{a}{A}0.75 \\ h_{11,\beta} = \frac{a}{A}0.5 \\ h_{20,\beta} = 0.333333 + \frac{a}{A}2 \\ h_{21,\beta} = \frac{a}{A}0.666668 \end{array}\right\} \tag{2.43}$$

2.3.3 Determination of a/A

The mean intensity is given by

$$B(\tau) = \frac{1}{2}\int_{-1}^{1} I_r(\tau,\mu')d\mu' = \frac{3}{4}F[\tau + q(\tau)] \tag{2.44}$$

But from equation (2.12) and (2.13) we have

$$\begin{aligned} B(\tau) &= 2A\tau + \frac{1}{4}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-} + I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \\ &= 2A\tau + \frac{1}{4}[A0.892714e^{-2.36701\tau} + a(2.333336 + \\ &\quad 2.352481e^{-2.36701\tau})] \end{aligned} \tag{2.45}$$

Comparing (2.44) and (2.45)

$$A = \frac{3}{8}F \tag{2.46}$$

Again

$$\begin{aligned} F &= 2\int_{-1}^{1} I_r(\tau, \mu')d\mu' \ , \ r = 1, 2. \\ &= 8A\tau + [A0.892714e^{-2.36701\tau} + a(2.333336 + \\ &\quad 2.352481e^{-2.36701\tau})] \end{aligned} \tag{2.47}$$

Thus (2.46) and (2.47) gives

$$\frac{a}{A} = \frac{(\frac{8}{3} - 8\tau - 0.892714e^{-2.36701\tau})}{(2.333336 + 2.352481e^{-2.36701\tau})} \tag{2.48}$$

2.4 Second approximation

We find the solution when $l_o = 2$ and name it second approximation. In this case we have from (2.18) and (2.19)

$$
\left.
\begin{array}{l}
\frac{4}{3}I_{10}^{+'} + 2I_{11}^{+'} + \frac{2}{3}I_{12}^{+'} - (\xi_1 I_{10}^{+} + \xi_2 I_{10}^{-} + \xi_1 I_{11}^{+} - \xi_2 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + \\
I_{21}^{+} + I_{21}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\
\frac{4}{3}I_{10}^{-'} - 2I_{11}^{-'} + \frac{2}{3}I_{12}^{-'} + (\xi_2 I_{10}^{+} + \xi_1 I_{10}^{-} + \xi_2 I_{11}^{+} - \xi_1 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + \\
I_{21}^{+} + I_{21}^{-}) = -2A - 4[(1+\epsilon\eta_1)(a+b\tau) + (1-\epsilon)\eta_1 A\tau - (1+\eta_1)A\tau] \\
2I_{10}^{+'} + \frac{24}{5}I_{11}^{+'} + 4I_{12}^{+'} - 2\xi_4(I_{10}^{+} + 3I_{11}^{+} + 2I_{12}^{+}) = -2A \\
-2I_{10}^{-'} + \frac{24}{5}I_{11}^{-'} - 4I_{12}^{-'} - 2\xi_4(I_{10}^{-} - 3I_{11}^{-} + 2I_{12}^{-}) = -2A \\
\frac{1}{3}I_{10}^{+'} + 2I_{11}^{+'} + \frac{80}{21}I_{12}^{+'} - \xi_4(2I_{11}^{+} + 5I_{12}^{+}) = 0 \\
\frac{1}{3}I_{10}^{-'} - 2I_{11}^{-'} + \frac{80}{21}I_{12}^{-'} - \xi_4(2I_{11}^{-} - 5I_{12}^{-}) = 0
\end{array}
\right\}
\qquad (2.49)
$$

and

$$\left.\begin{aligned}
&\tfrac{4}{3}I_{20}^{+'} + 2I_{21}^{+'} + \tfrac{2}{3}I_{22}^{+'} - (\lambda_1 I_{20}^{+} + \lambda_2 I_{20}^{-} + \lambda_1 I_{21}^{+} - \lambda_2 I_{21}^{-}) + \lambda_3(I_{10}^{+} - \\
&I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) = -2A - 4[(1+\epsilon\eta_2)(a+b\tau) + (1-\epsilon)\eta_2 A\tau - \\
&\qquad (1+\eta_2)A\tau] \\
&\tfrac{4}{3}I_{20}^{-'} - 2I_{21}^{-'} + \tfrac{2}{3}I_{22}^{-'} + (\lambda_2 I_{20}^{+} + \lambda_1 I_{20}^{-} + \lambda_2 I_{21}^{+} - \lambda_1 I_{21}^{-}) + \lambda_3(I_{10}^{+} - \\
&I_{10}^{-} + I_{11}^{+} + I_{11}^{-}) = -2A - 4[(1+\epsilon\eta_2)(a+b\tau) + (1-\epsilon)\eta_2 A\tau - \\
&\qquad (1+\eta_2)A\tau] \\
&2I_{20}^{+'} + \tfrac{24}{5}I_{21}^{+'} + 4I_{22}^{+'} - 2\lambda_4(I_{20}^{+} + 3I_{21}^{+} + 2I_{22}^{+}) = -2A \\
&-2I_{20}^{-'} + \tfrac{24}{5}I_{21}^{-'} - 4I_{22}^{-'} - 2\lambda_4(I_{20}^{-} - 3I_{21}^{-} + 2I_{22}^{-}) = -2A \\
&\tfrac{1}{3}I_{20}^{+'} + 2I_{21}^{+'} + \tfrac{80}{21}I_{22}^{+'} - \lambda_4(2I_{21}^{+} + 5I_{22}^{+}) = 0 \\
&\tfrac{1}{3}I_{20}^{-'} - 2I_{21}^{-'} + \tfrac{80}{21}I_{22}^{-'} - \lambda_4(2I_{21}^{-} - 5I_{22}^{-}) = 0
\end{aligned}\right\} \quad (2.50)$$

where ξ_i, $i = 1, 2, 3, 4.$ and λ_i, $i = 1, 2, 3, 4.$ are given by (2.31) and (2.34).

The above set of equations given by (2.49) and (2.50) are obtained (when $l_0 = 2$) by similar process described in section (2.1) considering the equation of transfer for $r = 1$ *and* 2 respectively.

and

$$\left.\begin{aligned}
&I_{il}^{+}(\tau,\mu) = A\tau + I_{i0}^{+}(\tau)\mu + 3I_{i1}^{+}(\tau)\mu P_1(2\mu-1) + 5I_{i2}^{+}(\tau)\mu P_2(2\mu-1), \\
&\qquad\qquad 0 \le \mu \le 1 \\
&I_{il}^{-}(\tau,\mu) = A\tau + I_{i0}^{-}(\tau)\mu + 3I_{i1}^{-}(\tau)\mu P_1(2\mu+1) + 5I_{i2}^{-}(\tau)\mu P_2(2\mu+1), r \\
&\qquad\qquad -1 \le \mu \le 0
\end{aligned}\right\} \quad (2.51)$$

Now taking the trial solution given by (2.22) and substituting these in (2.49) and (2.50) and then equating the coefficient of $e^{-k\tau}$ and constant term we obtain

$$
\left.
\begin{aligned}
&(\xi_1 + \tfrac{4k}{3})g_{10,\alpha} + (\xi_1 + 2k)g_{11,\alpha} + \tfrac{2k}{3}g_{12,\alpha} + \xi_2 h_{10,\alpha} - \xi_2 h_{11,\alpha} \\
&-\xi_3 g_{20,\alpha} - \xi_3 g_{21,\alpha} + \xi_3 h_{20,\alpha} - \xi_3 h_{21,\alpha} = 0 \\
&\xi_2 g_{10,\alpha} + \xi_2 g_{11,\alpha} + (\xi_1 - \tfrac{4k}{3})h_{10,\alpha} + (-\xi_1 + 2k)h_{11,\alpha} - \tfrac{2k}{3}h_{12,\alpha} \\
&+\xi_3 g_{20,\alpha} + \xi_3 g_{21,\alpha} - \xi_3 h_{20,\alpha} + \xi_3 h_{21,\alpha} = 0 \\
&2(k + \xi_4)g_{10,\alpha} + 6(\xi_4 + \tfrac{4k}{5})g_{11,\alpha} + 4(\xi_4 + k)g_{12,\alpha} = 0 \\
&2(-\xi_4 + k)h_{10,\alpha} + 6(\xi_4 - \tfrac{4k}{5})h_{11,\alpha} - 4(\xi_4 - k)h_{12,\alpha} = 0 \\
&\tfrac{k}{3}g_{10,\alpha} + 2(\xi_4 + k)g_{11,\alpha} + 5(\xi_4 + \tfrac{16k}{21})g_{21,\alpha} = 0 \\
&\tfrac{k}{3}h_{10,\alpha} + 2(\xi_4 - k)h_{11,\alpha} + 5(-\xi_4 + \tfrac{16k}{21})h_{12,\alpha} = 0 \\
&-\lambda_3 g_{10,\alpha} - \lambda_3 g_{11,\alpha} + \lambda_3 h_{10,\alpha} - \lambda_3 h_{11,\alpha} + (\lambda_1 + \tfrac{4k}{3})g_{20,\alpha} + \\
&(\lambda_1 + 2k)g_{21,\alpha} + \tfrac{2k}{3}g_{22,\alpha} + \lambda_3 h_{20,\alpha} - \lambda_3 h_{21,\alpha} = 0 \\
&-\lambda_3 g_{10,\alpha} + \lambda_3 g_{11,\alpha} - \lambda_3 h_{10,\alpha} + \lambda_3 h_{11,\alpha} + \lambda_2 g_{20,\alpha} + \lambda_2 g_{21,\alpha} \\
&+(\lambda_1 - \tfrac{4k}{3})h_{20,\alpha} + (-\lambda_1 + 2k)h_{21,\alpha} - \tfrac{2k}{3}h_{22,\alpha} = 0 \\
&2(k + \lambda_4)g_{20,\alpha} + 6(\lambda_4 + \tfrac{4k}{5})g_{21,\alpha} + 4(\lambda_4 + k)g_{22,\alpha} = 0 \\
&2(-\lambda_4 + k)h_{20,\alpha} + 6(\lambda_4 - \tfrac{4k}{5})h_{21,\alpha} - 4(\lambda_4 - k)h_{22,\alpha} = 0 \\
&\tfrac{k}{3}g_{20,\alpha} + 2(\lambda_4 + k)g_{21,\alpha} + 5(\lambda_4 + \tfrac{16k}{21})g_{22,\alpha} = 0 \\
&\tfrac{k}{3}h_{20,\alpha} + 2(\lambda_4 - k)h_{21,\alpha} + 5(-\lambda_4 + \tfrac{16k}{21})h_{22,\alpha} = 0
\end{aligned}
\right\} \quad (2.52)
$$

and

$$\left.\begin{array}{l}
\xi_1(g_{10,\beta}+g_{11,\beta})+\xi_2(h_{10,\beta}-h_{11,\beta})-\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}\\
+h_{21,\beta})=2+4(1+\epsilon\eta_1)\frac{a}{A}\\
\xi_2(g_{10,\beta}+g_{11,\beta})+\xi_1(h_{10,\beta}-h_{11,\beta})+\xi_3(g_{20,\beta}+g_{21,\beta}-h_{20,\beta}\\
+h_{21,\beta})=2-4(1+\epsilon\eta_1)\frac{a}{A}\\
2\xi_4g_{10,\beta}+6\xi_4g_{11,\beta}+4\xi_4g_{12,\beta}=2\\
2\xi_4h_{10,\beta}-6\xi_4h_{11,\beta}+4\xi_4h_{12,\beta}=2\\
2\xi_4g_{11,\beta}+5\xi_4g_{12,\beta}=0\\
2\xi_4h_{11,\beta}-5\xi_4h_{12,\beta}=0\\
-\lambda_3(g_{10,\beta}+g_{11,\beta}-h_{10,\beta}+h_{11,\beta})+\lambda_1(g_{20,\beta}+g_{21,\beta})+\lambda_2(h_{20,\beta}\\
-h_{21,\beta})=2+4(1+\epsilon\eta_2)\frac{a}{A}\\
\lambda_3(g_{10,\beta}+g_{11,\beta}-h_{10,\beta}+h_{11,\beta})+\lambda_2(g_{20,\beta}+g_{21,\beta})+\lambda_1(h_{20,\beta}\\
-h_{21,\beta})=2-4(1+\epsilon\eta_2)\frac{a}{A}\\
2\lambda_4g_{20,\beta}+6\lambda_4g_{21,\beta}+4\lambda_4g_{22,\beta}=2\\
2\lambda_4h_{20,\beta}-6\lambda_4h_{21,\beta}+4\lambda_4h_{22,\beta}=2\\
2\lambda_4g_{21,\beta}+5\lambda_4g_{22,\beta}=0\\
2\lambda_4h_{21,\beta}-5\lambda_4h_{22,\beta}=0
\end{array}\right\}\quad(2.53)$$

where λ_I and ξ_i $i=1,2,3,4.$ are same as (2.31) and (2.34).

The above equations (2.53) have a non trivial solution if the determinant of the coefficient of the constant $g_{il,\alpha}$ and $h_{il,\alpha}$ is zero, that is

$$\Delta(k) = \begin{vmatrix} M_1(k) & M_2(k) \\ M_3(k) & M_4(k) \end{vmatrix} = 0 \tag{2.54}$$

where

$$M_1(k) = \begin{pmatrix} (\xi_1+\frac{4k}{3}) & (\xi_1+2k) & \frac{2k}{3} & \xi_2 & -\xi_2 & 0 \\ \xi_2 & \xi_2 & 0 & (\xi_1-\frac{4k}{3}) & (-\xi_1+2k) & -\frac{2k}{3} \\ 2(\xi_4+k) & 6(\xi_4+\frac{4k}{5}) & 4(\xi_4+k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\xi_4+k) & 6(\xi_4-\frac{4k}{5}) & 4(\xi_4-k) \\ \frac{k}{3} & 2(\xi_4+k) & 5(\xi_4+\frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\xi_4-k) & 5(-\xi_4+\frac{16k}{21}) \end{pmatrix} \tag{2.55}$$

$$M_2(k) = \begin{pmatrix} -\xi_3 & -\xi_3 & 0 & \xi_3 & -\xi_3 & 0 \\ \xi_3 & \xi_3 & 0 & -\xi_3 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.56}$$

$$M_3(k) = \begin{pmatrix} -\lambda_3 & -\lambda_3 & 0 & \lambda_3 & -\lambda_3 & 0 \\ \lambda_3 & \lambda_3 & 0 & -\lambda_3 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.57}$$

$$M_4(k) = \begin{pmatrix} (\xi_1+\frac{4k}{3}) & (\xi_1+2k) & \frac{2k}{3} & \xi_2 & -\xi_2 & 0 \\ \xi_2 & \xi_2 & 0 & (\xi_1-\frac{4k}{3}) & (-\xi_1+2k) & -\frac{2k}{3} \\ 2(\xi_4+k) & 6(\xi_4+\frac{4k}{5}) & 4(\xi_4+k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\xi_4+k) & 6(\xi_4-\frac{4k}{5}) & 4(\xi_4-k) \\ \frac{k}{3} & 2(\xi_4+k) & 5(\xi_4+\frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\lambda_4-k) & 5(-\lambda_4+\frac{16k}{21}) \end{pmatrix} \tag{2.58}$$

Now we consider the case as previous i.e
$\eta_1 = 1$, $\eta_2 = \frac{1}{2}$, $\epsilon = 0$ and ξ_i and λ_i are given by (2.37)

Thus $\Delta(k)$ becomes

$$\Delta(k) = \left(\tfrac{556288}{3675}k^6 + \tfrac{500096}{2205}k^5 - \tfrac{55191488}{11025}k^4 + \tfrac{2105056}{2205}k^3 + \tfrac{72577488}{2205}k^2 - \tfrac{14977456}{21}k - 57344\right) \times \left(\tfrac{544}{245}k^6 - \tfrac{1152}{245}k^5 - \tfrac{1573648}{2205}k^4 + \tfrac{3486299}{4410}k^3 + \tfrac{3650659}{490}k^2 - \tfrac{146499}{56}k - \tfrac{124497}{8}\right) \tag{2.59}$$

Now $\Delta(k) = 0$ yields $k = 18, 281155, -7.269599, 8.794658, -9.770985.$

We take $k = 8.794658$.
Again using (2.37) in (2.53) we have in matrix form

$$\mathbf{A}\ \mathbf{X} = \mathbf{B} \tag{2.60}$$

where

$$\mathbf{A} = \begin{pmatrix}
\frac{10}{3} & \frac{10}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\
\frac{2}{3} & \frac{2}{3} & 0 & \frac{10}{3} & -\frac{10}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\
4 & 12 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & -12 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & \frac{17}{6} & \frac{17}{6} & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{17}{6} & -\frac{17}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 9 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -9 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & \frac{15}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -\frac{15}{2}
\end{pmatrix} \tag{2.61}$$

and

$$\mathbf{X}=\begin{pmatrix} g_{10,\beta}\\ g_{11,\beta}\\ g_{12,\beta}\\ h_{10,\beta}\\ h_{11,\beta}\\ h_{12,\beta}\\ g_{20,\beta}\\ g_{21,\beta}\\ g_{22,\beta}\\ h_{20,\beta}\\ h_{21,\beta}\\ h_{22,\beta}\end{pmatrix},\ \mathbf{B}=2\begin{pmatrix}1\\1\\1\\1\\0\\0\\1\\1\\1\\1\\0\\0\end{pmatrix}+4\frac{a}{A}\begin{pmatrix}1\\-1\\0\\0\\0\\0\\1\\-1\\0\\0\\0\\0\end{pmatrix} \tag{2.62}$$

Inverting (2.60) we obtain

$$\left.\begin{array}{l} g_{10,\beta}=0.283206+2.448164\frac{a}{A}\\ g_{11,\beta}=0.098542-1.1128\frac{a}{A}\\ g_{12,\beta}=-0.039416+0.44512\frac{a}{A}\\ g_{20,\beta}=0.638784+3.262356\frac{a}{A}\\ g_{21,\beta}=0.012674-1.482892\frac{a}{A}\\ g_{22,\beta}=-0.00507+0.593156\frac{a}{A}\\ \\ h_{10,\beta}=0.506274-1.650696\frac{a}{A}\\ h_{11,\beta}=0.002852-0.750316\frac{a}{A}\\ h_{12,\beta}=0.00114-0.300128\frac{a}{A}\\ h_{20,\beta}=0.69455-3.262356\frac{a}{A}\\ h_{21,\beta}=0.012674-1.482888\frac{a}{A}\\ h_{22,\beta}=0.00507-0.593156\frac{a}{A} \end{array}\right\} \tag{2.63}$$

and using the boundary conditions given by (2.42) and by (2.63) we obtain

$$\left.\begin{array}{l} h_{10,\alpha}=-0.506274+1.650696\frac{a}{A}\\ h_{11,\alpha}=-0.002852+0.750316\frac{a}{A}\\ h_{12,\alpha}=-0.00114+0.300128\frac{a}{A}\\ h_{20,\alpha}=-0.69455+3.262356\frac{a}{A}\\ h_{21,\alpha}=-0.012674+1.482888\frac{a}{A}\\ h_{22,\alpha}=-0.00507+0.0593156\frac{a}{A} \end{array}\right\} \tag{2.64}$$

Now using (2.63),(2.64) and $k = 8.794658$ in (2.52) we get

$$\begin{pmatrix} 15.059543 & 20.922649 & 5.863105 & -0.333333 & -0.333333 & 0 \\ 0.666666 & 0.666666 & 0 & 0.333333 & 0.333333 & 0 \\ 21.589315 & 54.214358 & 43.178631 & 0 & 0 & 0 \\ 2.931552 & 21.589315 & 34.802766 & 0 & 0 & 0 \\ -0.333333 & -0.333333 & 0 & 14.559543 & 20.422649 & 5.863105 \\ 0.333333 & 0.333333 & 0 & 0.166666 & 0.166666 & 0 \\ 0 & 0 & 0 & 20.589315 & 51.214357 & 41.178631 \\ 0 & 0 & 0 & 2.931552 & 20.589315 & 41.003458 \end{pmatrix}$$

$$\times \begin{pmatrix} g_{10,\alpha} \\ g_{11,\alpha} \\ g_{12,\alpha} \\ g_{20,\alpha} \\ g_{21,\alpha} \\ g_{22,\alpha} \end{pmatrix} = \begin{pmatrix} 0.56924666 \\ -4.44241339 \\ 0 \\ 0 \\ 0.28145334 \\ -6.188965148 \\ 0 \\ 0 \end{pmatrix} + \frac{a}{A} \begin{pmatrix} -1.193409334 \\ 5.510434518 \\ 0 \\ 0 \\ -0.596704667 \\ 10.90812471 \\ 0 \\ 0 \end{pmatrix} \qquad (2.65)$$

The above system of equation (2.65) is inconsistent. Now we have the following theorem[47].

Theorem 2.4.1 *Suppose that A is a $m \times n$ matrix whose columns are linearly independent and that $b \in R^m$. Then the vector x^* given $x^* = (A^TA)^{-1}A^Tb$ satisfies $||Ax^* - b|| \leq ||Ax - b||$ for all $x \in R^n$*

The vector $x^* = (A^TA)^{-1}A^Tb$ is called the *best least square solution* of the inconsistent system $Ax = b$. The matrix $(A^TA)^{-1}A^T$ is called the *generalized inverse* of the matrix A.

Applying the above theorem (2.4.1) in the inconsistent system of equation (2.65) we get

$$\begin{pmatrix} 702.148987 & 1549.493286 & 1122.518921 & -9.595251 & -11.549620 & -1.954368 \\ 1549.493286 & 3843.719238 & 3214.941406 & -11.549260 & -13.503988 & -1.954368 \\ 1122.518921 & 3214.941406 & 3110.002930 & -1.954368 & -1.954368 & 0 \\ -9.595251 & -11.549620 & -1.954368 & 644.744202 & 1412.421631 & 1053.407837 \\ -11.549620 & -13.503988 & -1.954368 & 1412.421631 & 3464.165039 & 3072.910400 \\ -1.954368 & -1.954368 & 0 & 1053.407837 & 3072.910400 & 3411.339355 \end{pmatrix}$$

$$\times \begin{pmatrix} g_{10,\alpha} \\ g_{11,\alpha} \\ g_{12,\alpha} \\ g_{20,\alpha} \\ g_{21,\alpha} \\ g_{22,\alpha} \end{pmatrix} = \begin{pmatrix} 3.358974 \\ 6.659461 \\ 3.300487 \\ 1.397892 \\ 3.048083 \\ 1.650191 \end{pmatrix} + \frac{a}{A} \begin{pmatrix} -10.463634 \\ -17.490716 \\ -6.997085 \\ -4.635112 \\ -8.133655 \\ -3.498542 \end{pmatrix} \qquad (2.66)$$

$$\left.\begin{aligned} g_{10,\alpha} &= -0.043688 - 0.052503\tfrac{a}{A} \\ g_{11,\alpha} &= 0.038890 + 0.019326\tfrac{a}{A} \\ g_{12,\alpha} &= -0.023374 - 0.003289\tfrac{a}{A} \\ g_{20,\alpha} &= -0.014174 - 0.028567\tfrac{a}{A} \\ g_{21,\alpha} &= 0.011657 + 0.011443\tfrac{a}{A} \\ g_{22,\alpha} &= -0.005643 - 0.002531\tfrac{a}{A} \end{aligned}\right\} \qquad (2.67)$$

As in first approximation (2.3.3) we find $A = \frac{3}{8}F$ and hence

$$B(\tau) = 2A\tau + \tfrac{1}{4}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-} + I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})$$

$$= A(2\tau + 0.29449575e^{-k\tau}) + [-0.038023 + a(1.448669 - 0.68253725e^{-k\tau})]$$

$$= \tfrac{3}{8}F(2\tau + 0.294495575e^{-8.794658\tau}) + [-0.038023 + a(1.448669-$$

$$0.68253725e^{-8.794658\tau})] \qquad (2.68)$$