Bibliography

[1] Abu-Shumays,I.K *Nucl.Sci and Engg.1967;27:607*

[2] Aronson, R *T.T.S.P ; 1986 : 15,829*

[3] Busbridge, I.W *Mon.Not.R.ast.Soc;1954:115,521*

[4] Busbridge, I.W and Stibbs,D.W.N *Mon.Not.R.ast.Soc;1954:114,2*

[5] Canosa,Jose & Penafiel, H.R *JQSRT, 1973: 13:21*

[6] Carlstedt, J.L & Mullikin, T.W *The Astrophysical Journal:1996 No.113. :12;449*

[7] Case, N.M *Ann. Phys ; 1960: 9 , I*

[8] Chandrasekhar, S *Ap.J , 1943 ; 99 : 180*

[9] Chandrasekhar, S *Ap.J , 1944 ; 100 : 117*

[10] Chandrasekhar, S *Ap.J , 1945 ; 101 : 95*

[11] Chandrasekhar, S *Radiative transfer, Dover publ. 1960 ; Inc.New York*

[12] DasGupta, S.R *Sai.Cult ; 1956 : 22,177*

[13] DasGupta, S.R and Karanjai, S *Astrophys. Space Sci ; 1972 : 18,246*

[14] DasGupta, S.R *Astrophys. Space Sci ;1978 : 56, 13-20*

[15] DasGupta, S.R *Astrophys. Space Sci ;1977 : 50, 187*

[16] Deb, T.K . Biswas,G and Karanjai, S *Astrophys. Space Sci ;1991 : 178, 107*

[17] Deb, T.K and Karanjai, S *Astrophys. Space Sci ;1991 : 184, 57-63*

[18] Degl'Innocenti, E.L *M.N.R.A.S ;1979 :186,369*

[19] Devaux,C.Y. , Fouqart,Y. & Herman,M. *JQSRT, 1973; 13:1421*

[20] Eddington,A.S *Internal Constituent of Stars, 1926; Cambridge,England*

[21] Eddington,A.S *M.N.R.A.S ; 1926 : 89,623*

[22] Garcia,R.D.M & Siewert, C.E *JQSRT ,1986 ; 36, 401*

[23] Ghosh, M & Karanjai, S *JQSRT ; 2006 , 98 : 277-289*

[24] Gratton, L *Soc. Astron. Italy, 1937 ; 10 : 309*

[25] Gross, E.P and Zering, S *Ap.J , 1955 ; 123:343*

[26] Karanjai, S *Pub.Astro.Soc.Japan ; 1968 :20(2),173-177*

[27] Karanjai, S and Barman, S.K *Astrophysics and Space Science 1981;77:271-278*

[28] Karanjai,S and Karanjai, M *Astrophysics and Space Science 1985;115:295-307*

[29] Karanjai, S & Karanjai, M *Astrophys. sp.Sc.1985;113:119*

[30] Karanjai, S and Biswas, L *Astrophys Space Science. 1992; 192:161*

[31] Karanjai, S and Biswas, L *Astrophys Space Science. 1992; 207:221*

[32] Karanjai, S & Talukdar, M *Astrophysics and Space Science 1992;197:309*

[33] Karanjai, S & Basak, S *NBU review(sci & tech), 1996 : vol.7, no.2*

[34] Karanjai, S & Basak, S *Proc WMVC-2001, 2001*

[35] Karanjai, S & Basak, S *Acta Ciencia Indica, 2005 : vol-XXXIM, no.2,317*

[36] Karp, Alan. H, Greenstadt, J & Fillmore, J.A *JQSRT. 1980 ; 24,391*

[37] Karp, Alan.H & Petrack, S *JQSRT. 1983; 30(4),351*

[38] Kourganoff, V *Basic methods in transfer problems,1963;Dover publication,New York*

[39] Krook, Max *Ap.J , 1955 ; 122:488*

[40] Krook, Max *Ap.J , 1959 ; 130:286*

[41] Mark, C *Phys. Rev. 1947 ; 78:558*

[42] Marshak, R.E *Phys. Rev. 1948 ; 71:443*

[43] Mertens, R *Simon Steven Supplement, 1954:30*

[44] Mihalas,D. , Kunasz, P.B., Hummer,D.G *Apj ,1975 : 202, 465*

[45] Muldashev,T.Z , Lyapustin,A.I & Sultangazin,U.M *JQSRT 1999; 61,3 :393-404*

[46] Peraiah, A *Kodaikanal Obs. Bul;1979 : Ser. A.2,230*

[47] Peressini,A.L. , Sullivan, F.E. , Uhl, J.J *The Mathematics Of Nonlinear Programming, 1988; Springer-Verlag,New York*

[48] Rons, N *Astron. Astrophys, 1996; 309: 840-848*

[49] Roychoudhury, A and Karanjai, . *J.Phys.Sci 1996;2: 10*

[50] Roychoudhury, A and Karanjai, . *Res.Bull.Punjab.Univ 1997;47:219*

[51] Roychoudhury, A and Karanjai, . *I.J.T.P 1998;46(4): 203*

[52] Sen, H.K *Ap. J; 1949 : 110,276*

[53] Sen, K.K and Wilson, S.J *Radiative Transfer In Curved Media ; 1990: World Scientific, Singapore*

[54] Siewert, C.E and McCormick, N.J *Astrophys. J ; 1967 :149 , 649*

[55] Siewert, C.E and Ozisik, M.N *M.N.R.A.S ; 1969 , 146, 351-360*

[56] Sobolev, V.V *AStron.J ;1951 : Bulletin 5,28*

[57] Sykes , J.B *M.N.R.A.S , 1951 ; 111:377*

[58] Takeuchi, Yoshiaki *JQSRT , 1988 ; 39;237*

[59] Tine,S. Aiello,S. Belleni,A & Pestellini,C.C *JQSRT , 1992 : 47,95*

[60] Wan,F.S; Wilson, S.J & Sen,K.K *JQSRT ,1977 ; 17: 577*

[61] Wan,F.S; Wilson, S.J & Sen,K.K *Astrophys. Sp. Sci 1986 : 121: 309*

[62] Wells,W.H & Sidorowich, J.J *JQSRT. 1985 ; 33(4),347*

[63] Wiener, N and Hopf, E *Preuss.Acad.Wiss ; 1931:696*

[64] White, H.E *Introduction to atomic spectra,1934;McGrawHill,London*

[65] Wilson, S. J and Sen, K . K *Publ. Astro. Soc. Japan 1963;15:351 - 67*

[66] Wilson, S. J and Sen, K . K *Ann. Astrophys. 1964 : 27:46*

[67] Wilson, S. J and Sen, K . K *Ann. Astrophys. 1964 : 27:654*

[68] Wilson, S. J and Sen, K . K *Ann. Astrophys. 1965 : 28:348*

[69] Wilson, S. J and Sen, K . K *Ann. Astrophys. 1965 : 28:855*

[70] Wilson, S. J and Sen, K . K *Canadian Jr. Phys. 1965 : 43:432*

[71] Woolley RvdR and Stibbs DWN *The outer layers of a star, 1953 : Oxford University Press ; 1953*

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Solution of the equation of radiative transfer for interlocked doublets by double interval spherical harmonic method

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Abstract

The double-ordinate spherical harmonic method presented by Wilson and Sen (Publ. Astron. Soc. Jpn. 15 (1963) 351) has been used to solve the equation of radiative transfer in the Milne–Eddington model for interlocked doublets. Solutions have been obtained in the first and second approximation in a particular case $\eta_1 = 1$; $\eta_2 = \frac{1}{2}$.

Keywords: Radiative transfer; Interlocked doublets; Double interval spherical harmonic method

1. Introduction

The equation of transfer in the Milne–Eddington (M–E) model for interlocking without redistribution had been developed by Woolley and Stibbs [1]. Busbridge and Stibbs [2] applies the principle of invariance governing the law of diffuse reflection with a slight modification to solve exactly the equation of transfer in the M–E model. Karanjai [3] calculated the residual intensities for doublet and triplet using his approximate form of H-function using the solution given by Busbridge and Stibbs [2]. DasGupta and Karanjai [4] applied Sovolev's Probabilistic method to solve the same problem. Karanjai and Barman [5] applied the extension of the method of Discrete ordinates to solve the problem. Karanjai [6] calculated Mg b lines from the solution of interlocked multiplets obtained by DasGupta and Karanjai [4]. DasGupta [7] obtained an exact solution of the problem with the linear Planck function by Laplace transform and Wiener–Hopf technique using a new representation of the H-function obtained by DasGupta [8]. The same problem with new linear Planck function has been solved by Karanjai and Karanjai [9] by the method used by DasGupta [7]. Deb et al. [10] solved the same problem by the method of discrete ordinate considering nonlinear form of Planck function.

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Deb and Karanjai [11] solved the problem of interlocked multiplets exactly by the method used by Busbridge and Stibbs [2]. Wilson and Sen [12] modified the spherical harmonic method and solved quite a number of radiative transfer problems with different law of angular distribution of radiation and in different geometries. In this paper, we solved the transfer equation for doublet by modified double-interval spherical harmonic method developed by Wilson and Sen [12], using Planck function $B_\nu(T)$ as an linear function of optical depth i.e. $B_\nu(T) = a + b\tau$.

2. Equation of transfer

The equation of transfer for the rth line of multiplets in the case of interlocking without redistribution is

$$\mu \frac{\mathrm{d}I_r(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)(a+b\tau) - (1-\varepsilon)\alpha_r \times \sum_{p=1}^{k} \frac{1}{2}\eta_p \int_{-1}^{1} I_p(\tau,\mu')\,\mathrm{d}\mu', \quad r=1,2,\ldots,k, \tag{1}$$

where

$$\alpha_r = \frac{\eta_r}{\eta_1+\eta_2+\cdots+\eta_k}, \tag{2}$$

so that

$$\alpha_1+\alpha_2+\cdots+\alpha_k = 1 \tag{3}$$

and η_r, the ratio of line to the continuum absorption coefficient for the rth line is independent of depth but is a function of frequency. ε is the coefficient of thermal emission, is independent of both frequency and depth.

For doublet (1) reduces to

$$\mu \frac{\mathrm{d}I_r(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)(a+b\tau) - (1-\varepsilon)\alpha_r \frac{1}{2}\left[\eta_p \int_{-1}^{1} I_p(\tau,\mu')\,\mathrm{d}\mu'\right], \quad r=1,2. \tag{4}$$

The above equation of transfer (4) is to be solved subject to the boundary conditions

$$I_r(0,\mu) \equiv 0 \quad \text{for } -1 \leqslant \mu \leqslant 0, \ r=1,2 \tag{5}$$

and

$$I_r(\tau,\mu)\mathrm{e}^{-\tau} \to 0 \quad \text{as } \tau\to\infty, \ r=1,2. \tag{6}$$

We shall seek a solution of Eq. (4) $I_r(\tau,\mu)$ can be expansions $I_r^+(\tau,\mu)$ and $I_r^-(\tau,\mu)$ for μ in the interval $(0,1)$ and $(-1,0)$, respectively, in the form [12]

$$I_r^+(\tau,\mu) = A\tau + \sum_{l=0}^{l_0} (2l+1)I_{rl}^+(\tau)\mu P_l(2\mu-1), \quad 0\leqslant\mu\leqslant 1, \ r=1,2, \tag{7}$$

$$I_r^-(\tau,\mu) = A\tau + \sum_{l=0}^{l_0} (2l+1) I_{rl}^-(\tau)\mu P_l(2\mu+1), \quad -1 \leqslant \mu \leqslant 0, \ r = 1,2, \tag{8}$$

where A is a constant (independent of μ) to be determined and the recurrence formulae

$$\mu P_l(2\mu \pm 1) = \frac{1}{(2l+1)}\left[\frac{l+1}{2}P_{l+1}(2\mu \pm 1) \mp \frac{2l+1}{2}P_l(2\mu \pm 1) + \frac{l}{2}P_{l-1}(2\mu \pm 1)\right] \tag{9}$$

has the advantages due to orthogonality of $P_l(2\mu-1)$ in $(0,1)$ and $P_l(2\mu+1)$ in $(-1,0)$.

The equation of transfer (4) in the present representation is equivalent to

$$\mu\frac{\mathrm{d}I_1(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_1)I_1(\tau,\mu) - (1+\varepsilon\eta_1)(a+b\tau)$$
$$-(1-\varepsilon)\alpha_1\frac{1}{2}\left[\eta_1\int_{-1}^{1} I_1(\tau,\mu')\,\mathrm{d}\mu' + \eta_2\int_{-1}^{1} I_2(\tau,\mu')\,\mathrm{d}\mu'\right] \tag{10}$$

and

$$\mu\frac{\mathrm{d}I_2(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_2)I_2(\tau,\mu) - (1+\varepsilon\eta_2)(a+b\tau)$$
$$-(1-\varepsilon)\alpha_2\frac{1}{2}\left[\eta_1\int_{-1}^{1} I_1(\tau,\mu')\,\mathrm{d}\mu' + \eta_2\int_{-1}^{1} I_2(\tau,\mu')\,\mathrm{d}\mu'\right], \tag{11}$$

also we have

$$\int_{-1}^{1} I_1(\tau,\mu')\,\mathrm{d}\mu' = 2A\tau + \tfrac{1}{2}(I_{10}^+ - I_{10}^- + I_{11}^+ + I_{11}^-), \tag{12}$$

$$\int_{-1}^{1} I_2(\tau,\mu')\,\mathrm{d}\mu' = 2A\tau + \tfrac{1}{2}(I_{20}^+ - I_{20}^- + I_{21}^+ + I_{21}^-). \tag{13}$$

Let us first consider the equation for $r = 1$.

The equation of transfer (10) can be written as

$$\mu\frac{\mathrm{d}I_1^+(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_1)I_1^+(\tau,\mu) - (1+\varepsilon\eta_1)(a+b\tau) - (1-\varepsilon)\eta_1 A\tau$$
$$-\frac{(1-\varepsilon)\alpha_1}{4}[\eta_1(I_{10}^+ - I_{10}^- + I_{11}^+ + I_{11}^-) + \eta_2(I_{20}^+ - I_{20}^- + I_{21}^+ + I_{21}^-)] \tag{14}$$

and

$$\mu\frac{\mathrm{d}I_1^-(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_1)I_1^-(\tau,\mu) - (1+\varepsilon\eta_1)(a+b\tau) - (1-\varepsilon)\eta_1 A\tau$$
$$-\frac{(1-\varepsilon)\alpha_1}{4}[\eta_1(I_{10}^+ - I_{10}^- + I_{11}^+ + I_{11}^-) + \eta_2(I_{20}^+ - I_{20}^- + I_{21}^+ + I_{21}^-)]. \tag{15}$$

Multiplying Eq. (14) by $P_l(2\mu-1)$ and Eq. (15) by $P_l(2\mu+1)$, respectively, and integrating over μ in their respective ranges and using the recurrence formulae (9), we have the following equations:

$$\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{1l-2}^{+'}+2lI_{1l-1}^{+'}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{+'}+2(l+1)I_{1l+1}^{+'}+\frac{l^2+3l+2}{2l+3}I_{1l+2}^{+'}\right]$$
$$+A\int_0^1 \mu P_l(2\mu-1)\,\mathrm{d}\mu$$
$$=\frac{(1+\eta_1)}{2(2l+1)}[lI_{1l-1}^{+}+(2l+1)I_{1l}^{+}+(l+1)I_{1l+1}^{+}]$$
$$-[(1+\varepsilon\eta_1)(a+b\tau)+(1-\varepsilon)\eta_1 A\tau-(1+\eta_1)A\tau]\int_0^1 P_l(2\mu-1)\,\mathrm{d}\mu$$
$$-\frac{(1-\varepsilon)\alpha_1}{4}[\eta_1(I_{10}^{+}-I_{10}^{-}+I_{11}^{+}+I_{11}^{-})+\eta_2(I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-})] \quad (16)$$

and

$$\frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{1l-2}^{-'}-2lI_{1l-1}^{-'}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{-'}-2(l+1)I_{1l+1}^{-'}+\frac{l^2+3l+2}{2l+3}I_{1l+2}^{-'}\right]$$
$$+A\int_{-1}^0 \mu P_l(2\mu+1)\,\mathrm{d}\mu$$
$$=\frac{(1+\eta_1)}{2(2l+1)}[lI_{1l-1}^{-}-(2l+1)I_{1l}^{-}+(l+1)I_{1l+1}^{-}]$$
$$-[(1+\varepsilon\eta_1)(a+b\tau)+(1-\varepsilon)\eta_1 A\tau-(1+\eta_1)A\tau]\int_{-1}^0 P_l(2\mu+1)\,\mathrm{d}\mu$$
$$-\frac{(1-\varepsilon)\alpha_1}{4}[\eta_1(I_{10}^{+}-I_{10}^{-}+I_{11}^{+}+I_{11}^{-})+\eta_2(I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-})], \quad (17)$$

where I_l' are differentials of I_l with respect to the optical thickness τ.

Separating the equations for $l=0$ and 1 from the Eqs. (16) and (17) we can write

for $l=0$:

$$\left(\frac{4}{3}I_{10}^{+'}+2I_{11}^{+'}+\frac{2}{3}I_{12}^{+'}\right)-[\{2(1+\eta_1)-\eta_1(1-\varepsilon)\alpha_1\}I_{10}^{+}+(1-\varepsilon)\alpha_1\eta_1 I_{10}^{-}$$
$$+\{2(1+\eta_1)-\eta_1(1-\varepsilon)\alpha_1\}I_{11}^{+}-(1-\varepsilon)\alpha_1\eta_1 I_{11}^{-}]+(1-\varepsilon)\alpha_1\eta_2[I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-}]$$
$$=-2A-4[(1+\varepsilon\eta_1)(a+b\tau)+(1-\varepsilon)\eta_1 A\tau-(1+\eta_1)A\tau],$$

for $l=1$:

$$\left(2I_{10}^{+'}+\frac{24}{5}I_{11}^{+'}+4I_{12}^{+'}+\frac{6}{5}I_{13}^{+'}\right)-2(1+\eta_1)(I_{10}^{+}+3I_{11}^{+}+2I_{12}^{+})=-2A,$$

for $l \neq 0, 1$:

$$\left[\frac{l^2-l}{2l-1}I_{1l-2}^{+'} + 2lI_{1l-1}^{+'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{+'} + 2(l+1)I_{1l+1}^{+'} + \frac{l^2+3l+2}{2l+3}I_{1l+2}^{+'}\right]$$
$$-2(1+\eta_1)[lI_{1l-1}^{+} + (2l+1)I_{1l}^{+} + (l+1)I_{1l+1}^{+}] = 0 \tag{18}$$

and

for $l = 0$:

$$\left(\frac{4}{3}I_{10}^{-'} + 2I_{11}^{-'} + \frac{2}{3}I_{12}^{-'}\right) - [(1-\varepsilon)\alpha_1\eta_1 I_{10}^{+} + \{2(1+\eta_1) - \eta_1(1-\varepsilon)\alpha_1\}I_{10}^{-}$$
$$+(1-\varepsilon)\alpha_1\eta_1 I_{11}^{+} - \{2(1+\eta_1) - \eta_1(1-\varepsilon)\alpha_1\}I_{11}^{-}] + (1-\varepsilon)\alpha_1\eta_2[I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}]$$
$$= 2A - 4[(1+\varepsilon\eta_1)(a+b\tau) + (1-\varepsilon)\eta_1 A\tau - (1+\eta_1)A\tau],$$

for $l = 1$:

$$\left(-2I_{10}^{-'} + \frac{24}{5}I_{11}^{-'} - 4I_{12}^{-'} + \frac{6}{5}I_{13}^{-'}\right) - 2(1+\eta_1)(I_{10}^{-} - 3I_{11}^{-} + 2I_{12}^{-}) = -2A,$$

for $l \neq 0, 1$:

$$\left[\frac{l^2-l}{2l-1}I_{1l-2}^{-'} - 2lI_{1l-1}^{-'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_{1l}^{-'} - 2(l+1)I_{1l+1}^{-'} + \frac{l^2+3l+2}{2l+3}I_{1l+2}^{-'}\right]$$
$$-2(1+\eta_1)[lI_{1l-1}^{-} - (2l+1)I_{1l}^{-} + (l+1)I_{1l+1}^{-}] = 0. \tag{19}$$

The Eqs. (18) and (19) are to be solved subject to the boundary conditions (5) and (6) which are restated below

$$I_{1l}^{-}(0) \equiv 0 \quad \text{and} \quad \left.\begin{matrix} I_{1l}^{+}(\tau)e^{-\tau} \to 0 \\ I_{1l}^{-}(\tau)e^{-\tau} \to 0 \end{matrix}\right\} \quad \text{as } \tau \to \infty \tag{20}$$

and

$$I_{2l}^{-}(0) \equiv 0 \quad \text{and} \quad \left.\begin{matrix} I_{2l}^{+}(\tau)e^{-\tau} \to 0 \\ I_{2l}^{-}(\tau)e^{-\tau} \to 0 \end{matrix}\right\} \quad \text{as } \tau \to \infty. \tag{21}$$

3. Solution

It is assumed that at the Nth approximation

$$I_{1N+1}^{+} = I_{1N+1}^{-} = 0, \tag{22}$$

$$I_{2N+1}^{+} = I_{2N+1}^{-} = 0. \tag{23}$$

We assume a trial solution of the form

$$I_{1l}^{+}(\tau) = A[g_{1l,\alpha}e^{-k\tau} + g_{1l,\beta}],$$
$$I_{1l}^{-}(\tau) = A[h_{1l,\alpha}e^{-k\tau} + h_{1l,\beta}] \tag{24}$$

and

$$I_{2l}^{+}(\tau)=A[g_{2l,\alpha}\mathrm{e}^{-k\tau}+g_{2l,\beta}],$$
$$I_{2l}^{-}(\tau)=A[h_{2l,\alpha}\mathrm{e}^{-k\tau}+h_{2l,\beta}], \tag{25}$$

where $g_{1l,\alpha}, g_{1l,\beta}, g_{2l,\alpha}, g_{2l,\beta}, h_{1l,\alpha}, h_{1l,\beta}, h_{2l,\alpha}, h_{2l,\beta}$ are constants to be determined.

Substituting these in (18) and (19) and equating the coefficients of $\mathrm{e}^{-k\tau}$ and constant term we obtain (26) and (27).

for $l=0$:

$$\left\{\frac{4k}{3}+2(1+\eta_1)-\alpha_1\eta_1(1-\varepsilon)\right\}g_{10,\alpha}+\{2k+2(1+\eta_1)-\alpha_1\eta_1(1-\varepsilon)\}g_{11,\alpha}$$
$$+\frac{2k}{3}g_{12,\alpha}-(1-\varepsilon)\alpha_1\eta_2 g_{20,\alpha}-(1-\varepsilon)\alpha_1\eta_2 g_{21,\alpha}+(1-\varepsilon)\alpha_1\eta_1 h_{10,\alpha}$$
$$-(1-\varepsilon)\alpha_1\eta_1 h_{11,\alpha}+(1-\varepsilon)\alpha_1\eta_2 h_{20,\alpha}-(1-\varepsilon)\alpha_1\eta_2 h_{21,\alpha}=0,$$

for $l=1$:

$$2(1+\eta_1+k)g_{10,\alpha}+6\left(1+\eta_1+\frac{4k}{5}\right)g_{11,\alpha}+4(1+\eta_1+k)g_{12,\alpha}+\frac{6k}{5}g_{13,\alpha}=0,$$

for $l\neq 0,1$:

$$\frac{l^2-l}{2l-1}g_{1l-2,\alpha}+2l(1+\eta_1+k)g_{1l-1,\alpha}+\left\{\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}k+2(1+\eta_1)(2l+1)\right\}$$
$$\times g_{1l,\alpha}+2(l+1)(1+\eta_1+k)g_{1l+1,\alpha}+\frac{l^2+3l+2}{2l+3}kg_{1l+2,\alpha}=0,$$

for $l=0$:

$$\left\{2(1+\eta_1)-\alpha_1\eta_1(1-\varepsilon)-\frac{4k}{3}\right\}h_{10,\alpha}+\{2(1+\eta_1)-\alpha_1\eta_1(1-\varepsilon)-2k\}h_{11,\alpha}$$
$$-\frac{2k}{3}h_{12,\alpha}-(1-\varepsilon)\alpha_1\eta_2 h_{20,\alpha}+(1-\varepsilon)\alpha_1\eta_2 h_{21,\alpha}+(1-\varepsilon)\alpha_1\eta_1 g_{10,\alpha}$$
$$+(1-\varepsilon)\alpha_1\eta_1 g_{11,\alpha}+(1-\varepsilon)\alpha_1\eta_2 g_{20,\alpha}+(1-\varepsilon)\alpha_1\eta_2 g_{21,\alpha}=0,$$

for $l=1$:

$$-2(1+\eta_1-k)h_{10,\alpha}+6\left(1+\eta_1-\frac{4k}{5}\right)h_{11,\alpha}-4(1+\eta_1-k)h_{12,\alpha}-\frac{6k}{5}g_{13,\alpha}=0,$$

for $l\neq 0,1$:

$$-\frac{l^2-l}{2l-1}h_{1l-2,\alpha}-2l(1+\eta_1-k)h_{1l-1,\alpha}-\left\{\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}k-2(1+\eta_1)(2l+1)\right\}$$
$$\times h_{1l,\alpha}-2(l+1)(1+\eta_1-k)h_{1l+1,\alpha}-\frac{l^2+3l+2}{2l+3}kh_{1l+2,\alpha}=0 \tag{26}$$

and

for $l=0$:

$$\{2(1+\eta_1)-(1-\varepsilon)\alpha_1\eta_1\}g_{10,\beta}+\{2(1+\eta_1)-(1-\varepsilon)\alpha_1\eta_1\}g_{11,\beta}$$
$$-(1-\varepsilon)\alpha_1\eta_2 g_{20,\beta}-(1-\varepsilon)\alpha_1\eta_2 g_{21,\beta}+(1-\varepsilon)\alpha_1\eta_1 h_{10,\beta}-(1-\varepsilon)\alpha_1\eta_1$$
$$\times h_{11,\beta}-(1-\varepsilon)\alpha_1\eta_2 h_{20,\beta}-(1-\varepsilon)\alpha_1\eta_2 h_{21,\beta}=2+4\frac{a}{A}(1+\varepsilon\eta_1),$$

for $l=1$:

$$g_{10,\beta}+3g_{11,\beta}+2g_{12,\beta}=1,$$

for $l\neq 0,1$:

$$lg_{1l-1,\beta}+(2l+1)g_{1l,\beta}+(l+1)g_{1l+1,\beta}=0,$$

for $l=0$:

$$\{2(1+\eta_1)-(1-\varepsilon)\alpha_1\eta_1\}h_{10,\beta}-\{2(1+\eta_1)-(1-\varepsilon)\alpha_1\eta_1\}h_{11,\beta}$$
$$+(1-\varepsilon)\alpha_1\eta_1 g_{10,\beta}+(1-\varepsilon)\alpha_1\eta_1 g_{11,\beta}+(1-\varepsilon)\alpha_1\eta_2 h_{20,\beta}+(1-\varepsilon)\alpha_1\eta_2$$
$$\times g_{21,\beta}-(1-\varepsilon)\alpha_1\eta_2 h_{20,\beta}+(1-\varepsilon)\alpha_1\eta_2 h_{21,\beta}=2-4\frac{a}{A}(1+\varepsilon\eta_1),$$

for $l=1$:

$$h_{10,\beta}-3h_{11,\beta}+2h_{12,\beta}=1,$$

for $l\neq 0,1$:

$$lh_{1l-1,\beta}-(2l+1)h_{1l,\beta}+(l+1)h_{1l+1,\beta}=0. \qquad (27)$$

A similar set of equations like (26) and (27) will be obtained considering $r=2$.

Solving (26) and the similar set of equation obtained when $r=2$ combinedly by the method described by Wilson and Sen [12] we obtain $k=k_r$; $r=8,12,\ldots$.

Using boundary conditions (20) and (21) we obtain

$$\sum_{r=1}^{n-1} h_{il,\alpha}^{(r)}+h_{il,\beta}=0, \quad i=1,2. \qquad (28)$$

Thus Eqs. (26) and (27) and similar set of equations (for $r=2$) and (28) are sufficient to determine the unknowns $g_{il,j}, h_{il,j}$; $i=1,2$; $j=\alpha,\beta$; $l=1,2,\ldots,n$.

Thus, we have

$$I_{il}^{+}(\tau)=A[g_{il,\alpha}^{(r)}\mathrm{e}^{-k\tau}+g_{il,\beta}],$$
$$I_{il}^{-}(\tau)=A[h_{il,\alpha}^{(r)}\mathrm{e}^{-k\tau}+h_{il,\beta}], \qquad (29)$$

where $i=1,2$; $l=1,2,\ldots,n$.

Now we will consider two approximation, viz. $l_0=1$ and 2.

4. First approximation

4.1. First approximation when $r = 1$

We name the solution first approximation when $l_0 = 1$.

In this case, we have from Eqs. (18) and (19)

$$\frac{4}{3}I_{10}^{+'} + 2I_{11}^{+'} - (\xi_1 I_{10}^{+} + \xi_2 I_{10}^{-} + \xi_1 I_{11}^{+} - \xi_2 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})$$
$$= -2A - 4[(1+\varepsilon\eta_1)(a+b\tau) + (1-\varepsilon)\eta_1 A\tau - (1+\eta_1)A\tau]$$
$$\frac{4}{3}I_{10}^{-'} - 2I_{11}^{-'} + (\xi_2 I_{10}^{+} + \xi_1 I_{10}^{-} + \xi_2 I_{11}^{+} - \xi_1 I_{11}^{-}) + \xi_3(I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})$$
$$= -2A - 4[(1+\varepsilon\eta_1)(a+b\tau) + (1-\varepsilon)\eta_1 A\tau - (1+\eta_1)A\tau]$$
$$2I_{10}^{+'} + \frac{24}{5}I_{11}^{+'} - 2(1+\eta_1)(I_{10}^{+} + 3I_{11}^{+}) = -2A,$$
$$-2I_{10}^{-'} + \frac{24}{5}I_{11}^{-'} - 2(1+\eta_1)(I_{10}^{-} - 3I_{11}^{-}) = -2A, \tag{30}$$

where

$$I_{1l}^{+}(\tau,\mu) = A\tau + I_{10}^{+}(\tau)\mu + 3I_{11}^{+}(\tau)\mu P_1(2\mu - 1), \quad 0 \leqslant \mu \leqslant 1,$$
$$I_{1l}^{-}(\tau,\mu) = A\tau + I_{10}^{-}(\tau)\mu + 3I_{11}^{-}(\tau)\mu P_1(2\mu + 1), \quad -1 \leqslant \mu \leqslant 0. \tag{31}$$

We now take the trial solution given by (24) and (25) and substituting these in (30) and then equating the coefficient of $e^{k\tau}$ and constant term we obtain

$$\left(\xi_1 + \frac{4k}{3}\right) g_{10,\alpha} + (\xi_1 + 2k)g_{11,\alpha} + \xi_2 h_{10,\alpha} - \xi_2 h_{11,\alpha} - \xi_3 g_{20,\alpha}$$
$$- \xi_3 g_{21,\alpha} + \xi_3 h_{20,\alpha} - \xi_3 h_{21,\alpha} = 0,$$
$$2(\xi_4 + k)g_{10,\alpha} + 6\left(\xi_4 + \frac{4k}{5}\right) g_{11,\alpha} = 0,$$
$$\xi_2 g_{10,\alpha} + \xi_2 g_{11,\alpha} + \left(\xi_1 - \frac{4k}{3}\right) h_{10,\alpha} + (-\xi_1 + 2k)h_{11,\alpha} + \xi_3 g_{20,\alpha}$$
$$+ \xi_3 g_{21,\alpha} - \xi_3 h_{20,\alpha} + \xi_3 h_{21,\alpha} = 0,$$
$$2(-\xi_4 + k)h_{10,\alpha} + 6\left(\xi_4 - \frac{4k}{5}\right) = 0 \tag{32}$$

and

$$\xi_1 g_{10,\beta} + \xi_1 g_{11,\beta} + \xi_2 h_{10,\beta} - \xi_2 h_{11,\beta} - \xi_3(g_{20,\beta} + g_{21,\beta} - h_{20,\beta} + h_{21,\beta}) = 2 + \frac{4a}{A}(1+\varepsilon\eta_1),$$
$$2\xi_4 g_{10,\beta} + 6\xi_4 g_{11,\beta} = 2,$$

$$\xi_2 g_{10,\beta} + \xi_2 g_{11,\beta} + \xi_1 h_{10,\beta} - \xi_1 h_{11,\beta} + \xi_3(g_{20,\beta} + g_{21,\beta} - h_{20,\beta} + h_{21,\beta}) = 2 - \frac{4a}{A}(1 + \varepsilon\eta_1),$$

$$2\xi_4 h_{10,\beta} - 6\xi_4 h_{11,\beta} = 2, \tag{33}$$

where we make the abbreviation

$$\xi_1 = 2(1 + \eta_1) - \alpha_1\eta_1(1 - \varepsilon),$$

$$\xi_2 = (1 - \varepsilon)\alpha_1\eta_1,$$

$$\xi_2 = (1 - \varepsilon)\alpha_2\eta_2,$$

$$\xi_4 = (1 + \eta_1). \tag{34}$$

4.2. First approximation when $r = 2$

Similarly, considering the rest of the Eq. (11) and proceeding in the same manner described in Sections 3 and 4.1 and taking $l_0 = 1$, we have, by equating coefficients of $e^{-k\tau}$ and constant term, the following set of Eqs. (35) and (36)

$$-\lambda_3 g_{10,\alpha} - \lambda_3 g_{11,\alpha} + \lambda_3 h_{10,\alpha} - \lambda_3 h_{11,\alpha} + \left(\lambda_1 + \frac{4k}{3}\right) g_{20,\alpha}$$

$$+ (\lambda_1 + 2k) g_{21,\alpha} + \lambda_2 h_{20,\alpha} - \lambda_2 h_{21,\alpha} = 0,$$

$$2(\lambda_4 + k) g_{20,\alpha} + 6\left(\lambda_4 + \frac{4k}{3}\right) g_{21,\alpha} = 0,$$

$$\lambda_3 g_{10,\alpha} + \lambda_3 g_{11,\alpha} - \lambda_3 h_{10,\alpha} + \lambda_3 h_{11,\alpha} + \lambda_2 g_{20,\alpha} + \lambda_2 g_{21,\alpha}$$

$$+ \left(\lambda_1 - \frac{4k}{3}\right) h_{20,\alpha} + (-\lambda_1 + 2k) h_{21,\alpha} = 0,$$

$$2(-\lambda_4 + k) h_{20,\alpha} + 6\left(\lambda_4 - \frac{4k}{5}\right) h_{21,\alpha} = 0 \tag{35}$$

and

$$\lambda_1 g_{20,\beta} + \lambda_1 g_{21,\beta} + \lambda_2 h_{20,\beta} - \lambda_2 h_{21,\beta} - \lambda_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} + h_{11,\beta}) = 2 + \frac{4a}{A}(1 + \varepsilon\eta_2),$$

$$2\lambda_4 g_{20,\beta} + 6\lambda_4 g_{21,\beta} = 2,$$

$$\lambda_2 g_{20,\beta} + \lambda_2 g_{21,\beta} + \lambda_1 h_{20,\beta} - \lambda_1 h_{21,\beta} + \lambda_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} + h_{11,\beta}) = 2 - \frac{4a}{A}(1 + \varepsilon\eta_2),$$

$$2\lambda_4 h_{20,\beta} - 6\lambda_4 h_{21,\beta} = 2, \tag{36}$$

where

$$\lambda_1 = 2(1 + \eta_2) - (1 - \varepsilon)\alpha_2\eta_2,$$

$$\lambda_2 = (1 - \varepsilon)\alpha_2\eta_2,$$

$$\lambda_3 = (1 - \varepsilon)\alpha_2\eta_1,$$

$$\lambda_4 = (1 + \eta_2). \tag{37}$$

Now the set of Eqs. (32) and (35) have a nontrivial solution if

$$\Delta(k) = 0, \tag{38}$$

where

$$\Delta(k) = \begin{vmatrix} \xi_1 + \frac{4k}{3} & \xi_1 + 2k & \xi_2 & -\xi_2 & -\xi_3 & -\xi_3 & \xi_3 & -\xi_3 \\ 2\xi_4 + 2k & 6\xi_4 + \frac{24k}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_2 & \xi_2 & \xi_1 - \frac{4k}{3} & -\xi_1 + 2k & \xi_3 & \xi_3 & -\xi_3 & \xi_3 \\ 0 & 0 & -2\xi_4 + 2k & 6\xi_4 - \frac{24k}{5} & 0 & 0 & 0 & 0 \\ -\lambda_3 & -\lambda_3 & \lambda_3 & -\lambda_3 & \lambda_1 + \frac{4k}{3} & \lambda_1 + 2k & \lambda_2 & -\lambda_2 \\ 0 & 0 & 0 & 0 & 2\lambda_4 + 2k & 6\lambda_4 + \frac{8k}{3} & 0 & 0 \\ \lambda_3 & \lambda_3 & -\lambda_3 & \lambda_3 & \lambda_2 & \lambda_2 & \lambda_1 - \frac{4k}{3} & -\lambda_1 + 2k \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda_4 + 2k & 6\lambda_4 - \frac{24k}{5} \end{vmatrix} \tag{39}$$

Now we consider the case [3] i.e. $\eta_1 = 1$, $\eta_2 = \frac{1}{2}$, $\varepsilon = 0$, so $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{1}{3}$.
Thus,

$$\xi_1 = \tfrac{10}{3}, \quad \lambda_1 = \tfrac{17}{6},$$
$$\xi_2 = \tfrac{2}{3}, \quad \lambda_2 = \tfrac{1}{6},$$
$$\xi_3 = \tfrac{1}{3}, \quad \lambda_3 = \tfrac{1}{3},$$
$$\xi_4 = 2 \quad \text{and} \quad \lambda_4 = \tfrac{3}{2}. \tag{40}$$

Therefore (39) becomes,

$$\Delta(k) = \begin{vmatrix} \frac{10}{3} + \frac{4k}{3} & \frac{10}{3} + 2k & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 4 + 2k & 12 + \frac{24k}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{10}{3} - \frac{4k}{3} & -\frac{10}{3} + 2k & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -4 + 2k & 12 - \frac{24k}{5} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{17}{6} + \frac{4k}{3} & \frac{17}{6} + 2k & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 3 + 2k & 9 + \frac{8k}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{17}{6} - \frac{4k}{3} & -\frac{17}{6} + 2k \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 + 2k & 9 - \frac{24k}{5} \end{vmatrix} \tag{41}$$

and $\Delta(k) = 0$ yields
$k = \pm 2.36701, \pm 5.63299, -1.12729, -2.14644, 1.63853, 4.67409.$

We take $k = 2.36701$

Using (40) in (33) and (36) we obtain in matrix form

$$\mathbf{AX} = \mathbf{B}, \tag{42}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{10}{3} & \frac{10}{3} & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 4 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{10}{3} & -\frac{10}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 4 & -12 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{17}{6} & \frac{17}{6} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 3 & 9 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{17}{6} & -\frac{17}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -9 \end{pmatrix} \tag{43}$$

and

$$\mathbf{X} = \begin{pmatrix} g_{10,\beta} \\ g_{11,\beta} \\ h_{10,\beta} \\ h_{11,\beta} \\ g_{20,\beta} \\ g_{21,\beta} \\ h_{20,\beta} \\ h_{21,\beta} \end{pmatrix}, \quad \mathbf{B} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 4\frac{a}{A}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{44}$$

Using the boundary conditions we obtain

$$h_{1l,\alpha} + h_{1l,\beta} = 0$$

$$h_{2l,\alpha} + h_{2l,\beta} = 0 \quad \text{(for all } l\text{)}. \tag{45}$$

Now from (32), (35), (42) and (45) we obtain

$$g_{10,\alpha} = 0.171344 + \frac{a}{A}0.235598,$$

$$g_{11,\alpha} = -0.064059 - \frac{a}{A}0.088081,$$

$$g_{20,\alpha} = 2.159601 + \frac{a}{A}2.933029,$$

$$g_{21,\alpha} = -1.707505 - \frac{a}{A}2.011397,$$

$$g_{10,\beta} = 0.5 + \frac{a}{A}0.75,$$

$$g_{11,\beta} = \frac{a}{A},$$

$$g_{20,\beta} = 0.333333 + \frac{a}{A}2,$$

$$g_{21,\beta} = \frac{a}{A}0.666668,$$

$$h_{10,\alpha} = -0.5 - \frac{a}{A}0.75,$$

$$h_{11,\alpha} = -\frac{a}{A}0.5,$$

$$h_{20,\alpha} = -0.333333 - \frac{a}{A}2,$$

$$h_{21,\alpha} = -\frac{a}{A}0.666668,$$

$$h_{10,\beta} = 0.5 + \frac{a}{A}0.75,$$

$$h_{11,\beta} = \frac{a}{A}0.5,$$

$$h_{20,\beta} = 0.333333 + \frac{a}{A}2,$$

$$h_{21,\beta} = \frac{a}{A}0.666668. \tag{46}$$

4.3. Determination of a/A

The mean intensity is given by

$$B(\tau) = \frac{1}{2}\int_{-1}^{1} I_r(\tau,\mu')\,\mathrm{d}\mu' = \frac{3}{4}F[\tau + q(\tau)]. \tag{47}$$

But from Eqs. (12) and (13) we have

$$B(\tau) = 2A\tau + \tfrac{1}{4}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-} + I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-})$$

$$= 2A\tau + \tfrac{1}{4}[A0.892714\mathrm{e}^{-2.36701\tau} + a(2.333336 + 2.352481\mathrm{e}^{-2.36701\tau})]. \tag{48}$$

Comparing (47) and (48)

$$A = \tfrac{3}{8}F. \tag{49}$$

Again

$$F=2\int_{-1}^{1} I_r(\tau,\mu')\,\mathrm{d}\mu', \quad r=1,2$$

$$=8A\tau+[A0.892714\mathrm{e}^{-2.36701\tau}+a(2.333336+2.352481\mathrm{e}^{-2.36701\tau})]. \tag{50}$$

Thus, (49) and (50) gives

$$\frac{a}{A}=\frac{(\frac{8}{3}-8\tau-0.892714\mathrm{e}^{-2.36701\tau})}{(2.333336+2.352481\mathrm{e}^{-2.36701\tau})}. \tag{51}$$

5. Second approximation

We find the solution when $l_0=2$ and name it second approximation. In this case, we have from (18) and (19)

$$\begin{aligned}
&\tfrac{4}{3}I_{10}^{+'}+2I_{11}^{+'}+\tfrac{2}{3}I_{12}^{+'}-(\xi_1 I_{10}^{+}+\xi_2 I_{10}^{-}+\xi_1 I_{11}^{+}-\xi_2 I_{11}^{-})+\xi_3(I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-})\\
&\quad=-2A-4[(1+\varepsilon\eta_1)(a+b\tau)+(1-\varepsilon)\eta_1 A\tau-(1+\eta_1)A\tau]\\
&\quad\tfrac{4}{3}I_{10}^{-'}-2I_{11}^{-'}+\tfrac{2}{3}I_{12}^{-'}+(\xi_2 I_{10}^{+}+\xi_1 I_{10}^{-}+\xi_2 I_{11}^{+}-\xi_1 I_{11}^{-})+\xi_3(I_{20}^{+}-I_{20}^{-}+I_{21}^{+}+I_{21}^{-})\\
&\quad=-2A-4[(1+\varepsilon\eta_1)(a+b\tau)+(1-\varepsilon)\eta_1 A\tau-(1+\eta_1)A\tau],\\
&2I_{10}^{+'}+\tfrac{24}{5}I_{11}^{+'}+4I_{12}^{+'}-2\xi_4(I_{10}^{+}+3I_{11}^{+}+2I_{12}^{+})=-2A,\\
&-2I_{10}^{-'}+\tfrac{24}{5}I_{11}^{-'}-4I_{12}^{-'}-2\xi_4(I_{10}^{-}-3I_{11}^{-}+2I_{12}^{-})=-2A,\\
&\tfrac{1}{3}I_{10}^{+'}+2I_{11}^{+'}+\tfrac{80}{21}I_{12}^{+'}-\xi_4(2I_{11}^{+}+5I_{12}^{+})=0,\\
&\tfrac{1}{3}I_{10}^{-'}-2I_{11}^{-'}+\tfrac{80}{21}I_{12}^{-'}-\xi_4(2I_{11}^{-}-5I_{12}^{-})=0
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\tfrac{4}{3}I_{20}^{+'}+2I_{21}^{+'}+\tfrac{2}{3}I_{22}^{+'}-(\lambda_1 I_{20}^{+}+\lambda_2 I_{20}^{-}+\lambda_1 I_{21}^{+}-\lambda_2 I_{21}^{-})+\lambda_3(I_{10}^{+}-I_{10}^{-}+I_{11}^{+}+I_{11}^{-})\\
&\quad=-2A-4[(1+\varepsilon\eta_2)(a+b\tau)+(1-\varepsilon)\eta_2 A\tau-(1+\eta_2)A\tau],\\
&\tfrac{4}{3}I_{20}^{-'}-2I_{21}^{-'}+\tfrac{2}{3}I_{22}^{-'}+(\lambda_2 I_{20}^{+}+\lambda_1 I_{20}^{-}+\lambda_2 I_{21}^{+}-\lambda_1 I_{21}^{-})+\lambda_3(I_{10}^{+}-I_{10}^{-}+I_{11}^{+}+I_{11}^{-})\\
&\quad=-2A-4[(1+\varepsilon\eta_2)(a+b\tau)+(1-\varepsilon)\eta_2 A\tau-(1+\eta_2)A\tau],\\
&2I_{20}^{+'}+\tfrac{24}{5}I_{21}^{+'}+4I_{22}^{+'}-2\lambda_4(I_{20}^{+}+3I_{21}^{+}+2I_{22}^{+})=-2A,\\
&-2I_{20}^{-'}+\tfrac{24}{5}I_{21}^{-'}-4I_{22}^{-'}-2\lambda_4(I_{20}^{-}-3I_{21}^{-}+2I_{22}^{-})=-2A,\\
&\tfrac{1}{3}I_{20}^{+'}+2I_{21}^{+'}+\tfrac{80}{21}I_{22}^{+'}-\lambda_4(2I_{21}^{+}+5I_{22}^{+})=0,\\
&\tfrac{1}{3}I_{20}^{-'}-2I_{21}^{-'}+\tfrac{80}{21}I_{22}^{-'}-\lambda_4(2I_{21}^{-}-5I_{22}^{-})=0,
\end{aligned} \tag{53}$$

where ξ_i, $i=1,2,3,4$ and λ_i, $i=1,2,3,4$ are given by (34) and (37).

The above set of equations given by (52) and (53) are obtained (when $l_0=2$) by similar process described in Section 2 considering the equation of transfer for $r=1$ and 2, respectively, and

$$\begin{aligned}
I_{1l}^{+}(\tau,\mu) &= A\tau + I_{10}^{+}(\tau)\mu + 3I_{11}^{+}(\tau)\mu P_1(2\mu-1) + 5I_{12}^{+}(\tau)\mu P_2(2\mu-1), \quad 0 \leqslant \mu \leqslant 1,\\
I_{1l}^{-}(\tau,\mu) &= A\tau + I_{10}^{-}(\tau)\mu + 3I_{11}^{-}(\tau)\mu P_1(2\mu+1) + 5I_{12}^{-}(\tau)\mu P_2(2\mu+1), \quad -1 \leqslant \mu \leqslant 0,\\
I_{2l}^{+}(\tau,\mu) &= A\tau + I_{20}^{+}(\tau)\mu + 3I_{21}^{+}(\tau)\mu P_1(2\mu-1) + 5I_{22}^{+}(\tau)\mu P_2(2\mu-1), \quad 0 \leqslant \mu \leqslant 1,\\
I_{2l}^{-}(\tau,\mu) &= A\tau + I_{20}^{-}(\tau)\mu + 3I_{21}^{-}(\tau)\mu P_1(2\mu+1) + 5I_{22}^{-}(\tau)\mu P_2(2\mu+1), \quad -1 \leqslant \mu \leqslant 0.
\end{aligned} \tag{54}$$

Now taking the trial solution given by (22) and (23) and substituting these in (52) and (53) and then equating the coefficient of $e^{-k\tau}$ and constant term we obtain

$$\left(\xi_1+\frac{4k}{3}\right)g_{10,\alpha}+(\xi_1+2k)g_{11,\alpha}+\frac{2k}{3}g_{12,\alpha}+\xi_2h_{10,\alpha}-\xi_2h_{11,\alpha}$$
$$-\xi_3g_{20,\alpha}-\xi_3g_{21,\alpha}+\xi_3h_{20,\alpha}-\xi_3h_{21,\alpha}=0,$$

$$\xi_2g_{10,\alpha}+\xi_2g_{11,\alpha}+\left(\xi_1-\frac{4k}{3}\right)h_{10,\alpha}+(-\xi_1+2k)h_{11,\alpha}-\frac{2k}{3}h_{12,\alpha}$$
$$+\xi_3g_{20,\alpha}+\xi_3g_{21,\alpha}-\xi_3h_{20,\alpha}+\xi_3h_{21,\alpha}=0,$$

$$2(k+\xi_4)g_{10,\alpha}+6\left(\xi_4+\frac{4k}{5}\right)g_{11,\alpha}+4(\xi_4+k)g_{12,\alpha}=0,$$

$$2(-\xi_4+k)h_{10,\alpha}+6\left(\xi_4-\frac{4k}{5}\right)h_{11,\alpha}-4(\xi_4-k)h_{12,\alpha}=0,$$

$$\frac{k}{3}g_{10,\alpha}+2(\xi_4+k)g_{11,\alpha}+5\left(\xi_4+\frac{16k}{21}\right)g_{21,\alpha}=0,$$

$$\frac{k}{3}h_{10,\alpha}+2(\xi_4-k)h_{11,\alpha}+5\left(-\xi_4+\frac{16k}{21}\right)h_{12,\alpha}=0,$$

$$-\lambda_3g_{10,\alpha}-\lambda_3g_{11,\alpha}+\lambda_3h_{10,\alpha}-\lambda_3h_{11,\alpha}+\left(\lambda_1+\frac{4k}{3}\right)g_{20,\alpha}$$
$$+(\lambda_1+2k)g_{21,\alpha}+\frac{2k}{3}g_{22,\alpha}+\lambda_3h_{20,\alpha}-\lambda_3h_{21,\alpha}=0,$$

$$-\lambda_3g_{10,\alpha}+\lambda_3g_{11,\alpha}-\lambda_3h_{10,\alpha}+\lambda_3h_{11,\alpha}+\lambda_2g_{20,\alpha}+\lambda_2g_{21,\alpha}$$
$$+\left(\lambda_1-\frac{4k}{3}\right)h_{20,\alpha}+(-\lambda_1+2k)h_{21,\alpha}-\frac{2k}{3}h_{22,\alpha}=0,$$

$$2(k+\lambda_4)g_{20,\alpha}+6\left(\lambda_4+\frac{4k}{5}\right)g_{21,\alpha}+4(\lambda_4+k)g_{22,\alpha}=0,$$

$$2(-\lambda_4 + k)h_{20,\alpha} + 6\left(\lambda_4 - \frac{4k}{5}\right)h_{21,\alpha} - 4(\lambda_4 - k)h_{22,\alpha} = 0,$$

$$\frac{k}{3}g_{20,\alpha} + 2(\lambda_4 + k)g_{21,\alpha} + 5\left(\lambda_4 + \frac{16k}{21}\right)G_{22,\alpha} = 0,$$

$$\frac{k}{3}h_{20,\alpha} + 2(\lambda_4 - k)h_{21,\alpha} + 5\left(-\lambda_4 + \frac{16k}{21}\right)h_{22,\alpha} = 0 \tag{55}$$

and

$$\xi_1(g_{10,\beta} + g_{11,\beta}) + \xi_2(h_{10,\beta} - h_{11,\beta}) - \xi_3(g_{20,\beta} + g_{21,\beta} - h_{20,\beta} + h_{21,\beta}) = 2 + 4(1 + \varepsilon\eta_1)\frac{a}{A},$$

$$\xi_2(g_{10,\beta} + g_{11,\beta}) + \xi_1(h_{10,\beta} - h_{11,\beta}) + \xi_3(g_{20,\beta} + g_{21,\beta} - h_{20,\beta} + h_{21,\beta}) = 2 - 4(1 + \varepsilon\eta_1)\frac{a}{A},$$

$$2\xi_4 g_{10,\beta} + 6\xi_4 g_{11,\beta} + 4\xi_4 g_{12,\beta} = 2,$$

$$2\xi_4 h_{10,\beta} - 6\xi_4 h_{11,\beta} + 4\xi_4 h_{12,\beta} = 2,$$

$$2\xi_4 g_{11,\beta} + 5\xi_4 g_{12,\beta} = 0,$$

$$2\xi_4 h_{11,\beta} - 5\xi_4 h_{12,\beta} = 0,$$

$$-\lambda_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} + h_{11,\beta}) + \lambda_1(g_{20,\beta} + g_{21,\beta}) + \lambda_2(h_{20,\beta} - h_{21,\beta}) = 2 + 4(1 + \varepsilon\eta_2)\frac{a}{A},$$

$$\lambda_3(g_{10,\beta} + g_{11,\beta} - h_{10,\beta} + h_{11,\beta}) + \lambda_2(g_{20,\beta} + g_{21,\beta}) + \lambda_1(h_{20,\beta} - h_{21,\beta}) = 2 - 4(1 + \varepsilon\eta_2)\frac{a}{A},$$

$$2\lambda_4 g_{20,\beta} + 6\lambda_4 g_{21,\beta} + 4\lambda_4 g_{22,\beta} = 2,$$

$$2\lambda_4 h_{20,\beta} - 6\lambda_4 h_{21,\beta} + 4\lambda_4 h_{22,\beta} = 2,$$

$$2\lambda_4 g_{21,\beta} + 5\lambda_4 g_{22,\beta} = 0,$$

$$2\lambda_4 h_{21,\beta} - 5\lambda_4 h_{22,\beta} = 0, \tag{56}$$

where λ_I and ξ_i, $i = 1, 2, 3, 4$ are same as (34) and (37).

The above Eqs. (55) have a nontrivial solution if the determinant of the coefficient of the constant $g_{il,\alpha}$ and $h_{il,\alpha}$ is zero, i.e.,

$$\Delta(k) = \begin{vmatrix} M_1(k) & M_2(k) \\ M_3(k) & M_4(k) \end{vmatrix} = 0, \tag{57}$$

where

$$M_1(k) = \begin{pmatrix} (\xi_1 + \frac{4k}{3}) & (\xi_1 + 2k) & \frac{2k}{3} & \xi_2 & -\xi_2 & 0 \\ \xi_2 & \xi_2 & 0 & (\xi_1 - \frac{4k}{3}) & (-\xi_1 + 2k) & -\frac{2k}{3} \\ 2(\xi_4 + k) & 6(\xi_4 + \frac{4k}{5}) & 4(\xi_4 + k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\xi_4 + k) & 6(\xi_4 - \frac{4k}{5}) & 4(\xi_4 - k) \\ \frac{k}{3} & 2(\xi_4 + k) & 5(\xi_4 + \frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\xi_4 - k) & 5(-\xi_4 + \frac{16k}{21}) \end{pmatrix}, \tag{58}$$

$$M_2(k) = \begin{pmatrix} -\xi_3 & -\xi_3 & 0 & \xi_3 & -\xi_3 & 0 \\ \xi_3 & \xi_3 & 0 & -\xi_3 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{59}$$

$$M_3(k) = \begin{pmatrix} -\lambda_3 & -\lambda_3 & 0 & \lambda_3 & -\lambda_3 & 0 \\ \lambda_3 & \lambda_3 & 0 & -\lambda_3 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{60}$$

$$M_4(k) = \begin{pmatrix} (\xi_1 + \frac{4k}{3}) & (\xi_1 + 2k) & \frac{2k}{3} & \xi_2 & -\xi_2 & 0 \\ \xi_2 & \xi_2 & 0 & (\xi_1 - \frac{4k}{3}) & (-\xi_1 + 2k) & -\frac{2k}{3} \\ 2(\xi_4 + k) & 6(\xi_4 + \frac{4k}{5}) & 4(\xi_4 + k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(-\xi_4 + k) & 6(\xi_4 - \frac{4k}{5}) & 4(\xi_4 - k) \\ \frac{k}{3} & 2(\xi_4 + k) & 5(\xi_4 + \frac{16k}{21}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{3} & 2(\lambda_4 - k) & 5(-\lambda_4 + \frac{16k}{21}) \end{pmatrix}. \tag{61}$$

Now, we consider the case as previous i.e., $\eta_1 = 1$, $\eta_2 = \frac{1}{2}$, $\varepsilon = 0$ and ξ_i and λ_i are given by (40)

Thus $\Delta(k)$ becomes

$$\Delta(k) = \left(\frac{556288}{3675}k^6 + \frac{500096}{2205}k^5 - \frac{55191488}{11025}k^4 + \frac{2105056}{2205}k^3 + \frac{72577488}{2205}k^2 - \frac{14977456}{21}k - 57344\right) \times \left(\frac{544}{245}k^6 - \frac{1152}{245}k^5 - \frac{1573648}{2205}k^4 + \frac{3486299}{4410}k^3 + \frac{3650659}{490}k^2 - \frac{146499}{56}k - \frac{124497}{8}\right). \quad (62)$$

Now $\Delta(k) = 0$ yields $k = 18,281155, -7.269599, 8.794658, -9.770985.$

We take $k = 8.794658$.

Again using (40) in (56) and using the boundary conditions given by (45) and (55), we obtain

$$g_{10,\alpha} = -0.043688 - 0.052503\,\frac{a}{A},$$

$$g_{11,\alpha} = 0.038890 + 0.019326\,\frac{a}{A},$$

$$g_{12,\alpha} = -0.023374 - 0.003289\,\frac{a}{A},$$

$$g_{20,\alpha} = -0.014174 - 0.028567\,\frac{a}{A},$$

$$g_{21,\alpha} = 0.011657 + 0.011443\,\frac{a}{A},$$

$$g_{22,\alpha} = -0.005643 - 0.002531\,\frac{a}{A},$$

$$h_{10,\alpha} = -0.506274 + 1.650696\,\frac{a}{A},$$

$$h_{11,\alpha} = -0.002852 + 0.750316\,\frac{a}{A},$$

$$h_{12,\alpha} = -0.00114 + 0.300128\,\frac{a}{A},$$

$$h_{20,\alpha} = -0.69455 + 3.262356\,\frac{a}{A},$$

$$h_{21,\alpha} = -0.012674 + 1.482888\,\frac{a}{A},$$

$$h_{22,\alpha} = -0.00507 + 0.0593156\,\frac{a}{A},$$

$$g_{10,\beta} = 0.283206 + 2.448164\,\frac{a}{A},$$

$$g_{11,\beta} = 0.098542 - 1.1128\,\frac{a}{A},$$

$$g_{12,\beta} = -0.039416 + 0.44512\,\frac{a}{A},$$

$$g_{20,\beta} = 0.638784 + 3.262356\,\frac{a}{A},$$

$$g_{21,\beta} = 0.012674 - 1.482892\,\frac{a}{A},$$

$$g_{22,\beta} = -0.00507 + 0.593156\,\frac{a}{A},$$

$$h_{10,\beta} = 0.506274 - 1.650696\,\frac{a}{A},$$

$$h_{11,\beta} = 0.002852 - 0.750316\,\frac{a}{A},$$

$$h_{12,\beta} = 0.00114 - 0.300128\,\frac{a}{A},$$

$$h_{20,\beta} = 0.69455 - 3.262356\,\frac{a}{A},$$

$$h_{21,\beta} = 0.012674 - 1.482888\,\frac{a}{A},$$

$$h_{22,\beta} = 0.00507 - 0.593156\,\frac{a}{A}. \tag{63}$$

As in first approximation (Section 4.3) we find $A = \frac{3}{8}F$ and hence

$$\begin{aligned} B(\tau) &= 2A\tau + \tfrac{1}{4}(I_{10}^{+} - I_{10}^{-} + I_{11}^{+} + I_{11}^{-} + I_{20}^{+} - I_{20}^{-} + I_{21}^{+} + I_{21}^{-}) \\ &= A(2\tau + 0.29449575\mathrm{e}^{-k\tau}) + [-0.038023 + a(1.448669 - 0.68253725\mathrm{e}^{-k\tau})] \\ &= \tfrac{3}{8}F(2\tau + 0.294495575\mathrm{e}^{-8.794658\tau}) + [-0.038023 + a(1.448669 \\ &\quad - 0.68253725\mathrm{e}^{-8.794658\tau})]. \end{aligned} \tag{64}$$

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References

[1] Woolley RvdR, Stibbs DWN. The outer layers of a star. London: Oxford University Press; 1953.
[2] Busbridge IW, Stibbs DWN. Mon Not R Astron Soc 1954;114:2–16.
[3] Karanjai S. Publ Astron Soc Japan 1968;20:173–77.
[4] DasGupta SR, Karanjai S. Astrophys Space Sci 1972;18:246–53.

[5] Karanjai S, Barman SK. Astrophys Space Sci 1981;77:271–78.
[6] Karanjai S. North Bengal University Review 1982;3:57–63.
[7] DasGupta SR. Astrophys Space Sci 1977;50:187.
[8] DasGupta SR. Astrophys Space Sci 1978;56:13.
[9] Karanjai S, Karanjai M. Astrophys Space Sci 1985;115:295–307.
[10] Deb TK, et al. Astrophys Space Sci 1991;178:107–17.
[11] Deb TK, Karanjai S. Astrophys Space Sci 1991;184:57–63.
[12] Wilson SJ, Sen KK. Publ Astron Soc Japan 1963;15:351–67.