Chapter 6

Approximate H-function for anisotropic scattering : Carlstedt and Mullikin's phase function

6.1 Formation of the problem

The Chandrasekhar's H-function[11] is the solution of a non-linear integral equation

$$\frac{1}{H(\mu,\omega)} = 1 - \mu \int_0^1 \frac{\Psi(\mu')H(\mu',\omega)}{\mu+\mu'} d\mu' \tag{6.1}$$

The H-function satisfies the following equations

$$\int_0^1 H(\mu,\omega)\Psi(\mu)d\mu = 1 - \left[1 - 2\int_0^1 \Psi(\mu)d\mu\right]^{\frac{1}{2}} \tag{6.2}$$

$$\left[1 - 2\int_0^1 \Psi(\mu)d\mu\right]^{\frac{1}{2}} \int_0^1 H(\mu,\omega)\Psi(\mu)\mu^2 d\mu + \frac{1}{2}\left[\int_0^1 H(\mu,\omega)\Psi(\mu)\mu d\mu\right]^2$$

$$= \int_0^1 \Psi(\mu)\mu^2 d\mu \tag{6.3}$$

$$\int_0^1 \frac{H(\mu,\omega)\Psi(\mu)}{1-\kappa\mu} d\mu = 1 \tag{6.4}$$

and Ψ is the characteristic function such that

$$\int_0^1 \Psi(\mu)d\mu \leq \frac{1}{2} \tag{6.5}$$

Carlstedt and Mullikin[6] introduced the phase function

$$P(\mu,\mu') = 1 + b_0 P_4(\mu) \tag{6.6}$$

where b_0 is a constant and can have value $1 \le b_0 \le 2$ and for each b_0 $1 + b_0 P_4(\mu)$ is positive function describing scattering without absorption.

The characteristic function corresponding to the phase function given by (6.6) is obtained , following Karanjai & Karanjai[29] as

$$\Psi(\mu) = \frac{1}{2}\left[1 - \frac{b_0}{8}(\frac{35}{3}\mu^2 - 3)P_4(\mu)\right] \tag{6.7}$$

or

$$\Psi(\mu) = A_1 + A_2\mu^2 + A_3\mu^4 + A_4\mu^6 \tag{6.8}$$

where

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2}(1 + \tfrac{9}{64}b_0) \\ A_2 &= -\tfrac{125}{128}b_0 \\ A_3 &= \tfrac{35\times13}{128}b_0 \\ A_4 &= -\tfrac{35\times35}{128\times3}b_0 \end{aligned}\right\} \tag{6.9}$$

The term κ in equation (6.4) is the solution of the trancedental equation given by Chandrasekhar[11]

$$1 = 2\int_0^1 \frac{\Psi(\mu)d\mu}{1-\kappa^2\mu^2} \tag{6.10}$$

using the form of $\Psi(\mu)$ one can from equation (6.8) in equation (6.10) have

$$\begin{aligned} 1 = &-\frac{2}{\kappa^2}\left\{A_2 + \frac{A_3}{3}\right\} - \frac{2}{\kappa^4}\left\{A_3 + \frac{A_4}{3}\right\} - \frac{A_4}{\kappa^5} - 2\frac{A_4}{\kappa^6} \\ &+ \left\{A_1 + \frac{A_2}{\kappa^2} + \frac{A_3}{\kappa^4} + \frac{A_4}{\kappa^6}\right\}\frac{1}{\kappa} ln\left\{\frac{1+\kappa}{1-\kappa}\right\} \end{aligned} \tag{6.11}$$

where A_1, A_2, A_3 and A_4 are given by (6.9).

6.1.1 Case I

Consider the approximate form of the H-function as

$$H(\mu,\omega) = 1 + a\mu + b\mu^2 \tag{6.12}$$

where a and b are function of ω.

Substituting for $H(\mu,\omega)$ and $\Psi(\mu)$ from equation (6.12) and (6.8) into equations (6.2), (6.3) we have

$$aB_1 + bB_2 = 1 \tag{6.13}$$

where

$$\left.\begin{aligned} B_1 &= \tfrac{1}{2}\left(1 - \tfrac{47b_0}{768}\right) \\ B_2 &= \tfrac{1}{3}\left(1 - \tfrac{b_0}{9}\right) \end{aligned}\right\} \tag{6.14}$$

and

$$b^2c_1 + bc_2c_3 = 0 \tag{6.15}$$

where

$$\left.\begin{aligned} c_1 &= B_3 - B_2^2 \\ c_2 &= 2(B_1^2 + B_2)^2(B_2^2 + B_1) \\ c_3 &= (B_1^2 + B_2)^2 - 4B_1^3 \\ B_3 &= 2\left(\tfrac{1}{8} - \tfrac{61b_0}{3072}\right) \end{aligned}\right\} \tag{6.16}$$

Now equation (6.13) and (6.16) gives a and b explicitly.

Second Approximation: Substituting for $H(\mu,\omega)$ from equation (6.12) and $\Psi(\mu)$ from equation (6.7) into (6.1), we obtain the second approximation as

$$\frac{1}{H(\mu,\omega)} = 1 - \mu(\xi_0 + a\xi_1 + b\xi_2) \tag{6.17}$$

where

$$\left.\begin{aligned} \xi_0 &= A_1S_0 + A_2S_2 + A_3S_4 + A_4S_6 \\ \xi_1 &= A_1S_1 + A_2S_3 + A_3S_5 + A_4S_7 \\ \xi_2 &= A_1S_2 + A_2S_4 + A_3S_6 + A_4S_8 \end{aligned}\right\} \tag{6.18}$$

and

$$S_n = \int_0^1 \frac{\mu'^n}{\mu + \mu'} d\mu \tag{6.19}$$

with a and b given by (6.13) and (6.16)

6.1.2 Case II

Now we consider the approximate form of the H-function as

$$H(\mu, \omega) = 1 + \frac{a\mu + b\mu^2}{A + 2\mu} \tag{6.20}$$

where $A = \sqrt{1 - \omega}$.

Substituting for $H(\mu, \omega)$ from equation (6.20) and $\Psi(\mu)$ from (6.8) into equation (6.2) and (6.3) we have

$$a = \frac{1 - 2bQ}{2P} \tag{6.21}$$

where

$$\left.\begin{aligned} &P = A_1I_1 + A_2I_3 + A_3I_5 + A_4I_7 \\ &Q = A_1I_2 + A_2I_4 + A_3I_6 + A_4I_8 \\ &I_n = \textstyle\int_0^1 \frac{\mu^n}{A+2\mu} d\mu \\ &\left[\tfrac{1}{2}B_1 + aQ + bR\right]^2 = B_2 \end{aligned}\right\} \tag{6.22}$$

where B_1 and B_2 are given by (6.14) and

$$R = A_1I_3 + A_2I_5 + A_3I_7 + A_4I_9 \tag{6.23}$$

Now one can get

$$b^2T^2 + 2bST + S^2 - 2B_2 = 0 \tag{6.24}$$

where

$$\left.\begin{aligned} S &= \tfrac{1}{2}B_1 + \tfrac{Q}{2P} \\ T &= R - \tfrac{Q^2}{P} \end{aligned}\right\} \tag{6.25}$$

Now equation (6.21) and (6.24) gives a , b explicitly.

Second Approximation: Substituting for $H(\mu,\omega)$ from equation (6.20) and $\Psi(\mu)$ from equation (6.8) into (6.1), one obtains the second approximation as

$$\frac{1}{H(\mu,\omega)} = 1 - \mu(\xi_0 + a\eta_1 + b\eta_2) \tag{6.26}$$

where

$$\left.\begin{aligned} \eta_1 &= A_1T_1 + A_2T_3 + A_3T_5 + A_4T_7 \\ \eta_2 &= A_1T_2 + A_2T_4 + A_3T_6 + A_4T_8 \end{aligned}\right\} \tag{6.27}$$

and

$$T_n = \int_0^1 \frac{\mu'^n}{(\mu+\mu')(A+2\mu')} d\mu' \tag{6.28}$$

with ξ_0 , a, b and A have been described before.

6.1.3 Case III

Next considering the approximate form of H-function as

$$H(\mu,\omega) = 1 + \frac{a\mu + b\mu^2}{1+\kappa\mu} \tag{6.29}$$

where κ is given by equation (6.11).

Substituting for $H(\mu,\omega)$ from equation (6.29) and $\Psi(\mu)$ from equation (6.8) into equation (6.2) and (6.4) we have

$$a = \frac{1-2bP_2}{2P_1} \tag{6.30}$$

where

$$\left.\begin{aligned} P_1 &= A_1J_1 + A_2J_3 + A_3J_5 + A_4J_7 \\ P_2 &= A_1J_2 + A_2J_4 + A_3J_6 + A_4J_8 \end{aligned}\right\} \tag{6.31}$$

and

$$J_n = \int_0^1 \frac{\mu^n}{1+\kappa\mu} d\mu \tag{6.32}$$

and

$$a\overline{P}_1 + b\overline{P}_2 = \overline{P}_3 \tag{6.33}$$

where

$$\left.\begin{aligned} \overline{P}_1 &= A_1L_1 + A_2L_3 + A_3L_5 + A_4L_7 \\ \overline{P}_2 &= A_1L_2 + A_2L_4 + A_3L_6 + A_4L_8 \\ \overline{P}_3 &= 1 - (A_1L_0 + A_2L_2 + A_3L_4 + A_4L_6) + \kappa(A_1L_1 + A_2L_3 + A_3L_5 + A_4L_7) \end{aligned}\right\} \tag{6.34}$$

and

$$L_n = \int_0^1 \frac{\mu^n}{(1-\kappa\mu)(1+\kappa\mu)} d\mu \tag{6.35}$$

Thus we get

$$b = \frac{P_1\overline{P}_3 - \overline{P}_1}{P_1\overline{P}_2 - P_2\overline{P}_1} \tag{6.36}$$

Thus equation (6.30) and equation (6.36) gives a, b explicitly.

Second Approximation: Substituting for $H(\mu,\omega)$ from equation (6.29) and $\Psi(\mu)$ from equation (6.8) into (6.1) one obtains second approximation as

$$\frac{1}{H(\mu,\omega)} = 1 - \mu(\xi_0 + aU_1 + bU_2) \tag{6.37}$$

where

$$\left.\begin{aligned} U_1 &= A_1f_1 + A_2f_3 + A_3f_5 + A_4f_7 \\ U_2 &= A_1f_2 + A_2f_4 + A_3f_6 + A_4f_8 \end{aligned}\right\} \tag{6.38}$$

where

$$f_n = \int_0^1 \frac{\mu'^n}{(\mu+\mu')(1+\kappa\mu')} d\mu' \tag{6.39}$$

with ξ_0, a , b and κ are given by (6.18),(6.30),(6.33)and(6.11)

6.1.4 Case IV

Consider the approximate form of the H-function as

$$H(\mu,\omega) = 1 + a\mu + b\mu^2 + c\mu^3 \tag{6.40}$$

where a and b are function of ω. Substituting for $H(\mu,\omega)$ and $\Psi(\mu)$ from equation (6.40) and (6.8) into equations (6.2), (6.3) and (6.4) we have

$$aB_1 + bB_2 + cB_3 = 1 \tag{6.41}$$

where

$$\left.\begin{aligned} B_1 &= \tfrac{1}{2}\left(1 - \tfrac{47b_0}{768}\right) \\ B_2 &= \tfrac{1}{3}\left(1 - \tfrac{b_0}{9}\right) \\ B_3 &= \tfrac{1}{4}\left(1 - \tfrac{61b_0}{384}\right) \end{aligned}\right\} \tag{6.42}$$

and

$$[B_1 + aB_2 + bB_3 + cB_4]^2 = 8B_2 \tag{6.43}$$

where

$$b_4 = \frac{1}{5}\left(1 - \frac{2851b_0}{88704}\right) \tag{6.44}$$

and

$$aR_1 + bR_2 + cR_3 = R_4 \tag{6.45}$$

where

$$\left.\begin{aligned} R_1 &= A_1Z_1 + A_2Z_3 + A_3Z_5 + A_4Z_7 \\ R_2 &= A_1Z_2 + A_2Z_4 + A_3Z_6 + A_4Z_8 \\ R_3 &= A_1Z_3 + A_2Z_5 + A_3Z_7 + A_4Z_9 \\ R_4 &= 1 - (A_1Z_0 + A_2Z_2 + A_3Z_4 + A_4Z_6) \end{aligned}\right\} \tag{6.46}$$

and

$$Z_n = \int_0^1 \frac{\mu^n}{1-\kappa\mu} d\mu \tag{6.47}$$

from eq(6.41) one have

$$\left.\begin{aligned} a &= K_1 + cK_2 \\ b &= K_3 + cK_4 \end{aligned}\right\} \quad (6.48)$$

where

$$\left.\begin{aligned} K_1 &= \tfrac{R_2 - B_2R_4}{B_1R_2 - B_2R_1} \\ K_2 &= \tfrac{R_2R_3 - R_3R_2}{B_1R_2 - B_2R_1} \\ K_3 &= \tfrac{1-K_1}{B_2} \\ K_4 &= \tfrac{B_3 + K_2B_1}{B_2} \end{aligned}\right\} \quad (6.49)$$

putting (6.42) in (6.43) one have

$$c^2Q_2^2 + cQ_1Q_2 + Q_1^2 - 8B_2 = 0 \quad (6.50)$$

where

$$\left.\begin{aligned} Q_1 &= B_1 + K_1B_2 + K_3B_3 \\ Q_2 &= K_2B_2 + K_4B_3 + B_4 \end{aligned}\right\} \quad (6.51)$$

Thus equations (6.48) and (6.50) gives , a, b and c explicitly.

Second Approximation: Substituting for $H(\mu,\omega)$ from equation (6.40) and $\Psi(\mu)$ from equation (6.8) into (6.1) one obtains second approximation as

$$\frac{1}{H(\mu,\omega)} = 1 - \mu(\xi_0 + a\xi_1 + b\xi_2 + c\xi_3) \quad (6.52)$$

where

$$\left.\begin{aligned} \xi_0 &= A_1S_0 + A_2S_2 + A_3S_4 + A_4S_6 \\ \xi_1 &= A_1S_1 + A_2S_3 + A_3S_5 + A_4S_7 \\ \xi_2 &= A_1S_2 + A_2S_4 + A_3S_6 + A_4S_8 \\ \xi_3 &= A_1S_3 + A_2S_5 + A_3S_7 + A_4S_9 \end{aligned}\right\} \quad (6.53)$$

where

$$S_n = \int_0^1 \frac{\mu'^n}{\mu + \mu'} d\mu' \quad (6.54)$$

with a , b and c are given by (6.48) and (6.50).

6.1.5 Case V

Consider the approximate form of the H-function as

$$H(\mu,\omega) = 1 + \frac{a\mu + b\mu^2 + c\mu^3}{A + 2\mu} \tag{6.55}$$

where $A = \sqrt{1-\omega}$. Substituting for $H(\mu,\omega)$ and $\Psi(\mu)$ from equation (6.55) and (6.8) into equations (6.2), (6.3) and (6.4) we have

$$aP + bQ + cR = \frac{1}{2} \tag{6.56}$$

where

$$\left.\begin{aligned} P &= A_1I_1 + A_2I_3 + A_3I_5 + A_4I_7 \\ Q &= A_1I_2 + A_2I_4 + A_3I_6 + A_4I_8 \\ R &= A_1I_3 + A_2I_5 + A_3I_7 + A_4I_9 \end{aligned}\right\} \tag{6.57}$$

$$I_n = \int_0^1 \frac{\mu^n}{A + 2\mu} d\mu \tag{6.58}$$

and

$$\left[\frac{1}{2}B_1 + aQ + bR + cS\right]^2 = B_2 \tag{6.59}$$

where

$$S = A_1I_4 + A_2I_6 + A_3I_8 + A_4I_10 \tag{6.60}$$

and B_1, B_2, Q as given above and

$$aG_1 + bG_2 + cG_3 = R_4 \tag{6.61}$$

where

$$\left.\begin{aligned} G_1 &= A_1V_1 + A_2V_3 + A_3V_5 + A_4V_7 \\ G_2 &= A_1V_2 + A_2V_4 + A_3V_6 + A_4V_8 \\ G_3 &= A_1V_3 + A_2V_5 + A_3V_7 + A_4V_9 \end{aligned}\right\} \tag{6.62}$$

and

$$V_n = \int_0^1 \frac{\mu^n}{(A+2\mu)(1-\kappa\mu)} d\mu \tag{6.63}$$

from equation (6.56) and (6.61) we have

$$\left.\begin{aligned} a &= K_5 + cK_6 \\ b &= K_7 + cK_8 \end{aligned}\right\} \tag{6.64}$$

where

$$\left.\begin{aligned} K_5 &= \tfrac{\frac{1}{2}G_2 - QR_4}{PG_2 - QG_1} \\ K_6 &= \tfrac{QG_3 - RG_2}{PG_2 - QG_1} \\ K_7 &= \tfrac{\frac{1}{2} - PK_5}{Q} \\ K_8 &= -\tfrac{PK_6 + R}{Q} \end{aligned}\right\} \tag{6.65}$$

putting these values of a, b in (6.61) one have

$$c^2 Q_4^2 + 2cQ_3Q_4 + Q_3^2 - B_2 = 0 \tag{6.66}$$

where

$$\left.\begin{aligned} Q_3 &= \tfrac{B_1}{2} + K_5Q + K_7R \\ Q_4 &= K_6Q + K_8R + S \end{aligned}\right\} \tag{6.67}$$

Thus equation (6.64) and (6.66) gives a , b and c explicitly.

Second Approximation: Substituting for $H(\mu, \omega)$ from equation (6.55) and $\Psi(\mu)$ from equation (6.8) into (6.1) one obtains second approximation as

$$\frac{1}{H(\mu,\omega)} = 1 - \mu(\xi_0 + a\eta_1 + b\eta_2 + c\eta_3) \tag{6.68}$$

where

$$\left.\begin{aligned} \eta_1 &= A_1T_1 + A_2T_3 + A_3T_5 + A_4T_7 \\ \eta_2 &= A_1T_2 + A_2T_4 + A_3T_6 + A_4T_8 \\ \eta_3 &= A_1T_3 + A_2T_5 + A_3T_7 + A_4T_9 \end{aligned}\right\} \tag{6.69}$$

where

$$T_n = \int_0^1 \frac{\mu'^n}{(\mu+\mu')(A+2\mu')} d\mu' \tag{6.70}$$

with ξ_0,a , b and c are given by (6.53),(6.64) and (6.66).

6.1.6 Case VI

Next considering the approximate form of H-function as

$$H(\mu,\omega) = 1 + \frac{a\mu + b\mu^2 + c\mu^3}{1+\kappa\mu} \tag{6.71}$$

where κ is given by equation (6.11).

Substituting for $H(\mu,\omega)$ from equation (6.71) and $\Psi(\mu)$ from equation (6.8) into equation (6.2), (6.3) and (6.4) one have

$$aP_1 + bP_2 + cP_3 = \frac{1}{2} \tag{6.72}$$

where

$$\left.\begin{aligned} P_1 &= A_1J_1 + A_2J_3 + A_3J_5 + A_4J_7 \\ P_2 &= A_1J_2 + A_2J_4 + A_3J_6 + A_4J_8 \\ P_3 &= A_1J_3 + A_2J_5 + A_3J_7 + A_4J_9 \end{aligned}\right\} \tag{6.73}$$

$$J_n = \int_0^1 \frac{\mu^n}{1+\kappa\mu} d\mu \tag{6.74}$$

and

$$\left[\frac{1}{2}B_1 + aP_2 + bP_3 + cP_4\right]^2 = B_2 \tag{6.75}$$

where

$$P_4 = A_1I_4 + A_2I_6 + A_3I_8 + A_4I_10 \tag{6.76}$$

and

$$a\overline{P_1} + b\overline{P_2} + c\overline{P_3} = \overline{P_4} \tag{6.77}$$

where

$$\left.\begin{aligned}\overline{P_1} &= A_1L_1 + A_2L_3 + A_3L_5 + A_4L_7\\ \overline{P_2} &= A_1L_2 + A_2L_4 + A_3L_6 + A_4L_8\\ \overline{P_3} &= A_1L_3 + A_2L_5 + A_3L_7 + A_4L_9\\ \overline{P_4} &= 1 - (A_1L_0 + A_2L_2 + A_3L_4 + A_4L_6) - \kappa\overline{P_1}\end{aligned}\right\} \quad (6.78)$$

from equation (6.72) and (6.77) one have

$$\left.\begin{aligned}a &= K_9 + cK_{10}\\ b &= K_11 + cK_{12}\end{aligned}\right\} \quad (6.79)$$

where

$$\left.\begin{aligned}K_9 &= \tfrac{\frac{1}{2}\overline{P_2} - \overline{P_3}P_2}{P_1\overline{P_2} - \overline{P_1}P_2}\\ K_{10} &= \tfrac{P_2\overline{P_4} - \overline{P_3}P_2}{P_1\overline{P_2} - \overline{P_1}P_2}\\ K_{11} &= \tfrac{\frac{1}{2} - K_9}{P_2}\\ K_{12} &= -\tfrac{P_3 + K_{10}}{P_2}\end{aligned}\right\} \quad (6.80)$$

putting these values of a, b in (6.75) one have

$$c^2Q_6^2 + 2cQ_6Q_5 + Q_5^2 - B_2 = 0 \quad (6.81)$$

where

$$\left.\begin{aligned}Q_5 &= \tfrac{B_1}{2} + K_9P_2 + K_{11}P_3\\ Q_6 &= K_{10}P_2 + K_{12}P_3 + P_4\end{aligned}\right\} \quad (6.82)$$

Thus equation (6.79) and (6.82) gives a, b and c explicitly.

Second Approximation: Substituting for $H(\mu,\omega)$ from equation (6.71) and $\Psi(\mu)$ from equation (6.8) into (6.1) one obtains second approximation as

$$\frac{1}{H(\mu,\omega)} = 1 - \mu(\xi_0 + aU_1 + bU_2 + cU_3) \quad (6.83)$$

where

$$\left.\begin{aligned} U_1 &= A_1 f_1 + A_2 f_3 + A_3 f_5 + A_4 f_7 \\ U_2 &= A_1 f_2 + A_2 f_4 + A_3 f_6 + A_4 f_8 \\ U_3 &= A_1 f_3 + A_2 f_5 + A_3 f_7 + A_4 f_9 \end{aligned}\right\} \tag{6.84}$$

where

$$f_n = \int_0^1 \frac{\mu'^n}{(\mu + \mu')(1 + \kappa\mu')} d\mu' \tag{6.85}$$

with ξ_0, a , b ,c and κ are given by (6.18),(6.79),(6.82)and(6.11).