Chapter 3

SCALAR FIELD DRESSING OF BLACK HOLES IN THE BERGMANN-NORDTVEDT-WAGONER THEORY

3.1. Introduction

An integral part of black hole physics is the so-called no scalar hair theorem [NSHT] which deals with the question of whether the exterior of a spherically symmetric black hole admits nontrivial scalar field(s) [1-8]. Several physical scenarios warrant such considerations. For examples, the scalar field couples to gravitation in a natural way in superstring theories [9, 10]. In the Brans-Dicke theory, such a coupling is necessary to incorporate Mach's principle [11]. In general, the coupling could be of minimal, non-minimal or conformal type. Hence, NSHT is of direct relevance to black hole solutions of such theories.

Recently, Saa [12] has proposed a new NSHT which rules out a larger class of nonminimally coupled theories admitting nontrivial, finite scalar field dressing of asymptotically flat, static, spherically symmetric black hole. The basic procedure in this theorem is to take the solution of the Einstein-minimally coupled scalar field (EMCSF) theory as the seed solution (e.g. Buchdahl solution [13]) and generate solutions of the Einstein-nonminimally coupled scalar field (ENCSF) theory. The next step is to look for the black hole solutions in the latter theory which are then found to correspond to only constant scalar fields. One then says that the NSHT is satisfied in ENCSF. A specific example of this is the Brans-Dicke theory with the coupling parameter ω = constant.

More recently, Sen and Banerjee [14] have examined, in the light of Saa's theorem, a few examples in the general Bergmann-Nordtvedt-Wagoner (BNW) theory [15-17] for which the coupling parameter ω is no longer a constant but a function of the scalar field. They showed that the only black hole solution in those examples correspond again to only constant scalar fields. There is admittedly a caveat here: Saa's theorem does not cover the Einstein-conformally coupled scalar field (ECCSF) due to the fact that it does not satisfy one of the crucial conditions, stated later, of the theorem. Consequently, a black hole solution dressed by a CCSF $\varphi \neq$ constant is permitted to exist in violation of NSHT. This is actually the case with the well - known Bekenstein solution [18, 19].

On the other hand, despite the failure of Saa's theorem in the case of ECCSF theory, its solutions can always be generated from EMCSF theory by special types of conformal maps discovered by Xanthopoulos and Dialynas [20]. They have shown that, among such solutions in the ECCSF theory in 4- dimensions, the only black hole solution with nontrivial scalar hair is the Bekenstein solution. This result constitutes a major step towards proving the uniqueness of the Bekenstein black hole.

Our aim in this chapter is to approach the question of uniqueness from a more general viewpoint provided by the massless nonminimally coupled BNW theory. The first step in this direction is to portray ECCSF theory as a BNW theory with a specific form of coupling function. The next step is to generate a new class of solutions within the BNW theory for a more general class of coupling functions. This is achieved through a conformal map of the metric tensor and a redefinition of the scalar field. An examination of the generated class of solutions reveals that the only black hole solution is again the Bekenstein one, confirming the results of Ref. [20].

3.2. The action, field equations and the Bekenstein solution

The most general matter-free action of the ENCSF theory including an electromagnetic field $F_{\mu\nu}$ is of the form

$$S[g,\varphi]=\int\left[f(\varphi)R-h(\varphi)g^{\mu\nu}\nabla_{\mu}\varphi\nabla_{\nu}\varphi-F_{\mu\nu}F^{\mu\nu}\right]\sqrt{-g}\,d^{4}x \tag{1}$$

$$h(\varphi)=\omega(\varphi)/\varphi \tag{2}$$

where $f(\varphi)$ and $h(\varphi)$ are arbitrary functions, ∇_μ denotes covariant derivative with respect to $g_{\mu\nu}$ and R is the Ricci scalar. Saa's procedure requires $f>0$ and $h>0$ for all φ . Brans-Dicke theory corresponds to $f(\varphi)=\varphi$ and $h(\varphi)=\omega/\varphi$, ω = constant. The BNW theory corresponds to $f(\varphi)=\varphi$ and $\omega=\omega(\varphi)$, while the EMCSF theory has the values $f(\varphi)=1$ and $h(\varphi)=1$. The ECCSF theory corresponds to the choices $f(\varphi)=1-\frac{1}{6}\varphi^{2}$, $h(\varphi)=1$, so that the condition $f>0$ is not satisfied for all φ ,and thus escapes Saa's theorem, as mentioned before. The resulting ECCSF equations, in suitable units, are

$$\begin{aligned}G_{\mu\nu}&=T^{(\varphi)}_{\mu\nu}+T^{(em)}_{\mu\nu}\\&=(1-\frac{1}{6}\varphi^{2})^{-1}[\nabla_{\mu}\varphi\nabla_{\nu}\varphi-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_{\alpha}\varphi\nabla_{\beta}\varphi-\frac{1}{6}\nabla_{\mu}\nabla_{\nu}(\varphi^{2})\\&+\frac{1}{6}g_{\mu\nu}g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}(\varphi^{2})-g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}+\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}]\end{aligned} \tag{3}$$

$$g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\varphi-\frac{1}{6}R\varphi=0, \tag{4}$$

$$\nabla_{\mu}F^{\mu\nu}=0, \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor, $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$. The Bekenstein black hole solution following from Eqs.(3)-(5) with a nontrivial φ is:

$$\begin{aligned}ds^{2}&=g_{\mu\nu}dx^{\mu}dx^{\nu}\\&=-(1-\frac{M}{r})^{2}dt^{2}+(1-\frac{M}{r})^{-2}dr^{2}+r^{2}d\theta^{2}+r^{2}\sin^{2}\theta d\phi^{2},\end{aligned} \tag{6}$$

$$\phi = \frac{q}{r-M}, \quad F_{\mu\nu} = er^{-2}\left(\partial^r_\mu \partial^t_\nu - \partial^t_\mu \partial^r_\nu\right),$$

$$M^2 = e^2 + \frac{1}{3}q^2 \tag{7}$$

where q and e are the scalar and electric charges respectively. The metric has the same form as that of extreme Reissner-Nordstrom. The divergence of φ at r=M has been shown to be physically innocuous in the sense that test objects interacting with the scalar field reach the horizon in finite proper time [18,19]. Note further that, both $T_{\mu\nu}^{(\varphi)}$ and $T_{\mu\nu}^{(em)}$ are traceless and hence R=0. Since Maxwell's equations are already conformally invariant, we need not repeatedly include $F_{\mu\nu}F^{\mu\nu}$ in the actions that will follow.

3.3. Reformulation in the BNW theory

Now define

$$\chi = 1 - \frac{1}{6}\varphi^2, \omega(\chi) = \left(\frac{3}{2}\right)\frac{\chi}{1-\chi} \tag{8}$$

Then the action (1) for the ECCSF can be rewritten in the form of that of BNW theory:

$$S(g,\chi) = \int [\chi R - \chi^{-1}\omega(\chi) g^{\alpha\beta}\nabla_\alpha \chi \nabla_\beta \chi]\sqrt{-g}\, d^4x \tag{9}$$

The resulting field equations are

$$G_{\mu\nu} = T_{\mu\nu}(\chi) = \chi^{-1}(\nabla_\mu \nabla_\nu \chi - g_{\mu\nu}\Box\chi) + \chi^{-2}\omega(\chi)\left[\nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}\nabla_\alpha \chi \nabla_\beta \chi\right] \tag{10}$$

$$\Box\chi = -[2\omega(\chi)+3]^{-1} \frac{d\omega}{d\chi} g^{\alpha\beta}\nabla_\alpha \chi \nabla_\beta \chi \tag{11}$$

where $\Box \equiv g^{\alpha\beta}\nabla_\alpha \nabla_\beta$ .The solution consists of the same ds^2 as in (6) but now with the scalar field being redefined as χ .The scalar curvature R follows from Eqs.(10) and (11);

$$R = \left[\omega\chi^{-2} - 3(2\omega+3)^{-1}\chi^{-1}\frac{d\omega}{d\chi} \right] g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi \tag{12}$$

which vanishes identically for $\omega(\chi)$ defined above. This is to be expected as the action (9) is essentially the same as the ECCSF action, only redefined via Eqs.(8) as a BNW action.

3.4. The conformal transformation

Consider the BNW action (9) with $\omega(\chi)$ as above. Apply the transformation

$$\tilde{g}_{\mu\nu} = \Omega^2(\chi) g_{\mu\nu} \tag{13}$$

where $\Omega(\chi)$ is a nonvanishing smooth function. Then

$$(-\tilde{g})^{\frac{1}{2}} = \Omega^4(\chi)(-g)^{\frac{1}{2}} \tag{14}$$

$$\tilde{R} = \Omega^{-2}[R + 6\Omega^{-1}\Box\Omega] \tag{15}$$

so that the action (9) becomes

$$S[\tilde{g},\chi] = \int [\Omega^{-2}\chi\tilde{R} - 6\chi\Omega^{-5}\Box\Omega + \chi^{-1}\omega(\chi)\Omega^{-2}\tilde{g}^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi](-\tilde{g})^{1/2} d^4x \tag{16}$$

Taking

$$\Omega = \chi^{\xi}, \quad \sigma = \chi^{1-2\xi}, \tag{17}$$

Where ξ is an arbitrary constant parameter, we find

$$6\,\chi^{1-5\xi}\Box\Omega = (1-2\xi)^{-2}[6\xi(\xi-1) - 2\xi\omega(\chi)]\sigma^{-2}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\sigma\tilde{\nabla}_\beta\sigma \tag{18}$$

$$\chi^{-1}\omega(\chi)\Omega^{-2}\tilde{g}^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi = (1-2\xi)^{-2}\omega(\chi)\sigma^{-1}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\sigma\tilde{\nabla}_\beta\sigma. \tag{19}$$

Putting (18) and (19) in (16), we get

$$S[\tilde{g},\sigma] = \int [\sigma\tilde{R} - \sigma^{-1}\tilde{\omega}(\sigma)\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\sigma\tilde{\nabla}_\beta\sigma](-\tilde{g})^{1/2} d^4x \tag{20}$$

which has again the form of a BNW action with $\tilde{\omega}(\sigma)$ given explicitly by

$$\tilde{\omega}(\sigma)=\frac{1}{2}(1-2\xi)^{-2}[1-\sigma^{1/(1-2\xi)}]^{-1}.[3(1+2\xi)\sigma^{1/(1-2\xi)}-12\xi(\xi-1)(1-\sigma^{1/(1-2\xi)})] \quad (21)$$

where $\xi \neq 1/2$. Clearly, the resulting field equations will be the same as Eqs.(10) and (11) but with $g_{\mu\nu}$ and $\omega(\chi)$ replaced by $\tilde{g}_{\mu\nu}$ and $\tilde{\omega}(\sigma)$ respectively. They will be identical only if $\xi = 0$. For different numerical values of ξ, rational or irrational, we obtain different functional forms for $\tilde{\omega}(\sigma)$. The standard method of solving Eqs.(10) and (11) for the obtained class of coupling functions $\tilde{\omega}(\sigma)$ would be quite tedious if not intractable.

3.5. The new classes of solutions

The present method enables us to immediately write down the solutions if we use

$$\tilde{g}_{\alpha\beta}=\chi^{2\xi}g_{\alpha\beta}, \sigma=\chi^{1-2\xi}, \quad (22)$$

where χ is computed from Eqs. (7) and (8) while $g_{\alpha\beta}$ is provided by the metric (6). Thus, typical solutions $[\tilde{g}_{\mu\nu},\sigma]$, parametrized by ξ, are:

$$\tilde{g}_{00}=6^{-2\xi}[6(r-M)^2-q^2]^{2\xi}r^{-2}(r-M)^{2(1-2\xi)} \quad (23)$$

$$\tilde{g}_{11}=6^{-2\xi}[6(r-M)^2-q^2]^{2\xi}r^{2}(r-M)^{-2(1+2\xi)} \quad (24)$$

$$\tilde{g}_{22}=6^{-2\xi}[6(r-M)^2-q^2]^{2\xi}r^{2}(r-M)^{-4\xi} \quad (25)$$

$$\tilde{g}_{33}=\tilde{g}_{22}\sin^2\theta \quad (26)$$

$$\sigma=6^{2\xi-1}[6(r-M)^2-q^2]^{1-2\xi}(r-M)^{-2(1-2\xi)} \quad (27)$$

The scalar curvature $\tilde{R}$ can be calculated using Eqs. (15) and (18) and the fact that $R=0$:

$$\tilde{R}=6\chi^{-2(1+\chi)}[\xi(\xi-1)-\frac{\xi\omega}{3}]g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi. \quad (28)$$

The solutions (23)-(26) are asymptotically flat and $\sigma \to 1$ as $r \to \infty$. For $\xi < 0$, all components of the metric tensor $\tilde{g}_{\alpha\beta}$ blow off at $r=r_0=(M+6^{-1/2}q) > M$, and for $\xi > 0$, they vanish at $r = r_0$. In either case, the solutions do not represent black holes since the surface $r = r_0$ is not a regular one. If ξ=0, then R = 0 and only in this case, one retrieves the scalar hair Bekenstein black hole solution .

Two important points should be noted here. First, using the transformation $[\tilde{g}_{\alpha\beta,}\sigma] \rightarrow [\bar{g}_{\alpha\beta,}\bar{\sigma}]$ given by

$$\bar{g}_{\alpha\beta} = \sigma\tilde{g}_{\alpha\beta}, \bar{\sigma} = \int \left|\tilde{\omega}(\sigma) + \frac{3}{2}\right|^{1/2} \frac{d\sigma}{\sigma}, \tag{29}$$

it is possible to reduce the non-Einsteinian BNW action (20) into the Einsteinian form given by

$$S_{Ein} = \int [\bar{R} - \bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\sigma}\bar{\nabla}_\beta\bar{\sigma}]\sqrt{-\bar{g}}d^4x. \tag{30}$$

This is just the action of the EMCSF theory and it no longer contains the coupling function. The field equations are

$$\bar{G}_{\alpha\beta} = \bar{T}_{\alpha\beta}(\bar{\sigma}) = \bar{\nabla}_\alpha\bar{\sigma}\bar{\nabla}_\beta\bar{\sigma} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{\nabla}_\lambda\bar{\sigma}\bar{\nabla}^\lambda\bar{\sigma}, \tag{31}$$

$$\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{\sigma} = 0, \tag{32}$$

where $\bar{\nabla}_\alpha$ denotes covariant differentiation with respect to $\bar{g}_{\alpha\beta}$. Physically interpretations of the theories resulting from the two actions (20) and (30) differ widely (21). For instance, $\tilde{T}_{\mu\nu}(\sigma)$ following from the action (20) is not necessary positive definite (signaling a violation of energy conditions) while $\bar{T}_{\mu\nu}(\bar{\sigma})$, following from the action (30), is. In the former case, usual singularity theorems do not apply and hence nontrivial scalar field(s) in the spherical black hole exterior is allowed to exist. In the latter case, by a theorem of Bekenstein [3] it is disallowed.

Second, spherically symmetric solutions $[\bar{g}_{\alpha\beta}, \bar{\sigma}]$ of Eqs. (31) and (32) are indeed well known [13,16,22]. Using these known solutions, it is now in principle possible to generate BNW solutions $[\tilde{g}_{\alpha\beta}, \sigma]$ via the prescription (29). However, to do this, one needs to plug in some choice of $\tilde{\omega}(\sigma)$ by hand, integrate, and see if σ could be inverted explicitly in terms of the seed function $\bar{\sigma}$. If not, the procedure fails. These restrictions of integrability and invertibility severely narrow down the choices of $\tilde{\omega}(\sigma)$ to some suitable functions only (see, for example,. Ref.[14]). On the other hand, the present method has the advantage that it does not depend upon either of the restrictions while Eq.(21) offers a fairly wide variety of coupling functions $\tilde{\omega}(\sigma)$ obtained by simply varying the parameter ξ.

3.6. Conclusions and remarks

Let us now summarize what we have achieved in the above: (1) The question of uniqueness of the Bekenstein black hole remains partially unresolved due to the fact that, for arbitrary choices of coupling function $\tilde{\omega}(\sigma)$, exact solutions of BNW theory are not guaranteed. In practice, solutions are found using Saa's method only on a case by case basis as was actually done by Sen and Banerjee [14] for specific choices of $\tilde{\omega}(\sigma)$, namely, those of Schwinger's [23] and Barker's [24]. Moreover, as mentioned before, the method works only for $f > 0$ and not for $f < 0$. We have here obtained a new class of ξ- parameter solutions $[\tilde{g}, \sigma]$ for $f < 0$. As shown above, the only black hole solution with nontrivial scalar hair ($\sigma \neq$ constant) is again the Bekenstein solution ($\xi = 0$) even for $f < 0$. We believe that these results provide a fair indication that this black hole might actually be a unique one. (2) Starting from EMCSF theory for which the Ricci scalar R $\neq 0$, Xanthopoulos and Dialynas [20] used conformal transformations to generate solutions for ECCSF theory for which the Ricci scalar vanishes identically. The difference here is that we have started from ECCSF theory and ended up with BNW solutions for which $\widetilde{R} \neq 0$. However, the route is not just exactly the reverse as the BNW solutions following from the action (20) are both mathematically and physically different from those of the EMCSF theory. In this sense, the present method is similar in spirit to but different in content from that of Ref.[20].

Finally, we wish to direct attention to a recent work by Sudarsky and Zannias [25] in which they argue that the Bekenstein solution cannot be interpreted as a genuine black hole and thus it does not merit as a counterexample to NFHT. The authors' main contention is that the stress tensor of the Bekenstein scalar field (r.h.s. of Eq.(3)) is ill defined at the horizon r = M. They attempt to regularize the divergence of the stress tensor at the horizon by according a distributional meaning to φ. However, the regularizing prescriptions employed by them do not work and hence the above claim.

In our opinion, the claim is untenable for the following reasons: (i) The ingoing Eddington-Finkelstein coordinates, which the authors use, do not describe outgoing motions satisfactorily and hence are not really fully well behaved [26, 27]. Any meaningful analysis should be carried out only in the Kruskal-Szekeres coordinate chart. But most importantly, (ii) There exist other regularizing prescriptions (test functions) for which the horizon is distributional character to φ leads to any conclusion one wants! In that sense, the method used by Sudarsky and Zannias [25] is a failure.

The value of the stress tensor at r = M should be computed only in the limit $r \to M$, and then the alleged divergence disappears. However, Bekenstein solution is suspected to be physically unstable under perturbations [28]. Nonetheless, from a nonperturbative analytical standpoint normally adopted in any discussion of NSHT, it seems only fair to accord a black hole status to the solution. A full discussion of our points (i) and (ii) above will take us out of the context of the present work and hence is reserved for the future.

References:

[1]. R. Ruffini,J.A. Wheeler, Phys. Today **24**, 30 (1971)

[2]. K.K. Nandi, S.M.K. Alam, B. Bhattacharjee and J. Evans, Gen. Relat. Grav. **30**, 877 (1998).

[3] J.D. Bekenstein, Phys. Rev. D **51**, R6608 (1995).

[4] A.E. Mayo, J.D. Bekenstein, Phys. Rev. D **54**, 5059 (1996).

[5] R.G. Cai and Y.S. Myung, Phys. Rev. D **56**, 3466 (1997).

[6] L. Sokolosski, B. Carr, Phys. Lett. B **176**, 334 (1986).

[7] T. Koikawa, M. Yoshimura, Phys. Lett. B **189**, 29 (1987).

[8] T. Zannias, J. Math. Phys. **36**, 6970 (1995)

[9] B. Green, J.M. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1987).

[10] Y.M. Cho, Phys. Rev. Lett. **68**, 3133 (1992).

[11] C.H. Brans and R.H. Dicke, Phys. Rev. **124**, 925 (1961).

[12] S. Saa, J. Math. Phys. **37**, 2346 (1996)

[13] H.A. Buchdahl, Phys. Rev. **115**, 1325 (1959).

[14] N. Bannerjee and S.Sen, Phys. Rev. D **58**, 104024 (1998).

[15]. O.Bergmann and R.Leipnik, Phys. Rev. **107**, 1157 (1957)

[16] K. Nordtvedt, Ap.J. **161**, 1059 (1970)

[17] R.V. Wagoner, Phys. Rev. D **1**, 3209 (1970).

[18]. J.D.Bekenstein, Ann.Phys. (N.Y.) **82**, 535 (1975).

[19]. J.D.Bekenstein, Ann.Phys. (N.Y.) **91**, 75 (1975).

[20]. B.C.Xanthopoulos and T.E. Dialynas, J. Math. Phys. **33**, 1463 (1992).

[21]. G. Magnano and L.M. Sokolowski, Phys. Rev. **D 50**, 5039 (1994).

[22] K.K. Nandi, B. Bhattacharjee, S.M.K. Alam and J. Evans, Phys.Rev. **D 57**, 823(1998)

[23] J. Schwinger, Particles, Sources and Fields, Addison-Wesley, Reading MA, 1970

[24] B. M. Barkar, Ap.J. **219**, 5 (1978)

[25] D. Sudarsky and T. Zannias, Phys. Rev. D **58**, 087502 (1998).

[26] W. Rindler, Essential Relativiy, Springer, NY, 1977

[27]. C. W.Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, Freeman, San Francisco (1973).

[28] K.A. Bronnikov, Yu. N. Kireyev, Phys. Lett. A **67**, 95 (1978).