Chapter 6

BRIEF SUMMARY OF THE WORK

We provide here chapter wise brief summary of the work and the new results reported in the thesis.

Chapter 2

We have investigated here the behavior of a few spherically symmetric static acclaimed black hole solutions in respect of tidal forces in the geodesic frame. It turns out that the forces diverge on the horizon of cold black holes (CBH) while for ordinary ones, they do not. It is pointed out that Kruskal-like extensions do not render the CBH metrics nonsingular. We have presented a CBH that is available in the Brans-Dicke theory for which the tidal forces do not diverge on the horizon and in that sense it is a better one.

Chapter 3

A new class of solution is obtained in the Bergmann-Nordtvedt-Wagoner scalar-tensor theory of gravity. An examination of this class reveals that the only black hole solution admitting a nontrivial scalar field in the exterior is the Bekenstein solution.

Chapter 4

The *exact* formulation for the effect of the Brans-Dicke scalar field on the gravitational corrections to the Sagnac delay in the Jordan and Einstein frames is presented for the first time. The results completely agree with the known PPN factors in the weak field region. The calculations also reveal how the Brans-Dicke

coupling parameter ϖ appears in various correction terms for different types of source/observer orbits. A first order correction of roughly 2.83×10^{-1} fringe shift for visible light is introduced by the gravity-scalar field combination for Earth bound equatorial orbits. It is also demonstrated that the final predictions in the two frames do not differ. The effect of the scalar field on the geodetic and Lense-Thirring precession of a spherical gyroscope in circular polar orbit around the Earth is also computed with an eye towards the Stanford Gravity Probe-B experiment currently in progress. The feasibility of optical and matter-wave interferometric measurements is discussed briefly.

Chapter 5

In an effort to investigate string effects in physical observations, we have analyzed the rotating Kerr-Sen metric in a Sagnac type experiment and have deduced exact expressions for the delay. For an Earth bound configuration, it turns out that a correction to the basic Sagnac delay by an order of $\sim 10^{-14}$ seconds leads to a terrestrial dilatonic charge of amount $\sim 10^{24}$ esu, a value nearly 200 times larger than the electronic charge of the Earth's magnetosphere.

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TIDAL FORCES IN COLD BLACK HOLE SPACETIMES

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We investigate here the behavior of a few spherically symmetric static acclaimed black hole solutions in respect of tidal forces in the geodesic frame. It turns out that the forces diverge on the horizon of cold black holes (CBH) while for ordinary ones, they do not. It is pointed out that Kruskal-like extensions do not render the CBH metrics nonsingular. We present a CBH that is available in the Brans–Dicke theory for which the tidal forces do not diverge on the horizon and in that sense it is a better one.

1. Introduction

In vacuum Einstein's General Relativity (EGR), it is well known that the only static, spherically symmetric (SSS) black hole solutions are those given by the charged Reissner–Nordström (RN) and uncharged Schwarzschild ones. In the non-vacuum EGR, several black hole solutions are known. For instance: (i) The one discovered by Bekenstein,[1] which is a solution of the Einstein conformally coupled scalar field theory. (ii) The dilaton-Maxwell solution discovered by Garfinkle, Horowitz and Strominger.[2] (iii) The solution for a nonlinear electromagnetic source found recently by Ayon-Beato and Garcia.[3] In the non-Einsteinian theories of gravity, too, some acclaimed black hole solutions exist in the literature. These theories include Brans–Dicke (BD) scalar–tensor theory[4] and the Weyl Integrable theory.[5] In the former, we have black hole solutions discussed by Campanelli and

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Lousto,[6] Bronnikov *et al.*[7] and in the latter, we have the Salim–Sautu[8] solutions. The list however is by no means exhaustive. The considered non-EGR family of solutions has infinite horizon area and zero Hawking temperature. Hence, such solutions have been baptized as cold black holes (CBH) by Bronnikov *et al.*[7]

Let us now recall a very fundamental criterion which a given solution must satisfy in order that it can represent a black hole spacetime: The curvature tensor components, computed in the observer's (static or freely falling) orthonormal frame must be finite everywhere including the horizon.[9] This condition comes from the physical requirement that the tidal forces do not crush or tear apart an extended observer falling freely through the horizon.

We wish to examine in this paper how many of the above solutions satisfy the condition of finiteness of tidal forces near the horizon. It turns out that all the considered solutions except the CBH satisfy this criterion. We shall then examine a particular solution in the BD theory which also turns out to be a CBH but in which the malady of infinite tidal forces does not appear. In that sense, the solution merits as a better CBH than the available ones.

In Sec. 2, the general expression for tidal forces in a freely falling frame is laid down. In Secs. 3 and 4, different solutions in EGR and non-EGR respectively are tested. It is pointed out in Sec. 5 that the Kruskal-like extensions do not render the CBH metrics nonsingular. Finally, in Sec. 6, a better CBH in BD theory is discussed.

2. Tidal Forces in a Geodesic Frame

Following the notations of Horowitz and Ross,[10] consider the general form of a SSS metric:

$$ds^2 = -\frac{F(r)}{G(r)}\,dt^2 + \frac{dr^2}{F(r)} + R^2(r)d\Omega^2\,, \qquad d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2\,. \tag{1}$$

In a static observer's orthonormal basis, the only nonvanishing components of the curvature tensor are R_{0101}, R_{0202}, R_{0303}, R_{1212}, R_{1313} and R_{2323}. Radially freely falling observers with conserved energy E are connected to the static orthonormal frame by a local Lorentz boost with an instantaneous velocity given by

$$v = \left[1 - \frac{FE^{-2}}{G}\right]^{1/2}\,. \tag{2}$$

Then the nonvanishing curvature components in the Lorentz-boosted frame ($\wedge$) are:[10]

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = R_{0101}\,, \tag{3}$$

$$R_{\hat{0}\hat{k}\hat{0}\hat{k}} = R_{0k0k} + \sinh^2\alpha(R_{0k0k} + R_{1k1k})\,, \tag{4}$$

$$R_{\hat{0}\hat{k}\hat{0}\hat{k}} = \cosh\alpha\sinh\alpha R_{0k0k} + R_{1k1k}\,, \tag{5}$$

$$R_{\hat{1}\hat{k}\hat{1}\hat{k}} = \sinh\alpha R_{0k0k} + R_{1k1k}\,, \tag{6}$$

where k, $l = 2, 3$ and $\sinh\alpha = v/\sqrt{1-v^2}$. The tidal acceleration between two parts of the traveler's body is given by:[11]

$$\Delta a_{\hat{j}} = -R_{\hat{0}\hat{j}\hat{0}\hat{p}}\xi^{\hat{p}}\,, \tag{7}$$

where $\boldsymbol{\xi}$ is the vector separation between two parts of the body. All that we have to do now is to calculate the components in Eqs. (3)–(6) for a given metric. If *any* of the components diverges as the horizon is approached, we say that the tidal forces physically disrupt the falling observer. Our strategy then is to first compute any one, say, $R_{\hat{0}\hat{2}\hat{0}\hat{2}}$, using Eq. (4). If it is well behaved, then proceed to check if the same behavior is obtained for the rest of the components. If the answer is positive, we say that the solution represents an ordinary black hole. If $R_{\hat{0}\hat{2}\hat{0}\hat{2}}$ itself is not well behaved, we check no further and conclude that the solution might at best represent a cold black hole.

From the metric (1), we can rewrite Eq. (4) as ($k = 2$):

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}} = -\frac{1}{R}\left[R''(E^2G - F) + \frac{R'}{2}(E^2G' - F')\right], \tag{8}$$

where primes on the right denote derivatives with respect to r. Now note that the conserved energy E can be decomposed as

$$E^2 = \left(\frac{F}{G}\right) + v^2(1-v^2)^{-1}\left(\frac{F}{G}\right) = E^2{}_{\mathrm{s}} + E_{\mathrm{ex}}{}^2\,. \tag{9}$$

The first term represents the value of E^2 in the static frame (${E_{\mathrm{s}}}^2$) and the second term represents the enhancement in ${E_{\mathrm{s}}}^2$ due to geodesic motion. Incorporating this, we can decompose $R_{\hat{0}\hat{2}\hat{0}\hat{2}}$ as follows:

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}} = -\frac{1}{R}\left[\frac{R'}{2}({E_{\mathrm{s}}}^2G' - F')\right] - \frac{1}{R}\left(R''G + \frac{R'G'}{2}\right){E_{\mathrm{ex}}}^2 = R^{(\mathrm{s})}_{0202} + R^{(\mathrm{ex})}_{0202}\,. \tag{10}$$

It is easy to verify that the term $|R^{(\mathrm{s})}_{0202}|$ actually represents the curvature component in the static frame, viz., $R^{(\mathrm{s})}_{0202} = R_{0202}$. Thus, only the term $R^{(\mathrm{ex})}_{0202}$ $(\equiv \sinh^2\alpha(R_{0202} + R_{1212}))$ represents overall enhancement in curvature in the Lorentz-boosted frame over the static frame. It is this part that needs to be particularly examined as the observer approaches the horizon. Note also that the energy E^2 is finite (it can be normalized to unity) and so are ${E_{\mathrm{s}}}^2$ and ${E_{\mathrm{ex}}}^2$. As the horizon is approached, $(F/G) \to 0$, $v \to 1$ such that $E^2 \to {E_{\mathrm{ex}}}^2$. Let us now proceed to test a few solutions.

3. EGR Solutions (Planck Units)

3.1. *RN solution*

The metric is given by

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dr^2 + r^2d\Omega^2\,, \tag{11}$$

where Q is the electric charge. It can be readily verified that, due to a remarkable cancellation, $R^{(\mathrm{ex})}_{0202}$ is identically zero. In fact, since $R^{(\mathrm{ex})}_{0k0k}$ is the same as $\sinh^2\alpha(R_{0k0k}+R_{1k1k})$, one can say that all the tensor components remain invariant under the Lorentz-boost. This is a peculiar feature of the RN geometry and is also shared by usual Schwarzschild geometry which is obtained by merely putting $Q=0$. All components in the static frame remain finite as $r\to 2m$. For example,

$$R_{0101} = -\frac{2m}{r^3}+\frac{3e^2}{r^4} \tag{12}$$

and so on. Hence it is concluded that the tidal forces do not diverge near the horizon either for the static or for moving observers.

3.2. *Bekenstein solution*

The metric is given by

$$\begin{aligned} ds^2 &= -\left(1-\frac{m}{r}\right)^2 dt^2 + \left(1-\frac{m}{r}\right)^{-2} dr^2 + r^2 d\Omega^2\,, \\ m^2 &= Q^2 + \frac{1}{3}q^2\,, \\ F_{\mu\nu} &= Qr^{-2}(\delta^r_\mu\delta^t_\nu - \delta^t_\mu\delta^r_\nu)\,, \end{aligned} \tag{13}$$

and the conformal scalar field is given by $\phi = q/(r-m)$. As far as the metric is concerned, it has the extreme RN form and the same conclusions as above apply.

3.3. *Garfinkle–Horowitz–Strominger (GHS) solutions*

These are solutions to the low-energy string theory representing SSS charged black holes. The action is given by

$$S = \int d^4x\sqrt{-g}\,\lfloor -R + 2(\nabla\phi)^2 + e^{-2\phi}F^2\rfloor\,, \tag{14}$$

where ϕ and $F_{\mu\nu}$ are dilatonic and Maxwell fields respectively. A class of solutions is given by

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r\left(r-\frac{Q^2e^{-2\phi_0}}{mr}\right)d\Omega^2\,, \tag{15}$$

$$e^{-2\phi} = e^{-2\phi_0}\left[1-\frac{Q^2e^{-2\phi_0}}{mr}\right]\,, \qquad F = Q\sin\theta\, d\theta\wedge d\varphi\,. \tag{16}$$

It follows from Eq. (4) that $R^{(\mathrm{ex})}_{0202}$ is simply proportional to R''/R which, at the horizon $r_{\mathrm{h}} = 2m$, is

$$\frac{R''}{R} = -\frac{1}{2m(m+D)}\,, \tag{17}$$

where D is the dilaton charge given by $D = -Q^2e^{-2\phi_0}/2m$. If $Q^2 = 2m^2e^{2\phi_0}$, then $(R''/R)|_{r_{\mathrm{h}}}\to\infty$, indicating that the tidal forces diverge at $r=r_{\mathrm{h}}$. That is quite consistent with the conclusion of GHS that this value of Q^2 actually represents a transition between black holes and naked singularities. For $Q^2 \neq 2m^2 e^{2\phi_0}$, the solution (15) does represent a black hole. In the string frame, the metric is obtained by a conformal transformation $e^{2\phi} g_{\mu\nu}$, and it has been shown that, for $Q^2 < 2m^2 e^{2\phi_0}$, the new metric also represents a black hole.[2] According to our criterion, $(R''/R)|_{r_h} \to$ *finite*, which can be readily verified.

3.4. *Black hole solution for nonlinear source*

Recently, Ayon-Beato and Garcia[3] have proposed an exact SSS solution of EGR when the source is a nonlinear electrodynamic field. The resulting metric is

$$ds^2 = -\left[1 - \frac{2mr^2 e^{-q^2/2mr}}{(r^2+q^2)^{3/2}}\right] dt^2 + \left[1 - \frac{2mr^2 e^{-q^2/2mr}}{(r^2+q^2)^{3/2}}\right] dr^2 + r^2 d\Omega^2 , \tag{18}$$

where q is interpreted as the electric charge as the electric field expands asymptotically as

$$E = \frac{q}{r^2} + O(r^{-3}) . \tag{19}$$

All curvature invariants are bounded everywhere *including* the origin. Evidently, $R^{(\mathrm{ex})}_{0202}$ is zero identically, indicating that the corresponding component of the tidal force is bounded too. In fact, it can be verified that all other components are also bounded. However, an undesirable feature of the solution, in our opinion, is that the horizon cannot be *precisely* located in the spacetime as $g_{00} = 0$ does not have an exact solution.

4. Non-EGR Solutions (Planck Units)

4.1. *Brans–Dicke black holes*

Campanelli and Lousto[6] have shown that the BD theory admits a black hole solution which is different from the Schwarzschild one, the metric being given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{\alpha+1} dt^2 + \left(1 - \frac{2m}{r}\right)^{\beta-1} dr^2 + \left(1 - \frac{2m}{r}\right)^{\beta} r^2 d\Omega^2 , \tag{20}$$

$$\phi(r) = \phi_0 \left(1 - \frac{2m}{r}\right)^{-\frac{\alpha+\beta}{2}} , \tag{21}$$

the coupling constant being given by

$$\varpi = -\frac{2(\alpha^2 + \beta^2 + \alpha\beta + \alpha - \beta)}{(\alpha+\beta)^2} . \tag{22}$$

According to Ref. 6, the solution represents a regular black hole for $\beta \leq -1$ and $\alpha - \beta + 1 > 0$. In addition, if one requires that the metric should coincide with the

PPN expansion, then we need to take $\alpha+\beta\to 0$, which implies $\varpi\to-\infty$. In this case, the solution takes on the form

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{1-\beta} dt^2 + \left(1-\frac{2m}{r}\right)^{\beta-1} dr^2 + \left(1-\frac{2m}{r}\right)^{\beta} r^2 d\Omega^2\,, \tag{23}$$

and $\phi(r)=\phi_0=\text{const}$. The parameter β plays the role of BD scalar hair. For this metric, all curvature invariants are finite, for $\beta\le -1$, as $r\to 2m$. However, it turns out that

$$\begin{aligned} R_{\hat{0}\hat{2}\hat{0}\hat{2}} &= \frac{(1-\beta)m}{2r^3}\left[1+\frac{(\beta-2)m}{r}\right]^{-\beta-1} - \frac{2\beta(\beta-2)m^2}{r^4}\left(1-\frac{2m}{r}\right)^{-2} E_{\mathrm{ex}}^2 \\ &= R^{(\mathrm{s})}_{0202} + R^{(\mathrm{ex})}_{0202}\,. \end{aligned} \tag{24}$$

We see that $R^{(\mathrm{ex})}_{0202}\to\infty$ as $r\to 2m$. Thus the horizon is singular. However, one might still argue that the value $\beta=2$ removes the divergence. But then the scalar invariant

$$I = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = O[(r-2m)^{-2-2\beta}]\to\infty\,, \tag{25}$$

as $r\to 2m$. Black hole solutions (type B1), called CBHs, proposed by Bronnikov, Clement, Fabris and Constantinidis[7] also exhibit similar properties.

4.2. *Black holes in Weyl integrable spacetime (WIST)*

The spacetime described by Weyl integrable geometry follows from the action

$$S = \int\left(R + \xi\omega^{\alpha}_{;\alpha} - \frac{1}{2}e^{-2\omega}F_{\alpha\beta}F^{\alpha\beta} + e^{2\omega}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)\sqrt{-g}\,d^4x\,, \tag{26}$$

where R is the scalar associated with the Weyl geometry, ω is the geometric scalar field and ϕ is an external scalar field. Salim and Sautu[8] proposed three classes of solutions. Let us consider only one of them:

$$ds^2 = -\left(1-\frac{\eta}{r}\right)^{m/\eta} dt^2 + \left(1-\frac{\eta}{r}\right)^{-m/\eta} dr^2 + \left(1-\frac{\eta}{r}\right)^{2m/\eta} r^2 d\Omega^2\,, \tag{27}$$

$$e^{\omega(r)} = e^{\omega_0}\left(1-\frac{\eta}{r}\right)^{-\sigma/\eta}, \qquad \frac{d\phi}{dr} = \phi_0 e^{2\omega_0}\frac{\left(1-\frac{\eta}{r}\right)^{-4m/\eta}}{r^2}\,, \tag{28}$$

$$\sigma^2 = \frac{4m^2-\eta^2}{2\lambda}\,, \qquad \lambda = \frac{1}{2}(4\xi-3)\,. \tag{29}$$

The solution (27) looks pretty similar to Eq. (23), but not quite. The horizon appears at $r=\eta$. The curvature in the static frame is finite, but in the moving frame, we find

$$\left|\frac{R''}{R}\right| = \frac{m(m-\eta)}{r^4(1-\eta/r)^2}\,. \tag{30}$$

Therefore, $R^{(\mathrm{ex})}_{0202} \to \infty$ as $r \to \eta$, unless $m = \eta$. Let us examine what happens to the Weyl scalar R. It is given by

$$R = -r^{-(2+2m/\eta)}(r-\eta)^{-3+2m/\eta}p(r)\,, \tag{31}$$

where $p(r)$ is a polynomial of $O(r) > 0$. If this scalar is finite, so is the Riemann scalar. If $m = \eta$, then $R \to \infty$ as $r \to \eta$. For $m \neq \eta$, tidal forces in the freely falling frame becomes infinitely large. In either case, the horizon is singular.

The two types of solutions (23) and (27) following from two entirely different theories exhibit a remarkable similarity as far as the behavior of tidal forces are concerned. In the examples, considered so far, it is clear that the black hole solutions of EGR (with or without source) are indeed black holes while those from the non-EGR are different, at least as far as tidal force considerations are concerned. Can we generalize our conclusion to include all SSS solutions of EGR? Perhaps not. For instance, consider the Janis–Newman–Winnicour[12] solution of the Einstein minimally coupled theory. It has exactly the same form as that of Eq. (23) with only a different (logarithmic) scalar field. Consequently, the tidal forces in the geodesic frame are infinite near the horizon. That explains why the JNW solution is said to have a naked singularity,[13] but it can also be interpreted as having the features of a CBH. At any rate, it follows that every EGR solution should be tested on a case by case basis.

5. Kruskal-like Extension of CBH

It is well known that Kruskal–Szekeres extension[14,15] offers the advantage that it removes the coordinate singularity from the Schwarzschild metric in the standard form. Let us now ask if similar advantages obtain in the Kruskal-like extension of the metric (20) performed by Campanelli and Lousto.[6] The answer seems to be in the negative. Defining the null variables $\bar{u}$ and $\bar{\nu}$ by[16]

$$d\bar{u} = dt - dr^*\,, \qquad d\bar{\nu} = dt + dr^*\,, \tag{32}$$

where the tortoise-like variable r^* is given by

$$dr^* = \left(1 - \frac{2m}{r}\right)^{\frac{\beta-\alpha}{2}-1} dr\,, \tag{33}$$

and applying further the transformations

$$U = -\exp(-\kappa_{\mathrm{H}})\bar{u}\,, \qquad V = \exp(\kappa_{\mathrm{H}}\bar{\nu})\,, \tag{34}$$

in which the surface gravity κ is given by

$$\kappa = (\alpha+1)\,\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{(\alpha-\beta)/2}\,, \tag{35}$$

the final form of the metric (20) becomes:[17]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{\alpha+1} \kappa_{\rm H}{}^{-2} \exp(-2\kappa_{\rm H} r^*) dU\, dV + r^2 \left(1 - \frac{2m}{r}\right)^{\beta} d\Omega^2 , \qquad (36)$$

where $\kappa_{\rm H}$ is the value of κ at $r = 2m$. Thus, $\kappa_{\rm H} = 0$ for $\alpha > \beta$, $\kappa_{\rm H} = \infty$ for $\alpha < \beta$, $\kappa_{\rm H} = (\alpha+1)/4m$ for $\alpha = \beta$. One may apply a further transformation to spacelike and timelike variables $u = (V - U)/2$, $v = (V + U)/2$ respectively on metric (36) to make it look partly familiar. For the special case $\alpha = \beta$, one has only $g_{\rm UV}(r = 2m)$ finite. For $\alpha \neq \beta$, which includes $\alpha = -\beta$ [leading to the metric (23)], $g_{\rm UV}(r = 2m)$ is no longer finite. But most importantly, the last term in the metric (36) continue to have a singularity at $r = 2m$ for $\beta \leq -1$. Thus, the Kruskal-like extension, Eq. (36), does *not* offer any advantage as such. The surface area of the horizon still remains infinite. One may then equally well use metrics (20) and (23) for computing curvature tensors and invariants, which was actually done in Sec. 4.1. Similar considerations apply to the metric (27).

The metric representing type B1 CBHs proposed by Bronnikov *et al.*[7] is conformal to the metric (20). As shown in Ref. 7, the tidal forces are infinite at the horizon in this case, too. The surface area of the horizon remains infinite even in the extended form of the metric. It is not possible to reduce the area by Kruskal-like extensions.

6. A Better CBH

We have seen that Kruskal-like extensions cannot do away with the singularities in $g_{\theta\theta}$ and $g_{\varphi\varphi}$ as is evident from the metric (36). Consequently, one has infinite horizon areas and entropies. Hence the name CBH, as mentioned before. Accepting these facts, we enquire if there exists a CBH for which the tidal forces are *finite* on the horizon. It seems that there indeed is one: The class IV solutions of the BD theory[18] provide just such a CBH.

Class IV solutions are:

$$ds^2 = -e^{2\mu(r)} dt^2 + e^{2\nu(r)} \lfloor dr^2 + r^2 d\Omega^2 \rfloor , \qquad (37)$$

$$\mu(r) = \mu_0 - \frac{1}{Br} , \qquad (38)$$

$$\nu(r) = \nu_0 + \frac{C+1}{Br} , \qquad (39)$$

$$C = \frac{-1 \pm \sqrt{-2\varpi - 3}}{\varpi + 2} , \qquad (40)$$

$$\phi = \phi_0\, e^{-C/Br} . \qquad (41)$$

Usual asymptotic flatness and weak field conditions fix the constants as

$$\mu_0 = \nu_0 = 0 , \qquad B = 1/m > 0 , \qquad (42)$$

where m is the mass of the configuration. The horizon appears at $r = r_{\mathrm{h}} = 0$ and its area is infinite. Other features of this solution are discussed in Refs. 19 and 20. In the static orthonormal frame, the curvature components are:

$$R_{0101} = -\frac{1}{Br^3 e^{2(C+1)/Br}}\left[\frac{C+2}{Br} - 2\right], \tag{43}$$

$$R_{0202} = R_{0303} = -\frac{1}{Br^3 e^{2(C+1)/Br}}\left[1 - \frac{C+1}{Br}\right], \tag{44}$$

$$R_{1212} = R_{1313} = -\frac{C+1}{Br^3 e^{2(C+1)/Br}}, \tag{45}$$

$$R_{2323} = -\frac{C+1}{Br^3 e^{2(C+1)/Br}}\left[\frac{C+1}{Br} - 2\right], \tag{46}$$

All these components tend to zero as $r \to r_{\mathrm{h}}$, provided $C + 1 \geq 0$. This happens only if $\varpi < -2$. It should be recalled that the CBH solutions proposed in Refs. 6 and 7 also correspond to negative values of ϖ. As discussed in Ref. 6, it is the numerical value of ϖ, rather than its sign, that is more relevant. Also, the EGR effects are recovered for $|\varpi| \to \infty$.

The Ricci scalar for the considered solution is given by

$$R \equiv g_{\alpha\beta}R^{\alpha\beta} = \frac{-2(1 + C + C^2)}{B^2 r^4}\, e^{-2(C+1)/Br} \tag{47}$$

which goes to zero as $r \to r_{\mathrm{h}}$. It may be verified that all other Riemann invariants are also finite at the horizon. The metric (37)–(41) has a remarkable feature in that, for $C = 0$, it describes all the solar system tests exactly as does the EGR Schwarzschild metric. The curvature components in the moving frame also remains finite as the horizon is approached. For example,

$$\begin{aligned} R_{\hat{0}\hat{2}\hat{0}\hat{2}} = {} & \left(\frac{C+1}{Br^2} - \frac{1}{r}\right)\left(\frac{e^{-2C/Br}}{Br^2}\right)\left(CE_s^2 - (C+1)e^{-2/Br}\right) \\ & - \left(\frac{e^{-2C/Br}}{Br^2}\right)\left(\frac{C+1}{Br^2} + \frac{C}{r}\right)E_{\mathrm{ex}}^2, \end{aligned} \tag{48}$$

tends to zero as $r \to r_{\mathrm{h}}$. This implies that the tidal forces in the geodesic frame do not diverge. In many ways, therefore, the class IV metric resembles the ones discussed in Refs. 6–8 but it has an added merit as indicated above. Hence, it seems to represent a better CBH than the ones proposed so far.

Summarizing, we have to say the following: We examined a few EGR solutions for which the tidal forces do not diverge on the horizon. However, it was also indicated that a general conclusion to that effect cannot be drawn because of the existence of the JNW solution. Non-EGR CBH solutions exhibit infinite tidal forces on the horizon. Extended solutions too fail to remove this divergence as the CBH metric continues to remain singular. We then presented a better CBH in the BD theory. Bronnikov *et al.*[7] conjectured that infinite horizon areas could be related to infinite tidal forces. Our example in Sec. 6 indicates that this is not necessarily the case. In that sense, class IV solutions may be interpreted as providing a counterexample to the conjecture.

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Scalar field dressing of black holes in the Bergmann–Nordtvedt–Wagoner theory

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Abstract

A new class of solutions is obtained in the Bergmann–Nordtvedt–Wagoner scalar–tensor theory of gravity. An examination of this class reveals that the only black hole solution admitting a nontrivial scalar field in the exterior is the Bekenstein solution. © 2000 Published by Elsevier Science B.V.

Keywords: Bekenstein black hole; Scalar–tensor theory

An integral part of black hole physics is the so-called no scalar hair theorem (NSHT) which deals with the question of whether the exterior of a spherically symmetric black hole admits nontrivial scalar field(s) [1–8]. Several physical scenarios warrant such considerations. For example, the scalar field couples to gravitation in a natural way in superstring theories [9,10]. In the Brans–Dicke theory, such a coupling is necessary to incorporate Mach's principle [11]. In general, the coupling could be of minimal, nonminimal or conformal type. Hence, NSHT is of direct relevance to black hole solutions of such theories.

Recently, Saa [12] has proposed a new NSHT which rules out a large class of nonminimally coupled theories admitting nontrivial, finite scalar field dressing of asymptotically flat, static, spherically symmetric black hole. The basic procedure in this theorem is to take the solution of the Einstein–minimally coupled scalar field (EMCSF) theory as the seed solution (e.g., Buchdahl solution [13]) and generate solutions of the Einstein–nonminimally coupled scalar field (ENCSF) theory. The next step is to look for the black hole solutions in the latter theory which are then found to correspond to only constant scalar fields. One then says that the NSHT is satisfied in ENCSF. A specific example of this is the Brans–Dicke theory with the coupling parameter $\omega =$ constant.

More recently, Sen and Banerjee [14] have examined, in the light of Saa's theorem, a few examples in the more general Bergmann–Nordtvedt–Wagoner (BNW) theory [15–17] for which the coupling parameter ω is no longer a constant but a function of

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the scalar field. They showed that the only black hole solution in those examples correspond again to only constant scalar fields. There is admittedly a caveat here: Saa's theorem does not cover the Einstein–conformally coupled scalar field (ECCSF) due to the fact that it does not satisfy one of the crucial conditions, stated later, of the theorem. Consequently, a black hole solution dressed by a CCSF $\varphi \neq$ constant is permitted to exist in violation of NSHT. This is actually the case with the well known Bekenstein solution [18,19].

On the other hand, despite the failure of Saa's theorem in the case of ECCSF theory, its solutions can always be generated from EMCSF theory by special types of conformal maps discovered by Xanthopoulos and Dialynas [20]. They have shown that, among such solutions in the ECCSF theory in 4 dimensions, the only black hole solution with nontrivial scalar hair is the Bekenstein solution. This result constitutes a major step towards proving the uniqueness of the Bekenstein black hole.

Our aim in this Letter is to approach the question of uniqueness from a more general viewpoint provided by the massless nonminimally coupled BNW theory. The first step in this direction is to portray ECCSF theory as a BNW theory with a specific form of coupling function. The next step is to generate a new class of solutions within the BNW theory for a more general class of coupling functions. This is achieved through a conformal map of the metric tensor and a redefinition of the scalar field. An examination of the generated class of solutions reveals that the only black hole solution is again the Bekenstein one, confirming the results of Ref. [20].

The most general matter-free action of the ENCSF theory including an electromagnetic field $F_{\mu\nu}$ is of the form

$$S[g,\varphi] = \int \left[f(\varphi) R - h(\varphi) g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - F_{\mu\nu} F^{\mu\nu} \right] (-g)^{1/2} d^4x, \tag{1}$$

$$h(\varphi) = \omega(\varphi)/\varphi, \tag{2}$$

where $f(\varphi)$ and $h(\varphi)$ are arbitrary functions, ∇_μ denotes covariant derivative with respect to $g_{\mu\nu}$ and R is the Ricci scalar. Saa's procedure requires $f > 0$ and $h > 0$ for all φ. Brans–Dicke theory corresponds to $f(\varphi) = \varphi$ and $h(\varphi) = \omega/\varphi$, $\omega =$ constant. The BNW theory corresponds to $f(\varphi) = \varphi$ and $\omega = \omega(\varphi)$, while the EMCSF theory has the values $f(\varphi) = 1$ and $h(\varphi) = 1$. The ECCSF theory corresponds to the choices $f(\varphi) = 1 - \frac{1}{6}\varphi^2$, $h(\varphi) = 1$, so that the condition $f > 0$ is not satisfied for *all* φ, and thus escapes Saa's theorem, as mentioned before. The resulting ECCSF equations, in suitable units, are

$$\begin{aligned} G_{\mu\nu} = T^{(\varphi)}_{\mu\nu} + T^{(\mathrm{em})}_{\mu\nu} = \left(1 - \tfrac{1}{6}\varphi^2\right)^{-1} \big[& \nabla_\mu \varphi \nabla_\nu \varphi \\ & - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi - \tfrac{1}{6} \nabla_\mu \nabla_\nu (\varphi^2) \\ & + \tfrac{1}{6} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \nabla_\beta (\varphi^2) \\ & - g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} + \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \end{aligned} \tag{3}$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi - \tfrac{1}{6} R \varphi = 0, \tag{4}$$

$$\nabla_\mu F^{\mu\nu} = 0, \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. The Bekenstein black hole solution following from Eqs. (3)–(5) with a nontrivial φ is ($8\pi G = c = 1$):

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 \\ &\quad + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \end{aligned} \tag{6}$$

$$\varphi = \frac{q}{r - M}, \quad F_{\mu\nu} = e r^{-2} \left(\delta^r_\mu \delta^t_\nu - \delta^t_\mu \delta^r_\nu \right),$$

$$M^2 = e^2 + \tfrac{1}{3} q^2. \tag{7}$$

where q and e are the scalar and electric charges respectively. The metric has the same form as that of extreme Reissner–Nordstrom. The divergence of φ at $r = M$ has been shown to be physically innocuous in the sense that test objects interacting with the scalar field reach the horizon in finite proper time [18,19]. Note further that, both $T^{(\varphi)}_{\mu\nu}$ and $T^{(\mathrm{em})}_{\mu\nu}$ are traceless and hence $R = 0$. Since Maxwell's equations are already conformally invariant, we need not repeatedly include $F_{\mu\nu} F^{\mu\nu}$ in the actions that will follow.

Now define

$$\chi = 1 - \tfrac{1}{6}\varphi^2, \quad \omega(\chi) = \left(\frac{3}{2}\right) \frac{\chi}{1 - \chi}. \tag{8}$$

Then the action (1) for the ECCSF can be rewritten in the form of that of BNW theory:

$$S(g,\chi)=\int\left[\chi R-\chi^{-1}\omega(\chi)g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi\right]\times(-g)^{1/2}d^4x. \tag{9}$$

The field equations are

$$G_{\mu\nu}=T_{\mu\nu}(\chi)=\chi^{-1}(\nabla_\mu\nabla_\nu\chi-g_{\mu\nu}\Box\chi)+\chi^{-2}\omega(\chi)\left(\nabla_\mu\chi\nabla_\nu\chi+\tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi\right), \tag{10}$$

$$\Box\chi=-[2\omega(\chi)+3]^{-1}\frac{d\omega}{d\chi}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi, \tag{11}$$

where $\Box\equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. The solution consists of the same ds^2 as in (6) but now with the scalar field being redefined as χ. The scalar curvature R follows from Eqs. (10) and (11):

$$R=\left[\omega\chi^{-2}-3(2\omega+3)^{-1}\chi^{-1}\frac{d\omega}{d\chi}\right]g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi, \tag{12}$$

which vanishes identically for $\omega(\chi)$ defined above. This is to be expected as the action (9) is essentially the same as the ECCSF action, only redefined via Eqs. (8) as a BNW action.

Consider the BNW action (9) with $\omega(\chi)$ as above. Apply the transformation

$$\tilde{g}_{\mu\nu}=\Omega^2(\chi)g_{\mu\nu}, \tag{13}$$

where $\Omega(\chi)$ is a nonvanishing smooth function. Then

$$(-\tilde{g})^{1/2}=\Omega^4(\chi)(-g)^{1/2}, \tag{14}$$

$$\tilde{R}=\Omega^{-2}[R+6\Omega^{-1}\Box\Omega], \tag{15}$$

so that the action (9) becomes

$$S[\tilde{g},\chi]=\int\left[\Omega^{-2}\chi\tilde{R}-6\chi\Omega^{-5}\Box\Omega+\chi^{-1}\omega(\chi)\Omega^{-2}\tilde{g}^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi\right]\times(-\tilde{g})^{1/2}d^4x. \tag{16}$$

Taking

$$\Omega=\chi^{\xi},\quad \sigma=\chi^{1-2\xi}, \tag{17}$$

where ξ is an arbitrary constant parameter, we find

$$6\chi^{1-5\xi}\Box\Omega=(1-2\xi)^{-2}[6\xi(\xi-1)-2\xi\omega(\chi)]\sigma^{-2}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\sigma\tilde{\nabla}_\beta\sigma, \tag{18}$$

$$\chi^{-1}\omega(\chi)\Omega^{-2}\tilde{g}^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi=(1-2\xi)^{-2}\omega(\chi)\sigma^{-1}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\sigma\tilde{\nabla}_\beta\sigma. \tag{19}$$

Putting (18) and (19) in (16), we get

$$S[\tilde{g},\sigma]=\int\left[\sigma\tilde{R}-\sigma^{-1}\tilde{\omega}(\sigma)\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\sigma\tilde{\nabla}_\beta\sigma\right]\times(-\tilde{g})^{1/2}d^4x, \tag{20}$$

which has again the form of a BNW action with $\tilde{\omega}(\sigma)$ given explicitly by

$$\tilde{\omega}(\sigma)=\tfrac{1}{2}(1-2\xi)^{-2}\left[1-\sigma^{1/(1-2\xi)}\right]^{-1}\times\left[3(1+2\xi)\sigma^{1/(1-2\xi)}-12\xi(\xi-1)(1-\sigma^{1/(1-2\xi)})\right], \tag{21}$$

where $\xi\neq 1/2$. Clearly, the resulting field equations will be the same as Eqs. (10) and (11) but with $g_{\mu\nu}$ and $\omega(\chi)$ replaced by $\tilde{g}_{\mu\nu}$ and $\tilde{\omega}(\sigma)$ respectively. They will be identical only if $\xi=0$. For different numerical values of ξ, rational or irrational, we obtain different functional forms for $\tilde{\omega}(\sigma)$. The standard method of solving Eqs. (10) and (11) for the obtained class of coupling functions $\tilde{\omega}(\sigma)$ would be quite tedious if not untractable. The present method enables us to immediately write down the solutions if we use

$$\tilde{g}_{\alpha\beta}=\chi^{2\xi}g_{\alpha\beta},\quad \sigma=\chi^{1-2\xi}, \tag{22}$$

where χ is computed from Eqs. (7) and (8) while $g\alpha\beta$ is provided by the metric (6). Thus, typical solutions $[\tilde{g}_{\mu\nu},\sigma]$, parametrized by ξ, are:

$$\tilde{g}_{00}=6^{-2\xi}\left[6(r-M)^2-q^2\right]^{2\xi}r^{-2}(r-M)^{2(1-2\xi)}, \tag{23}$$

$$\tilde{g}_{11} = 6^{-2\xi}\left[6(r-M)^2 - q^2\right]^{2\xi} r^2 (r-M)^{-2(1+2\xi)}, \tag{24}$$

$$\tilde{g}_{22} = 6^{-2\xi}\left[6(r-M)^2 - q^2\right]^{2\xi} r^2 (r-M)^{-4\xi}, \tag{25}$$

$$\tilde{g}_{33} = \tilde{g}_{22}\sin^2\theta, \tag{26}$$

$$\sigma = 6^{2\xi-1}\left[6(r-M)^2 - q^2\right]^{1-2\xi} (r-M)^{-2(1-2\xi)}. \tag{27}$$

The scalar curvature $\tilde{R}$ can be calculated using Eqs. (15) and (18) and the fact that $R = 0$:

$$\tilde{R} = 6\chi^{-2(1+x)}\left[\xi(\xi-1) - \frac{\xi\omega}{3}\right] g^{\alpha\beta}\nabla_\alpha \chi \nabla_\beta \chi. \tag{28}$$

The solutions (23)–(26) are asymptotically flat and $\sigma \to 1$ as $r \to \infty$. For $\xi < 0$, all components of the metric tensor $\tilde{g}_{\alpha\beta}$ blow off at $r = r_0 = (M + 6^{-1/2}q) > M$, and for $\xi > 0$, they vanish at $r = r_0$. In either case, the solutions do not represent black holes since the surface $r = r_0$ is not a regular one. If $\xi = 0$, then $R = 0$ and only in this case, one retrieves the scalar hair Bekenstein black hole solution.

Two important points should be noted here. First, using the transformation $[\tilde{g}_{\alpha\beta}, \sigma] \to [\bar{g}_{\alpha\beta}, \bar{\sigma}]$ given by

$$\bar{g}_{\alpha\beta} = \sigma \tilde{g}_{\alpha\beta}, \quad \bar{\sigma} = \int \left|\tilde{\omega}(\sigma) + \tfrac{3}{2}\right|^{1/2} \frac{d\sigma}{\sigma}, \tag{29}$$

it is possible to reduce the non-Einsteinian BNW action (20) into the Einsteinian representation given by

$$S_{\text{Ein}} = \int\left[\bar{R} - \bar{g}^{\alpha\beta}\bar{\nabla}_\alpha \bar{\sigma}\bar{\nabla}_\beta \bar{\sigma}\right]\sqrt{-\bar{g}}\, d^4x. \tag{30}$$

This is just the action of the EMCSF theory and it no longer contains the coupling function. The field equations are

$$\bar{G}_{\alpha\beta} = \bar{T}_{\alpha\beta}(\bar{\sigma}) = \bar{\nabla}_\alpha \bar{\sigma}\bar{\nabla}_\beta \bar{\sigma} - \tfrac{1}{2}\bar{g}_{\alpha\beta}\bar{\nabla}_\lambda \bar{\sigma}\bar{\nabla}^\lambda \bar{\sigma}, \tag{31}$$

$$\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\sigma} = 0, \tag{32}$$

where $\bar{\nabla}_\alpha$ denotes covariant differentiation with respect to $\bar{g}_{\alpha\beta}$. Physical interpretations of the theories resulting from the two actions (20) and (30) differ widely [21]. For instance, $\tilde{T}_{\mu\nu}(\sigma)$ following from the action (20) is not necessarily positive definite (signalling a violation of energy conditions) while $\bar{T}_{\mu\nu}(\bar{\sigma})$, following from the action (30), is. In the former case, usual singularity theorems do not apply and hence nontrivial scalar field(s) in the spherical black hole exterior is allowed to exist. In the latter case, by a theorem of Bekenstein [3], it is disallowed.

Second, spherically symmetric solutions $[\bar{g}_{\alpha\beta}, \bar{\sigma}]$ of Eqs. (31) and (32) are indeed wellknown [13,16,22]. Using these known solutions, it is now in principle possible to generate BNW solutions $[\tilde{g}_{\alpha\beta}, \sigma]$ via the prescription (29). However, to do this, one needs to plug in some choice of $\tilde{\omega}(\sigma)$ by hand, integrate, and see if σ could be inverted explicitly in terms of the seed function $\bar{\sigma}$. If not, the procedure fails. These restrictions of integrability and invertibility severely narrow down the choices of $\tilde{\omega}(\sigma)$ to some suitable functions only (see, for example, Ref. [14]). On the other hand, the present method has the advantage that it does not depend upon either of the restrictions while Eq. (21) offers a fairly wide variety of coupling functions $\tilde{\omega}(\sigma)$ obtained by simply varying the parameter ξ.

Let us now summarize what we have achieved in this Letter: (1) The question of uniqueness of the Bekenstein black hole remains partially unresolved due to the fact that, for arbitrary choices of coupling functions $\tilde{\omega}(\sigma)$, exact solutions of BNW theory are not guaranteed. In practice, solutions are found using Saa's method only on a case by case basis as was actually done by Sen and Banerjee [14] for specific choices of $\tilde{\omega}(\sigma)$, namely, those of Schwinger's [23] and Barker's [24]. Moreover, as mentioned before, the method works only for $f > 0$ and not for $f < 0$. We have here obtained a new class of ξ-parameter solutions $[\tilde{g}, \sigma]$ for $f < 0$. As shown above, the only black hole solution with nontrivial scalar hair ($\sigma \neq$ constant) is again the Bekenstein solution ($\xi = 0$) *even for* $f < 0$. We believe that these results provide a fair indication that this black hole might actually be a unique one. (2) Starting from EMCSF theory for which the Ricci scalar $R \neq 0$, Xanthopoulos and Dialynas [20] used conformal transformations to generate solutions for ECCSF theory for which the Ricci scalar vanishes identically. The difference here is that we have started from ECCSF theory and ended up with BNW solutions for which $\tilde{R} \neq 0$. However, the route is not just exactly the reverse as the BNW solutions following from the action (20) are both mathematically and physically different from those of the EMCSF theory. In this sense, the present method is similar in spirit to but different in content from that of Ref. [20].

Finally, we wish to direct attention to a recent work by Sudarsky and Zannias [25] in which they argue that the Bekenstein solution can not be interpreted as a genuine black hole and thus it does not merit as a counterexample to NSHT. The authors' main contention is that the stress tensor of the Bekenstein scalar field (r.h.s. of Eq. (3)) is ill defined *at* the horizon $r = M$. They attempt to regularize the divergence of the stress tensor at the horizon by according a distributional meaning to φ. However, the regularizing prescriptions employed by them do not work and hence the above claim.

In our opinion, the claim is untenable for the following reasons: (i) The ingoing Eddington–Finkelstein coordinates, which the authors use, do not describe outgoing motions satisfactorily and hence are not really *fully* well behaved [26,27]. Any meaningful analysis should be carried out only in the Kruskal–Szekeres coordinate chart. But most importantly, (ii) there exist *other* regularizing prescriptions (test functions) for which the horizon is perfectly regular. Therefore, the idea of giving a distributional character to φ leads to any conclusion one wants! In that sense, the method used by Sudarsky and Zannias [25] is a failure.

The value of the stress tensor at $r = M$ should be computed only in the limit $r \to M$, and then the alleged divergence disappears. However, Bekenstein solution is suspected to be physically unstable under perturbations [28]. Nonetheless, from a nonperturbative *analytical* standpoint normally adopted in any discussion of NSHT, it seems only fair to accord a black hole status to the solution. A full discussion of our points (i) and (ii) above will take us out of the context of the present Letter and hence is reserved for future communications.

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Brans-Dicke corrections to the gravitational Sagnac effect

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The *exact* formulation for the effect of the Brans-Dicke scalar field on the gravitational corrections to the Sagnac delay in the Jordan and Einstein frames is presented. The results completely agree with the known parametrized post Newtonian factors in the weak-field region. The calculations also reveal how the Brans-Dicke coupling parameter ϖ appears in various correction terms for different types of source or observer orbits. A first-order correction of roughly 2.83×10^{-1} fringe shift for visible light is introduced by the gravity–scalar field combination for Earth-bound equatorial orbits. It is also demonstrated that the final predictions in the two frames do not differ. The effect of the scalar field on the geodetic and Lense-Thirring precession of a spherical gyroscope in a circular polar orbit around the Earth is also computed with an eye towards the Stanford Gravity Probe-B experiment currently in progress. The feasibility of optical and matter-wave interferometric measurements is discussed briefly.

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I. INTRODUCTION

Ever since its discovery, the Sagnac effect [1] has played a very important role in the understanding and development of fundamental physics. For a recent review, see the works of Stedman [2]. The effect stems from the basic physical fact that the round-trip time of light around a closed contour, when its source is fixed on a turntable, depends upon the angular velocity, say Ω, of the turntable. Furthermore, this round-trip time is different for light corotating and counter-rotating with the turntable. Using the special theory of relativity (STR), and assuming $\Omega r \ll c$, one obtains the proper time difference $\delta\tau_S$ when the two beams meet again at the starting point as [3]

$$\delta\tau_S \cong \frac{4\Omega}{c^2} S, \tag{1}$$

where c is the vacuum speed of light and $S(=\pi r^2)$ is the projected area of the contour perpendicular to the axis of rotation. Note that the expression (1) represents a lack of simultaneity as recorded by a single rotating clock (from where the beams depart and reunite). It is thus a real physical effect in the sense that it does not involve any arbitrary synchronization convention which is required between two distant clocks [3,4]. Moreover, the effect is universal as it manifests not only for light rays but also for all kinds of waves, including matter waves [5–11].

The formula (1) has been tested to a good accuracy and the remarkable degree of precision attained lately by the advent of ring laser interferometry raises the hope that the measurements of higher-order corrections to this effect might be possible in the near future [2]. Motivated by this prospect, Tartaglia [12], in a recent interesting paper, has considered the Einsteinian general relativistic (EGR) effects on the proper delay time when the source or receiver orbits a massive rotating body (a "massive turntable," as it were). The author considered the Kerr metric for a rotating body and obtained the EGR corrections to the Sagnac effect in the cases when the light source or receiver executes equatorial, polar, and geodesic circular motions.

On the other hand, there is a recent surge of interest in the non-Einsteinian theories of gravity, such as the celebrated Brans-Dicke (BD) theory [13] or other scalar tensor theories. The motivation comes from the fact that the occurrence of scalar fields coupled to gravity seems inevitable in superstring theories [14], higher-order theories [15], as well as in the extended [16] and hyperextended [17] inflationary theories of the early universe. Moreover, scalar tensor theories provide a rich arena for investigations into wormhole physics [18–23]. One also recalls that the standard solar system tests of gravity were calculated in the BD theory that displayed the effect of the scalar field on those tests. Current experimental estimates place the BD coupling parameter $\varpi \geq 500$. In the same spirit, it seems quite desirable that the effect of the scalar field on the corrections to the Sagnac effect, geodetic, and Lense-Thirring precession be also calculated using a Kerr-like solution of the BD theory. This precisely is the aim of the present paper, and we follow exactly the same procedure as in Ref. [12] for the Sagnac part.

In dealing with scalar-tensor theories in general and BD theory in particular, one envisages two types of variables delineating two types of frames, viz., the Jordan and Einstein frames which are connected by the scalar field. In Sec. II we discuss the rotating solutions in the two frames. Sections III and IV derive, respectively, the exact and approximate expressions for the proper time delay $\delta\tau$ in the case of the

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equatorial trajectory of the source or observer. The polar and geodesic trajectories are considered in Secs. V and VI, respectively. In Sec. VII the relevant corrections in the Einstein frame are considered. Section VIII contains a broad discussion which is divided into various sections containing numerical estimates for the Sagnac delay in STR and BD theory for Earth-bound experiments, a comparison with the usual parametrized post Newtonian (PPN) factors as well as the possibility of using optical and matter-wave interferometers to measure the correction factors. In Section IX we calculate the geodetic and Lense-Thirring precession in the weak-field limit of the Kerr-like BD metric for a satellite in a circular polar orbit about the Earth. We end with a summary of our results in Sec. X.

II. ROTATING SOLUTIONS IN THE JORDAN AND EINSTEIN FRAMES

Let us first define what are meant by the Jordan and Einstein frames [15,20]. The pair of variables ($g_{\mu\nu}$, scalar ϕ) defined originally in the BD action constitute what is called a Jordan frame. Consider now the conformal rescaling

$$\tilde{g}_{\mu\nu}=f(\phi)g_{\mu\nu},\tilde{\phi}=h(\phi), \tag{2}$$

such that, in the redefined action, $\tilde{\phi}$ couples minimally to $\tilde{g}_{\mu\nu}$ for some functions $f(\phi)$ and $h(\phi)$. Then the new pair ($\tilde{g}_{\mu\nu}$, scalar $\tilde{\phi}$) is said to constitute an Einstein frame. Sometimes, it is mathematically preferable to use this latter frame for computation of experimental predictions. In the Jordan pair, the scalar field ϕ plays the role of a component of gravity in the sense that $\langle\phi\rangle\approx G^{-1}$, where G is the Newtonian constant of gravity, signifying the Machian character of the BD theory. On the other hand, in the Einstein pair, the scalar $\tilde{\phi}$ plays the role of some kind of matter source. These features will become evident from the field equations that follow. Throughout this paper, we take $G=c=1$ unless they are explicitly restored.

The matter-free Jordan frame BD action is given by

$$S_J[g_{\mu\nu},\phi]=\frac{1}{16\pi}\int\left(\phi R-\frac{\varpi}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)\sqrt{-g}\,d^4x, \tag{3}$$

where $\varpi=\text{const}$ is a dimensionless coupling parameter. The resultant field equations are

$$(\phi^{;\rho})_{;\rho}=0, \tag{4}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{\varpi}{\phi^2}\left[\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}\right]+\frac{1}{\phi}[\phi_{;\mu;\nu}-g_{\mu\nu}(\phi^{;\sigma})_{;\sigma}], \tag{5}$$

where the semicolon indicates a covariant derivative with respect to $g_{\mu\nu}$. Following the procedure of Newman and Janis [24], a two-parameter rotating solution of the above field equations has indeed been found by Krori and Bhattacharjee (KB) [25] from the static BD solution. They called it a Kerr-like solution but we choose to call it the KB solution in what follows. In order to see how the different arbitrary constants are related, it is necessary to display the static BD solution which, in "isotropic" coordinates $(t,\bar{\rho},\theta,\varphi)$, is

$$ds^2=\left[\frac{1-\frac{r_0}{2\bar{\rho}}}{1+\frac{r_0}{2\bar{\rho}}}\right]^{2/\lambda}dt^2-\left(1+\frac{r_0}{2\bar{\rho}}\right)^4\left[\frac{1-\frac{r_0}{2\bar{\rho}}}{1+\frac{r_0}{2\bar{\rho}}}\right]^{2(\lambda-C-1)/\lambda}\times[d\bar{\rho}^2+\bar{\rho}^2d\theta^2+\bar{\rho}^2\sin^2\theta\,d\varphi^2], \tag{6}$$

$$\phi=\phi_0\left[\frac{1-\frac{r_0}{2\bar{\rho}}}{1+\frac{r_0}{2\bar{\rho}}}\right]^{C/\lambda}, \tag{7}$$

where λ,C,ϕ_0,r_0 are constants, and the first two relate to ϖ as

$$\lambda^2\equiv(C+1)^2-C\left(1-\frac{\varpi C}{2}\right). \tag{8}$$

The KB solution generated from the above is given by

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=\left(1-\frac{2r_0r}{\rho}\right)^\eta(dt-\omega\,d\varphi)^2-\left(1-\frac{2r_0r}{\rho}\right)^\xi\rho\left(\frac{dr^2}{\Delta}+d\theta^2+\sin^2\theta\,d\varphi^2\right)+2\left(1-\frac{2r_0r}{\rho}\right)^\sigma\omega(dt-\omega\,d\varphi)d\varphi, \tag{9}$$

$$\phi=\phi_0\left(1-\frac{2r_0r}{\rho}\right)^{-\sigma},\quad \sigma=\frac{\xi+\eta-1}{2}=-\frac{C}{2\lambda}. \tag{10}$$

$$\omega=a\sin^2\theta,\quad \rho=r^2+a^2\cos^2\theta,$$

$$\Delta=r^2+a^2-2r_0r,\quad r=\bar{\rho}(1+r_0/2\bar{\rho})^2. \tag{11}$$

The solutions (9)–(11) represent the exterior metric due to a massive body rotating with respect to the fixed stars, the scalar field being given by Eq. (10). As one can see, the presence of the coupling parameter ϖ in the solution is manifested through the expressions (8) and (10). For $\xi=0$, $\sigma=0$, $\eta=1$, one recovers the usual Kerr metric in Boyer-Lindquist coordinates. Here $r_0=GM/c^2$, M is the mass of the source, and a is the ratio between the total angular momentum J and the mass M, that is, $a=J/M$.

The Einstein frame action is obtained from the BD action (3) by means of a particular conformal transformation, called the Dicke transformations, given by

$$\tilde{g}_{\mu\nu}=\frac{1}{16\pi}\phi g_{\mu\nu}, \tag{12}$$

$$d\tilde{\phi}=\left(\frac{\varpi+\frac{3}{2}}{\alpha}\right)^{1/2}\frac{d\phi}{\phi}, \tag{13}$$

where α is an arbitrary constant. The action then is

$$S_E[\tilde{g}_{\mu\nu},\tilde{\phi}]=\int [\tilde{R}-\alpha\tilde{g}^{\mu\nu}\tilde{\phi}_{,\mu}\tilde{\phi}_{,\nu}]\sqrt{-\tilde{g}}\,d^4x. \tag{14}$$

The resulting field equations are

$$\tilde{R}_{\mu\nu}-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}=\alpha\left[\tilde{\phi}_{,\mu}\tilde{\phi}_{,\nu}-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\phi}_{,\sigma}\tilde{\phi}^{,\sigma}\right], \tag{15}$$

$$(\tilde{\phi}^{;\rho})_{;\rho}=0. \tag{16}$$

The KB solutions of the above Einstein minimally coupled equations (15) and (16) can be explicitly written out as

$$\begin{aligned} ds^2&=\tilde{g}_{\mu\nu}\,dx^\mu\,dx^\nu \\ &=\left(1-\frac{2r_0r}{\rho}\right)^{\eta-\sigma}(dt-\omega\,d\varphi)^2 \\ &\quad-\left(1-\frac{2r_0r}{\rho}\right)^{\xi-\sigma}\rho\left(\frac{dr^2}{\Delta}+d\theta^2+\sin^2\theta\,d\varphi^2\right) \\ &\quad+2\omega(dt-\omega\,d\varphi)d\varphi, \end{aligned} \tag{17}$$

$$\tilde{\phi}=-\left[\frac{\varpi+\frac{3}{2}}{\alpha}\right]^{1/2}\sigma\ln\left(1-\frac{2r_0r}{\rho}\right), \tag{18}$$

$$\omega=a\sin^2\theta,\quad \rho=r^2+a^2\cos^2\theta,$$

$$\Delta=r^2+a^2-2r_0r.$$

Here also, for $a=0$, the solutions (17) and (18) go over to Buchdahl solutions [20,26] in "standard" coordinates under a suitable radial transformation defined below.

The vacuum KB solution (9) resembling the Kerr metric is defined for the radial coordinate r in the range $r_0+(r_0^2-a^2\cos^2\theta)^{1/2}<r<\infty$ which translates in "standard" radial coordinate $\bar{R}$ into the range $0<\bar{R}<\infty$ where $\bar{R}$ is defined by

$$\bar{R}^2=\rho\left(1-\frac{2r_0r}{\rho}\right)^{\xi}. \tag{19}$$

The solution does exhibit a curvature singularity at the origin $\bar{R}=0$ which is not clothed by an event horizon and hence is naked. In fact, the singularity has the topology of a point as the area of the equipotential surfaces and proper lengths of closed curves on these surfaces all reduce to zero size as $\bar{R}\to 0$. The coupling between gravity and a massless scalar field renders the event horizon to collapse to a point and one has gravitation without black holes [27]. At any rate, we are interested only in the effects due to a normal, uncollapsed rotating star coupled to a scalar field. Hence, the Penrose conjecture of cosmic censorship (preventing the occurrence of naked singularities), for which a precise formulation is yet unavailable, should not concern us here. Indeed, we will see that the PPN calculations precisely agree with those following from the KB metrics in both Jordan and Einstein frames.

III. EQUATORIAL TRAJECTORY

Consider that the source or receiver of two oppositely directed light beams is moving around the gravitating body, along a circumference at a radius $r=R=\text{const}$ $[R>r_0+(r_0^2-a^2\cos^2\theta)^{1/2}]$ on the equatorial plane $\theta=\pi/2$. Suitably placed mirrors send back to their origin both beams after a circular trip about the central body. Let us further assume that the source or receiver is moving with uniform orbital angular speed ω_0 with respect to distant stars such that the rotation angle is

$$\varphi_0=\omega_0 t. \tag{20}$$

Under these conditions, the KB metric (9) reduces to

$$ds^2=[\chi\omega_0^2+2a(P^\sigma-P^\eta)\omega_0+P^\eta]dt^2,$$

$$\chi\equiv P^\eta a^2-P^\xi R^2-2P^\sigma a^2, \tag{21}$$

$$P=\left(1-\frac{2r_0}{R}\right). \tag{22}$$

The trajectory of a light ray is given by $ds^2=0$ which immediately gives

$$0=\chi\omega^2+2a(P^\sigma-P^\eta)\omega+P^\eta\equiv\chi(\omega-\Omega_+)(\omega-\Omega_-), \tag{23}$$

where ω is the orbital angular speed of photons. The two roots $\Omega_\pm$ satisfy the following equations:

$$\Omega_++\Omega_-=-\frac{2a(P^\sigma-P^\eta)}{\chi},\quad \Omega_+\Omega_-=\frac{P^\eta}{\chi}. \tag{24}$$

The rotation angles for light are then

$$\varphi_\pm=\Omega_\pm t. \tag{25}$$

Eliminating t between Eqs. (20) and (25), we get

$$\varphi_\pm=\frac{\Omega_\pm}{\omega_0}\varphi_0. \tag{26}$$

The first intersection of the world lines of the two light rays with the world line of the orbiting observer after emission at time $t=0$ occurs when

$$\varphi_+=\varphi_0+2\pi,\quad \varphi_-=\varphi_0-2\pi,\quad \text{or}\quad \frac{\Omega_\pm}{\omega}\varphi_0=\varphi_0\pm 2\pi, \tag{27}$$

where + refers to corotating and − refers to counterrotating beams. Solving for φ_0, we get

$$\varphi_{0\pm}=\pm\frac{2\pi\omega_0}{\Omega_\pm-\omega_0}. \tag{28}$$

The proper time of the rotating observer is deduced from Eq. (21) as

$$d\tau=\sqrt{[\chi\omega_0^2+2a(P^\sigma-P^\eta)\omega_0+P^\eta]}\frac{d\varphi_0}{\omega_0}. \tag{29}$$

Therefore, integrating between φ_{0+} and φ_{0-}, we obtain the Sagnac delay

$$d\tau=\sqrt{[\chi\omega_0^2+2a(P^\sigma-P^\eta)\omega_0+P^\eta]}\frac{\varphi_{0+}-\varphi_{0-}}{\omega_0}. \quad (30)$$

From Eq. (28), we have

$$\varphi_{0+}-\varphi_{0-}=2\pi\omega_0\left[\frac{\Omega_++\Omega_--2\omega_0}{(\Omega_+-\omega_0)(\Omega_--\omega_0)}\right]. \quad (31)$$

Using this expression in Eq. (30), we find

$$\delta\tau=(2\pi)\frac{\chi[(\Omega_++\Omega_-)-2\omega_0]}{\sqrt{\chi\omega_0^2+2a(P^\sigma-P^\sigma)\omega_0+P^\eta}}. \quad (32)$$

We see that the delay $\delta\tau$ is zero if the angular speed of the orbiting observer is

$$\omega_0\equiv\omega_n=\frac{\Omega_++\Omega_-}{2}=\frac{a(P^\eta-P^\sigma)}{\chi}$$
$$=\frac{a(P^\eta-P^\sigma)}{P^\eta a^2-P^\xi R^2-2P^\sigma a^2}, \quad (33)$$

provided $a\neq 0$. In the usual Kerr case, the above reduces to $(r_0=M)$

$$\omega_n=\frac{2aM}{R^3+a^2R+2Ma^2}, \quad (34)$$

which is exactly the same as the one obtained by Tartaglia [12]. The observers having the angular speed ω_n are locally nonrotating and may be imagined to be equivalent to the static observers in the Schwarzschild geometry for whom no Sagnac effect exists. On the other hand, if the observers keep fixed positions with regard to distant stars so that $\omega_0=0$, then the Sagnac delay becomes

$$\delta\tau_0=\delta\tau(\omega_0=0)=(4\pi a)\frac{(P^\eta-P^\sigma)}{\sqrt{P^\eta}}. \quad (35)$$

In the usual Kerr case, one obtains from the above

$$\delta\tau_0=\frac{8\pi aM}{R\sqrt{1-\frac{2M}{R}}}=\frac{8\pi J}{R\sqrt{1-\frac{2M}{R}}}=\frac{8\pi I_0\Omega_0}{R\sqrt{1-\frac{2M}{R}}} \quad (36)$$

in which we have used the expression for the moment of inertia I_0 given by $J=aM=I_0\Omega_0$, where Ω_0 is the angular speed of the rotating source, assumed to be solid and spherical with uniform density. The expression (36) again is the same as in Ref. [12].

To the order in $1/R^2$ we have, from Eq. (35),

$$\delta\tau_0\cong\frac{8\pi ar_0}{R}(\sigma-\eta)\left[1+\frac{r_0}{R}(1-\sigma-\eta)\right]\left[1-\frac{2r_0}{R}\right]^{-\eta/2}$$
$$\cong\frac{8\pi ar_0}{R}(\sigma-\eta)\left[1+\frac{r_0}{R}(1-\sigma)\right]. \quad (37)$$

This too coincides with the calculations in the Kerr case when appropriate values σ and η are chosen. However, the effect of the scalar field is manifest in the determination of values for σ and η away from the Kerr values.

One may also reexpress the delay $\delta\tau_0$ in terms of the Lense-Thirring effect (see Sec. IX) given by (using $ar_0=I\Omega_0$)

$$\omega_{LT}=\frac{I\Omega_0}{R^3} \quad (38)$$

and the result is

$$\delta\tau_0\cong\frac{8\omega_{LT}}{R}(\sigma-\eta)(\pi R^3)\left[1+\frac{r_0}{R}(1-\sigma)\right]. \quad (39)$$

If the observer is fixed on the equator, then $\omega_0=\Omega_0$, and then the delay $\delta\tau$ can also be expressed in terms of I, r_0, and Ω_0,

$$\delta\tau=(4\pi\Omega_0)\frac{\frac{I}{r_0}(P^\eta-P^\sigma)-\chi}{\sqrt{\chi\Omega_0^2+\frac{2I\Omega_0^2}{r_0}(P^\sigma-P^\eta)+P^\eta}}, \quad (40)$$

where

$$\chi=\frac{I^2\Omega_0^2}{r_0^2}P^\eta-R^2P^\xi-2P^\sigma\frac{I^2\Omega_0^2}{r_0^2}. \quad (41)$$

All these reduce to the corresponding expressions in the Kerr case.

IV. APPROXIMATIONS

For our convenience, let us adopt the following abbreviations:

$$\zeta\equiv a/R,\quad \Psi\equiv\omega_0R,\quad \varepsilon\equiv r_0/R. \quad (42)$$

Since we shall be concerned mainly with Earth-bound experiments, it is useful to have an idea of how small the quantities ζ, ψ, and ε are. For Earth, these are (exact individual values of the pieces will be given later)

$$\zeta_\oplus=\frac{a_\oplus}{R_\oplus c}\sim 10^{-6},\quad \psi_\oplus=\frac{\omega_0R_\oplus}{c}\sim 10^{-7},$$
$$\varepsilon_\oplus=\frac{GM_\oplus}{R_\oplus c^2}\sim 10^{-9}, \quad (43)$$

and for Sun, these are

$$\zeta_\odot\sim\psi_\odot\sim\varepsilon_\odot\sim 10^{-6}. \quad (44)$$

Let us rewrite Eq. (32) as

$$\frac{\delta\tau}{4\pi R}=\frac{\psi(\chi/R^2)+\zeta(P^\sigma-P^\eta)}{[P^\eta+\psi^2(\chi/R^2)+2\zeta\psi(P^\sigma-P^\eta)]^{1/2}}. \quad (45)$$

With the values displayed in Eqs. (43) and (44) in mind, we use the expansions

$$\frac{\chi}{R^2}=\zeta^2P^\eta-P^\xi-2\zeta^2P^\sigma$$
$$\cong -1+2\xi\varepsilon+2\xi(\xi-1)\varepsilon^2+\zeta^2+O(\ )^3, \tag{46}$$

$$P^\sigma-P^\eta=\left[1-\frac{2r_0}{R}\right]^\sigma-\left[1-\frac{2r_0}{R}\right]^\eta$$
$$\cong -2(\sigma-\eta)\varepsilon[1+(1-\sigma-\eta)\varepsilon]O(\ )^3, \tag{47}$$

$$P^\eta=\left[1-\frac{2r_0}{R}\right]^\eta \cong 1-2\eta\varepsilon+2\eta(\eta-1)\varepsilon^2+O(\ )^3, \tag{48}$$

where $O(\ )^3$ stands for any cubic terms in the small quantities ζ, ψ, ε. Using these expansions, we obtain the delay, denoting it by $\delta\tau_E$,

$$\frac{\delta\tau_E}{4\pi R}\cong\psi+2(\sigma-\eta)\varepsilon\zeta+(\eta-2\xi)\varepsilon\psi$$
$$+\zeta^2\psi+2(\eta-\sigma)(\sigma-1)\varepsilon^2\psi-\tfrac{1}{2}(4\eta\xi+4\xi-4\xi^2$$
$$-2\eta-\eta^2)\varepsilon^2\psi+\tfrac{1}{2}\psi^3+O(\ )^4. \tag{49}$$

After cross multiplying and substituting in the definitions of small quantities in Eq. (49), we get

$$\delta\tau_E\cong\delta\tau_s+\frac{8\pi r_0 a}{R}(\sigma-\eta)+4\pi\omega_0 r_0 R(\eta-2\xi)+4\pi a^2\omega_0$$
$$+\frac{8\pi r_0^2 a}{R}(\eta-\sigma)(\sigma-1)-2\pi\omega_0^2 r_0^2(4\eta\xi+4\xi-4\xi^2$$
$$-\eta^2-2\eta)+2\pi\omega_0^3R^4+O(\ )^4. \tag{50}$$

The second term above represents the correction due to the moment of inertia I of the rotating source ($ar_0=I\Omega_0$), the third term represents the correction due to the mass parameter r_0, and the remaining higher-order terms represent variously combined effects of I, r_0, and Ω_0. Most importantly, one can now visualize the effects of the scalar field through the factors η, σ, and ξ.

In the absence of a scalar field and for a homogeneous spherical object whose radius is R_0, one has

$$I=\tfrac{8}{15}\pi\rho R_0^5=\tfrac{2}{5}MR_0^2. \tag{51}$$

ρ is the density (assumed to be uniform) of the object. Hence a for the sphere is approximately

$$a\cong\tfrac{2}{5}R_0^2\Omega_0. \tag{52}$$

V. POLAR (CIRCULAR) ORBITS

We shall now investigate the effect when the light rays move along a circular trajectory passing over the poles. In this case, too, we may take $r=R=\text{const}$ and $\varphi=\text{const}$. Assuming uniform motion again, we take $\theta=\omega_0 t$. Then, we have, using $dr=0$, $d\varphi=0$, $d\theta=\omega_0 dt$, and $ds^2=0$, from the metric (9)

$$\frac{d\theta}{dt}=\pm\frac{(R^2-2r_0R+a^2-a^2\sin^2\theta)^{(\eta-\xi)/2}}{(R^2+a^2\cos^2\theta)^{(\eta-\xi+1)/2}}. \tag{53}$$

Under the assumption that $a^2/R^2\ll 1$, and assuming $t=0$ when $\theta=0$, we have

$$t\cong\frac{R}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi)/2}}\theta+\frac{a^2}{2R}\frac{\left[\left(1-\frac{2r_0}{R}\right)\left(\frac{\eta-\xi+1}{2}\right)+\frac{\xi-\eta}{2}\right]}{\left(1-\frac{2r_0}{R}\right)^{\eta-\xi+2/2}}\int_0^\theta\cos\theta' d\theta'$$

$$=\frac{R}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi)/2}}\theta+\frac{a^2}{4R}\frac{\left[\left(1-\frac{2r_0}{R}\right)\left(\frac{\eta-\xi+1}{2}\right)+\frac{\xi-\eta}{2}\right]}{\left(1-\frac{2r_0}{R}\right)^{\eta-\xi+2/2}}\times(\cos\theta\sin\theta+\theta)$$

$$=\left(\frac{R}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi)/2}}+\frac{a^2}{4R}\frac{\left[\left(1-\frac{2r_0}{R}\right)\left(\frac{\eta-\xi+1}{2}\right)+\frac{\xi-\eta}{2}\right]}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi+2)/2}}\right)\theta$$
$$+\frac{a^2}{8R}\frac{\left[\left(1-\frac{2r_0}{R}\right)\left(\frac{\eta-\xi+1}{2}\right)+\frac{\xi-\eta}{2}\right]}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi+2)/2}}\sin 2\theta. \tag{54}$$

During this time, the rotating observer describes an angle θ_0 while light travels an angle $2\pi\pm\theta_0$ (once again, + for the corotating beam and − for the counterrotating beam) so that

$$\frac{\theta_0}{\omega_0}=(p+q)(2\pi+\theta_0)\pm\frac{q}{2}\sin 2\theta_0, \tag{55}$$

where

$$q=\frac{a^2}{4R}\frac{\left[\left(1-\frac{2r_0}{R}\right)\left(\frac{\eta-\xi+1}{2}\right)+\frac{\xi-\eta}{2}\right]}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi+2)/2}},$$

$$p=\frac{R}{\left(1-\frac{2r_0}{R}\right)^{(\eta-\xi)/2}}. \tag{56}$$

Assume, as we did already, a low speed observer and that the angle $2\theta_0$ is so small as to justify $\sin 2\theta_0 \cong 2\theta_0$. Then

$$\frac{\theta_0}{\omega_0}=(p+q)(2\pi\pm\theta_0)\pm q\theta_0. \tag{57}$$

Solving for θ_0, we get

$$\theta_{0\pm}=2\pi\frac{p+q}{\frac{1}{\omega_0}\mp(p+q)\mp q}. \tag{58}$$

Finally, the difference between two round trip "coordinate" times (recalling the approximations already used) comes to

$$\begin{aligned}t_+-t_-&=\frac{\theta_{0+}-\theta_{0-}}{\omega_0}\\&=4\pi\omega_0\frac{\left[X+\frac{a^2}{2R}Y\right]\left[X+\frac{a^2}{R}Y\right]}{Z^2-\omega_0^2\left[X+\frac{a^2}{R}Y\right]^2},\end{aligned} \tag{59}$$

where

$$X=R\left(1-\frac{2r_0}{R}\right), \tag{60}$$

$$Y=\left(1-\frac{2r_0}{R}\right)\left(\frac{\eta-\xi+1}{2}\right)+\frac{\xi-\eta}{2}, \tag{61}$$

$$Z=\left[1-\frac{2r_0}{R}\right]^{(\eta-\xi+2)/2}. \tag{62}$$

Neglecting terms of order R^{-3} and $\omega_0^2R^2$ and higher, we get

$$t_+-t_-\cong\pi\omega_0R^2\left\{4+\frac{3a^2}{R^2}+\frac{8r_0(\eta-\xi)}{R}\right\}. \tag{63}$$

Thus, the correction due to the angular momentum of the source is independent of R in this case. The term is in fact given by, using Eq. (52),

$$3\pi a^2\omega_0=\tfrac{12}{25}\pi R_0^4\Omega_0\omega_0, \tag{64}$$

where R_0 is the radius of a source sphere of uniform density.

In order to obtain what the rotating observer measures, we must calculate the proper time in his/her frame. This is done as follows: From the metric Eq. (9),

$$\tau=\int\left[\left(1-\frac{2r_0R}{\rho}\right)^{\eta}-\left(1-\frac{2r_0R}{\rho}\right)^{\xi}\rho\omega_0^2\right]^{1/2}dt,$$

$$\rho=R^2+a^2\cos^2(\omega_0t). \tag{65}$$

For short enough ω_0t, we have $\cos(\omega_0t)\cong 1$, $\sin(\omega_0t)\cong 0$. Further, neglecting terms of the order R^{-2} in the integrand, we have

$$\tau\cong\left(1-\frac{2r_0\eta}{R}-\omega_0^2R^2\right)^{1/2}t. \tag{66}$$

Therefore, the time delay in the polar case, denoted by $\delta\tau_P$, is given by

$$\begin{aligned}\delta\tau_P&\cong\left(1-\frac{2r_0\eta}{R}-\omega_0^2R^2\right)^{1/2}(t_+-t_-)\\&=\left(1-\frac{2r_0\eta}{R}-\omega_0^2R^2\right)^{1/2}\\&\quad\times\left[\pi\omega_0R^2\left\{4+\frac{3a^2}{R^2}+\frac{8r_0(\eta-\xi)}{R}\right\}\right].\end{aligned} \tag{67}$$

Therefore, to the first and second orders in ζ, ψ, and ε, we have

$$\begin{aligned}\delta\tau_P&\cong\delta\tau_S(1-\eta\varepsilon)[1+\tfrac{3}{4}\zeta^2+2(\eta-\xi)]\\&\cong\delta\tau_s[1+\tfrac{3}{4}\zeta^2+(\eta-2\xi)\varepsilon].\end{aligned}$$

Comparing with the equatorial case, the excess is, using Eq. (49),

$$\frac{\Delta\tau}{\delta\tau_S}=\frac{\delta\tau_E-\delta\tau_P}{\delta\tau_S}\cong\frac{2(\sigma-\eta)\zeta\varepsilon}{\psi}-\frac{3}{4}\zeta^2. \tag{68}$$

The term $(\eta-2\xi)\varepsilon$ cancels out due to the spherical symmetry of the orbits considered. After cross multiplying by $\delta\tau_S$, we get

$$\Delta\tau\equiv\delta\tau_E-\delta\tau_P\cong\frac{8\pi r_0a}{R}(\sigma-\eta)-3a^2\pi\omega_0. \tag{69}$$

It may be observed from Eqs. (50) and (69) that the scalar field appears only in the terms that contain the gravitating mass parameter r_0. This fact is quite consistent with the form of the KE metric which also has this property.

VI. GEODESICS

Let us now consider the geodesic motion of the source or receiver having a four-velocity $u^\mu(\equiv dx^\mu/ds)$. The geodesic equations are

$$\frac{\partial u^\mu}{\partial x^\nu}u^\nu+\Gamma^\mu_{\nu\alpha}u^\nu u^\alpha=0, \tag{70}$$

where $\Gamma^{\mu}_{\nu\alpha}$ are the Christoffel symbols formed from the KB metric (9). We can simplify the problem by taking $\theta=\pi/2$, that is, $u^{\theta}=0$. The geodesic equations do allow such a solution [12]. In this case, $\sin\theta=1$, $\cos\theta=0$, $\omega=a$, and $P=1-2r_0/r$. For a circular geodesic orbit with a constant radius $r=R$, the condition is $u^r=0$. Then the radial equation becomes

$$\Gamma^r_{tt}(u^t)^2+\Gamma^r_{\varphi\varphi}(u^{\varphi})^2+2\Gamma^r_{t\varphi}u^t u^{\varphi}=0. \tag{71}$$

Defining the angular speed of rotation of the source or receiver as $\omega=u^{\varphi}/u^t$, we get

$$\omega_{\pm}=\frac{1}{\Gamma^r_{\varphi\varphi}}-\Gamma^r_{t\varphi}\pm\sqrt{(\Gamma^r_{t\varphi})^2-\Gamma^r_{tt}\Gamma^r_{\varphi\varphi}}. \tag{72}$$

The above expression simply turns out to be

$$\omega_{\pm}=\frac{1}{\dfrac{\partial g_{\varphi\varphi}}{\partial r}}\left[-\frac{\partial g_{t\varphi}}{\partial r}\pm\sqrt{\left(\frac{\partial g_{t\varphi}}{\partial r}\right)^2-\frac{\partial g_{tt}}{\partial r}\frac{\partial g_{\varphi\varphi}}{\partial r}}\right], \tag{73}$$

where

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=\left(\frac{2}{r^2}\right)[\eta a^2 r_0 P^{\eta-1}-r^3P^{\xi}-\xi r^2 r_0 P^{\xi-1}$$
$$-2\sigma a^2 r_0 P^{\sigma-1}], \tag{74}$$

$$\frac{\partial g_{t\varphi}}{\partial r}=-\left(\frac{2}{r^2}\right)[a\eta r_0 P^{\eta-1}-a\sigma r_0 P^{\sigma-1}], \tag{75}$$

$$\frac{\partial g_{tt}}{\partial r}=\left(\frac{2}{r^2}\right)[\eta r_0 P^{\eta-1}]. \tag{76}$$

Thus, at $r=R$, we finally have $P=1-2r_0/R$ and

$$\omega_{\pm}=\frac{\tilde{P}}{\tilde{Q}}, \tag{77}$$

where

$$\tilde{P}\equiv aM(\eta P^{\eta-1}-\varphi P^{\sigma-1})\pm[\eta R^3 r_0 P^{\xi+\eta-1}+a^2\sigma^2 r_0^2 P^{2\sigma-2}$$
$$+\eta\xi R^2 r_0^2 P^{\xi+\eta-2}]^{1/2},$$

$$\tilde{Q}\equiv \eta a^2 r_0 P^{\eta-1}-R^3P^{\xi}-\xi R^2 r_0 P^{\xi-1}-2\sigma a^2 r_0 P^{\sigma-1}.$$

Dividing the numerator and denominator of $\omega_{\pm}$ by R^3P^{ξ} and retaining terms up to a/R, we find

$$\omega_{\pm}\approx\mp\frac{1}{R}\sqrt{\frac{\eta r_0}{R}}+\frac{ar_0}{R^3}(\sigma-\eta). \tag{78}$$

The sign flip in this equation can be rectified. Suppose we follow the convention that $\omega_{+}>0$ and $\omega_{-}<0$ in the Kerr limit, that is, the $\pm$ signs on $\omega_{\pm}$ indicate the sign of the frequency. Then, from Eq. (75), assuming $a>0$, we find that $\partial g_{t\varphi}/\partial r<0$, so that the numerator (the large square brackets) in Eq. (73) is positive. But $\partial g_{\varphi\varphi}/\partial\varphi$ in Eq. (74) has the leading term $-2rP^{\xi}<0$. Thus ω_{+} as defined by Eq. (73) is actually negative in the Kerr limit and similarly $\omega_{-}>0$. Thus if we were to change the $\pm\to\mp$ on the right-hand side of Eq. (73) (in the large square brackets) then Eq. (78) would read, using the notations of Sec. IV, as

$$\psi_{\pm}\equiv\omega_{\pm}R\cong\pm\sqrt{\eta\varepsilon}+(\sigma-\eta)\varepsilon\zeta. \tag{79}$$

On using this in Eq. (49), we get the delay

$$\delta\tau_{G\pm}\cong 4\pi R[\psi_{\pm}+2(\sigma-\eta)\varepsilon\zeta+(\eta-2\xi)\varepsilon\psi_{\pm}],$$

which yields, to the lowest order in ε,

$$\delta\tau_{G\pm}\cong 4\pi R[\pm\sqrt{\eta\varepsilon}+3(\sigma-\eta)\varepsilon\zeta+O(\varepsilon)^{3/2}]. \tag{80}$$

Now the traditional Sagnac effect is Ref. [12], obtained here by setting in Eq. (80), $a=0$, $\eta=1$, and $\sigma=0$,

$$\delta\tau_{s\pm}=4\pi R\psi_{\pm}=\pm 4\pi\sqrt{MR},$$

so that we have

$$\delta\tau_{G\pm}\cong\sqrt{\frac{\eta r_0}{M}}\,\delta\tau_{s\pm}+\frac{12\pi a r_0}{R}(\sigma-\eta)+O\left(\frac{r_0}{R}\right)^{3/2}. \tag{81}$$

Thus, unlike the case of polar or equatorial orbits, the traditional part of the Sagnac effect is multiplied by a factor $\sqrt{\eta r_0/M}$. Its value will be found from the PPN form of the metric (9) in Sec. VIII.

VII. EINSTEIN FRAME

It is instructive to calculate the relevant corrections in the Einstein frame as well, already defined in Sec. II. The metric to be used now is Eq. (17) and the steps to be followed are precisely the same as those in Secs. III–VI. However, it is not necessary to do them explicitly. Instead, one may simply use the replacements given by $\eta\to\eta-\sigma$, $\xi\to\xi-\sigma$, and $\sigma\to\sigma-\sigma$ in the desired expressions computed in the Jordan frame.

A. Equatorial orbits

As can be verified, ω_n of Eq. (33) remains completely unaffected, that is, $\omega_n^{(J)}=\omega_n^{(E)}$. This implies that the definition of "static" observers, for which no Sagnac delay exists, is preserved even though the physics in the two frames differs widely. However, $\delta\tau_0$ of Eq. (39) changes to

$$\delta\tau_0^{(E)}=\delta\tau_0^{(J)}\approx 8\pi\omega_{LT}R^2(\sigma-\eta). \tag{82}$$

The exact expression for the delay, that is, the $\delta\tau$ between the two frames are also related in the same way and under the approximations as before, we find from Eq. (50),

$$\delta\tau^{(E)} = \delta\tau^{(J)} \cong \delta\tau_s + \frac{8\pi r_0 a}{R}(\sigma - \eta) + 4\pi r_0 R\omega_0(\eta - 2\xi - \sigma). \tag{83}$$

B. Polar orbits

It can easily be noticed from Eqs. (56) that $p^{(E)} = p^{(J)}$, $q^{(E)} = q^{(J)}$ so that we have $(t_+ - t_-)^{(E)} = (t_+ - t_-)^{(J)}$ and consequently, from Eq. (68),

$$\delta\tau_P^{(E)} \cong \delta\tau_S[1 + \tfrac{3}{4}\zeta^2 + (\eta - 2\xi + \sigma)\varepsilon]. \tag{84}$$

The difference becomes, using Eq. (69),

$$\Delta\tau^{(E)} = (\delta\tau_E - \delta\tau_P)^{(E)} = \frac{8\pi a r_0}{R}(\sigma - \eta) - 3\pi a^2\omega_0. \tag{85}$$

C. Geodesics

The exact expression for $\omega_\pm^{(E)}$ can be easily obtained from Eq. (77) under the specified replacements. We shall here write only the approximated final result from Eq. (81),

$$\delta\tau_{G\pm}^{(E)} = \sqrt{\frac{(\eta - \sigma)r_0}{M}}\,\delta\tau_{s\pm} + \frac{12 r_0 a(\sigma - \eta)}{R} + O\left(\frac{r_0}{R}\right)^{3/2}. \tag{86}$$

Although some of the terms in Eqs. (83), (84), and (86) look different from the corresponding terms in the Jordan frame, a PPN approximation will show that they are actually the same. In fact, the coefficients in the first terms in Eqs. (81) and (86) are both unity.

VIII. DISCUSSIONS

A. STR numerical estimates

In the foregoing we calculated the effect of the BD scalar field on the gravitational corrections to the Sagnac effect in the Jordan and Einstein frames. Three types of source or observer trajectories were considered, viz., equatorial, polar, and geodesic. In the Jordan frame the corresponding expressions are Eqs. (50), (69), and (81), while in the Einstein frame, these are Eqs. (83), (85), and (86). All these expressions reveal the effect of the scalar field through the presence of η, ξ, and σ. Since these parameters are connected by Eq. (10), it is clear that the knowledge of any two would suffice in determining the remaining one. Measurements of the correction terms would place upper limits on the values of η and σ. These limits would translate into a limit on ϖ, via Eqs. (8) and (10), just as it happened in the static BD solutions with respect to solar system tests. Conversely, we can take the solar system value $\varpi \geqslant 500$ and calculate the expected numerical values of η, σ, and ξ.

For the sake of comparison, let us now estimate the numerical values of the basic as well as the correction terms in STR. Consider the exact proper time delay $\delta\tau$ from STR given by (under similar circumstances as in Sec. III)

$$\delta\tau_{\mathrm{STR}} = \frac{(4\pi R^2)(\omega_0 + \Omega)}{\sqrt{(1 - \Omega^2 R^2) - 2\omega_0\Omega R^2 - \omega_0^2 R^2}}, \tag{87}$$

where Ω and ω_0 are, respectively, the angular speed of the coordinate system rotating about the origin (turntable) and the orbital angular speed of the source or observer with respect to this turntable [28]. If the coordinate system is nonrotating, that is $\Omega = 0$ but $\omega_0 \neq 0$, then

$$\delta\tau_{\Omega=0} = (4\pi\omega_0 R^2)(1 - \omega_0^2 R^2)^{-1/2} \tag{88}$$

and conversely, if the source or observer is fixed to the turntable such that $\omega_0 = 0$ but $\Omega \neq 0$, then

$$\delta\tau_{\omega_0=0} = (4\pi\Omega R^2)(1 - \Omega^2 R^2)^{-1/2}. \tag{89}$$

The effect is doubled if the source or observer has $\omega_0 = \Omega \neq 0$

$$\delta\tau(\omega_0 = \Omega) = (8\pi\Omega R^2)(1 - 4\Omega^2 R^2)^{-1/2} \tag{90}$$

and is zero if $\omega_0 = -\Omega$, that is, when the source observer is moving on the turntable opposite to its rotation but with the same angular speed Ω.

Tartaglia [12] considers the case when the source or observer is fixed to the equator of the Earth, which means one has to consider Eq. (89) with $\Omega = \Omega_\oplus$ where the symbol $\oplus$ denotes Earth values. Expanding Eq. (89), and restoring c, we get

$$\delta\tau(\omega_0 = 0) = \frac{4\pi\Omega_\oplus R_\oplus^2}{c^2} + \frac{2\pi\Omega_\oplus^3 R_\oplus^4}{c} + \cdots, \tag{91}$$

where $R_\oplus$ denotes the radius of the Earth.

Now recall the relevant data for Earth,

$$R_\oplus = 6.37 \times 10^6\ \mathrm{m},$$

$$\Omega_\oplus = 7.27 \times 10^{-5}\ \mathrm{rad/s},$$

$$\frac{GM_\oplus}{c^2} = 4.4 \times 10^{-3}\ \mathrm{m},$$

$$a_\oplus = 9.81 \times 10^8\ \mathrm{m^2/s},$$

$$c = 3 \times 10^8\ \mathrm{m/s}.$$

Substituting these values into Eq. (89) we obtain

$$\delta\tau_{\mathrm{STR}}(\omega_0 = 0) = [4.12 \times 10^{-7} + 4.6 \times 10^{-19} + \cdots]\ \mathrm{s}. \tag{92}$$

Therefore, the basic Sagnac delay, Eq. (1), amounts to 4.12×10^{-7} s. To compare the above terms with the corresponding ones in the BD theory, we must first determine the unknown constants appearing there. This is achieved by using the PPN approximation, discussed below.

B. PPN approximation

Our aim in this section is to express the KB parameters η, σ, ξ in terms of the coupling constant ϖ. The first step in this direction is to rewrite Eq. (8) in the form

$$1-(\eta-\sigma)^2=(2\varpi+3)\sigma^2 \tag{93}$$

by noting that

$$\eta=\frac{1}{\lambda}, \quad \sigma=-\frac{C}{2\lambda}, \quad \xi=\frac{\lambda-C-1}{\lambda}. \tag{94}$$

The next step is to consider the PPN parameters α, β, γ which appear in the metric

$$ds^2\approx-\left[1-2\alpha\left(\frac{M}{\rho}\right)+2\beta\left(\frac{M}{\rho}\right)^2\right]dt^2 +\left[1+2\gamma\left(\frac{M}{\rho}\right)\right](d\rho^2+\rho^2 d\Omega^2). \tag{95}$$

Since η, σ, ξ already appear in the static form of the metric (9), and we are considering only the weak-field form of the metric, we can, for the moment, assume $a=0$. In isotropic coordinates (ρ, θ, φ) given by

$$r=\rho\left(1+\frac{r_0}{2\rho}\right)^2,$$

the reduced metric (9) becomes

$$ds^2=-\left[\frac{1-\frac{r_0}{2\rho}}{1+\frac{r_0}{2\rho}}\right]^{2\eta} dt^2+\left[\frac{1-\frac{r_0}{2\rho}}{1+\frac{r_0}{2\rho}}\right]^{2\xi-2}\left(1+\frac{r_0}{2\rho}\right)^4 \times(d\rho^2+\rho^2 d\Omega^2). \tag{96}$$

Comparing the corresponding orders, we get

$$\alpha=1, \quad \beta=1, \quad \gamma=1-\frac{2\sigma}{\eta}, \quad \eta r_0=M. \tag{97}$$

The usual PPN value of γ is $\gamma=(1+\varpi)/(2+\varpi)$ [29] and using Eq. (92) we get

$$\sigma=\frac{1}{\sqrt{(2\varpi+3)(2\varpi+4)}}, \quad \eta=\sqrt{\frac{2\varpi+4}{2\varpi+3}},$$
$$\xi=1-\eta+2\sigma. \tag{98}$$

Let us now consider the weak-field rotational part given by $4(\eta-\sigma)(r_0/\rho^3)(x\,dy-y\,dx)dt$ (see later in Sec. IX). Using $r_0=M/\eta$, we find that the effect of the scalar field is equivalent to multiplying the Kerr part by the factor $[(2\varpi+3)/(2\varpi+4)]$, which is exactly the PPN prediction as well.

Regarding the values given in Eqs. (98) as those determined from the weak-field boundary conditions, we can now rewrite the exact form of Sagnac delay given in Eq. (35)

$$|\delta\tau_0|=(4\pi a)\times\left[\frac{\left(1-\frac{2M}{\eta R}\right)^{\eta}-\left(1-\frac{2M}{\eta R}\right)^{\sigma}}{\left(1-\frac{2M}{\eta R}\right)^{\eta/2}}\right] \tag{99}$$

which yields, to second order in $(M/R)^2$,

$$|\delta\tau_0|\approx\frac{8\pi aM}{R}\left[\frac{2\varpi+3}{2\varpi+4}\right]\left[1+\frac{M}{R}\left(\frac{1}{2\varpi+4}-\sqrt{\frac{2\varpi+3}{2\varpi+4}}\right)\right]. \tag{100}$$

Equation (50) represents the corrections due to other physical factors (such as the moment of inertia, etc.), and using the boundary values in Eqs. (98) one can easily deduce how the scalar field combines with them through the appearance (or absence) of ϖ.

An exact expression for the Sagnac delay for polar orbits can be obtained by plugging in the value of (t_+-t_-) from Eq. (59) into Eq. (65). A similar expression can be obtained for the geodesic motion using Eqs. (45), (77), and (79). Expansion of these exact expressions would enable us to assess the influence of other physical factors as well as the involvement of the scalar field.

A simple demonstration will reveal that calculations in both the Jordan and Einstein frames lead to the *same* ϖ factors for the corrections. Turning to the calculations in the Einstein frame for which the KB metric is given by Eq. (18), we find from the PPN requirement that

$$(\eta-\sigma)r_0=M, \quad \sigma-\eta\rightarrow\sigma-\eta=\sqrt{\frac{2\varpi+3}{2\varpi+4}},$$
$$\eta\rightarrow\eta-\sigma=\sqrt{\frac{2\varpi+4}{2\varpi+3}}, \tag{101}$$
$$1-(\eta-\sigma)^2=(2\varpi+3)\sigma^2.$$

Then the first-order correction term in Eq. (83) reads

$$\frac{8\pi aM}{R}\left[\frac{2\varpi+3}{2\varpi+4}\right],$$

which is precisely the same as the first term in Eq. (100). Use of Eqs. (101) would enable us to see also that Eqs. (50) and (83), (69) and (85), and (81) and (86) are actually the same.

C. BD numerical estimates

In order to compare Eq. (92) with the corresponding situation in the BD theory, we should consider the case when the source or observer is fixed on the surface of the Earth, viz., $\omega_0=\Omega_\oplus$. The various correction terms are, for the equatorial orbit, setting $\omega_0=\Omega_\oplus$ in Eq. (50) and using the identification $r_0=M/\eta$,

$$\frac{4\pi GM_\oplus R_\oplus\Omega_\oplus}{c^4}\left(1-\frac{2\xi}{\eta}\right)\approx2.84\times10^{-16}\left(1-\frac{2\xi}{\eta}\right)\text{ s}, \tag{102}$$

$$\frac{8\pi GM_{\oplus}a_{\oplus}}{R_{\oplus}c^4}\left(\frac{\sigma}{\eta}-1\right)\approx 1.89\times 10^{-16}\left(\frac{\sigma}{\eta}-1\right)\text{ s.} \quad (103)$$

These estimates suggest that the first two corrections in Eq. (50) are at least three orders of magnitude higher than the STR one if η and σ assume nearly Kerr values. For visible light, $v\sim 10^{14}$ Hz, and ignoring for the moment the BD parameters $(1-2\xi/\eta)$ and $(\sigma/\eta-1)$, the expected fringe shift would be $\sim 10^{-2}$ and the parameters would alter the above multiplicative coefficients. Thus, depending on the deviation of the observed shift from this resulting value, we might conclude about the existence of BD scalar field.

In computing the polar and geodesic cases, Tartaglia [12] considers polar and geodesic trajectories of the same radius $R=7\times 10^6$ m. Then, our Eq. (68) for polar orbits reveals the following: If we take $\omega_0=(1/R)\sqrt{GM/R}$, the first and the second terms are of order $\sim 10^{-15}(1-2\xi/\eta)$ s and $\sim 10^{-18}$ s, respectively. Considering the first term, one has an expected fringe shift of order $\sim 10^{-1}(1-2\xi/\eta)$ s for visible light. From the difference in Eq. (69), we find that the first term on the right-hand side (rhs) is of $\sim 10^{-16}(\sigma/\eta-1)$ s, or equivalent to a $10^{-2}(\sigma/\eta-1)$ fringe shift, but the advantage of this equation is that one need not fix a "zero" or a "pure" Sagnac term (that is, the one unaffected by either gravity or scalar field).

For a circularly orbiting geodesic source or observer (Earth-bound satellites, for example) with an orbit radius, say, $R=7\times 10^6$ m, the first term on the rhs of Eq. (81) is 7.35×10^{-6} s. This delay corresponds to a fringe shift of $\sim 10^8$ for visible light, which should be immensely measurable. A first-order correction to this, namely, the second term in Eq. (81) is of the order $\sim 10^{-16}(\sigma/\eta-1)$ s. Therefore, a better correction term still follows from Eqs. (50) [which is of the order of $\sim 10^{-1}(1-2\xi/\eta)$] and it would put bounds on ϖ. One then has to compare these bounds with the Kerr values in order to determine whether a BD scalar field is feasible or not. Even if we take the lowest value for ϖ, viz., $\varpi=500$, the coefficients in Eqs. (102) and (103), respectively, would change only very minutely. Accordingly, the required measurement has to be very precise so that such small deviations are detectable. Feasibilities of such measurements are discussed next.

D. Optical and matter-wave interferometric measurements

Bounds on ϖ at least from the leading term $(8\pi aM/R)[(2\varpi+3)/(2\varpi+4)]$ should be within the realm of experimental feasibility. The discussion in Sec. VIII C reveals that Earth-bound verification of the Kerr and/or BD corrections to the basic Sagnac effect requires the detection of delays $O(10^{-14}-10^{-18}\text{ s})$ or $O(1-10^{-4})$ fringes, or equivalently, $O(10^{-6}-10^{-10})\Omega_{\oplus}$ in interferometry experiments. In single-input-port optical gyroscopes and rotation sensors the minimal detectable phase scales as $\Delta\phi=O(1/\sqrt{N})$, where N is the number of particles passing through the device per unit time [30]. Currently devices are operating near this shot noise limit and can detect angular velocities of $O(10^{-10})\Omega_{\oplus}$ [31].

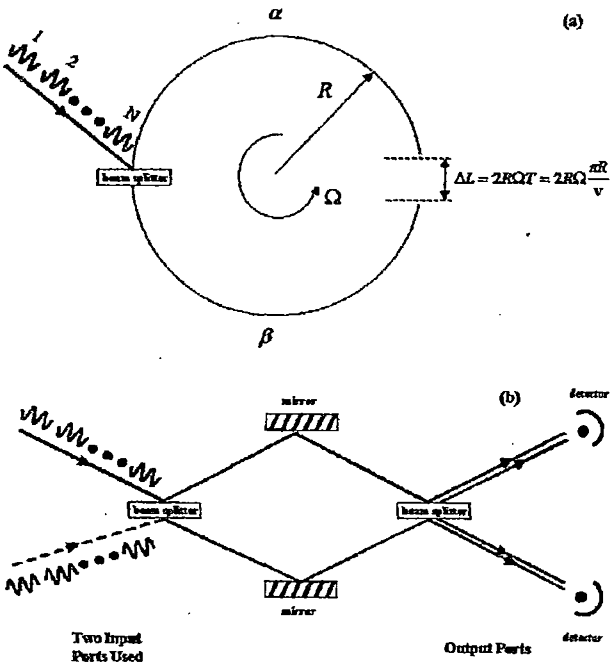

FIG. 1. (a) A schematic illustration of an idealized light or matter-wave interferometer used as a rotation sensor or gyroscope (after [30]). The interferometer has circular arms of length R and rotates with angular velocity Ω with N atoms passing one-at-a-time through a beam splitter. The path difference between the upper and lower branches α and β is given by $\Delta L=2R\Omega T$, where $T=2\pi R/c$ for light and $T=\pi R/v$ for matter. (b) A two-input-port quantum interferometer. Quantum states are entangled (correlated) at the input ports and phase shifts are measured at the output ports. The use of correlated quantum states in the interferometer allows for minimum phase sensitivities which scale as $\Delta\phi=O(1/N)$ versus the uncorrelated state shot-noise limit of $\Delta\phi=O(1/\sqrt{N})$.

On the other hand, the use of material particles instead of light holds great promise in the field of interferometry and rotational sensors. The advantage of using matter over light in interferometers can be seen as follows: consider an interferometer with semicircular arms rotating with angular frequency Ω about an axis through its center and normal to the loop plane depicted in Fig. 1(a). In a given time T, particles traversing in the same and opposite rotational sense as the interferometer will travel a distance $L_+=2\pi R+R\Omega T$ and $L_-=2\pi R-R\Omega T$, respectively, yielding a path difference of $\Delta L=2R\Omega T$. For light with a single beam-splitter input/output port we have $T=2\pi R/c$, so that we recover Eq. (1) via $\delta\tau_s=\Delta L/c$. However, for particles of mass m traveling at velocity v, with a beam-splitter output port located diametrically opposite the input port, we have $T=\pi R/v$. This leads to $\delta\phi_{matter}=k\Delta L=2A\Omega/\lambda_r v$, where $\lambda_r=\lambda/2\pi$ is the reduced wavelength. For matter, $\lambda_r=\hbar/mv$ is the de Broglie wavelength and the phase signal is given by $\delta\phi_{matter}=2A\Omega m/\hbar$. For light, we can define the "photon mass" by $m_\gamma c^2=\hbar\omega$. Thus the inherent sensitivity of a matter-wave interferometer exceeds that of a photon-based system by the mass-enhancement factor $mc^2/\hbar\omega\approx 10^{10-11}$. This impressive mass-enhancement factor for matter-wave interferometers is offset by a factor of $O(10^4)$ for smaller particle fluxes and $O(10^4)$ smaller number of cavity round trips (usually 1 for matter and 10^4 for light). Matter-wave interferometry experiments have seen to date a sensitivity of 2×10^{-8} (rad/s)$\sqrt{Hz}$ [32], which is comparable to the best active ring laser gyroscopes, and they are getting better.

The use of quantum entangled input states or correlated-two-input-port interferometers offers exciting possibilities for the future [30]. A single input-port interferometer can be considered as a two-input-port device where light or matter enters in one port (i.e., one side of a beam splitter) as the source and the ever present vacuum enters the second (empty) port. The minimal detectable phase scales as $\Delta\phi = O(1/\sqrt{N})$, where N is the number of particles passing through the device in unit time. In a two-input-port device, a nonvacuum state is presented to each port and is correlated at the input beam splitter as shown in Fig. 1(b). The use of quantum entangled states (for both matter and light) leads to minimal detectable phase sensitivity scales as $\Delta\phi = O(1/N)$. It can be shown that a two-input-port matter-wave interferometer can be 10^6 more sensitive than a single-input-port matter-wave interferometer, a two-input-port optical interferometer can be 10^8 times more sensitive than a single-port optical interferometer, and a two-input-port matter-wave interferometer can be an impressive 10^{10} times more sensitive than a single-input port optical interferometer.

Clearly there are considerable technical challenges to overcome in bringing such devices to fruition. *Decoherence*, the intrinsic quantum decay that ensues when a quantum system is coupled to undesired states, can degrade the performance of matter-wave or entangled quantum detectors and reduce the phase sensitivity back down to $\Delta\phi = O(1/\sqrt{N})$ [33]. This result can sometimes occur since, although the phase sensitivity increases with the number of particles N used in the interferometer, the decoherence rate grows commensurately. However, even with decoherence issues considered, current experiments are already making significant strides towards realizations of matter-wave and entangled quantum state interferometers useful for measuring the Sagnac effect [32]. With such promise, we may someday soon be able to experimentally detect the higher-order general relativistic corrections to the Sagnac effect and be able to place tighter bounds on the BD parameters.

IX. GEODETIC AND LENSE-THIRRING PRECESSION

We can also investigate the effects of the KB metric on the precession of a spherical gyroscope in a circular polar orbit around the Earth as a means to experimentally measure or bound the values of the parameters η, ξ, σ or just ϖ. The Stanford Gravity Probe-*B* experiment [34] is just such an experiment which will use a superconducting niobium-coated quartz spherical gyroscope (machined to a precision greater than 10^{-6} cm) to detect gravitational precession effects arising from the geodetic motion of the satellite and due to the rotation of the Earth (the Lense-Thirring effect). In the following, we follow the calculation of Ohanian and Ruffini [35] by writing the KB metric to first order in $\varepsilon = r_0/r$ and $\zeta = a/r$, converting to isotropic coordinates, and then computing the parallel transport equation for the spin S^μ of the gyroscope as it is carried about the polar circular orbit. Isotropic coordinates (x,y,z) are used since a change in the rectangular components of the spin vector can be immediately attributed to the curvature of space time, whereas a change in curvilinear components contains contributions both from the curvature of the coordinates and the curvature of space time.

We begin with the KB metric in the Jordan frame, Eq. (9), and expand it to first order in ε, ζ to obtain

$$\begin{aligned} ds^2 \cong & (1-2\eta\varepsilon)dt^2 - [1-2(\xi-1)\varepsilon]dr^2 \\ & -(1-2\xi\varepsilon)(r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2) \\ & +4(\eta-\sigma)\varepsilon\zeta\sin^2\theta r\, d\phi\, dt. \end{aligned} \tag{104}$$

The change to a radial isotropic coordinate is the same as in the Schwarzschild case (see [36], p. 196ff and p. 256ff) and is given by $r=\rho(1+r_0/2\rho)^2 \approx \rho(1+r_0/\rho)$, where ρ is the radial isotropic marker. To lowest order $\varepsilon \to \varepsilon' \equiv r_0/\rho$, $\zeta \to \zeta' = a/\rho$, and from now on we drop the primes on ε, ζ. Carrying out the change to a radial isotropic coordinate and using coordinates $x=\rho\sin\theta\cos\phi$, $y=\rho\sin\theta\sin\phi$, $z=\rho\cos\theta$, $|d\vec{\rho}|^2 = dx^2+dy^2+dz^2$ fixed to the center of the Earth and nonrotating with respect to the distant stars, and noting $\rho^2\sin^2\theta d\phi = x\,dy - y\,dx$, we arrive at

$$\begin{aligned} ds^2 \cong & \left(1-\frac{2\eta r_0}{\rho}\right)dt^2 - \left(1+2(1-2\xi)\frac{r_0}{\rho}\right)|d\vec{\rho}|^2 \\ & +4(\eta-\sigma)\frac{r_0 a}{\rho^3}(x\,dy - y\,dx)dt. \end{aligned} \tag{105}$$

Comparison with the Kerr metric [35,36] allows us to identify the last term of Eq. (105) with the rotation of the mass M (where $r_0 = GM/c^2$). In going from the Kerr to the KB metric we have the identification $a_{\rm KB} = (1-\sigma/\eta)a_{\rm Kerr}$, where $a=-J/Mc$ is the angular momentum per unit mass of the rotating body (for a body rotating in the positive sense $J>0$, a is negative, see [36], p. 258).

We are now interested in computing the change in the spatial components of the spin S^μ of a gyroscope in a circular polar orbit, as depicted in Fig. 2. We will first evaluate the parallel transport equations for the spin at a single point $\vec{\rho} = (0,\rho,0)$ of the orbit where the four-velocity is given by $\dot{x}^\mu \equiv dx^\mu/d\tau = (1,0,0,v)$ and where the velocity v of the satellite has a value on the order of $\sqrt{GM/\rho}$. The equation for the parallel transport of the spin is given by

$$\dot{S}^\mu \equiv \frac{dS^\mu}{d\tau} = -\Gamma^\mu_{\alpha\beta} S^\alpha \dot{x}^\beta. \tag{106}$$

A lengthy, though straightforward, calculation yields the Christoffel symbols evaluated at the point $\vec{\rho} = (0,\rho,0)$ to be

$$\Gamma^0_{02} = \eta r_0/\rho^2, \quad \Gamma^0_{12} = -3(\eta-\sigma)r_0 a/\rho^3,$$

$$\Gamma^1_{02} = -(\eta-\sigma)r_0 a/\rho^3, \quad \Gamma^1_{12} = -(1-2\xi)r_0 a/\rho^2,$$

$$\Gamma^2_{00} = \eta r_0/\rho^2, \quad \Gamma^2_{01} = (\eta-\sigma)r_0 a/\rho^3,$$

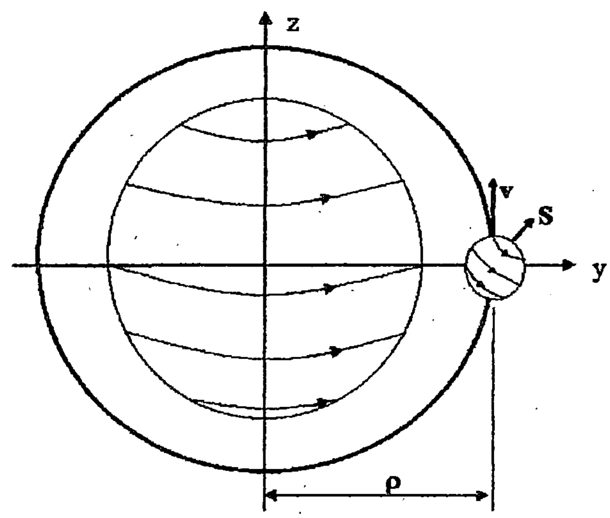

FIG. 2. A spherical gyroscope in a circular polar orbit about the Earth. At one instant, the gyroscope is at the position $x=0$, $y=\rho$, $z=0$ with instantaneous spatial velocity v along the $\hat{z}$ direction.

$$\Gamma^2_{11}=-\Gamma^2_{22}=\Gamma^2_{33}=(1-2\xi)r_0/\rho^2,$$

$$\Gamma^3_{23}=-(1-2\xi)r_0/\rho^2. \tag{107}$$

We note that in the comoving reference frame of the satellite the spin is purely spatial $S'^0=0$, and the four-velocity is purely temporal, $\dot{x}'^{\mu}=(1,0,0,0)$ so that the relationship $g'_{\alpha\beta}S'^{\alpha}\dot{x}'^{\beta}=0$ holds. Since this is a tensor equation, it must also hold in the reference frame centered on the Earth, $g_{\alpha\beta}S^{\alpha}\dot{x}^{\beta}=0$. This constraint allows us to solve for $S^0=-1/(g_{00}+vg_{03})\Sigma_{i=1,3}(g_{i0}+vg_{i3})S^i\simeq vS^3+O(\varepsilon)$. Substituting this and the Christoffel symbols into Eq. (106) yields the equations

$$\dot{S}^1=\frac{r_0a}{\rho^3}S^2,$$

$$\dot{S}^2=-2\frac{r_0v}{\rho^2}S^3-\frac{r_0a}{\rho^3}S^1,$$

$$\dot{S}^3=\frac{r_0v}{\rho^2}S^2. \tag{108}$$

The terms proportional to v give rise to the geodetic precession while those proportional to a give rise to the Lense-Thirring precession. Although Eq. (108) was derived for a specific point on the orbit, we can generalize to any point on the orbit as follows. For $a=0$ we can write Eq. (108) as

$$\dot{\vec{S}}_g=-(1+\eta-2\xi)(\vec{v}\cdot\vec{S}_g)\vec{\nabla}\Phi+(1-2\xi)(\vec{S}_g\cdot\vec{\nabla}\Phi)\vec{v}, \tag{109}$$

where $\vec{S}_g$ refers to the geodetic contribution to the spin and $\Phi=-GM/\rho$ is the Newtonian gravitational potential. We are interested in the long-term secular change in the spin. As such we express the orbit of the satellite as $\vec{\rho}=\rho(0,\cos\omega_s t,\sin\omega_s t)$, where ω_s is the angular velocity of the satellite. Inserting $\vec{v}=d\vec{\rho}/dt=v(0,-\sin\omega_s t,\cos\omega_s t)$, where $v=\rho\omega_s$ and $\vec{\nabla}\Phi=r_0/\rho^2(0,\cos\omega_s t,\sin\omega_s t)$ into Eq. (109) and averaging over one period yields

$$\langle\dot{\vec{S}}_g\rangle=(1+\eta/2-2\xi)\frac{r_0v}{\rho^2}(-S^3\hat{x}+S^2\hat{y})\equiv\vec{\Omega}^{\rm KB}_g\times\vec{S}_g,$$

$$\vec{\Omega}^{\rm KB}_g\equiv(1+\eta/2-2\xi)\frac{r_0}{\rho^3}\vec{\rho}\times\vec{v}=\frac{2}{3}(3/2-2\xi/\eta)\vec{\Omega}_g, \tag{110}$$

where $\vec{\Omega}^{\rm KB}_g$ is the geodetic precession which reduces to the Schwarzschild and Kerr form $\vec{\Omega}_g=3M/2\rho^3\vec{\rho}\times\vec{v}$ [35] in the limit $\{\eta\to1,\xi=\sigma\to0\}$. The geodetic precession of the spin $\vec{\Omega}_g$ is in the plane of the orbit and in the direction of the orbital motion of the satellite.

A similar calculation can be performed for the "gravitomagnetic" terms proportional to a in Eq. (108). These Lense-Thirring terms lead to the precession of the spin in the direction perpendicular to the orbit and in the same sense as the rotation of the Earth ("frame dragging"),

$$\vec{\Omega}^{\rm KB}_{LT}=\frac{(\eta-\sigma)ar_0}{\rho^3}\left\langle\frac{3}{2}(\vec{\rho}\cdot\hat{S}_{\oplus})\vec{\rho}\right\rangle-\hat{S}_{\oplus}=(1-\sigma/\eta)\vec{\Omega}_{LT}, \tag{111}$$

where $\hat{S}_{\oplus}$ is a unit vector in the direction of the spin of the Earth (here $\hat{S}_{\oplus}=\hat{z}$). As we observed earlier from the metric Eq. (105), this is just the usual Kerr Lense-Thirring precession $\vec{\Omega}_{LT}$ [35] with $a_{\rm KB}=(1-\sigma/\eta)a_{\rm kerr}$. Performing the time average as above one obtains

$$\langle\vec{\Omega}^{\rm KB}_{LT}\rangle=\frac{(\eta-\sigma)ar_0}{2\rho^3}\hat{S}_{\oplus}=(1-\sigma/\eta)\langle\vec{\Omega}_{LT}\rangle. \tag{112}$$

For a 650-km circular polar orbit, as depicted in Fig. 2, with the spin of the satellite in the plane of the orbit, $v=\sqrt{GM/\rho}$ and $\{\eta\to1,\xi=\sigma\to0\}$ we obtain the values $|\vec{\Omega}_g|=6.6''/\mathrm{yr}$, $|\vec{\Omega}_{LT}|=0.042''/\mathrm{yr}$ [35]. Thus, for the KB metric in both the frames, these values would be multiplied by $2/3(3/2-2\xi/\eta)$ and $(1-\sigma/\eta)$, respectively [obtained by using $r_0=M/\eta$ or $r_0=M/(\eta-\sigma)$]. Since the Gravity Probe-B experiment is capable of measuring the bare $\{\eta\to1,\xi=\sigma\to0\}$ values of these precessions, any possible deviations due to the Kerr-like BD scalar field should be detectable.

X. SUMMARY

In the foregoing, our aim was to examine how the presence of a BD scalar field modifies the gravitational correction terms to the Sagnac effect. To our knowledge, such an analysis has not been undertaken heretofore. A first-order effect on the geodetic and Lense-Thirring precession was also computed. It was found that the presence of the scalar field introduces a combination of different BD factors η, σ, ξ

into the correction terms. The obtained results are of both theoretical and practical importance: The values of η and σ away from the Kerr values would indicate the presence of the BD scalar field.

The paper derives *exact* expressions for the scalar field modified Sagnac delay. The unknown BD factors can be determined in terms of ϖ by using an input from the PPN analysis, viz., $\gamma=(1+\varpi)/(2+\varpi)$, as a *boundary* condition. From the expansion of the exact expressions, it is possible to directly find out corrections to *all* orders, visualize the physical characters of these terms, and assess how the scalar field modifies each of them. Thus, the present formulation offers two distinct theoretical advantages: (1) It is applicable also in the strong field where the usual PPN analysis fails. (2) It has a flexibility in the sense that *any* functional choice of $\gamma(\varpi)$ is admissible leading to forms of $\eta(\varpi)$ and $\sigma(\varpi)$ different from those in Eqs. (98). The possibility of a non-PPN γ and its physical implications are discussed in Ref. [39], but are not pursued in this paper.

From a practical standpoint, a first-order fringe shift of $\sim 10^{-1}(1-2\xi/\eta)$ is predicted for the Sagnac delay for Earth-bound equational orbits ($R=7\times 10^6$ m), which should be measurable given the accuracy being attained by the current technology. The most exciting promise is offered by the Stanford Gravity Probe-B experiment which is attempting to measure the geodetic and Lense-Thirring precessions for Earth-bound orbits. As shown above, the multiplying factors to the first-order corrections are, respectively, $\frac{2}{3}[\frac{3}{2}-(2\xi/\eta)]$ and $1-(\sigma/\eta)$. For an estimate, taking $\varpi=500$, we find, using the PPN values in Eq. (98), that $|2\xi/\eta|\approx 2.98\times 10^{-3}$, $|\sigma/\eta|\approx 9.96\times 10^{-4}$.

It was demonstrated that the observable predictions in the two frames are identical, as expected. All the equations presented in this work reduce to those in the Kerr case. Lastly, Eq. (35) represents the exact BD expression for the gravitational analog of the Aharonov-Bohm effect [10,37,38]. We will have more to say about this in a forthcoming paper.

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String corrections to the Sagnac effect

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Abstract

In an effort to investigate string effects in physical observations, we have analyzed the rotating Kerr–Sen metric in a Sagnac type experiment and have deduced exact expressions for the delay. For an Earth bound configuration, it turns out that a correction to the basic Sagnac delay by an order of $\sim 10^{-14}$ s leads to a terrestrial dilatonic charge of amount $\sim 10^{24}$ esu, a value nearly 200 times larger than the electronic charge of the Earth's magnetosphere.

1. Introduction

Low energy effective field theory describing heterotic string theory has now become an indispensable part of the frontiers of theoretical physics [1]. An interesting result is that black hole solutions exist also in the string theory and that they exhibit qualitatively different properties than those of Einstein's general relativity [2]. A rotating black hole solution, that reduces to the Kerr solution for a constant dilaton field, has been constructed and analyzed by Horne and Horowitz [3].

A more general classical exact solution has been found by Sen [4] which we refer to here as the Kerr–Sen metric. The action underlying the theory has a $U(1)$ gauge symmetry and contains antisymmetric tensor gauge field. Also, 6 of the 10 dimensions are compactified to a suitable manifold, but the resulting massless fields are not included in the action. The absence of these fields enhances the possibility of the black hole nature of the solution; otherwise, naked singularities could arise. Kerr–Sen solution describes a rotating black hole carrying finite amount of charge and angular momentum and it differs from the Horowitz–Horne black hole even in the limit of small angular momentum. This difference arises due to the coupling of the antisymmetric tensor gauge field to the Chern–Simons term in the action considered by Sen [4]. These, and other developments taken together, indicate that there have been tremendous theoretical advances in the understanding and utility of the string theory. However, relatively much less is discussed in the literature as to how a black hole in the string theory could possibly affect physical observations in practice. It has been shown by Gegenberg [5] that *static* spherically symmetric solutions do not lead to string effects in the PPN approximation of the solar system scenario

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but that, he conjectured, *rotating* solutions might lead to observable string effects, however tiny.

The present Letter aims to undertake an investigation precisely in this direction. For this purpose, we consider the Kerr–Sen metric and examine how the black hole parameters appear in the correction terms in a Sagnac-type experiment. We shall find *exact* expressions for the time delay by following the procedure of Tartaglia [6] which we had also adopted in our recent investigation of the Brans–Dicke correction factors for different types of orbits [7]. An estimate of the possible terrestrial dilatonic charge and some remarks are also added.

We have chosen the Sagnac effect because of its simplicity and its easy adaptability to rotating sources. The effect stems from the basic physical fact that the round trip time of light around a closed contour, when the source is fixed on a turntable, depends on the angular velocity, say Ω, of the turntable. Using special theory of relativity, and assuming $\Omega r \ll c$, one obtains the proper time difference $\delta\tau_s$ when the two beams meet again at the starting point as [6]

$$\delta\tau_s \cong \frac{4\Omega}{c^2}S, \tag{1}$$

where c is the vacuum speed of light, S $(=\pi r^2)$ is the projected area of the contour perpendicular to the axis of rotation. It is a real physical effect in the sense that it does not involve any arbitrary synchronization convention that is required between two distant clocks [8]. The effect is also universal as it manifests not only for light rays but also for all kinds of waves including matter waves [9]. Formula (1) has been tested to a good accuracy and the remarkable degree of precision attained lately by the advent of ring laser interferometry raises the hope that measurements of higher-order corrections to this effect might be possible in near future [6,10].

The string theory effective action in 4 dimensions, considered by Sen [4], is

$$S = -\int d^4x\,\sqrt{-g}\,e^{-\Phi} \times \left(-R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi + \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\right), \tag{2}$$

where $g_{\mu\nu}$ is the metric that arises naturally in the σ-model, R is the Ricci scalar, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the Maxwell field A_μ, Φ is the dilaton field, and

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} - \left[\Omega_3(A)\right]_{\mu\nu\rho}, \tag{3}$$

where $B_{\mu\nu}$ is the antisymmetric tensor gauge field and

$$\left[\Omega_3(A)\right]_{\mu\nu\rho} \equiv \frac{1}{4}\left(A_\mu F_{\nu\rho} + A_\nu F_{\rho\mu} + A_\rho F_{\mu\nu}\right) \tag{4}$$

is the gauge Chern–Simons term. The Einstein frame metric is obtained from the relation

$$g^{(E)}_{\mu\nu} = e^{-\Phi}g_{\mu\nu}.$$

For our purposes, we recast the Einstein frame Kerr–Sen metric into a form that closely resembles the familiar Kerr solution in Boyer–Lindquist coordinates. The result is $(G = c = 1)$

$$d\tau^2 = \left(1 - \frac{2M\rho}{\Sigma}\right)dt^2 - \Sigma\left(\frac{d\rho^2}{\Delta} + d\theta^2\right) - \left[\rho(\rho+\xi) + a^2 + \frac{2M\rho a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\varphi^2 + \frac{4M\rho a\sin^2\theta}{\Sigma}\,dt\,d\varphi, \tag{5}$$

$$\Phi = -\ln\left[\frac{\Sigma}{\rho^2 + a^2\cos^2\theta}\right],$$

$$A_\varphi = -\frac{2\sqrt{2}a\rho Q\sin^2\theta}{\Sigma}, \qquad A_t = \frac{2\sqrt{2}\rho Q}{\Sigma},$$

$$B_{t\varphi} = \frac{a\rho Q^2\sin^2\theta}{M\Sigma}, \qquad \Sigma = \rho(\rho+\xi) + a^2\cos^2\theta,$$

$$\Delta = \rho(\rho+\xi) + a^2 - 2M\rho, \qquad \xi = \frac{Q^2}{M}. \tag{6}$$

The metric describes a black hole with mass M, dilatonic charge Q, angular momentum aM, and magnetic dipole moment aQ. For $Q = 0$, the metric reduces to the Kerr solution of GR and for $a = 0$, it reduces to the Gibbons–Garfinkle–Horowitz–Strominger black hole solution [2] with the redefinition: $\rho \to r - \xi$. Using metric (5), we proceed to calculate the proper Sagnac delay for three types of source/receiver orbits: equatorial, polar and geodesic circular orbits.

2. Equatorial orbit

Suppose that the source/receiver of two oppositely directed light beams is moving around an uncollapsed normal gravitating body, along a circumference at a radius $\rho = R = \text{const}$, on the equatorial plane $\theta = \pi/2$. Suitably placed mirrors send back to their origin both beams after a circular trip about the central rotating body. Let us further assume that the source/receiver is moving with uniform orbital angular speed ω_0 with respect to distant stars such that the rotation angle is

$$\varphi_0 = \omega_0 t. \tag{7}$$

Under these circumstances, metric (5) reduces to

$$d\tau^2 = \left[\left(1 - \frac{2M}{X}\right) + \frac{4Ma\omega_0}{X} - \left\{X(X-\xi) + a^2 + \frac{2Ma^2}{X}\right\}\omega_0^2\right] dt^2, \tag{8}$$

where $X \equiv R + \xi$. The trajectory of a light ray is given by $d\tau^2 = 0$ which gives

$$\left(1 - \frac{2M}{X}\right) + \frac{4Ma\omega}{X} - \left\{X(X-\xi) + a^2 + \frac{2Ma^2}{X}\right\}\omega^2 = 0, \tag{9}$$

where ω is the angular speed of photons. The two roots $\Omega_\pm$ of the quadratic equation (9) provide the rotation angles for the light rays:

$$\varphi_\pm = \Omega_\pm t = \frac{\Omega_\pm}{\omega_0}\varphi_0. \tag{10}$$

The first intersection of the world lines of the two light rays with the world line of the orbiting observer after emission at time $t = 0$ occurs when

$$\varphi_+ = \varphi_0 + 2\pi, \qquad \varphi_- = \varphi_0 - 2\pi, \quad \text{or,}$$
$$\frac{\Omega_\pm}{\omega_0}\varphi_0 = \varphi_0 \pm 2\pi, \tag{11}$$

where $+$ and $-$ refer to co-rotating and counter-rotating beams. Solving for φ_0, one finds from Eqs. (11) for the two $\pm$ beams:

$$\varphi_{0\pm} = \pm\frac{2\pi\omega_0}{\Omega_\pm - \omega_0}. \tag{12}$$

The proper time as measured the orbiting observer is found from Eq. (8) by using $dt = d\varphi_0/\omega_0$, and integrating between φ_{0+} and φ_{0-}. The final result, which is the Sagnac delay, is given by

$$\delta\tau_E = \frac{4\pi}{X} \times \frac{\omega_0(X^3 + a^2X + 2Ma^2 - X^2\xi) - 2Ma}{[(1 - 2M/X) + 4Ma\omega_0/X - \{X(X-\xi) + a^2 + 2Ma^2/X\}\omega_0^2]^{1/2}}. \tag{13}$$

This is an exact expression for the delay and it is zero if the angular speed ω_0 of the orbiting observer is such that the numerator in Eq. (13) is zero. On the other hand, if the observer keeps fixed positions with respect to distant stars so that $\omega_0 = 0$, then the Sagnac delay becomes

$$\delta\tau_0 = -\frac{8\pi aM}{X(1 - 2M/X)^{1/2}}. \tag{14}$$

The effect of the dilatonic charge Q is evident in Eqs. (13) and (14) through the appearance of the factor ξ and the values of $\delta\tau_E$, $\delta\tau_0$ are different from the Kerr case if $\xi \neq 0$.

Assuming $\varepsilon = M/R \ll 1$ and $\beta = \omega_0 R \ll 1$, the correction terms to the basic Sagnac delay $\delta\tau_S$ ($\equiv 4 \times \pi\beta R$) to first order in ε and β are obtained from Eq. (13):

$$\delta\tau_E \cong \delta\tau_S - 8\pi a\varepsilon + 4\pi\beta R(3\eta - 2\varsigma), \tag{15}$$

where

$$\eta = \frac{M}{R + \xi}, \qquad \varsigma = \frac{2M - \xi}{2R}. \tag{16}$$

The second term in Eq. (15) is a contribution purely due to the angular momentum J while the last term displays the contribution from ξ ($\equiv Q^2/M$).

3. Polar orbit

We shall now investigate the effect when the light rays move along a circular trajectory passing over the poles. In this case, too, we may take $\rho = R = \text{const}$ and $\varphi = \text{const}$. Assuming uniform motion again, we take $\theta = \omega_0 t$. Then, we have, using $d\rho = 0$, $d\varphi = 0$, $d\theta = \omega_0\, dt$ and $d\tau^2 = 0$, from metric (5):

$$\frac{d\theta}{dt} = \pm\frac{(R^2 - 2MR + a^2\cos^2\theta)^{1/2}}{R(R + \xi) + a^2\cos^2\theta}. \tag{17}$$

Assuming that $a^2/R^2 \ll 1$, $t = 0$ when $\theta = 0$, we have, on integration,

$$t \cong \frac{R+\xi}{(1-2M/R)^{1/2}}\theta + \frac{a^2}{4R}\frac{1-(4M+\xi)/R}{(1-2M/R)^{3/2}}(\cos\theta\sin\theta+\theta). \tag{18}$$

During this time t, the rotating observer describes an angle θ_0 (say) while light travels an angle $2\pi \pm \theta_0$ (once again, $+$ for co-rotating beam and $-$ for the counter-rotating beam) so that one obtains, after some manipulations [7],

$$\frac{\theta_0}{\omega_0} = (p+q)(2\pi \pm \theta_0) \pm \frac{q}{2}\sin 2\theta_0, \tag{19}$$

where

$$p \equiv \frac{R+\xi}{(1-2M/R)^{1/2}},$$
$$q \equiv \frac{a^2}{4R}\frac{1-(4M+\xi)/R}{(1-2M/R)^{3/2}}. \tag{20}$$

Assuming a low speed observer and that the angle $2\theta_0$ be so small as to justify $\sin 2\theta_0 \cong 2\theta_0$, we get, on solving for θ_0, from Eq. (19):

$$\theta_{0\pm} = 2\pi\frac{p+q}{1/\omega_0 \mp (p+q) \mp q}. \tag{21}$$

Finally, the difference between two round trip "co-ordinate" times (recalling the approximations already used) comes to

$$t_+ - t_- = \frac{\theta_{0+}-\theta_{0-}}{\omega_0} \cong \pi\omega_0(R+\xi)^2\left\{4+8\varepsilon+\frac{3a^2}{R(R+\xi)}\right\}, \tag{22}$$

where we have retained terms of the order of ω_0 and ε only.

It is now necessary to express the above time difference in terms of the proper time τ of the rotating observer. This is done using in metric (5), $\theta = \omega_0 t$, $d\theta = \omega_0\, dt$:

$$\tau = \int\left[\frac{\Sigma-2MR}{\Sigma} - \omega_0^2\Sigma\right]^{1/2} dt,$$
$$\Sigma \equiv R(R+\xi) + a^2\cos^2(\omega_0 t). \tag{23}$$

Under the assumption of small $\omega_0 t$ such that $\cos(\omega_0 t) \cong 1$, $\sin(\omega_0 t) \cong 0$ and $a^2/R(R+\xi) \ll 1$, we get

$$\tau \cong \left(1 - \frac{2M}{R+\xi} - \omega_0^2 R(R+\xi)\right)^{1/2} t. \tag{24}$$

Therefore, the time delay in the polar case, denoted by $\delta\tau_P$, is given by

$$\delta\tau_P \cong \left(1 - \frac{2M}{R(R+\xi)} - \omega_0^2 R(R+\xi)\right)^{1/2}(t_+ - t_-). \tag{25}$$

Using the result in Eq. (23), we get

$$\begin{aligned}\delta\tau_P &\cong \pi\omega_0(R+\xi)^2 \times\left[4+\frac{3a^2}{(R+\xi)}+4M\left\{\frac{R+2\xi}{R(R+\xi)}\right\}\right] \\ &\cong 4\pi\omega_0 R^2 + \pi\omega_0(3a^2+4MR) \\ &\quad + \pi\omega_0\left(8R\xi + 4\xi^2 + \frac{3a^2\xi}{R} + 12M\xi + \frac{8M\xi^2}{R}\right).\end{aligned} \tag{26}$$

The first term is the basic Sagnac delay $\delta\tau_s$, the second term is the Kerr contribution and the third term is the string correction.

The absolute difference between equatorial and polar Sagnac delay is

$$|\delta\tau_E - \delta\tau_P| \cong 8\pi a\varepsilon + 3\pi\omega_0 a^2 + 4\pi\omega_0\xi(R+6M). \tag{27}$$

4. Geodesic orbit

Consider the geodesic motion of the source/receiver having a 4-velocity u^μ ($\equiv dx^\mu/d\tau$) so that the equations are

$$\frac{\partial u^\mu}{\partial x^\nu}u^\nu + \Gamma^\mu_{\nu\alpha}u^\nu u^\alpha = 0, \tag{28}$$

where $\Gamma^\mu_{\nu\alpha}$ are the Christoffel symbols formed from metric (5). The problem can be simplified by choosing $\theta = \pi/2$, so that $u^\theta = 0$. The geodesic equations this case, do allow such a solution [11]. Then, for a circular geodesic orbit with a constant radius $\rho = R$, the condition is $u^\rho = 0$. Then the radial equation turns out to be

$$\Gamma^\rho_{tt}(u^t)^2 + \Gamma^\rho_{\varphi\varphi}(u^\varphi)^2 + 2\Gamma^\rho_{t\varphi}u^t u^\varphi = 0. \tag{29}$$

Defining the angular speed of rotation of the source/receiver as $\omega = u^\varphi/u^t$, we get, after a simplification, the two roots $\omega_\pm$ from Eq. (29):

$$\omega_\pm = \frac{1}{\partial g_{\varphi\varphi}/\partial\rho} \times \left[-\frac{\partial g_{t\varphi}}{\partial\rho} \pm \sqrt{\left(\frac{\partial g_{t\varphi}}{\partial\rho}\right)^2 - \frac{\partial g_{tt}}{\partial\rho}\frac{\partial g_{\varphi\varphi}}{\partial\rho}} \right]. \quad (30)$$

Using the metric tensor from (5), we get

$$\omega_\pm = \left[\frac{1}{2Ma^2 - (R+\xi)^2(2R+\xi)}\right] \times \left[2Ma \pm \left\{2M(R+\xi)^2(2R+\xi)\right\}^{1/2}\right]. \quad (31)$$

We can put this $\omega_\pm$ in place of ω_0 in Eq. (13) to get the exact expression for the Sagnac delay. The approximation follows from Eq. (15):

$$\delta\tau_G \cong -\frac{8\pi a M}{R} + 4\pi\omega_\pm R^2(1 + 3\eta - 2\varsigma). \quad (32)$$

The leading terms are

$$\delta\tau_G \cong -\frac{8\pi a M}{R} \mp 4\pi\sqrt{MR} \pm \frac{5\pi Q^2}{\sqrt{RM}} = \mp\delta\tau_s - \frac{8\pi a M}{R} - \frac{4\pi a M^2}{R^2} \pm 5\pi\xi\sqrt{\frac{M}{R}}, \quad (33)$$

where $\delta\tau_s$ ($\equiv 4\pi\sqrt{MR}$) is the basic Sagnac delay in the polar case [6]. In the above approximation, we have neglected terms of the order $(\xi/R)^2$, a^2/R^3, etc. because of their smallness.

5. An estimate of the dilatonic charge

In the above, we provided exact formulations of the Sagnac delay for three types of orbits. These lead to correction terms embodied in Eqs. (15), (26) and (33), which reveal, especially, the role of the dilaton parameter ξ. The terms involving only a and M are the Kerr corrections which could be easily evaluated for a given configuration. However, this aside, we could try to have an idea of the possible estimate of the dilatonic charge Q from its contribution to the delay.

One possibility is the following: The existence of a horizon in the Kerr–Sen metric requires that $M^2 > J + (1/2)Q^2$. Thus, if $J \geqslant 0$, one has $Q < M$, that is, the mass of the black hole itself provides the upper limit on Q. But this need not be the case when one considers orbits not around a black hole but around an ordinary uncollapsed object like the Earth. Therefore, the other possibility is to look for values of Q in Earth bound experiments.

Consider, for instance, a polar orbit around the Earth, say at a radius $R = 7 \times 10^6$ m. We also restore G and c in the expressions that follow and recall the relevant Earth values, designated by the subscript $\oplus$:

$$R_\oplus = 6.37 \times 10^6 \text{ m}, \qquad \Omega_\oplus = 7.27 \times 10^{-5} \text{ m/s},$$
$$GM_\oplus/c^2 = 4.4 \times 10^{-3} \text{ m},$$
$$a_\oplus = 9.81 \times 10^8 \text{ m}^2/\text{s},$$
$$c = 3 \times 10^8 \text{ m/s}. \quad (34)$$

For the angular velocity $\omega_0 = (c/R)\sqrt{GM_\oplus/Rc^2}$, the main Sagnac term becomes $\delta\tau_S = 4\pi\omega_0 R^2/c^2 \approx 7.35 \times 10^{-6}$ s, while the Kerr contribution, designated by $\delta\tau_K$, is $\delta\tau_K = (\pi\omega_0/c^2)(3a^2 + 4MGR)$. The first term is $\approx 1.39 \times 10^{-18}$ s, while the second term is $\approx 4.84 \times 10^{-15}$ s, respectively. The dilatonic contribution, designated by $\delta\tau_\xi$, is

$$\delta\tau_\xi = \frac{\pi\omega_0}{c^2}\left(8R + \frac{12M_\oplus G}{c^2} + \frac{3a_\oplus^2}{Rc^2}\right)\xi + \frac{\pi\omega_0}{c^2}\left(4 + \frac{8M_\oplus G}{Rc^2}\right)\xi^2. \quad (35)$$

Note that ξ has the dimension of length, and so has Q. For the above expansion to be meaningful, we have assumed $\xi < 1$ m.

For the values cited in (34), the leading term in Eq. (35) is

$$\delta\tau_\xi = \frac{8\pi\omega_0 R}{c^2}\xi \approx 4.75 \times 10^{-10} Q^2 \text{ s/m}^2. \quad (36)$$

Since $\xi \equiv Q^2c^2/GM_\oplus = 2.27 \times 10^2 Q^2$ m^{-1} and $\xi < 1$ m, we can, in general, take $Q^2 \approx 10^{-\beta}$ m^2, $\beta > 2$ so that $Q \approx 10^{26-\beta/2}$ esu noting that 1 m $\approx 10^{26}$ esu. Then, $\delta\tau_\xi \approx 4.75 \times 10^{-(10+\beta)}$ s. This implies that $Q \approx 10^{26-\beta/2}$ esu. Obviously, the value of β depends on the accuracy of observation of $\delta\tau_P$. Suppose, we are able to measure $\delta\tau_P$ up to an accuracy of 10^{-14} s, which might be technologically possible in the near future. Then, if we wish to ascribe this correction to the dilatonic charge, we must have $\beta = 4$ and it leads to $Q \approx 10^{24}$ esu. In this unit, the value of the terrestrial dilatonic charge looks huge, in fact exceeding the electronic charge of the Earth's magnetosphere (2×10^{22} esu) by 200 times! A large value of $\beta \gg 4$

would certainly reduce the dilatonic charge, but then the corrections to $\delta\tau_P$ would be extremely small and would go far beyond the Kerr corrections.

6. Some remarks

Corrections to the Sagnac delay beyond the basic value $\delta\tau_S$ could also come from the non-Einsteinian theories such as the Brans–Dicke or other nonminimally coupled scalar tensor theories [7] but, unfortunately, these theories produce only naked singularities instead of black holes. This situation leaves us only with a very few physically viable options in the form of either Kerr–Newman or dilaton gravity. The best Kerr corrections to the Earth bound experiments is of the order of $\sim 10^{-15}$ s and beyond, as we saw above. Therefore, a good candidate for the interpretation of corrections within the intermediate range (between 10^{-6} and 10^{-15} s) could be the terrestrial dilatonic charge which, however, seems rather huge in electrostatic units. On the other hand, we still do not know of a basic unit of the dilatonic charge given by an independent theory. Consequently, it is really not known what the actual magnitude would be in that basic unit.

It has come to our attention that in a recent paper, Bini, Jantzen and Mashhoon [12] have elaborately studied, from the relative observer point of view, the gravitomagnetic clock effect and the Sagnac effect for circularly rotating orbits in a general class of axisymmetric spacetimes. Although not specifically mentioned therein, the Kerr–Sen spacetime is automatically a member of this class due to its symmetries and much of the calculations in the present Letter follow directly from Ref. [12]. For instance, consider the metric of the circular orbit cylinder in the threading and slicing notation:

$$d\tau_{(t,\varphi)}^2 = N^2\,dt^2 - g_{\varphi\varphi}\left(d\varphi + N^{\varphi}\,dt\right)^2, \tag{37}$$

with

$$N^{\varphi} = -2Ma/\left(X^3 + a^2X + 2Ma^2 - X^2\xi\right).$$

Eq. (8) turns out to be exactly the same as Eq. (3.5) of Ref. [12]. Also, note that Eq. (12) gives the coordinate time difference as

$$\Delta t \equiv t_+ - t_- = \frac{\varphi_{0+} - \varphi_{0-}}{\omega_0}$$

$$= -2\pi\left[\frac{2\omega_0 - (\Omega_+ + \Omega_-)}{(\omega_0 - \Omega_+)(\omega_0 - \Omega_-)}\right] \tag{38}$$

which is precisely the same as Eq. (6.2) of Ref. [12] under the identifications $\varsigma \equiv \omega_0$, $\varsigma_\pm \equiv \Omega_\pm$. Several other similarities could also be cited but the point is that we only considered here what Bini et al. [12] called the observer-dependent "single-clock clock effect". That is, we calculated the *proper* time $\delta\tau$ as measured by a single clock and examined specifically the role of the dilaton charge Q in the correction terms. On the other hand, the work in Ref. [12] contains, in addition, a comprehensive analysis of two other distinct gravitomagnetic clock effects and their relation to the Sagnac effect in a general class of spacetimes. It would indeed be interesting to work out, for example, the "two-clock clock effects" in the Kerr–Sen spacetime for various observers. This we plan to do in a future communication.

It has also been pointed out [13] to us that, by defining

$$Z^2 = \frac{(R+\xi)^2(2R+\xi)}{2M}, \tag{39}$$

one could rewrite Eq. (31) as

$$\omega_\pm = \frac{1}{a \mp Z}. \tag{40}$$

In terms of coordinate periods, one then has

$$t_+ - t_- = 4\pi a. \tag{41}$$

Remarkably, the difference is proportional only to the angular momentum of the source, exactly as in the Kerr–Newman spacetime. The (single clock) proper time $\delta\tau_G$ (Eq. (33)) however reveals the effect of Q. The difference $(\tau_+ - \tau_-)$ between the proper periods gathered by two clocks ("two-clock clock effect") in the Kerr–Newman spacetime has been calculated by Mashhoon et al. [14]. The equivalence of this gravitomagnetic effect with NASA's gravity probe-B concept is also elucidated there.

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