CHAPTER 4*

DIGRAPHS REPRESENTED BY INTERVALS HAVING BASE POINTS

4.1 Introduction

We recall from chapter 1, that intersection, overlap and containment model for digraphs were introduced and characterized by Sen *et al.* In [Sen *et al.*, 1989a] it was shown that a digraph D is an interval-point digraph iff its adjacency matrix $A(D)$ has consecutive ones property for rows. Again Sen and Sanyal [1994] has shown that a digraph is an indifference digraph iff it's $A(D)$ has monotone consecutive arrangement property. By this we mean that there exists independent row and column permutations exhibiting the following structure of $A(D)$: the 0's of the resulting matrix can be labeled R or C such that every position above or to the right of an R is an R, and every position below or to the left of a C is a C.

Since an indifference digraph has also consecutive one's property for rows, a question immediately arises as to under what conditions an interval-point digraph reduces to an indifference digraph. In section 4.2. We answer this question and show that the class of indifference digraphs is the same as the class of interval-point digraphs, where the source intervals are of unit length.

The notion of 'base interval' was introduced by Sanyal [1994]. If S_v is a closed interval and p_v is a point of S_v, then the ordered pair (S_v, p_v) is called a *base interval*. Replacing a pair of intervals by a pair of base intervals $\{(S_v, p_v), (T_v, q_u) : v \in V\}$, a digraph was obtained in the following manner : $uv \in E$ *iff*

(i) S_u and T_v overlap (no containment),

(ii) $\inf S_u < \inf T_v$ and

(iii) $p_u, q_v \in S_u \cap T_v$. Such a digraph was termed *right overlap base interval (Robin)* digraph and was characterized by Sanyal interms of its adjacency matrix as follows : A

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digraph is a *Robin digraph* iff its adjacency matrix has a 4-directable property (defined earlier in chapter 1, fig. 1.7). Replacing the condition of overlapping of two intervals S_u and T_v in Robin digraph, by the conditions of intersection, Sanyal introduced the idea of base interval digraphs. Also the problem of characterization of these digraphs was initiated by him. In the section 4.3 we characterize these digraphs in terms of their adjacency matrices. As a particular case of 4-directable matrix we consider a binary matrix where 0's have a partition into two classes, say X and Y; a binary matrix will be said to have an *X-Y partition* if its rows and columns can be labeled either X or Y such that (i) the positions to the right or the position above any X are 0's labeled X, (ii) the positions to the left or positions below any Y are also zeros labeled Y, and moreover (iii) if any column contains both X and Y which have all X's and Y's to their right and left respectively then the row corresponding to Y must occur below the row corresponding to X. Similarly if any row contains both X and Y which have all X's and Y's to the above and below respectively then the column corresponding to Y must occur to the left of the column corresponding to X (fig. 4.1). It may be noted that this definition is a modified form of the definition of X-Y partition given by Sanyal [1994].

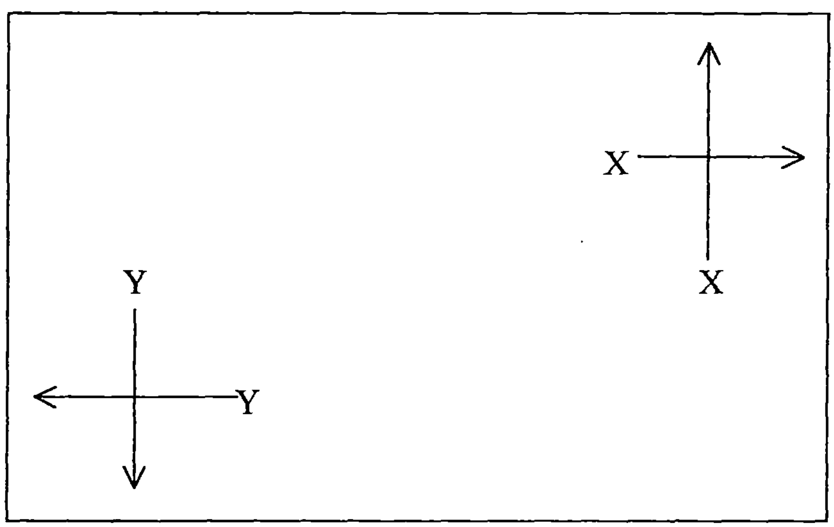


Fig 4.1 *X-Y* Partition

In section 4.3 we will show that these binary matrices characterize the adjacency matrices of base interval digraphs.

Now we draw our attention to an X-Y partitioned binary matrix. If we form a binary matrix $A(D_1)$ consist only of those 0's which correspond to the X's occurring consecutively in a row and the Y's occurring consecutively in a column, we note that since $X \cap Y = \varphi$, this new matrix is actually the adjacency matrix of an interval digraph. Similarly considering the X's occurring in a column and the Y's occurring in a row we get another interval digraph. Thus we see that a base interval digraph is practically the intersection of two interval digraphs. In the same section we present detailed proof of the characterization. In the same section we also introduce the concept of base interval in an undirected graph and obtain an analogous characterization of what has been termed a base interval graph.

In section 4.4, we deal with an overlap base interval digraph and show how closely it is related to the notion of an interval containment digraph. The interval containment digraph $D(V, E)$ of a family $\mathfrak{I} = \{(S_v, T_v) : v \in V\}$ of an ordered pairs of intervals is the digraph with vertex set V in which $uv \in E$ iff S_u contains T_v. These digraphs were characterized by Sen, Sanyal and West [1995] and it was shown that this class is equivalent to the class of digraphs of Ferrers dimension 2. Also it was proved by Sanyal [1994] that a digraph is a Robin digraph iff its adjacency matrix is 4-directable. Noting that this digraph is of Ferrers dimension 4 and that a digraph of Ferrers dimension 2 is nothing but an interval containment digraph, it immediately follows that a Robin digraph must be the intersection of two interval containment digraphs. Here we probe into the necessary and sufficient condition for a Robin digraph to be the intersection of two particular interval containment digraphs.

In the last section we deal with Robin digraph where the intervals are of unit length. For this we first observe that an overlap digraph though of Ferrers dimension 3 is such that its adjacency matrix is 4-directable. So, it should be expressed in terms of a Robin digraph. Probing this question we indeed get the result that an overlap digraph is actually a Robin digraph with unit length intervals.

In this chapter sometime we use the symbol $u \to v$ to mean that uv is an edge of the digraph $D(V, E)$.

4.2 Interval-point digraph and indifference digraph

An interval-point digraph is an interval digraph where the sink interval reduces to a point. In other word every vertex v is assigned a pair (S_v, p_v) where S_v is an interval and p_v is a point and $uv \in E$ iff $p_v \in S_v$. Adjacency matrix of the interval-point digraphs were characterized by Sen *et al* [1989a] in the following way : adjacency matrix of the interval point digraph has consecutive ones property for rows and conversely. Since adjacency matrix of an indifference digraph has also consecutive ones property for rows, the immediate question that arises is under what condition an interval point digraph reduces to an indifference digraph.

For this, we first observe that if the source intervals $[a_i, b_i]$ corresponding to the vertex v_i are such that

$$a_i \leq a_{i+1}, \qquad bi \leq b_{i+1}, \qquad i = 1, 2, \ldots, n-1$$

then arranging the columns in increasing order of c_i's [the terminal points corresponding to v_i's] and the rows in the increasing order of a_i's (or b_i's) the adjacency matrix $A(D)$ of D exhibits an MCA and interval-point digraph becomes an indifference digraph.

That the converse of the above statement is also true, as follows from the proof of Theorem 2 of [Sen *et al,* 1989a]. As a matter of fact repeating the arguments as it is there, the assignments

$$a(v_k) = min\ \{ i,\ v_i v_k \in E\}$$

and $\quad b(v_k) = \max\ \{i,\ v_i u_k \in E\}$

and the condition of MCA guarantees that the indifference digraph is an interval-point digraph with the above property. Hence we have the following proposition :

Proposition 4.1 *An interval point digraph $D(V, E)$ where v_i corresponds to the pair $([a_i, b_i], c_i)$ is an indifference digraph iff the intervals are such that*

$$a_i \leq a_{i+1}, \qquad bi \leq b_{i+i}, \qquad i = 1, 2, \ldots, n-1$$

We define a *unit interval-point* digraph as one where all the source intervals are of unit length and a *proper interval-point* digraph is one there a source interval does not contain properly another source interval.

Now we use the above proposition to prove the following :

Theorem 4.1 *For a digraph D(V, E) the following conditions are equivalent :*

1) *D is an indifference digraph;*
2) *D is a unit interval-point digraph ;*
3) *D is a proper interval-point digraph.*

Proof. 1) $\Rightarrow$ 2) Let $\{f(v), g(v)) : v \in V\}$ be an indifference representation of a digraph $D(V, E)$. Corresponding to the vertex v of D, construct a unit interval-point representation $(S_v, g(v)/2)$ where

$$S_v = \left[\frac{f(v)}{2} - \frac{1}{2}, \frac{f(v)}{2} + \frac{1}{2}\right]$$

Now $uv \in E \Leftrightarrow |f(u) - g(v)| \leq 1 \Leftrightarrow \left|\frac{f(u)}{2} - \frac{g(v)}{2}\right| \leq \frac{1}{2} \Leftrightarrow \frac{g(v)}{2} \in S_u \Leftrightarrow uv$ is an edge of

of the unit interval-point digraph.

2) $\Rightarrow$ 3) follows obviously.

3) $\Rightarrow$ 1) follows from the proposition 1.

As a matter of fact we note that if $\{([a_v, b_v], c_v) : v \in V\}$ is a unit interval-point representation of $D(V, E)$ then $f(v) = a_v + b_v$, $g(v) = 2c_v$ will be an indifference representation of $D(V,E)$.

4.3 Base interval digraph

A digraph $D(V, E)$ is a base interval digraph if its vertex set V has one-to-one correspondence with a family of ordered pairs of base intervals $\{S_u, p_v), (T_v, q_v) : v \in V\}$, $p_v \in S_v$, $q_v \in T_v$ and $u \to v$ if and only if $p_u, q_v \in S_u \cap T_v (\neq \varphi)$. The following theorem characterizes the adjacency matrix of a base interval digraph.

Theorem 4.2 *A necessary and sufficient condition that a digraph is a base interval digraph is that its adjacency matrix has an X-Y partition.*

Proof (necessary) Let $\{(S_v, p_v), (T_v, q_v) : v \in V\}$ be a base interval representation of a digraph $D(V, E)$ where $S_v = [a_v, b_v], p_v \in S_v$ and $T_v = [C_v, d_v], q_v \in T_v$; then we have $u \to v$ if and only if $(p_v, q_v) \in S_u \cap T_v$. So $uv \notin E$ if and only if one of the four inequalities (i) $b_u < q_v$, (ii) $p_u < c_v$, (iii) $q_v < a_u$, and (iv) $d_v < p_u$ holds.

From section 4.1 it is clear that D is of Ferrers dimension 4; that is D is the union of four Ferrers digraphs. These four Ferrers digraphs are obtained from the pairs (u, v) satisfying the above four inequalities. Arranging the rows of the adjacency matrix in increasing order of the values of p_u and its column in the increasing order of the values of q_v we see that the matrix exhibits 4-directable property.

Denote a 0 in the adjacency matrix by X if it corresponds to a position uv which satisfies the inequalities (i) or (ii), and otherwise by Y. It is to be noted that a position uv may satisfy both the inequalities (i) and (ii); that is, they are not mutually exclusive. Similarly for (iii) and (iv). Now it is a matter of verification that the sets X and Y satisfy the conditions of the X-Y partition and that $X \cap Y = \varphi$. This gives the required X-Y partition of the adjacency matrix.

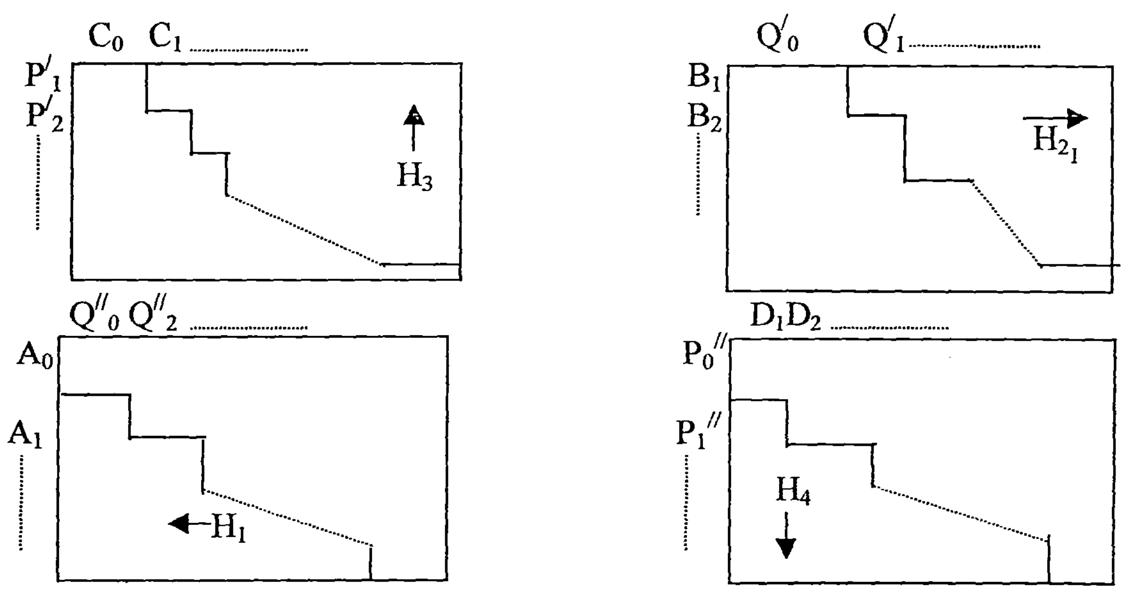


Fig. 4.2 decomposition of a matrix having X-Y partition.

For sufficiency, consider a permutation of the rows and columns of the adjacency matrix $A(D)$ that exhibits an X-Y partition. As observed earlier, D is the union of four Ferrers digraphs, which we view as sets of positions in the adjacency matrix given by

i) H_1 consisting of the Y's in the $A(D)$ that have only Y's to their left.

ii) H_2 consisting of the X's in $A(D)$ that have only X's to their right.

iii) H_3 consisting of the X's in $A(D)$ that have only X's above them.

iv) H_4 consisting of the Y's in $A(D)$ that have only Y's below them.

Now we want to construct base intervals (S_v, p_v) and (T_v, q_v) where $p_v \in S_v = [a_v, b_v]$ and $q_v \in T_v = [c_v, d_v]$ for all $v \in V$, such that uv is outside of H_i, $i = 1, 2, 3, 4$ if and only if $a_u < q_v < b_u$ and $c_v < p_u < d_v$, which gives the base interval representation of the digraph D. The values for the end points of the intervals and the base points within them will come from a topological ordering of an auxiliary acyclic digraph. We shall use H_1, H_2, H_3, H_4 to define eight partitions of V. Because the successor sets of a Ferrers digraph is ordered by inclusion, we can define a natural partition of the rows of the adjacency matrix with two rows in the same block if and only if the successor sets of the two corresponding vertices are identical. Furthermore, the blocks of the partition are indexed naturally by the inclusion ordering on the successor sets. The same is true of the predecessor sets and the columns of the adjacency matrix.

For H_1 we can permute the rows to achieve Ferrers diagram in the lower left. Also for H_2, we can permute the rows to achieve this in the upper right. So the natural terminal partitions of H_1 and H_2 have the columns in the same order. Similarly for H_3 and H_4 we can permute the columns to active Ferrers diagram in the upper right and lower left respectively. Here again the natural source partitions of H_3 and H_4 have rows in the same order. This is illustrated in the figure 4.2, where we have given the names to the blocks of the partitions.

Let $A = \{A_i\}$, $B = \{B_i\}$, $C = \{C_i\}$ and $D = \{D_i\}$. Since the rows of H_3, H_4 and columns of H_1, H_2 are in the same order, we can define additional partitions P_0, P_1, ..., P_s and Q_0, Q_1,

...., Q_t that maintain the order of the rows, where each block P_i is the intersection of one P'_j and one P''_k, and each Q_i is the intersection of one Q'_j and one Q''_l. In other words, the partition $P = \{P_i\}$ is the common refinement of $\{P'_j\}$ and $\{P''_k\}$ with fewest blocks, indexed by the shared order on the rows, and similarly for $Q = \{Q_i\}$. Note that the indexing of the various types of P's agrees with the row order for H_3 and H_4, and the indexing for the Q's agrees with the column order for H_1 and H_2.

We construct an auxiliary digraph $Z=Z_1\cup Z_2$ with vertices $A\cup B\cup C\cup D\cup P\cup Q$, which we call nodes to distinguish them from the vertices of the original digraph. We will assign distinct integers to these nodes via a map f Each $v\in A_i$ will receive $f(A_i)$ as the value of of a_v; similarly b_v, c_v, d_v, p_v and q_v are set from the values of f on B, C, D, P and Q respectively. We put an edge in Z from one node to another when we want the number assigned to the first node to be less than the number assigned to the second, and then f will be chosen to increase along every edge. Since we want the p-values and q-values to be increasing in rows and columns in accordance with the discussion of the X-Y partition above, we put $Q_i\to Q_j$ in Z if $i<j$, and similarly $P_i\to P_j$ if $i<j$.

First we construct the digraph Z_1 in the following way :

If $u\in A_i$ and $v\in Q''_j$ with $i\geq j$ then $uv\notin E(D)$ $(uv\in H_1)$ and we want $q_v<a_u$; on the other hand if $i<j$ then possibly $u\to v$ and we need to allow this by $q_v>a_u$. Hence for the pair Ai, Q_l with $Q_l\subseteq Q''_j$ we put $A_i\to Q_l$ if $i<j$ but $Q_l\to A_i$ if $i\geq j$. This defines a linear ordering on $A\cup Q$. Similarly for the pair B_i, Q_l with $Q_l\subseteq Q'_j$ we put $B_i\to Q_l$ if $i\leq j$ but $Q_l\to B_i$ if $i>j$. This again defines a linear ordering on $B\cup Q$.

But the interaction between this two ordering, we first observe that in the ordering $A\cup Q$ (or $B\cup Q$) between two Q node we have at most one A (or B) node. Combining these two ordering $A\cup Q$ and $B\cup Q$ we form a linear ordering on $A\cup B\cup Q$ such that between two Q node if there exist two node A_i and B_j with $A_i\cap B_j\neq\varphi$ then we place B_j succeeding A_i, but if $A_i\cap B_j=\varphi$ then we place them in any order.

Now f-values of the nodes are increasing from left to right in the linear ordering $A \cup B \cup Q$; so if B_k precede A_i we must have $A_i \cap B_k = \varphi$, (since a_v's and b_v's are end points of real interval). Thus we must require $f(A_i) < f(B_k)$ if $v \in A_i \cap B_k$. To check this let in the linear ordering $A \cup B \cup Q$, B_k precede A_i and $u \in A_i \cap B_k$. Also let Q_j be a node between A_i and B_k and $v \in Q_j$. Now $B_k \to Q_j$ with $u \in B_k$ and $v \in Q_j$ implies $uv \in H_2$ i.e. uv is an X. Again $Q_j \to A_i$ with $u \in A_i$ and $v \in Q_j$ implies $uv \in H_1$ that is uv is an Y; so $X \cap Y \neq \varphi$ which is impossible.

Similarly we construct the linear ordering Z_2 with notes $C \cup D \cup P$.

For the interaction between Z_1 and Z_2 so as to obtain $Z = Z_1 \cup Z_2$, we note that the value of base point of a source interval must lies within it, so we must require $f(A_i) \leq f(P_l) \leq f(B_j)$ if there is a vertex $v \in A_i \cap P_l \cap B_j$. We represent this by placing edges from A_i to P_l and P_l to B_j. Similarly for base point of terminal interval we require $f(C_i) \leq f(Q_k) \leq f(D_j)$ if $v \in C_i \cap Q_k \cap D_j$. And we represent this by placing edges from C_i to Q_k and Q_k to D_j.

Our problem now is to show that $Z = Z_1 \cup Z_2$ is acyclic. If it be so, consider a numbering $f : V(Z) \to R$ such that $XY \in E(Z)$ implies $f(X) < f(Y)$. Then using the values of f to determine a_v, b_v, c_v, d_v, p_v, q_v as described above, we have created base intervals (S_v, p_v) and (T_v, q_v) where $S_v = [a_v, b_v]$ $T_v = [c_v, d_v]$, $a_v < p_v < b_v$ and $c_v < q_v < d_v$ such that $uv \in E(D)$ if and only if $a_u < q_v < b_u$ and $c_v < p_u < d_v$.

Here we shall show that the auxiliary digraph Z has no cycle. We note that in each of the matrix $H_i (i=1, 2, 3, 4)$, the indices of the blocks as we go down the row or go to the right along the column are in increasing order. Also in each ordering, the indices on a particular type of node appear in increasing order. For example the block. P_i occures above the block P_j in the matrix H_3 (or H_4) if $i<j$; and a block C_i occures earlier than the block C_j in the matrix H_3 if $i<j$.

Now we claim that the directed graph Z with nodes A, B, C, D, P, Q has no cycle. Let if possible the digraph Z has a cycle. Below we consider the following possibilities in which a cycle may occur.

Case 1 Let $B_j\, A_i\, P_k\, P_m$ be a cycle (fig 4.3(i)).Then we must have $k<m$. If there is no node Q_r between B_j and A_i, then without loss of generality we can interchange the node B_j and A_i and get rid of the cycle (Fig. 4.3(ii)) . So let there be a node Q_r between B_j and A_i (Fig. 4.3(iii)). The edge $B_j \to Q_r$ implies that the edges uw

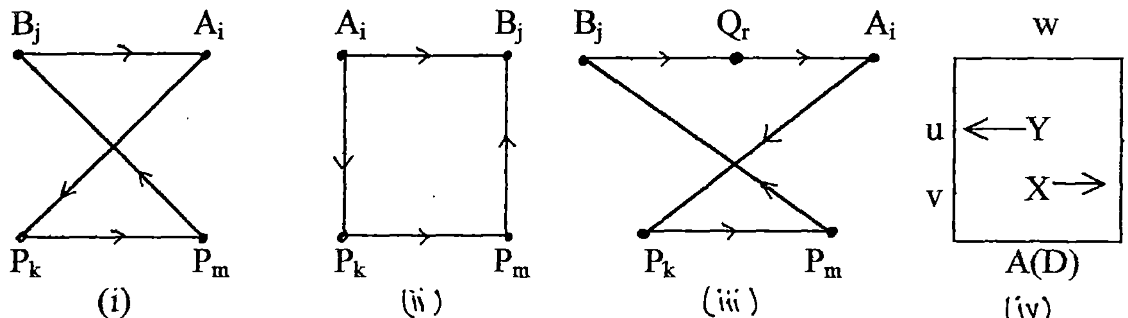


Fig 4.3

for which $u \in B_j$ and $w \in Q_r$ all belongs to H_2. This means that the position (u, w) in the matrix $A(D)$ is X and positions to its right are all X. Similarly $Q_r \to A_i$ implies that for $w \in Q_r$ and $v \in A_i$ the position vw in the matrix $A(D)$ is Y and the positions to the left in the row are all Y (see fig 4.3(iv)). Again $A_i \to P_k$ implies that $v \in P_k$ and $P_m \to B_j$ implies that $u \in P_m$, and since $k<m$, the u-row occur below the v-row which violates the condition (iii) of the X-Y partition.

Case 2. Let there be a cycle of the form B_i ....., $Q_j\, D_r$.... $P_s\, B_i$, (Fig. 4.4). Now $P_s \to B_i$ implies that $u \in P_s \cap B_i$ and $Q_j \to D_r$ implies $v \in Q_j \cap D_r$. Again $B_i \to Q_j$ with $u \in B_i$ and $v \in Q_j$ implies $uv \in H_2$ i.e. uv is an X. Similarly $D_r \to P_s$ with $u \in P_s$ and $v \in D_r$ implies $uv \in H_4$ i.e. uv is an Y. Therefore $uv \in X \cap Y$, which is impossible since $X \cap Y = \varphi$.

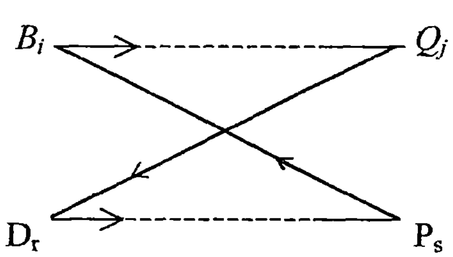


Fig. 4.4

Similarly we can arrive at a contradiction from the other cases. ■

Example 4.1. Here we illustrate the above method of construction of base intervals from an *X-Y* partitionable binary matrix.

Consider the *X-Y* partitionable binary matrix M.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	1	1	1	0	1	0	0
v_2	1	1	1	0	0	0	0
v_3	0	0	1	1	0	0	0
v_4	0	0	1	0	1	0	0
v_5	0	0	0	0	1	1	0
v_6	0	1	1	0	1	0	0
v_7	0	0	0	0	1	1	0

Here the four Ferrers digraphs H_i $(i = 1,2,3,4)$ are as follows :

Q''

$H_1 =$ A

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1							
v_2							
v_6	0						
v_3	0	0					
v_4	0	0					
v_5	0	0	0	0			
v_7	0	0	0	0			

Q'

$H_2 =$ B

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_2				0	0	0	0
v_3					0	0	0
v_1						0	0
v_4						0	0
v_6						0	0
v_5							0
v_7							0

C

$H_3 =$ P'

	v_1	v_2	v_3	v_5	v_4	v_6	v_7
v_1					0	0	0
v_2					0	0	0
v_3						0	0
v_4						0	0
v_5							0
v_6							0
v_7							0

D

$H_4 =$ P''

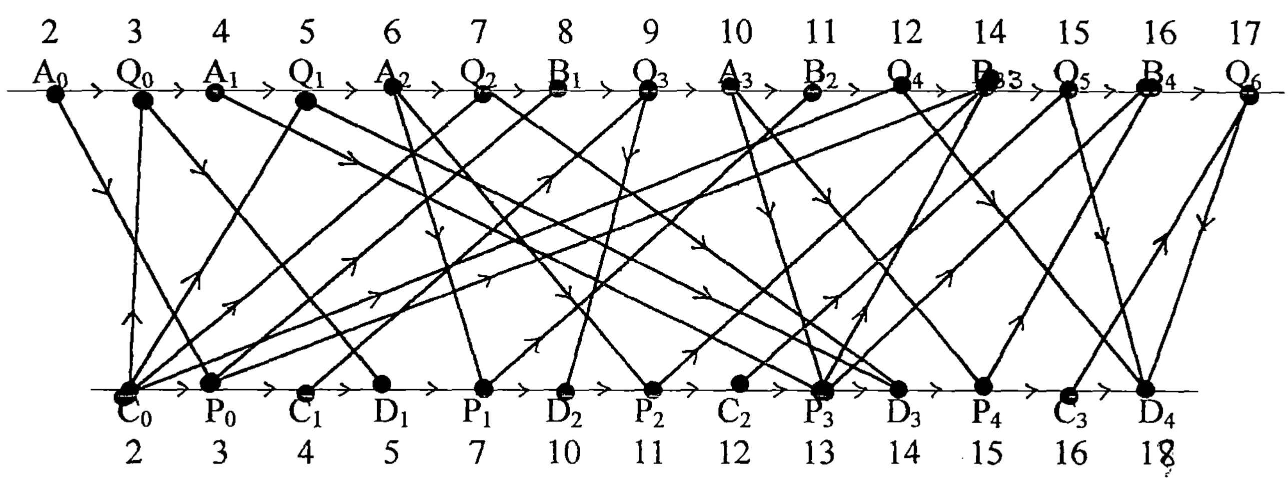
	v_1	v_4	v_2	v_3	v_5	v_6	v_7
v_1							
v_2							
v_3	0						
v_4	0	0					
v_5	0	0					
v_6	0	0					
v_7	0	0	0	0			

The vertex partitions from H_1 & H_2 are given by

i	0	1	2	3	4	5	6
A_I	v_1, v_2	v_6	v_3, v_4	v_5, v_7			
B_I		v_2	v_3	v_1, v_4, v_6	v_5, v_7		
Q'_i	v_1, v_2, v_3	v_4	v_5	v_6	v_7		
Q''_i		v_1	v_2	v_3, v_4	v_5, v_6, v_7		
$Q_i = Q'_i \cap Q''_i$	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Again the vertex partitions from H_3 & H_4 are given by

i	0	1	2	3	4
C_i	v_1, v_2, v_3, v_4	v_4	v_6	v_7	
D_i		$v_1,$	v_4	v_2, v_3	v_5, v_6, v_7
P'_i		v_1, v_2	v_3, v_4	v_5, v_6, v_7	
P''_i	v_1, v_2	v_3	v_4, v_5, v_6	v_7	
$P_i = P'_i \cap P''_i$	v_1, v_2	v_3	v_4	v_5, v_6	v_7



The auxiliary Digraph Z

Fig 4.5

Then the resulting topological ordering yields the following sequences

i	0	1	2	3	4	5	6
Ai	2	4	6	10			
Bi		8	11	14	16		
Ci	2	4	12	16			
Di		5	10	14	18		
Pi	3	7	11	13	15		
Qi	3	5	7	9	12	15	17

Now picking out *a(v)*, *b(v)*, *c(v)*, *d(v)*, *p(v)* and *q(v)* for each vertex v, we have the following base interval representation for M.

i	1	2	3	4	5	6	7
(S_{v_i}, p_{v_i})	[2, 14], 3	[2, 8], 3	[6, 11], 7	[6, 14], 11	[10, 16], 13	[4, 14], 13	[10, 16], 15
(T_{v_i}, q_{v_i})	[2, 5], 3	[2, 14], 5	[2, 14], 7	[4, 10], 9	[2, 18], 12	[12, 18], 15	[16, 18], 17

So for we have characterized the adjacency matrix of a base interval digraph in terms of an *X-Y* partition of the matrix. Below we take a look into the *X-Y* partition of the matrix again to obtain yet another characterization of a base interval digraph.

Let R_1 denote the zeros of *A(D)* where a zero has all positions zero to its right and let R_2 denote the zeros of *A(D)* where a zero has all positions zero to its left. Similarly let $C_1(C_2)$ denote the zeros where zero has all positions zero above (below) it. Note that $R_1 \cap C_1$ and $R_2 \cap C_2$ are not necessarily empty but $R_1 \cap C_2 = \varphi$ and $R_2 \cap C_1 = \varphi$ (Fig 4.6).

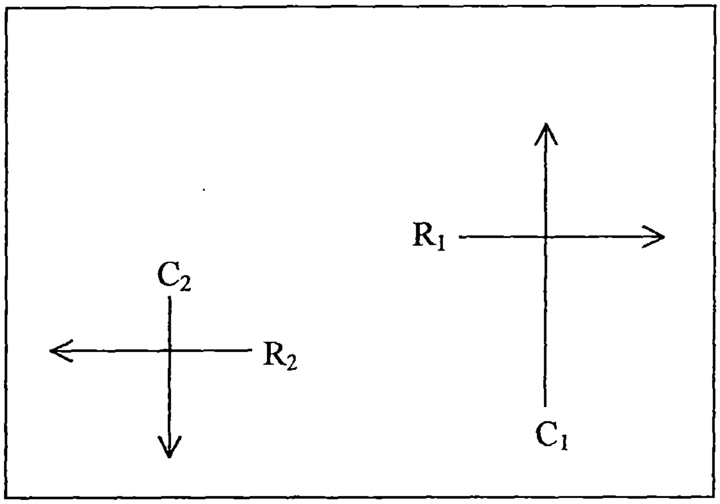


Fig. 4.6

Now construct a binary matrix from $A(D)$ which has zeros corresponding to R_1 and R_2 only. This matrix has consecutive ones property for rows and accordingly the corresponding digraph is an interval-point digraph. Similarly construct a binary matrix from $A(D)$ which has zeros corresponding to C_1 and C_2 only. This matrix has consecutive ones property for columns and so converse of this matrix is the adjacency matrix of an interval-point digraph. Accordingly D is the intersection of these two digraphs.

Again construct a binary matrix from $A(D)$ which has zeros corresponding to R_1 and C_2 only. This matrix has partitionable zeros property and so the corresponding digraph is an interval digraph. Similarly construct a binary matrix from $A(D)$ which has zeros corresponding to C_1 and R_2 only. Reversing the order of rows and columns of this matrix we observe that it has again partitionable zeros property and the corresponding digraph is an interval digraph. Thus D is the intersection of two interval digraphs. These observations motivate us to prove the following.

Theorem 4.3. *For a digraph $D(V, E)$ the following conditions are equivalent :*

1) *D is a base interval digraph with base interval representation $\{([a_v, b_v], p_v), ([c_v, d_v], q_v) : v \in V\}$.*

2) *$D = D_1 \cap D'_2$ where D_1 is the interval-point digraph $\{([a_v, b_v], q_v) : v \in V\}$ and D_2 is the interval-point digraph $\{ ([c_v, d_v], p_v) : v \in V\}$ with $p_v \in [a_v, b_v]$ and $q_v \in [c_v, d_v]$ and D'_2 is the converse of D_2.*

3) *$D = F_1 \cap F_2$ where F_1 and F_2 are two interval digraphs, the pairs of intervals corresponding to a vertex v in F_1 and F_2 being of the form $\{([a_v, p_v], [c_v, q_v]) : v \in V\}$ and $\{([p_v, b_v], [q_v, d_v]) : v \in V\}$ respectively.*

Proof. 1) $\Rightarrow$ 2). Let $D(V, E)$ be a base interval digraph

Then $uv \in E \Rightarrow p_u, q_v \in S_u \cap T_v$ where $S_u = [a_u, b_u]$, $T_v = [c_v, d_v]$

$\Rightarrow p_u \in [c_v, d_v]$ and $q_v \in [a_u, b_u]$

$\Rightarrow uv \in D'_2$ and $uv \in D_1$

$\Rightarrow uv \in D_1 \cap D'_2$

where D_1 and D_2 are two interval-point digraphs $\{([a_v, b_v], q_v) : v \in V\}$ and $\{([c_v, d_v], p_v) : v \in V\}$ respectively.

Again when $uv \notin E$ then easily we see that $uv \notin D_1$ or $uv \notin D'_2$ or both i.e. $uv \notin D_1 \cap D'_2$. Thus if D has a base interval respective on then D is intersection of interval point digraph.

2) $\Rightarrow$ 3) Let $D = D_1 \cap D'_2$ where D_1 and D_2 are two interval-point digraphs with the given representation.

Consider two interval digraphs F_1 and F_2 where the pairs of intervals corresponding to the vertex v are $([a_v, p_v], [c_v, q_v])$ and $([p_v, b_v], [q_v, d_v])$ respectively for F_1 and F_2. Let $uv \in E$ and let $p_u < q_v$.

Then uv is an edge of D'_2

$\Rightarrow p_u \in [c_v, d_v]$

$\Rightarrow p_u \in [c_v, q_v]$

$\Rightarrow [a_u, p_u] \cap [c_v, q_v] \neq \varphi$

$\Rightarrow uv \in F_1$.

Again uv is an edge of D_1

$$\Rightarrow q_v \in [a_u, b_u]$$

$$\Rightarrow q_v \in [p_u, b_u]$$

$$\Rightarrow [p_u, b_u] \cap [q_v, d_v] \neq \varphi$$

$$\Rightarrow uv \in F_2$$

Thus $uv \in F_1 \cap F_2$

Similarly, when $p_u \geq q_v$ we can prove $uv \in F_1 \cap F_2$. When $uv \notin E$, then it is easy to see that either $uv \notin F_1$ or $uv \notin F_2$ or both; that is $uv \notin F_1 \cap F_2$.

So $\quad D = F_1 \cap F_2$..

3) $\Rightarrow$ 1) Let $D = F_1 \cap F_2$ where the pairs of interval in F_1 and F_2 corresponding to a vertex v are of the form $([a_v, p_v], [c_v, q_v])$ and $([p_v, b_v], [q_v, d_v])$ respectively. Corresponding to a vertex v of D consider a base interval representation $([a_v, b_v], p_v)$, $([c_v, d_v], q_v)$ where $p_v \in [a_v, b_v]$ and $q_v \in [c_v, d_v]$.

Now $uv \in F_1 \cap F_2 \Rightarrow uv \in F_1$ and $uv \in F_2$

$$\Rightarrow [a_u, p_u] \cap [c_v, q_v] \neq \varphi \text{ and } [p_u, b_u] \cap [q_v, d_v] \neq \varphi$$

$$\Rightarrow p_u \in [c_v, q_v] \text{ or } q_v \in [a_u, p_u] \text{ and } p_u \in [q_v, d_v] \text{ or } q_v \in [p_u, b_u]$$

First and last cases are simultaneously possible and second and third cases are simultaneously possible. So in either case $[a_u, b_u]$ and $[c_v, d_v]$ intersect and $p_u, q_v \in [a_u, b_u] \cap [c_v, d_v]$.

Also if $uv \notin D$ then either $uv \notin F_1$ or F_2 or both. Let $uv \notin F_1$ then with $p_u \notin [c_v, d_v]$ or $q_v \notin [a_u, b_u]$. So uv is not an edge of the base interval digraph. ■

Unit base interval digraphs are those base interval digraphs in which all the source base intervals and sink base intervals are of unit length. As a immediate consequence of

Theorem 4.3, we have the following corollary which shows that such digraphs are the intersection of two indifference digraphs.

Corollary 4.1. *$D(V, E)$ is a unit base interval digraph with unit base interval representation $\{(S_v, p_v), (T_v, q_v) : v \in V\}$ iff $D = D_1 \cap D'_2$ where D_1 is the interval-point digraph $\{(S_v, q_v) : v \in V)$ and D'_2 is the interval point digraph $\{(T_v, p_v) : v \in V\}$ with $p_v \in S_v$ and $q_v \in T_v$.*

4.3.1. Base interval graph

We introduce the notion of base interval graph for an undirected graph and then characterize its adjacency matrix.

Let $\mathfrak{I}=\{(S_v, p_v) : v \in V\}$ be a family of base intervals with $p_v \in S_v$. We say that a graph $G(V, E)$ is a base interval graph when $uv \in E \Leftrightarrow p_u \in S_v$ and $p_v \in S_u$.

If G is a base interval graph then the corresponding symmetric digraph with loops at each vertex is a base interval digraph, as can be seen by taking

$$(T_v, q_v) = (S_v, p_v),\ v \in V.$$

Consequently its adjacency matrix has an X-Y partition. While characterizing a base interval graph we will observe in the next theorem how the symmetry of the adjacency matrix fits in with the X-Y partition. In fact we will see that an (i, j) entry in the adjacency matrix of G is an X iff (j, i) entry is an Y.

Theorem 4.4 *The following statements are equivalent for a graph $G(V, E)$:*

1) G is a base interval graph

2) There exists a simultaneous permutation of the rows and columns of the augmented adjacency matrix $A(G)$ of G such that if an entry in the upper triangular matrix is zero, then all entries to the right of it or above it are zeros and if an entry in the lower triangular matrix is zero then all entry to the left of it or below it are zeros.

Proof. As indicated above, the part 1) $\Rightarrow$ 2) follows from the X-Y partition of the corresponding digraph with loops at every vertex and the symmetry of the adjacency matrix. So below we prove the other part only. Let after a suitable simultaneous permutation of rows and columns, $A(G)$ has the stated properties. From $A(G)$ we form a matrix $A(D_I)$ which has a consecutive ones property for rows by converting all the 0 entry to 1 which lie between the first one and last one in any row. Then D_I is an interval-point digraph. We now show that if G^* denotes the symmetric digraph with loop at every vertex, corresponding to the graph G, then

$$G^* = D_I \cap D'_I$$

We note that while forming D_I from G^* we have not deleted any edge from G^* and so $G^* \subset D_I$. Again since G^* is a symmetric digraph, $G^* \subset D'_I$. Thus $G^* \subset D_I \cap D'_I$.

On the other hand, if some entry (i, j) is a 1 in $A(D_I)$ which is a zero in $A(G)$ then by the construction of $A(D_I)$ there is at least a 1 to the left or to the right of this position in $A(G)$ (according as this entry is in the lower triangle or in the upper triangle). Consequently by the hypothesis all the entries below it or above it in $A(D)$ must be 0's. Hence all the positions to the right or left of (j, i) entry in $A(G)$ are 0's. So the entry 0 in the (j, i) position does not come in the way of consecutive one's property of D_I and hence the position remains 0 in $A(D_I)$.

This means that (i, j) entry in $A(D'_I)$ is 0 which implies in turn that the (i, j) entry in $D_I \cap D'_I$ is a zero. Thus $D_I \cap D'_I \subset G^*$.

Now from the consecutive one's arrangement of the rows of D_I construct an interval-point digraph $\{(S_v, p_v) : v \in V\}$ and since every element in the main diagonal of D_I is 1 we have $p_v \in S_v$ for all $v \in V$. This is the base interval representation for G. ∎

Again proceeding along the same line as in Theorem 4.3, we can prove the following theorem.

Theorem 4.5. *The following statements are equivalent* :

1) *$G(V, E)$ is a base interval graph where a vertex $v \in V$ is assigned a pair $([a_v, b_v], p_v)$, $p_v \in [a_v, b_v]$.*

2) *If G^* denotes the symmetric digraph corresponding to G with a loop at each vertex then $G^* = D_1 \cap D'_1$. Where D_1 is the interval-point digraph, the vertex v being assigned the same pair $([a_v, b_v], p_v)$, $p_v \in [a_v, b_v]$.*

3) *$G(V, E)$ is the intersection of two interval graphs G_1 and G_2 where the intervals corresponding to a vertex v in G_1 and G_2 are $[a_v, p_v]$ and $[p_v, b_v]$ respectively.*

4.4 Robin digraph and interval containment digraph

Motivated by the facts that a Robin digraph is of Ferrers dimension 4 and an interval containment digraph is of Ferrers dimension 2, we show below that how a Robin digraph can be characterized in terms of the intersection of two interval containment digraphs.

Let $D(V,E)$ be a Robin digraph with Robin representation $\{(S_v, p_v), (T_v, q_v)) : v \in V\}$ where $S_v = [a_v, b_v]$, $p_v \in S_v$ and $T_v = [c_v, d_v]$, $q_v \in T_v$

Then $uv \in E \Rightarrow a_u < c_v < b_u < d_v$ and $c_v \leq p_u$, $q_v \leq b_u$

$$\Rightarrow [a_u, b_u] \supset [c_v, q_v] \text{ and } [p_u, b_u] \subset [c_v, d_v]$$

$$\Rightarrow uv \in D_1 \text{ and } uv \in D'_2.$$

$$\Rightarrow uv \in D_1 \cap D'_2.$$

where D_1 and D_2 are two interval containment digraphs with representation $\{([a_v, b_v], [c_v, q_v]) : v \in v\}$ and $\{([c_v, d_v], [p_v, b_v]) : v \in V\}$ respectively.

Conversely, let

$$uv \in D_1 \cap D'_2.$$

$$\Rightarrow uv \in D_1 \text{ and } uv \in D'_2.$$

Corresponding to the vertex v construct the pair of intervals $([a_v, b_v], [c_v, d_v])$, then $a_u<c_v$, $b_u <d_v$ and $p_u \in S_u$, $q_v \in T_v$; also $q_v \in S_u$, $p_u \in T_v$.

Therefore $[a_u, b_u]$ and $[c_v, d_v]$ overlap with $\inf S_u < \inf T_v$ and $p_u, q_v \in S_u \cap T_v$.

Thus D is a Robin digraph with the above representation. This proves the following :

Theorem 4.6 *A digraph $D(V, E)$ is a Robin digraph with Robin representation $\{(S_v, p_v), (T_v, q_v) : v \in V\}$ if and only if $D = D_1 \cap D'_2$ where D_1 and D_2 are two interval containment digraphs with representations $\{([a_v, b_v], [c_v, q_v] : v \in V\}$ and $\{([c_v, d_v], [p_v, b_v]) : v \in V\}$ respectively.*

4.5 Unit Robin digraph

In this section we study a Robin digraph where the intervals are of unit length. In this case we will observe that Ferrers dimension of its adjacency matrix reduces to 3 and moreover this class becomes equivalent to the class of overlap digraph.

Theorem 4.7 *If D is a digraph then the following conditions are equivalent :*

1) D is a Robin digraph with intervals of unit length.

2) The rows and columns of the adjacency matrix of D can be permuted independently so that its 0's can be labeled R or P such that (i) the positions to the right and positions above any R are also 0's labeled R and (ii) the positions to the left or positions below any P are also 0's labeled P.

3) *D is a right overlap interval digraph.*

Proof. 1) $\Rightarrow$ 2). Let $\{((S_v, p_v), (T_v, q_v)) : v \in V\}$ be a right overlap base interval representation of a digraph D, where $S_v = [a_v, b_v]$, $p_v \in S_v$, $T_v = [c_v, d_v]$, $q_v \in T_v$ and $|S_v|=1$, $|T_v|=1$. Let m_v and n_v be the midpoints of S_v and T_v respectively. If $uv \notin E$, then $m_u \geq n_v$ or $p_u < c_v$ *or* $q_v > b_u$. The last two possibilities are not mutually exclusive. We label the u-v position of the adjacency matrix by R if $m_u \geq n_v$ and by P if $p_u < c_v$ *or* $q_v < b_u$. Now we arrange the rows of the matrix in decreasing order of values m_u and columns in decreasing order of values n_v. If $(u, v) \in R$; then every position to the right and every position above (u, v) is also R. If $(u, v) \in P$ and if $p_u < c_v$ then every position to the left of (u, v) is P and if $(u, v) \in P$ with $q_v > b_u$ then every position below (u, v) is P.

2) $\Rightarrow$ 3) See [Sen, Sanyal and West, 1995] for proof.

3) $\Rightarrow$ 1) We observed in the section 1.10 that the class of ROI digraph and of LOI-digraph are the same. So we may consider a LOI-digraph. Let $\{(S_v, T_v) : v \in V\}$ be a given LOI representation of D, where $S_v = [a_v, b_v]$, $T_v = [c_v, d_v]$.

We want to construct a Robin representation (S'_v, p_v), (T'_v, q_v) where $S'_v = [a'_v, b'_v]$, $T'_v = [c'_v, d'_v]$, $p_v \in S'_v$ and $q'_v \in T'_v$ and S'_v, T'_v are of the same length. From the LOI-representation of D, we have for $u \to v$ the following inequality holds :

$$c_v < a_u < d_v < b_u$$

Choose a number $l > \max_{v \in V} \{d_v - c_v, b_v - a_v\}$, that is, l is greater than the length of any interval of the LOI-representation.

Now setting

$$b'_u = a_u, \; a'_u = b'_u - l$$

$$d'_u = d_u, \; c'_u = d'_u - l$$

$$q_u = c_u, \; p_u = b_u - l,$$

we easily verify that all the intervals S'_u, T'_u are all of the same length l, $p_u \in S'_u$, $q_u \in T'_u$ and $u \to v$ iff (i) $\inf S'_u < \inf T'_v$ and (ii) $p_u, q_v \in S_u \cap T_v \neq \varphi$. ∎

Example 4.2 To illustrate the above method of construction of Robin digraph of same length interval from a LOI-digraph, consider the following LOI-representation of a digraph

$$v_1 \to ([11, 18], [10, 14])$$

$$v_2 \to ([15, 21], [10, 12])$$

$$v_3 \to ([13, 18], [14, 20])$$

$$v_4 \to ([19, 25], [10, 16])$$

$$v_5 \to ([21, 23], [20, 22])$$

$$v_6 \to ([17, 21], [18, 24])$$

$$v_7 \to ([17, 21], [14, 20])$$

Here we may take $l = 10$. And using the formula described in the proof we have the following Robin representation of the digraph with interval of the same length.

$v_1 \to S_1 = [1, 11], \; p_1 = 8, \quad T_1 = [4, 14], \; q_1 = 10$

$v_2 \to S_2 = [5, 15], \; p_2 = 11, \quad T_2 = [2, 12], \; q_2 = 10$

$v_3 \to S_3 = [3, 13], \; p_3 = 8, \quad T_3 = [10, 20], \; q_3 = 14$

$v_4 \to S_4 = [9, 19], \; p_4 = 15, \quad T_4 = [6, 16], \; q_4 = 10$

$v_5 \to S_5 = [11, 21], \; p_5 = 13, \quad T_5 = [12, 22], \; q_5 = 20$

$v_6 \to S_6 = [7, 17], \; p_6 = 11, \quad T_6 = [14, 24], \; q_6 = 18$

$v_7 \to S_7 = [7, 17], \; p_7 = 11, \quad T_7 = [10, 20], \; q_7 = 14$