CHAPTER 3

BIGRAPHS/DIGRAPHS OF FERRERS DIMENSION 2 AND ASTEROIDAL TRIPLE OF EDGES (ATE)

3.1 Introduction

Previously, we have noted that the concepts of interval digraphs and interval bigraphs are closely related and, in fact, basically equivalent. Analogous to the notion of containment graph, a containment digraph was introduced and studied in [Sen, Sanyal and West, 1995]. Equivalently, a *containment bigraph* is a bipartite graph $B(U, V, E)$ for which there are two families of intervals $\{S_u : u \in U\}$ and $\{T_v : v \in V\}$ such that $u \in U$ and $v \in V$ are adjacent if and only if $S_u \supset T_v$. In this chapter the digraph D and the corresponding bigraph $B=B(D)$ (obtained from D by vertex splitting operation [Müller, 1997]) will be used interchangeably. The adjacency matrix of D is the *biadjacency matrix* of B. The intervals corresponding to the members of U and V and the *source intervals* and *sink intervals* respectively.

A pair of edges x_1y_1 & x_2y_2 of a bipartite graph $H(X,Y,E)$ is separable [Golumbic, 1980] if the corresponding vertices induce the subgraph $2K_2$ in H; in this case its biadjacency matrix contains a 2×2 permutation submatrix. A bigraph containing at least a pair of separable edges is *a separable bigraph.* Otherwise it is *non-separable*.

It is clear that the bipartite analogue of a Ferrers digraph is a non-separable bigraph and that of a digraph D of higher $f(D)$ is separable. The Ferrers dimension of a digraph will also be called the *Ferrers dimension of the corresponding bigraph.* It was proved [Sen *et al.*, 1989a] that a bigraph is an interval bigraph if and only if it is the intersection of two non-separable bigraphs whose union is complete. It was also proved that a bigraph is an interval bigraph if and only if the rows and columns of its biadjacency matrix can be permuted independently so that each 0 can be replaced by one of $\{R, C\}$ in such a way that every R has only R's to its right and every C has only C's below it. The matrix is said to have an *$\{R, C\}$ partition of zeros* or *zero partitionable property.*

While an interval bigraph is necessarily of Ferrers dimension at most 2, a containment bigraph is exactly what characterizes a bigraph of *f(D)* at most 2. The following theorem is a characterization of a containment bigraph translated from its digraph version.

Theorem 3.1 [Sen *et al.*, 1989a] and [Sen, Sanyal & West, 1995]. *The following conditions are equivalent :*

i) *B is a containment bigraph;*

ii) *B is of Ferrers dimension at most 2;*

iii) *The rows and columns of the adjacency matrix of B have an independent permutation so that no 0 has a 1 both to its right and below it.*

The rearranged matrix with this permutation of rows and columns is referred to as F_2 *matrix*. Also we shall refer to this property as F_2 *property* for the rearranged matrix.

It is clear that the class of interval bigraphs form a proper sub-class of the class of containment bigraphs. In fact, the bigraph in Fig. 3.1. is an example of a containment bigraph which is not an interval bigraph.

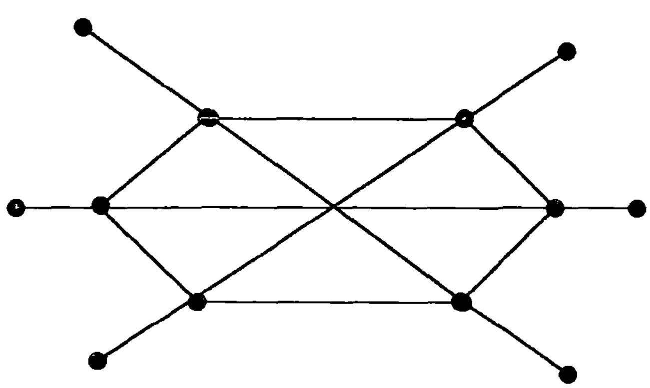


Fig. 3.1

Its adjacency matrix has the characterizations of Ferrers dimension 2 but does not have $\{R, C\}$ –partition of zeros.

Motivated by the study of Gaussian elimination in *(0, 1)*–matrices, Golumbic and Goss [1978] introduced the notion of chordal bipartite graph, an analogue of chordal graph. It was characterizd by Hammer *et al.*in [1989].

A bigraph is called *chordal bipartite* or simply *bichordal,* if every cycle of length ≥ 6 has a chord. Müller [1997] showed that an interval bigraph is a bichordal graph. It is now relevant to observe that every (even) cycle of length ≥ 6 is of Ferrers dimension ≥ 3; consequently it follows that a bigraph B having Ferrers dimension at most 2 does not contain a cycle of length ≥ 6 and equivalently a containment bigraph is necessarily bichordal.

Lekkerkerker and Boland [1962] used the notion of asteroidal triple of vertices to charac e trize an interval graph and then obtained a complete list of forbidden subgraphs of an interval graph. Analogously in the present chapter, we define an *asteroidal triple of edges* (*ATE*), as follows : Three mutually separable edges e_1, e_2, e_3 of a graph G are said to form an *asteroidal triple of edges,* if for any two of them, there is a path from the vertex set of one to the vertex set of the other that avoids the neighbours of the third edge. This definition is a slightly modified version of the definition given by Müller [1997]. Here we consider the three edges to be mutually separable.

Analogous to the characterization of an interval graph in terms of asteroidal triples, Müller in the same paper conjectured that a chordal bipartite graph is an interval digraph iff it is ATE-free and also free from a class of bigraphs, he termed insects. In section 3.2 we give a counter example to show that the conjecture is not true. In the next section we study the notion of strong and weak bisimplicial edges in a bigraph. Then we try to obtain the significance of an ATE in a bigraph. In section 3.4 we first show that a bigraph B having $f(B) = 2$ is also ATE-free which strengthens the result by Müller that an interval bigraph is ATE -free. Next we address the problem of characterizing an ATE-free bigraph and in this endeavor, we first give a counter-example (Fig. 3.5) to show that an ATE-free bichordal graph is not necessarily of Ferrers dimension 2. Finally in this section, we show that when a bighraph contains a strong bisimplicial edge, the bigraph of example 3.1 (Fig. 3.5) is the only forbidden subgraph of a bichordal, ATE-free bigraph having Ferrers dimension 2 and this is the main result of this chapter.

One final remark : of the two equivalent directed graph and bipartite graph models, the latter one seems preferable. Because, as we have seen, the adjacency matrix and the permutation of its rows and columns play an important role in their characterization; but the (independent) permutation of rows and columns for the adjacency matrix of a bigraph gives us the (same) graph up to isomorphism, where as this is not the same for the digraph case. Moreover, results on the well studied notion of chordal bipartite graphs fits in very well and we rely heavily on them in our present study.

3.2 Müller's conjecture : a counterexample

Müller considered two bigraphs B and B^* whose biadjacency matrices are

$$\begin{array}{|cccccc|}\hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline\end{array}$$

$Adj(B)$

$$\begin{array}{|cccccc|}\hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \hline\end{array}$$

$Adj(B^{*})$

and then defined an insect to be a bigraph G such that $B \subseteq G \subseteq B^*$. He showed that an interval bigraph is ATE-free and also insect-free. He then conjectured that a bichordal graph is an interval bigraph iff it is ATE-free and insect-free. The following example shows that the conjecture is not true.

Consider the bigraph of Figure 3.2.

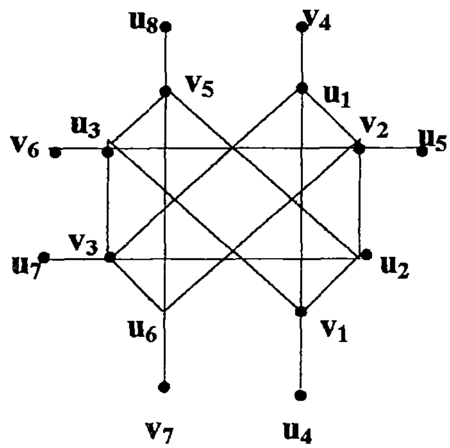


Fig. 3.2

The biadjacency matrix in terms of a bicolouring of $H_b(D)$ of the above graph is given by

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1	1	1	1	1	R	R	R
u_2	1	1	1	C	1	I	R
u_3	1	1	1	C	1	1	R
u_4	1	R	R	I	R	I_r	R
u_5	C	1	R	I_c	R	I_r	I_r
u_6	C	1	1	C	1	C	1
u_7	C	C	1	I_c	R	I_c	I_r
u_8	C	C	C	C	1	I_c	I

It is clear that for this bigraph $I_r \cap I_c = \phi$ and it contains the configuration

	v_1	v_5	v_6
u_2	1	1	I
u_5	C	R	I_r
u_7	C	R	I_c

Hence from Theorem 2.4 it immediately follows that D is not an interval digraph and thereby the bigraph is not an interval bigraph.

It is to be noted that in [Das and Sen, 1993] the digraph in example 1 was shown to be an example of a digraph which is not an interval digraph. The bigraph in Figure 3.2 that we have considered in present section is a slight variant of that example and can be obtained by deleting the column 4 of the adjacency matrix.

3.3 Bisimplical edges : Strong & Weak

Throughout this chapter, we will assume that no vertex of $H(X, Y, E)$ is a copy of one another. Let $e = xy$ be an edge of a bipartite graph $H(X,Y,E)$. Also let $B(e) = B(xy)$ denote the subgrph induced by $adj(x) + adj(y)$. An edge $e = xy$ of the bipartite graph H is called *bisimplicial* if $B(e)$ or $B(xy)$ is complete. Analogous to the notion of strong and weak simplicial vertices, a bisimplicial edge $e=xy$ of H is said to be *strong* if $H \backslash B(e)$ is connected; other wise it is *weak*. Note that a graph may contain strong bigimplicial edges as well as weak bisimplical edges. (For example in the graph of example 3.1, the edge xy is a strong bisimplical edge where as the edge x_3y_2 is a weak bisimplical edge). We begin with the following theorems which guarantee the existence of bisimplicial edges in a bichordal graph.

Theorem 3.2 [Golumbic, 1980] *Let H be a connected bichordal graph. If H is separable, then it has at least two separable bisimplical edges.*

Theorem 3.3 [Golumbic and Goss, 1978] *Let H be a connected bichordal graph. If H is non-separable then every vertex of H is incident with some bisimplicial edge of H.*

Regarding Theorem 3.3, we have the following interesting observation to make. It has been seen in the introduction that a non-separable bigraph is equivalent to a Ferrers digraph and so its adjacency matrix can be arranged in the form of a Ferrers diagram with 1's blocked in the upper right corner as in the following Figure.

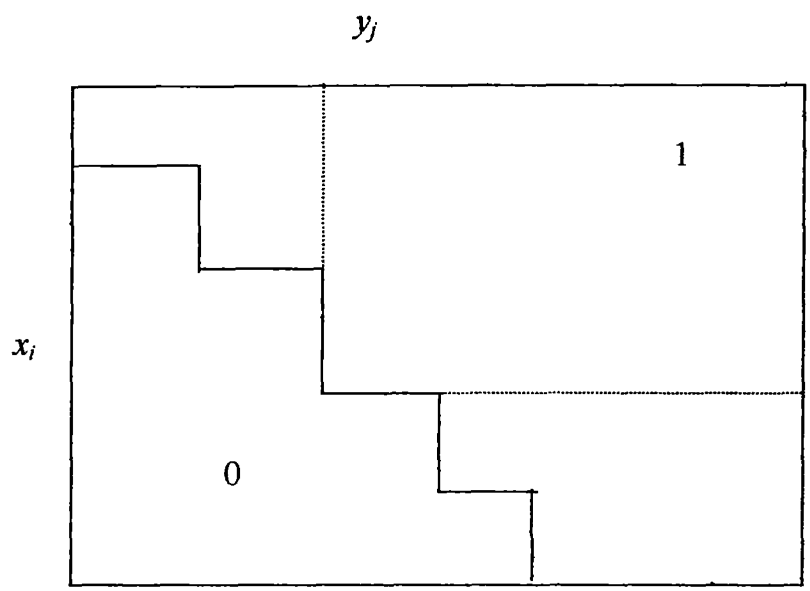


Fig. 3.3

So given any vertex say x_i, of the bigraph *H(X, Y, E),* there is a permutation of the members of its *X*-set; so that x_i becomes the last member in its equivalent class. Now consider the vertex y_j in $Adj(x_i)$, such that y_j is the first member in its equivalent class. Then we can easily see the $Adj(x_i)$ and $Adj(y_j)$ form a rectangular block of 1's in the diagram, which induces a complete bipartite subgraph and by the way leave the vertices of its complement totally isolated. Consequently it follows that the bisimplicial edges of a non-separable connected bigraph are all weak.

The converse, however, is not true, as can be seen from the following graph (Fig 3.4)

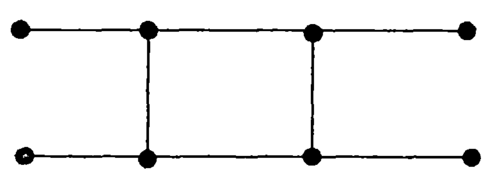


Fig. 3.4

The above graph is separable with no strong bisimplicial edge.

Proposition 3.1 *If a chordal bipartite graph G has exactly two bisimplicial edges, then they must be strong.*

Proof. Let $e=xy$ be a weak bisimplicial edge of H and let $C_1, C_2, .., C_k (k\geq 2)$ be the components of $H\backslash B(e)$. Now consider the subgraph H_1 induced by the vertices of $B(e)$ and C_1. If C_1 consists of a single vertex v, it is a leaf of H and so the edge incident to it is a bisimplicial edge of H. If C_1 is non-trivial component, then H_1 is separable and so has at least two bisimplicial edges; the bisimplicial edges other than e must belong entirely to C_1 and since C_i's are all disconnected, they are also bisimplicial edges of H. So H has more than k (at least $k+1$, $k\geq 2$) bisimplicial edges, which contradicts the hypothesis. ■

Let $H\ (X,\ Y,\ E)$ be a containment bigraph. In its containment representation let δ_v $(v=x \text{ or } y)$ be the interval corresponding to the vertex v and $r(v)$, $l(v)$ be its right and left end point respectively.we call δ_v as end interval if

i) $r(v') > l(v)$

or ii) $l(v') < r(v)$ for all vertex v' belongs to the same partite set as v. In case (i) δ_v is the *left end interval* and in case (ii) δ_v is the *right end interval.*

3.4 ATE-free bigraphs and bigraphs of Ferrers dimension 2

In this section, first we prove that a bigraph having Ferrers dimension at most 2 is necessarily bichordal and ATE-free.

Proposition 3.2 A bigraph of Ferrers dimension ≤ 2 is bichordal and ATE-free.

Proof. We have already observed in the introduction that a cycle of length ≥ 6 is of Ferrers dimension 3 and accordingly a bigraph H with $f(H) \leq 2$ is necessarily bichordal. To show that a bigraph with $f(H) \leq 2$ is ATE-free, we rely on the characterization of a bigraph with Ferrers dimension ≤ 2 which states that H is of $f(H) \leq 2$ iff after suitable arrangement of rows and columns, its (bi)adjacency matrix becomes such that for any 0 in the matrix either every position to its right is a 0 or every position below it is 0.

Let $e_i = x_iy_i$ $(i=1, 2, 3)$ be three mutually separable edges of H. From the characterization of a containment bigraph as stated above, the rearranged F_2-matrix has the submatrix in the following form

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \end{array}$$

(because otherwise, any different permutation will violate its F_2 matrix characterization).

Now we consider a path P joining one end of e_1 to one end of e_3 say, $u_1, \dots\, x_iy_p \dots x_jy_q \dots u_3$, where $u_1 = x_1$ or y_1 and $u_3 = x_3$ or y_3. To reach e_3 from e_1, a close look into the rearranged matrix show that the path has a subpath from one edge x_iy_p to x_jy_q which calls for either of the two possibilities :

1) there must exists two vertices, say x_i and x_j on P such that x_2 lie between x_i and x_j in the rearranged matrix.

2) there must exists two vertices, say, y_p and y_q on P such that y_2 lie between y_p and y_q.

Then the sequence of vertices of the subpath is either $x_iy_px_jy_q$ or $y_px_iy_qx_j$. In the first case y_p precedes y_2 and so the position x_2y_p cannot be 0 (because in that case this 0 will leave 1 both to its right and below). So x_2y_p position must be 1. This means that x_2 is adjacent to the path. Similarly for the second case we see that y_2 is adjacent to the path. This proves that H is ATE-free. ∎

Note that if a bichordal graph has three strong bisimplicial edges, then they must form an ATE. Since a containment bigraph is ATE-free, it follows that a containment bigraph can not contain more than 2 strong bisimplical edges. In this context, we have the following interesting result concerning the relative position of the intervals representing a strong bisimplicial edge in a containment bigraph.

Proposition 3.3 *Let $e=xy$ be a strong bisimplicial edge of a containment bigraph $H(X,Y,E)$. Then for any containment model of H, the intervals representing x and y are the end intervals (on the same end).*

Proof. Let $e = xy$ be a strong bisimplicial edge of H, so that $H\backslash B(e)$ is connected. Then in any representation of H

$$\delta_y \subset \delta_x' \quad \text{for the vertices } x' \text{ of } B(e) \text{ only,}$$

$$\text{and} \quad \delta_x \supset \delta_y' \quad \text{for the vertices } y' \text{ of } B(e) \text{ only,}$$

Let, if possible, there is a representation of H for which δ_y is not an end (say, left) interval. Then there exists vertices y_1, y_2 of H such that

$$l(y) > r(y_1) \text{ and } r(y) < r(y_2).$$

Since no vertices of H is copy of another, there exist vertices x_1, x_2 of $H\backslash B(e)$ for which $x_1y_1 \in E$ and $x_2y_2 \in E$.

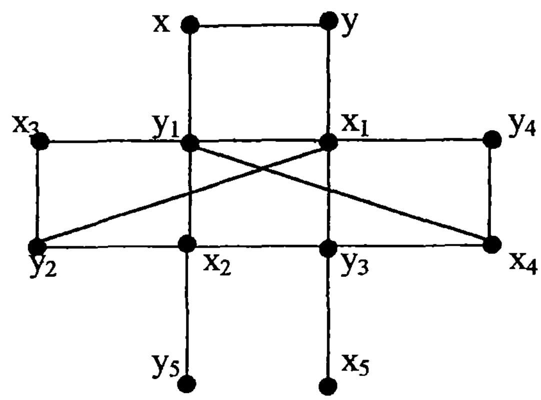
Again since $x_1, x_2 \in H\backslash B(e)$,

$$r(x_1) < r(y) \text{ and } l(y) < l(x_2);$$

because otherwise x_1 or x_2 would belong to $B(e)$. but then the edges x_1y_1 and x_2y_2 can not be connected by any path in $H\backslash B(e)$, meaning that $H\backslash B(e)$ is disconnected. This contradiction shows that δ_y is an end interval. It follows that δ_x along with other source intervals of $B(e)$ are the end intervals (on the same end as δ_y). ■

Now we are in a position to address the converse problem to Proposition 3.2. Our question is whether a bichordal and ATE-free bigraph is of Ferrers dimension at most 2. The answer is, infact, negative. Below we give an example of bigraph which is bichordal, ATE free but of Ferrers dimension 3. The process in which we have obtained this example is however a matter of long deliberation and will be clear when we come to the next theorem.

Example 3.1 The following bigraph (Figure 3.5) is bichordal, ATE-free but of Ferrers dimension 3.



The bigraph H_0

Fig. 3.5

It can be seen that the associated graph $H(H_0)$ of the above bigraph H_0 has an odd cycle and so from Cogis [1979], it follows that $f(H_0) > 2$. If, however, we delete any vertex from H_0, its Ferrers dimension becomes equal to 2. So clearly $f(H_0) = 3$.

Below we state the central result of this chapter in Theorem 3.4.

Let $e=xy$ be a bisimplicial edge of the bipartite graph $H=H(X,Y,E)$.

We write $H_1(X',Y',E') = H \setminus \{x,y\}$, $B_1 = B_1(X_1,Y_1,E_1) = B(xy)\setminus\{x,y\}$,

$H_2(X_2, Y_2, E_2) = H\setminus B(xy) = H_1 \setminus B_1$.

$N(e)$ = vertex set of $B(e)$ i.e., the set of neighbours of x and y.

X'_2 be the set of those members of X_2 which are adjacent to some member of B_1 and $X''_2 = X_2 - X'_2$. So $X_2 = X'_2 \cup X''_2$. Similarly we can define Y'_2 and Y''_2 so that $Y_2 = Y'_2 \cup Y''_2$.

Next we denote the subgraphs induced by the vertices $X_1 \cup Y'_2$ and $X'_2 \cup Y_1$ by P and Q respectively.

Theorem 3.4. *Let a bipartite graph $H(X,Y,E)$ be bichordal and ATE-free and contains a strong bisimplicial edge. Then either $f(H) = 2$ or H contains the bigraph H_0 of Fig. 3.5 as an induced subgraph.*

The proof of the theorem is actually very long and requires a very careful and meticulous reading.

To prove the above theorem we first prove the following lemma.

Lemma 3.1 *Let H be a bichordal and ATE-free bigraph and let $e=xy$ be a strong bisimplicial edge of H. then the induced subgraphs P and Q as defined above are non-separable.*

Proof of the lemma. If possible, let the induced subgraph P have separable edges x_iy_i and x_jy_j where x_i, $x_j \in X_1$, and y_i, $y_j \in Y'_2$. Since H_2 is connected, it must contain a path between y_i and y_j. Now if this path is of length 2, say $y_ix'y_j$, $x' \in X_1$, then we have a 6-cycle $x_iyx_jy_jx'y_ix_i$ in H. On the other hand if the path is of length > 2, say $y_ix' \ldots x''y_j$, then three edges $e=xy$, $e' = x'y_i$ and $e''=x''y_j$ are mutually separable and constitute an ATE of H; for we have paths yx_iy_i between e and e' and yx_jy_j between e and e'' which avoid $N(e'')$ and $N(e')$ respectively and $N(e)$ does not contain any vertex of the component H_2.

Similarly it can be shown that the subgraph Q is also non-separable.

Proof of Theorem 3.4. We recall that no vertex of H is a copy of one another. By the above lemma, the subgraphs P and Q are non-separable (i.e. their biadjacency matrices are Ferrers bigraphs) so we can order the vertices of Y'_2 and X'_2 such that, $Adj(H_1)$, the biadjaceney matrix of H_1 has the following configuration (Fig. 3.6).

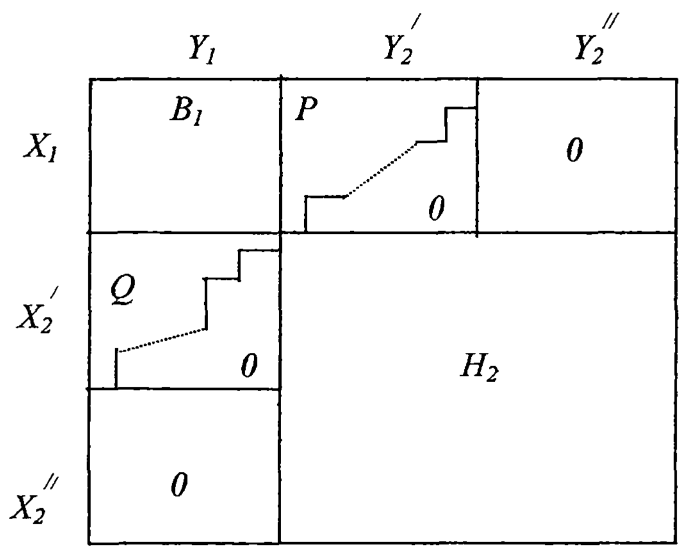


Fig. 3.6 Biadjacency matrix of H_1

We recall that the vertices $X_2 = X'_2 \cup X''_2$ and $Y_2 = Y'_2 \cup Y''_2$ induce the subgraph H_2. Consequently $Adj(H_2)$, the biadjacency matrix of H_2 is a submatrix of the Fig 3.6, where the row and column arrangement are the same as in $Adj(H_1)$ (i.e., of Fig 3.6). Also it is to be noted that in the $Adj(H_2)$ we can permute the rows (columns) of X''_2 (Y''_2), and the rows (columns) of X'_2 (Y'_2) which belong to the same partitioned class, without changing the structure of Fig 3.6. These permutations will be referred to as the *permissible permutations* in this chapter.

We will show that when a bigraph H satisfies the given conditions of the theorem and free from H_0, $Adj(H_1)$ with its rows and columns arranged as in Fig 3.6 will exhibit the characteristics of a bigraph of Ferrers dimension 2 (Theorem 3.1) and once this is established, we place the x-row and y-column of the bisimplical edge $e=xy$ to the top and extreme left of $Adj(H_1)$ and this will prove the theorem. This will be established if we can show that $Adj(H_2)$ has the property that no 0 has a 1 both to its right and below it. For this we will show below that, if $Adj(H_2)$ for any permissible permutation of its rows and columns, contains a configuration of the form

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array}$$

Where '-' positions is either 0 or 1, then the bigraph H contains either an ATE or a 6-cycle or the forbidden graph H_0 as induced subgraph.

To complete the proof of the theorem, we need to consider following cases :

case 1. Neither P nor Q is complete bipartite graph;

case 2. P is complete but Q is not ;

case 3. Q is complete but P is not;

case 4. Both P and Q are complete.

We will observe that while a 6-cycle or an ATE will be present in all the four cases, the forbidden graph H_0 will occur only in case 4, when P and Q are both complete bipartite graphs.

Case 1. Suppose for any permissible permutation of its rows and columns, $Adj(H_2)$ contains the configuration

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array}$$

This case is to be divided again into four subcases subject to whether the vertices x_i, x_j and the vertices y_i, y_j belong to the same partitioned class or to distinct partitioned classes.

Subcase 1a. x_i and x_j belong to two distinct partitioned classes of X_2 and so do y_i and y_j belong to two distinct classes of Y_2.

Subcase 1b. y_i and y_j belong to the same partitioned class, whereas x_i and x_j belong to two distinct classes.

Subcase 1c. x_i and x_j belong to the same partitioned class of X_2 whereas y_i and y_j belong to two distinct classes of Y_2.

Suncase 1d. x_i and x_j belong to the same partitioned class and y_i and y_j belong to the same partitioned class.

Subcase 1a. In this case clearly $Adj(H_1)$ contains a configuration

$$\begin{array}{c|ccc} & y_1 & y_i & y_j \\ \hline x_1 & 1 & 1 & 0 \\ x_i & 1 & 0 & 1 \\ x_j & 0 & 1 & - \end{array}$$

Where $x_1, y_1 \in V(B_1) = X_1 \cup Y_1$.

Now if the '-' position is 1, then the above configuration is a 6-cycle. So we suppose that '-' position is a 0. Then the three edges $e=xy$, $e_i=x_iy_j$, $e_2=x_jy_i$ of H are mutually separable. Also we have path xy_1x_i between e and e_1 and path yx_1y_i between e and e_2 which avoid respectively $N(e_2)$ and $N(e_1)$. Also there exist path between e_1 and e_2 which avoids $N(e)$ (since e_1 and e_2 are two edges of the connected component H_2 and no vertex of it is adjacent to e). So $\{e, e_1, e_2\}$ constitute an ATE of H.

Sub case 1b. In this case $Adj(H_1)$ contains a configuration

$$\begin{array}{c|ccc}
 & y_1 & y_i & y_j \\
\hline
x_1 & 1 & - & - \\
x_i & 1 & 0 & 1 \\
x_j & 0 & 1 & -
\end{array}$$

where $x_1 \in X_1$, $y_1 \in Y_1$ and x_1y_i, x_1y_j positions are both 1 or both 0.

It is possible that by permuting the vertices y_i an y_j, we can get a F_2-matrix, except of course when we are confronted with a vertex x_k belonging to still another partitioned class of X_2 and having the following configuration in $Adj(H_1)$.

$$\begin{array}{c|cccc}
 & y_1 & y_2 & y_i & y_j \\
\hline
x_1 & 1 & 1 & - & - \\
x_i & 1 & 1 & 0 & 1 \\
x_j & 1 & 0 & 1 & 0 \\
x_k & 0 & 0 & 0 & 1
\end{array}$$

where $x_1 \in X_1$ and $y_1, y_2 \in Y_1$.

Here also we find three mutually separable edges $e=xy$, $e_1=x_jy_i$ and $e_2 = x_ky_j$. And these three edges constitute an ATE of H; for the paths xy_1x_j, $xy_2x_iy_j$ between e, e_1 and e, e_2 avoid the neighbour of e_2 and e_1 respectively and because H_2 is connected the path between e_1 and e_2 avoids neighbours of e.

Sub case 1c. This case is similar to case 1b and so is omitted.

Subcase 1d First suppose that x_i, $x_j \in X'_2$ and y_i, $y_j \in Y'_2$. Now one possibility is that we can permute x_i, x_j and/or y_i, y_j and get $Adj(H_1)$ as F_2-matrix straightway without facing any obstruction elsewhere. Otherwise, the four configurations are the instances in $Adj(H_1)$, where there are vertices x_k, belonging to a class other than that of x_i, x_j and y_k, belonging to a class other than that of y_i and y_j, when we fail to derive the F_2-matrix directly.

	y_1	y_i	y_j	y_k
x_1	1	1	1	0
x_i	1	0	1	0
x_j	1	1	0	1
x_k	0	0	1	-

Fig 3.7 (i)

	y_1	y_k	y_i	y_j
x_1	1	1	0	0
x_k	1	-	1	0
x_i	0	1	0	1
x_j	0	0	1	-

Fig 3.7 (ii)

	y_1	y_i	y_j	y_k
x_1	1	1	1	0
x_k	1	1	0	-
x_i	0	0	1	0
x_j	0	1	0	1

Fig 3.7 (iii)

	y_1	y_k	y_i	y_j
x_1	1	1	0	0
x_i	1	1	0	1
x_j	1	0	1	0
x_k	0	-	0	1

Fig 3.7 (iv)

where $x_1 \in X_1$ and $y_1 \in Y_1$.

In these cases we can see after a careful scrutiny that H contains either an ATE or 6-cycle. (For example, if $x_j\,y_j$ position is 0 then $\{x_jy_i, x_iy_j, xy\}$ is an ATE of the graph that contains Fig (ii) or (iii) as a submatrix in its adjacency matrix.

The cases when x_i, $x_j \in X''_2$; y_i, $y_j \in Y'_2$ or when x_i, $x_j \in X'_2$; y_i, $y_j \in Y''_2$ are similar to the previous case so are omitted.

Finally to complete the subcase 1d, we suppose that x_i, $x_j \in X''_2$; y_i, $y_j \in Y''_2$. As earlier, two following configurations occur when we cannot get $Adj(H_1)$ as F_2-matrix directly by permuting x_i, x_j and y_i, y_j.

		Y_1	Y'_2 y'	Y''_2 y_i	y_j
X_1			1	0	
X'_2	x'	1	-	1	0
X''_2	x_i	0	1	0	1
	x_j		0	1	-

Fig 3.8 (i)

		Y_1 Y_1	Y'_2 y'	Y''_2 y_i	y_j
X_1			1	0	
X'_2	x'	1	-	0	1
X''_2	x_i	0	0	0	1
	x_j		1	1	-

Fig. 3.8 (ii)

Note that here $x' \in X'_2$ and $y' \in Y'_2$.

Now in each cases for any value in the '-' position the presence of ATE and 6-cycle can be easily checked.

Case 2. & 3. Proof in these cases are similar to the case 1 (i.e. if, for any permissible permutation of its rows and columns, $Adj(H_2)$ contains the configuration

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

then the bigraph H contains either an ATE or a 6-cycle) and so are omitted.

Case 4. Here both P and Q are complete bipartite graphs. Since no vertex of H is a copy of one another, it is clear that X_1 and Y_1 are singleton sets, let $X_1 = \{x_1\}$ and $Y_1 = \{y_1\}$. So $Adj(H)$ has the following structure :

$$\begin{array}{c|c|c|c|} & y & y_1 & Y_2 \\ \hline x & 1 & 1 & 0 - - - - - - - - - - - - - 0 \\ \hline x_1 & 1 & 1 & \\ \hline & 0 & & \\ X_2 & \vdots & & H_2 \\ & 0 & & \\ \hline \end{array}$$

Fig. 3.9

Suppose if possible $Adj(H_2)$ contains a configuration (1). Here we will show that H_o is the only graph which is of Ferrers dimension > 2 but does not contain a 6-cycle or an ATE.

We observe that the structure

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	1	0
x_i	0	1	0	1
x_j	0	0	1	-

of $Adj(H)$ with xy as strong bisimplicial edge implics that H must contain an ATE or C_6 according as '-' position is a *0* or *1*. So if $Adj(H_2)$ contains the configuration (1), (and since $Adj(H)$ is symmetric) $Adj(H)$ must have one of the following structures :

	y	Y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	-	-
x_i	0	1	0	1
x_j	0	1	1	-

and

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	-	-
x_i	0	1	0	1
x_j	0	0	1	-

Where the positions x_1y_i and x_1y_j are both *0* are both *1*.

Here we suppose that both the positions $x_1\,y_i$ and $x_1\,y_j$ are 1. (We are not considering the possibility that x_1y_i and x_1y_j are *0*, since later we add all possible row and / or column to the above matrices).

Not that the x_jy_j position in either of the matrices is *0* or *1*. To facilitate the mater, we replace the x_j row by two rows x_{j_1} and x_{j_2} to the matrices, one row taking the value '*0*' and the other taking the value '*1*' is the corresponding y_j column. So we get the matrices

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	1	1
x_i	0	1	0	1
x_{j_1}	0	1	1	0
x_{j_2}	0	1	1	1

Fig 3.10 (i)

and

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	1	1
x_i	0	1	0	1
x_{j_1}	0	0	1	0
x_{j_2}	0	0	1	1

Fig 3.10 (ii)

We label the vertices x_i, x_{j_1}, x_{j_2} by x_3, x_4 and x_2 and the vertices y_i, y_j *by* y_3, y_2 respectively in both the Figures for the sake of convenience.

Clearly, by permuting the x_3, x_4 and x_2 row and y_3, y_2 column of Fig 3.10 (i), *Adj(H)* gets the following F_2 –matrix structure (Fig 3.11(i)). Also permuting the rows and columns of the matrix in Fig. 3.10(ii) we get F_2-matrix of Fig 3.11(ii)

	y	y_1	y_2	y_3
x	1	1	0	0
x_1	1	1	1	1
x_2	0	1	1	1
x_3	0	1	1	0
x_4	0	1	0	1

Fig 3.11 (i)

and

	y	y_1	y_2	y_3
x	1	1	0	0
x_1	1	1	1	1
x_3	0	1	1	0
x_2	0	0	1	1
x_4	0	0	0	1

Fig 3.11 (ii)

Naturally, the question arises : is it possible that by adding a row / column to the rearranged matrices of the above Figures we will get a matrix which forbids its F_2-representation? And we have to address this important question every time, when ever we come across a matrix having F_2-characteristics.

We answer this question through a very long and exhaustive searching process, where we will show that

(a) in the case of Fig 3.11(ii) these new additions (to forbid its being an F_2-matrix) will always lead us to either a 6-cycle or an ATE, where as

(b) in the case of Fig. 3.11(i), these attempts yield in addition to a 6-cycle or an ATE, the only one forbidden graph H_o (Fig 3.5).

In the present chapter we will prove our point for graph of Fig. 3.11(i) through a detailed study. The proof for the graph of 3.11(ii) is of similar nature and so will be omitted.

We note that the bigraph H_o is a singularly important bigraph in the present chapter, in the sense that for the case when H contains a strong bisimplicial edge it will prove to be the only smallest bichordal ATE free bigraph with Ferrers dimension >2.

Because of this importance, we first take into account the particular means of adding suitable rows and columns to Fig 11(i) (to forbid F_2-matrix) that yields the graph H_o.

To the matrix of Fig. 3.11 (i), there are several alternatives for adding rows. Among these, we consider the particular row (name it x_5)

	y	y_1	y_2	y_3
x_5	0	0	0	1

Then the new matrix gets the followings F_2-representation.

	y	y_1	y_2	y_3
x	1	1	0	0
x_1	1	1	1	1
x_2	0	1	1	1
x_3	0	1	1	0
x_4	0	1	0	1
x_5	0	0	0	1

Fig 3.1 2

To this, we add two new columns, say y_4 and y_5 to get the matrix

	y	y_1	y_2	y_3	y_4	y_5
x	- 1	1	0	0	0	0
x_1	1	1	1	1	1	0
x_2	0	1	1	1	-	-
x_3	0	1	1	0	0	0
x_4	0	1	0	1	-	-
x_5	0	0	0	1	-	-

Fig 3.13

Now it is a matter of verification that when y_4 and y_5 column of the above matrix have the structure as in Fig 3.14 then we get the matrix which is the biadjacency matrix of the crucially important bigraph H_o. For the other structures of y_4 and y_5 columns, when Fig. 3.13 contain the submatrix

	y_4	y_5
x_i	0	1
x_j	1	-

where x_i, x_j are any two among the x_2, x_3, x_4 rows, then it can be checked that the graph H must contain either an ATE or a 6-cycle.

	y	y_1	y_2	y_3	y_4	y_5
x	1	1	0	0	0	0
x_1	1	1	1	1	1	0
x_2	0	1	1	1	0	1
x_3	0	1	1	0	0	0
x_4	0	1	0	1	1	0
x_5	0	0	0	1	0	0

Fig 3.14 biadjacency matrix of H_0

Now we come to the detailed and arduous task of considering all possible ways of adding different rows and columns to Fig 3.11(i) for forbidding F_2-characteristics.

We continue from Fig 3.12 obtained from Fig 11(i) by adding the row x_5

	y	y_1	y_2	y_3
x_5	0	0	0	1

We bypass the question of adding other possible rows to it (which will be taken care of later), and consider adding columns to Fig. 3.12.

To this matrix if we add a new column, say, y_i with $x_1y_i = 0$ and $x_3y_i=1$, it can be seen that $\{xy, x_3y_i, x_5y_2\}$ is an ATE or $x_1y_2x_3y_ix_5y_3x_1$ is a 6-cycle in H according as $x_5y_i = 0$ or 1. So in our search for new graphs other than a 6-cycle or an ATE, we are to consider the following possibilities :

$$\text{(A)}\quad \begin{array}{c|c} & y_i \\ \hline x & 0 \\ x_1 & 1 \\ x_3 & 0 \end{array} \qquad \text{(B)}\quad \begin{array}{c|c} & y_i \\ \hline x & 0 \\ x_1 & 0 \\ x_3 & 0 \end{array} \qquad \text{(C)}\quad \begin{array}{c|c} & y_i \\ \hline x & 0 \\ x_1 & 1 \\ x_3 & 1 \end{array}$$

Note that for each of the cases x_2y_i, x_4y_i and x_5y_i may take either of values *0* or *1* and so we have to consider 2^3 possibilities for each of the cases.

A little deliberation will reveal that we have actually reached the graph H_o from Fig. 3.12 by adding the two columns y_4 and y_5 having configurations of cases (A) and (B) respectively with particular values to the other positions.

Now for the detailed study, we first add a column y_6 having the configurations of case (C) to get the following matrix.

	y	y_1	y_2	y_6	y_3
x	1	1	0	0	0
x_1	1	1	1	1	1
x_3	0	1	1	1	0
x_2	0	1	1	-	1
x_4	0	1	0	-	1
x_5	0	0	0	-	1

Fig 3.15

In order that the graph of the above matrix retains its bichordality we observe that if any '-' position in the y_6 columns is *0* then all positions below it are also *0* (since otherwise, the graph has a C_6 as an induced subgraph).

It is very important to note here that for an exhaustive study of the matrix obtained by adding all possible combinations of columns to Fig 3.12, it is imperative that we should consider possibilities, when two or more of such new columns have the same conFigurations from either of the cases (A), (B) and (C).

Below we add two columns y_j and y_k to Fig. 3.15 where any column (or both the columns) has (have) the configurations of either (A) or (B), and get the matrix.

	y	y_1	y_2	y_6	y_3	y_i	y_j
x	1	1	0	0	0	0	0
x_1	1	1	1	1	1	-	-
x_3	0	1	1	1	0	0	0
x_2	0	1	1	-	1	-	-
x_4	0	1	0	-	1	-	-
x_5	0	0	0	-	1	-	-

Fig 3.16

Now we can easily verify (as in the matrices of Fig 3.13 and Fig 3.15) that when '-' positions in the above matrix takes either the values *0* or *1*, then either it is a F_2-matrxix or other wise it contains a 6-cycle or an ATE or the graph H_o as an induced subgraph.

With the above deliberations we have addressed the problem of additions of extra columns to Fig 3.15 completely. Now we come back to the question of adding new possible rows to Fig 3.15 (which was left earlier when we added only the x_5 row to Fig 3.11 (i)). It can be seen that the following possible rows remain to be added to Fig. 3.15.

	y	y_1	y_2	y_6	y_3
1. x_i	0	1	0	0	0
2. x_j	0	1	1	0	0
3. x_k	0	1	0	1	0
4. x_l	0	0	0	1	0
5. x_m	0	0	0	0	0
6. x_n	0	0	1	-	-

Note that in the last case we have an ATE $\{xy, x_5y_3, x_ny_2\}$ or a 6-cycle $x_3y_1x_4y_3x_ny_2x_3$ in the graph *H* according as x_ny_3 is *0* or *1*. So we omit this row from our consideration.

We already observed that when x_5y_6 position is 1, then x_2y_6 and x_4y_6 positions are also 1. So if x_5y_6 position is 0 and we add x_l row to the Fig 3.15, then the positions x_2y_6, x_4y_6 are 1. (since otherwise we have an ATE $\{x_5y_2, x_ly_6, xy\}$in the graph). Thus we distinguish two subcases :

i) when we add the above mentioned rows to the Fig 3.15 together with x_l row.

ii) When we add above rows to the Fig 3.15 but not the x_l row.

In the case (i) we have the matrix of Fig 3.17 (i) (or its submatrix), where as in the case (ii) we have the matrix of Fig 3.17 (ii) (or its submatrix).

Note that matrix of Fig 3.17(i) is a F_2-matrix. Also the matrix of Fig. 3.17(ii) is a F_2-matrix when its bigraph is free from ATE and 6-cycle.

At this point we make an important observation. To the matrix of Fig 3.11(i) if we add a row other than x_5 and then as before add the column y_6 and all other possible rows then we will arrive at the same matrices as Fig. 17(i) and 17(ii).

	y	y_1	y_3	y_6	y_2
x	1	1	0	0	0
x_1	1	1	1	1	1
x_2	0	1	1	1	1
x_3	0	1	1	1	0
x_j	0	1	1	0	0
x_k	0	1	0	1	0
x_4	0	1	0	1	1
x_i	0	1	0	0	0
x_l	0	0	0	1	0
x_5	0	0	0	-	1
x_m	0	0	0	0	0

Fig 3.17 (i)

	y	y_1	y_6	y_3	y_2
x	1	1	0	0	0
x_1	1	1	1	1	1
x_3	0	1	1	1	0
x_k	0	1	1	0	0
x_2	0	1	-	1	1
x_j	0	1	0	1	0
x_4	0	1	-	0	1
x_i	0	1	0	0	0
x_5	0	0	-	0	1
x_m	0	0	0	0	0

Fig 3.17 (ii)

It is clear that Fig 17(i) and 17(ii) exhausts all possibilities of adding rows to Fig. 15. So we are in the final stage, when we will address the problem of adding all possible columns to Fig 17(i) and 17(ii), wherefrom the final solution to the problem will emerge.

Our purpose is now to add various possible columns which will forbid its F_2-characteristics and arrive at possible new graphs other then a 6-cycle, an ATE or H_o.

Note that for such forbidding, a matrix obtained by adding new possible columns to Fig. 17(i) or 17(ii) must contain a submatrix

$$\begin{array}{c|cc} & y' & y'' \\ \hline x' & 0 & 1 \\ x'' & 1 & - \end{array}$$

where one at least of the two columns y' and y'' is a new one.

We observe that if x' and x'' rows are taken from the rows x_2, x_3 , x_4 and x_5 and all the other positions in the y' and y'' column are 0, we arrive at a situation similar to Fig. 3.16. So by similar arguments as therein, we conclude that either the graph is of Ferrers dimension 2 or otherwise it contains a 6-cycle, or an ATE or H_o. Thus we are left to the case when every added column has a 1 in either of the x_i, x_j, x_k, x_l, x_m-rows. We divide all the columns to be added to Fig 3.17(i) (or 3.17(ii)) into two types :

Type I. In an added column there is a1 in at least one of the three rows x_i, x_j and x_k. Denote the corresponding columns by y_i, y_j and y_k respectively. This means that x_iy_i position in the y_i column is 1 and so are x_jy_j and x_ky_k. Note that other position of these columns may have any value. So any two or all the three columns y_i, y_j and y_k may in some case become identical and the proof will follow the same course in such case.

Type II. All the positions (of the added columns) in x_i, x_j and x_k row are 0.

First we consider the case, when all the added columns are of Type–I. In this case we will prove the assertion that either $f(H) = 2$ or the bigraph H contains a 6-cycle or an ATE.

As mentioned earlier we consider the added columns in the matrix of Fig. 3.17(i). The other case for Fig. 3.17(ii) is similar and so will be omitted. Here the matrix is

	y	y_1	y_2	y_6	y_3	y_i	y_j	y_k
x	1	1	0	0	0	0	0	0
x_1	1	1	1	1	1	-	-	-
x_3	0	1	1	1	0	-	-	-
x_2	0	1	1	1	1	-	-	-
x_j	0	1	1	0	0	-	1	-
x_k	0	1	0	1	0	-	-	1
x_4	0	1	0	1	1	-	-	-
x_i	0	1	0	0	0	1	-	-
x_l	0	0	0	1	0	-	-	-
x_5	0	0	0	-	1	-	-	-
x_m	0	0	0	0	0	-	-	-

In the above matrix we first observe that all the positions x_1y_i , x_1y_j, x_1y_k must be 1. For example, if $x_1y_j = 0$ then we have a 6-cycle $x_1y_1x_jy_jx_5y_3x_1$ or an ATE $\{xy, x_5y_3, x_jy_j\}$ in the graph H according as $x_5y_j = 1$ or 0.

Now we permute the rows and columns of the above matrix and obtain the following

	y	y_1	y_i	y_j	y_2	y_6	y_k	y_3
x	1	1	0	0	0	0	0	0
x_1	1	1	1	1	1	1	1	1
x_i	0	1	1	-	0	0	-	0
x_j	0	1	-	1	1	0	-	0
x_2	0	1	-	-	1	1	-	1
x_3	0	1	-	-	1	1	-	0
x_k	0	1	-	-	0	1	1	0
x_4	0	1	-	-	0	1	-	1
x_l	0	0	-	-	0	1	-	0
x_5	0	0	-	-	0	-	-	1
x_m	0	0	-	-	0	0	-	0

Fig. 3.18

Consider a submatrix of Fig. 3.18, when only one column y_i (of Type I) is added to Fig. 3.17 (i). Note that if the graph of the submatrix is to be of Ferrers dimension >2, there must be a *1* below a *0* in the y_i column (and the positions of the two rows can not be interchanged). In this case we can verify the existence of a 6-cycle or an ATE. As an

if $x_jy_i = 1$, so $x_jy_i = 0$. Now we have a C_6 $x_iy_jx_3y_6x_ly_ix_i$

example, let $x_3y_i = 0$ and $x_ky_i = 1$. In this case we have a 6-cycle $x_jy_ix_k\, y_6x_3y_2x_j$ or an ATE $\{x_iy_i, x_jy_2, x_ly_6\}$ according as x_ly_i is ... *1* or 0.

Similar proof will follow, when instead of y_i, only one column y_j or y_k is added to Fig 3.17(i) to form a submatrix of Fig. 3.18.

Now we consider the case when we require adding not one column, but two columns, say y_i and y_j to Fig. 3.17(i) making the bigraph of Ferrers dimension >2. In this case the matrix must contain the forbidden submatrix.

	y_i	y_j
x'	0	1
x''	1	0

In this case also, some deliberation will reveal the existence of a 6-cycle or an ATE involving both the vertices y_i and y_j in it. We consider an example. Suppose $x''=x_5$ and $x'=x_l$ Now in this submatrix (of Fig 3.18) if $x_5y_6=0$ then we observe that $\{xy, x_5y_3, x_ly_6)$ is an ATE of the bigraph. So x_5y_6 position must be *1*. Next x_ly_j position must be *0*, since otherwise we have a 6-cycle $x_iy_ix_ly_6x_5y_jx_i$ in the bigraph. Now we verify that $x_jy_ix_ly_6x_5y_jx_j$ is a 6-cycle or $\{xy, x_5y_l, x_ly_j\}$ is an ATE of the bigraph according as x_jy_i is a *1* or *0* respectively.

Next we consider the matrix of Fig. 3.18. If its bigraph H is of $f(H) > 2$ then the cases just stated above will arise again and similarly we can verify that H must contain either an ATE or a 6-cycle.

Now we consider the case when addition of columns of type II only to Fig 3.17(i) or 3.17 (ii) leads us to a bigraph of Ferrers dimension >2. It is a matter of routine deliberation to verity our assertion in this case also.

The same assertion can be verified for the case when we have to add columns of both the type I and II. It will be very interesting to observe in this context, that for this bigraph to be of Ferrers dimension >2, the columns of type I has no role to play and as a consequence the ATE or 6-cycle present in the graph can be seen to be independent of the vertices y_i, y_j and y_k (columns of Type I).

One last case still remains to be considered. Note that during the process of expansion of Fig 17(i) or 17(ii), by adding new columns to them, two or more rows of the same form in Fig 17(i) or 17(ii) may be repeated with '-' s in the corresponding position of the added columns. Also two or more columns of the same form may get a repetition, with the result that the new bigraph turns out to be of Ferrers dimension >2. We just mention here that the exhaustive search in this case also will lead us to the same old story and no new graph. ■

Combining proposition 3.3 and theorem 3.4 we sum up the main result of this chapter in the following form :

Theorem 3.5. *A bi(di) graph H of Ferrers dimension ≤ 2 is bichordal and ATE free. On the other hand, in case when a bichordal and ATE free bigraph contains a strong bisimplicial edge them the graph H_0 of Fig 3.5 is the only forbidden subgraph for a bigraph of Ferrers dimension at most 2.*