# CHAPTER 2*

# FORBIDDEN CONFIGURATIONS AND RECOGNITION ALGORITHM OF INTERVAL DIGRAPH/BIGRAPH

## 2.1 Introduction

An *interval digraph* is a directed graph $D(V, E)$ for which every vertex $v \in V$ is assigned a pair of closed intervals $(S_v, T_v)$ such that $uv$ is an edge (arc) iff $S_u$ and $T_v$ have a non-empty intersection. An *interval bigraph* is a basically equivalent concept of an interval digraph. It is a bipartite graph $B(U, V, E)$ having bipartite sets $U$ and $V$, for which there are two families of intervals $\{S_u : u \in U\}$ and $\{T_v : v \in V\}$ such that $uv \in E$ iff $S_u \cap T_v \neq \phi$.

An interval bigraph was introduced in [Harray,Kabel, McMorris,1982] while an interval digraph was introduced in [Sen *et al.*,1989a]. That the two concepts are equivalent can be seen from the following .

Given a digraph $D(V, E)$, consider the bipartite graph $B=B(D)$ whose partite sets are two disjoint copies $U$ and $V$ of the set $V$ of $D$ and let two vertices $u$ and $v$ in $B(D)$ be adjacent iff $uv \in E$. Then it is not difficult to show that $D$ is an interval digraph iff $B(D)$ is an interval bigraph. On the other hand, $B(U, V, E)$ is an interval bigraph iff the directed graph $D(U \cup V, E)$, obtained from $B$ by directing all the edges from $U$ to $V$ is an interval digraph.

Several characterizations of an interval digraph / bigraph are known [Müller,1997; Sanyal&Sen,1996; Sen *et al.*, 1989a]. In [Sen *et al.*,1989a] it was characterized in terms of its adjacency matrix and in terms of Ferrers digraphs. We recall the following theorem that characterizes an interval digraph.

**Theorem** [Sen *et. al.*, 1989a] *The following conditions are equivalent.*

*(A) D is an interval digraph.*

*(B) The rows and columns of the adjacency matrix of D can be (independently) permuted so that each 0 can be replaced by one of {R, C} in such a way that every R has only R's to*

* This Chapter has been communicated to Discrete Appl. Math

*its right and every C has only C's below it.*

*(C) D is intersection of two Ferrers digraphs whose union is complete.*

From the above theorem it follows that the Ferrers dimension $f(D)$ of an interval digraph $D$ is at most $2$. It was also shown by them that the converse is not true and in fact there exists a digraph of $f(D) = 2$which is not an interval digraph. The Ferrers dimension of a digraph $D$ will also be referred to as the Ferrers dimension of its corresponding bigraph $B(D)$. A digraph with $f(D) = 2$ was characterized independently by Cogis [1979] and also in [Doignon, Ducamp, Falmagne,1984;Sen *et al.*,1989a; Sen, Sanyal and West ,1995] in different contexts.

Cogis [1979] introduced the concept of the *associated graph H(D)* corresponding to a digraph $D$. It is the graph whose vertices correspond to the $0$'s of the adjacency matrix $A(D)$ of $D$ with two such vertices are joined by an edge in $H(D)$ when the corresponding $0$'s form the permutation matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $A(D)$.The $0$'s are then said to form an *obstruction.* Alternatively $H(D)$ can be defined in following manner: let $D = (V, E)$ be a digraph, i.e., $E \subseteq V \times V$. Then $H(D)$ is an undirected graph with vertex set $(V \times V) \setminus E$ and two non-edges $(u, v)$ and $(x, y)$ of $D$ are adjacent in $H(D)$ if and only if $(x, v) \in E$ and $(u, y) \in E$.

Cogis [1979] proved that $f(D)$ of a digraph $D$ is at most $2$ iff $H(D)$ is bipartite. Then he used this result to obtain a recognition algorithm for a digraph of $f(D) = 2$ in a polynomial time.

Müller [1997] obtained a dynamic programming algorithm to recognize an interval bigraph in a polynomial time. He first observed that an interval bigraph is chordal bipartite. It is easily observed that a cycle of length at least 6 is of Ferrers dimension $3$. So a bigraph which contains an induced cycle of length $\geq 6$ is necessarily of $f(D) \geq 3$. Since an interval digraph (bigraph) is of $f(D)$ at most $2$, it follows that it must be bichordal. In order to obtain his algorithm, Müller relies on the theorem by Golumbic and Goss [6] that a bipartite graph is chordal bipartite iff each minimal vertex separator induces a complete bipartite subgraph. He then recursively constructs a bipartite interval representation of a graph from interval representations of its proper subgraphs.

Das and Sen [1993] tried to characterize an interval digraph in terms of forbidden configurations of its adjacency matrix. As a matter of fact, Müller had also made an attempt to solve this problem. In section 2.2, we continue from the paper by Das and Sen [1993] and obtain a complete list of forbidden configurations of the adjacency matrix of an interval digraph. In the process we obtain in section 2.3, a recognition algorithm of an interval digraph in a polynomial time $O(n^3)$.

## 2.2. Forbidden Configurations of Interval Digraphs/Bigraphs

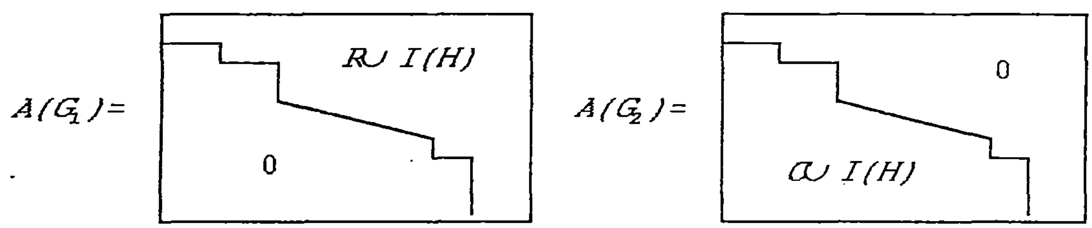
As noted in the introduction, an interval digraph is of Ferrers dimension at most *2*, but the converse is not true. In [Das and Sen, 1993], an effort was made to find out the forbidden configurations of an interval digraph from the perspective of its relations with the associated bipartite graph *H(D)* of *D*. The present paper is, in effect, a continuation of that paper. So it may be worth recalling the main results contained therein for the sake of motivation. Cogis [1979] proved that *f*(*D*) of a directed graph D is at most *2* iff *H(D)* is bipartite. The graph *H(D)* may have more than one connected component; besides it may have one or more isolated vertices (corresponding to the *0*'s which do not belong to any obstruction). The graph obtained by deleting the isolated vertices from *H(D)* is denoted by $H_b(D)$ and is called *the bare graph associated with D* [Doignon *et al.*, 1984].

Let *D* be a digraph of f(D) = 2 so that *H(D)* is bipartite. The set of all isolated vertices of *H(D)* is denoted by ***I****(H)* or ***I*** and a bicolouration of *H(D)* by *(**R**, **C**)*. Recall that a colouration of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colours. Naturally, a bicolourable graph uses two colours only. If $H_b(D)$ has more than one connected components $H_1, H_2, \ldots, H_p$, a bicolouration of $H_i$ will be denoted by $(\boldsymbol{R_i}, \boldsymbol{C_i})$. It is evident that $\boldsymbol{R} = \cup \boldsymbol{R}_i$ and $\boldsymbol{C} = \cup \boldsymbol{C}_i$ for any labelling of the bicolouration $(\boldsymbol{R_i}, \boldsymbol{C_i})$ of $H_i$. Also the elements of the set $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{R_i}$, $\boldsymbol{C_i}$, or $\boldsymbol{I}$ are denoted by the corresponding capital letters R, C, $R_i$, $C_i$, or *I* respectively. The stable sets $R_i$ and $C_i$ are called the *fragments* of $H_b(D)$. While proving his result, Cogis obtained the particular bicolouration *(**R**, **C**)* of $H_b(D)$ in such a way that adjoining all the edges of ***I****(H)* to each of $\boldsymbol{R}$ and $\boldsymbol{C}$ yielded the required Ferrers digraph realization $G_1$ and $G_2$ where $G_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$. Such a bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ for which $G_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$ are Ferrers digraphs, is called a *satisfactory bicolouration.* Clearly if $H(D)$ has no isolated vertex then $D$ is an interval digraph.

While the recognition of a digraph of f(D) = 2 requires the realization of its complement as the union of two Ferrers digraph $G_1$ and $G_2$, not necessarily disjoint, such that $D^c = G_1 \cup G_2$, the problem for an interval digraph recognition, however is to cover its complement by two Ferrers digraph which should necessarily be disjoint, $D^c = H_1 \cup H_2$, $H_1 \cap H_2 = \phi$. This is equivalent to adjoining every edge of $\boldsymbol{I}(H)$ into only one of two digraphs $G_1(V, \boldsymbol{R})$ and $G_2(V, \boldsymbol{C})$ so that they become two disjoint Ferrers digraphs.

To this end, the notion of interior edges was introduced in the same paper.

Let $(\boldsymbol{R}, \boldsymbol{C})$ be a satisfactory bicolouration of $H_b(D)$ leading to a realization of $D^c = G_1(V, E_1) \cup G_2(V, E_2)$ where $E_1 = \boldsymbol{R} \cup \boldsymbol{I}(\boldsymbol{H})$ and $E_2 = \boldsymbol{C} \cup \boldsymbol{I}(\boldsymbol{H})$. Let the rows and columns of $A(G_1)$ be so arranged that all the ones are clustered in the upper right. Similarly, the rows and columns of $A(G_2)$ are so arranged that all the ones are clustered in the lower left.




An edge $I \in \boldsymbol{I}$ is said to be an interior edge of $G_1$, denoted by $I_r$, if there exists a configuration of the form

| | |
|---|---|
| $R$ | $I$ |
| $0$ | $R$ |

in $A(G_1)$; similarly, an $I \in \boldsymbol{I}$ is said to be an interior edge of $G_2$ denoted by $I_c$, if there exist a configuration of the form

$$\begin{array}{|cc|} \hline C & 0 \\ I & C \\ \hline \end{array}$$

in $A(G_2)$. With reference to a particular realization of $D^c$ as the union of $G_1$ and $G_2$, $D^c = G_1 \cup G_2$, the set of all interior edges of $G_1$ is called interior of $G_1$ and is denoted by $\boldsymbol{I_r}(G_1)$ or $\boldsymbol{I_r}$ and all interior edges of $G_2$ is called interior of $G_2$ and is denoted by $\boldsymbol{I_c}(G_2)$ or $\boldsymbol{I_c}$. Note that the sets $\boldsymbol{I_r}$ and $\boldsymbol{I_c}$ are identified with reference to a particular realization of $D^c$ and will change if the realization changes. With these notions, it was proved in the same paper that for a digraph of f(D) = 2, the property $\boldsymbol{I_r \cap I_c \neq \phi}$ is invariant under any satisfactory bicolouration of $H_b(D)$. This means that if $\boldsymbol{I_r \cap I_c \neq \phi}$ for a certain satisfactory bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ of a digraph $D$ of f(D) =2 then the same is true for any satisfactory bicolouration of $H_b(D)$. As a matter of fact, the following proposition was proved in [Das and Sen, 1993].

**Proposition 2.1**[ Das and Sen, 1993]. *Let D be a digraph of f(D) = 2. If* $\boldsymbol{I_r \cap I_c \neq \phi}$ *for a certain satisfactory bicolouration* $(\boldsymbol{R}, \boldsymbol{C})$ *of* $H_b(D)$*, then the same is true for any satisfactory bicolouration of* $H_b(D)$.

Lastly the paper concluded with the following proposition.

**Proposition 2.2** [ Das and Sen ,1993]). *Let D be a digraph of f(D) = 2. If D is an interval digraph, then for any satisfactory bicolouration of* $H_b(D)$, $\boldsymbol{I_r \cap I_c = \phi}$*; but the converse is not true.*

In the Theorems 2.1 and 2.2 of the present chapter we do away with the restriction of a satisfactory bicolouration and prove the same result for any bicolouration $H_b(D)$. Thus the Theorems 2.1 and 2.2 of this chapter are improvements upon the previous one. This generalization, as we will later see, will have a lasting effect when we come to the question of recognition algorithm.

For this generalization, we require extending the definition of $I_r$ and $I_c$ for any bicolouration of $H_b(D)$. With reference to *any* bicolouration $(\boldsymbol{R},C)$ of $H_b(D)$, an $\mathcal{I}_0 \in \boldsymbol{I}$ will be termed $\mathcal{I}_r$ or $\mathcal{I}_c$ if $A(D)$ contains a configuration

| | |
|---|---|
| $R$ | $\mathcal{I}_0$ |
| $1$ | $R$ |

or

| | |
|---|---|
| $C$ | $\mathcal{I}_0$ |
| $1$ | $C$ |

respectively. We also need the following notions of a core matrix of a matrix and of the compatibility between two rows/columns in a matrix.

Let two rows (columns) of the adjacency matrix of a digraph $D$ be identical. It means that the out-neighbours (in-neighbours) of the two vertices are the same. Alternatively the two vertices of the bipartite graph obtained by vertex splitting operations [Müller, 1997] are copies of one another. Since an interval digraph (bigraph) property is a hereditary property, we are not interested in such identical row or columns (copies). Deleting those rows or columns of a matrix A which are identically equal to a previous row (or column) the resulting matrix will be called the core matrix of A and the corresponding digraph, the core digraph of $D$.

In a $(0,1)$ matrix, we will frequently use a '-' in any position to indicate that it is either $0$ or $1$. The rows (or columns) of a binary matrix are compatible, if for some combination of values of the '-' positions they become identical; otherwise they are incompatible. For example in the matrix $M$ below of Fig. 2.1, the rows 2 and 3 are compatible, because they become identical but putting the values 0 to the positions (2,5) and (3,7); but since (1,6) and (2,6) positions have values $0$ and $1$ respectively, the rows 1 and 2 are incompatible.

By a *configuration of an adjacency matrix A*, we shall mean a sub-matrix of $A$ obtained by any (independent) permutation of rows and of columns.

**Proposition 2.3** *Let D be a digraph of Ferrers dimension 2 and let* $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \phi$ *for a satisfactory bicolouration* $(\boldsymbol{R},C)$ *of* $H_b(D)$. *Then the same is true for any other bicolouration of* $H_b(D)$.

The proof of the proposition relies heavily on the following lemma.

**Lemma 2.1** *Let D be a digraph of* $f(D) = 2$ *and let* $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \phi$ *for a satisfactory bicolouration of* $H_b(D)$. *Then the adjacency matrix* $A(D)$ *of D must contain the core matrix of the matrix*

*M or its transpose $M^T$ (subject to independent permutations of rows and/or columns) where*

$$M = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & - & 1 & 0 \\ 3 & 1 & 1 & 1 & 1 & 0 & 1 & - \\ 4 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 5 & 1 & - & 0 & 0 & - & 0 & 0 \\ 6 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & - & 1 & 0 & 0 & - \\ \hline \end{array}$$

Fig. 2.1

**Proof of lemma 2.1.** We first make some observations on the matrix *M*. If the values of the '-' positions are all *0*'s, then the column 2 becomes identical with the column 3 and so also the rows 2 and 3. Then the core matrix of *M* with a bicolouration of the vertices of *H(D)* is of the form

$$\begin{array}{|cccccc|} \hline 1 & 1 & 1 & 1 & R_1 & R_2 \\ 1 & 1 & 1 & C_1 & 1 & R_3 \\ 1 & 1 & 1 & C_2 & C_3 & 1 \\ 1 & R_4 & R_5 & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ C_4 & 1 & R_6 & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ C_5 & C_6 & 1 & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ \hline \end{array}$$

But if the values of the (5,5) and (7,7) positions are both 1 then all the components coalesce into one component. Now we begin the proof of lemma 2.1. Since $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \phi$, there is an $\mathcal{I} \in$

$I_r \cap I_c$ for which the configurations

$$\begin{array}{|cc|} 1 & R \\ R & I \end{array} \quad \text{and} \quad \begin{array}{|cc|} 1 & C \\ C & I \end{array}$$

must be present in the adjacency matrix $A(D)$ of $D$.

So the adjacency matrix $A(D)$ must have configuration

$$\begin{array}{c|ccc} & 1 & 4 & 6 \\ \hline 1 & - & 1 & R \\ 4 & 1 & - & C \\ 6 & C & R & I \end{array} \qquad \ldots\ldots\ldots (1)$$

We label the rows and columns of the configuration conveniently so that they will ultimately coincide with those of the matrix $M$. Now every $R$ and $C$ in the above configuration must be in obstruction with some $C$ and $R$ respectively in the required matrix $A(D)$. We pay our attention to them.

The $R$ and $C$ of (1,6) and (4,6) position require the structure

$$\begin{array}{c|cc} & - & 6 \\ \hline 1 & 1 & R \\ - & C & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 6 & - \\ \hline - & 1 & R \\ 4 & C & 1 \end{array}$$

respectively. Note that we have not labelled the new rows and columns in the above two structures. Several possibilities may occur; we can give them different labels or we can identify two rows and/or two columns, whenever we find them compatible. Our aim is now to explore all possibilities and find out the forbidden configurations.

First we consider the case when two rows and columns in the above structure are given different, as in the following

| | 5 | 6 |
|---|---|---|
| 1 | 1 | R |
| 3 | C | 1 |

and

| | 6 | 7 |
|---|---|---|
| 2 | 1 | R |
| 4 | C | 1 |

In this case the configuration gets the form

| | 1 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 1 | - | 1 | 1 | R | - |
| 2 | - | - | - | 1 | R |
| 3 | - | - | C | 1 | - |
| 4 | 1 | - | - | C | 1 |
| 6 | C | R | - | I | - |

……… 1(a)

Next we explore other possibilities as regards the shape of the matrix when some rows/columns in the above configuration coincide.

For example, we first consider the case by identifying rows 2 and 3 in the configuration 1(a). Then the configuration is

| | 1 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 1 | - | 1 | 1 | R | - |
| 2=3 | - | - | C | 1 | R |
| 4 | 1 | - | - | C | 1 |
| 6 | C | R | - | I | - |

……… 1(b)

Note that all the four rows in 1(b) are incompatible to one another and so we try to identify the possible compatible columns. As an example, let us see what happens when column 1 becomes identical with column 7 and also column 4 with column 6. Then the configuration becomes

| | 1(7) | 2(5) | 6 |
|---|---|---|---|
| 1 | - | 1 | R |
| 4 | 1 | - | C |
| 6 | C | R | I |
| 2(3) | R | C | 1 |

In this case we look at the structure

| | 4(5) | 6 |
|---|---|---|
| 6 | R | I |
| 2(3) | C | 1 |

Here $C$ and $I$ are in obstruction in the matrix $\boldsymbol{C \cup I}$, which is contradictory to our hypothesis, because $\boldsymbol{C \cup I}$ is a Ferrers digraph in a satisfactory bicolouration. So this possibilities is ruled out.

By similar reasoning, we can check that in whatever way we identify the columns either in the configuration 1(b) or in 1(a) we will reach an impossible situation. Therefore we are now left to search for the matrix coming up from the configurations 1(a) and 1(b) only.

First we consider the configuration 1(a). In that configuration the structure

| | 6 | 7 |
|---|---|---|
| 1 | R | - |
| 2 | 1 | R |
| 6 | I | - |

implies that positions (1,7) and (6,7) must be $0$ (otherwise it would contradict the property that $\boldsymbol{R \cup I}$ is a Ferrers digraph for a satisfactory bicolouration). Again since $\boldsymbol{C \cup I}$ is a Ferrers digraph, the structure

| | 5 | 6 |
|---|---|---|
| 3 | *C* | *1* |
| 4 | - | *C* |
| 6 | - | *I* |

implies that the positions (4,5) and (6,5) are *0*.

Now by the same logic,

| | 1 | 6 |
|---|---|---|
| 1 | - | *R* |
| 4 | *1* | *C* |

implies that (1,1) position cannot *C* or *I*. Also

| | 1 | 4 |
|---|---|---|
| 1 | - | *1* |
| 6 | *C* | *R* |

means that (1,1) position cannot be *R* or *I*. So (1,1) position must be 1. Similarly (4,4) position is *1*. The configuration thus takes the shape

| | 1 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 1 | *1* | *1* | *1* | *R* | *0* |
| 2 | - | - | - | *1* | *R* |
| 3 | - | - | *C* | *1* | - |
| 4 | *1* | *1* | *0* | *C* | *1* |
| 6 | *C* | *R* | *0* | *I* | *0* |

The structure

| | 5 | 7 |
|---|---|---|
| 1 | *1* | *0* |
| 4 | *0* | *1* |

means that the two *0*'s are of two distinct colours and for a satisfactory bicolouration the position (1,7) must be *R* and position (4, 5) is *C*.

Next the structure

| | 1 | 6 |
|---|---|---|
| 1 | 1 | R |
| 2 | - | 1 |
| 4 | 1 | C |

implies that the position (2,1) must be *1*. On similar grounds the positions (3,1), (2,4) and (3,4) are all *1*.Thus the configuration gets the form

| | 1 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | R | R |
| 2 | 1 | 1 | - | 1 | R |
| 3 | 1 | 1 | C | 1 | - |
| 4 | 1 | 1 | 0 | C | 1 |
| 6 | C | R | 0 | I | 0 |

Now we consider the obstruction corresponding to *C* and *R* in the positions (6,1) and (6,4) respectively. For them we have the structures

| | 1 | - |
|---|---|---|
| - | 1 | R |
| 6 | C | 1 |

and

| | - | 4 |
|---|---|---|
| 6 | 1 | R |
| - | C | 1 |

Arguing similarly as before we arrive at the matrix

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | R | R |
| 2 | 1 | 1 | 1 | 1 | - | 1 | R |
| 3 | 1 | 1 | 1 | 1 | C | 1 | - |
| 4 | 1 | 1 | 1 | 1 | C | C | 1 |
| 5 | 1 | - | R | R | - | 0 | 0 |
| 6 | C | 1 | 1 | R | 0 | I | 0 |
| 7 | C | C | - | 1 | 0 | 0 | - |

We note that rows 2 and 3 are the only compatible rows and the columns 2 and 3 are the only compatible columns in the above matrix. We also note that the rows 5 and 7 constructed in the way can not be merged with any other row. Replacing *R*'s and *C*'s by *0*'s we arrive at the required matrix *M*.

If we now start with the configuration 1(b), and consider the obstructions of C and R in the positions (6,1) and (6,4) respectively , then exactly as before we will arrive only at the matrix M where with rows 2 and 3 identifying together. ■

**Proof of Proposition 2.3.** We divide the proof into three cases according to the values of the positions (5,5) and (7,7) in the above matrix *M*.

(i) When both of them are *1*;

(ii) When both of them are *0*; and

(iii) When one of them is *1* and the other is *0*.

In case (i), the graph $H_b(D)$ is connected and we have nothing to prove.

In case (ii), the number of components of graph $H_b(D)$ is 6 and a possible bicolouration is given by

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | $R_1$ | $R_2$ |
| 2 | 1 | 1 | 1 | 1 | - | 1 | $R_3$ |
| 3 | 1 | 1 | 1 | 1 | $C_1$ | 1 | - |
| 4 | 1 | 1 | 1 | 1 | $C_2$ | $C_3$ | 1 |
| 5 | 1 | - | $R_4$ | $R_5$ | I | I | I |
| 6 | $C_4$ | 1 | 1 | $R_6$ | I | I | I |
| 7 | $C_5$ | $C_6$ | - | 1 | I | I | I |

The above matrix has the interesting feature that if the matrix is divided into four blocks as in the figure, the upper left (UL) block has all its elements equal to *1*, while all the *I*s comprise the lower right (LR) block. The upper right (UR) block has the fragments of the components $H_1$, $H_2$ and $H_3$ whereas those of $H_4$, $H_5$ and $H_6$ all have their places in the lower left (LL) block.

We now come to the question of interchange of colours. Suppose, as a case in point, we interchange the colours of $H_1$ only without changing the colours of the other components. Here we observe that the column 6 has two fragments of the same colour and so the $\mathcal{I}$ of the (6,6) position loses the property that it is both an $\mathcal{I}_r$ and $\mathcal{I}_c$. But then the column 5 has fragments of two colours and so the $\mathcal{I}$ in the (6,5) position becomes both $\mathcal{I}_r$ and $\mathcal{I}_c$. If we interchange the fragment of colours of any number of components in the UR-block, keeping the colours of the components in the other block unchanged, we find that at least one column in the UR-block must have fragments of two different colours and one of the $\mathcal{I}$s in the 6th row becomes both an $\mathcal{I}_r$ and an $\mathcal{I}_c$.

Similar things happen when we interchange the fragment colours of the components in the LL-block, this time the $\mathcal{I}$s of the 6th column coming to satisfy the requirement.

This loss and gain property of an $\mathcal{I} \in \mathbf{I}_r \cap \mathbf{I}_c$ is clearly manifest in the block diagram of the matrix and remains the motivating spirit behind our assertion.

Considering the other possibility, if we interchange the fragment colours in both the blocks, then all because of the in-built pattern of the different blocks in the matrix, one column in the UR-block and one row in the LL-block have fragments of both the colours and the corresponding row and the column intersect at the required $\mathcal{I}$

For an instance, suppose the colours of components $H_2$, $H_5$ and $H_6$ are interchanged, the first lying in the UR-block and the other two in the LL-block. Then the two blocks UR and LL take the look

| | 5 | 6 | 7 |
|---|---|---|---|
| 1 | 1 | $R_1$ | $C_2$ |
| 2 | - | 1 | $R_3$ |
| 3 | $C_1$ | 1 | - |
| 4 | $R_2$ | $C_3$ | 1 |

UR-block

and

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 5 | 1 | - | $R_4$ | $C_5$ |
| 6 | $C_4$ | 1 | 1 | $C_6$ |
| 7 | $R_5$ | $R_6$ | - | 1 |

LL-block

Here we see that all the columns in the UR-block and the 5th row in the LL-block have the features of containing fragments of both colours and the corresponding $I$s get the required criterion.

Lastly we come to case (iii) when of the two positions (5,5) and (7,7) one has the value *1* and other *0*. Specifically, suppose (5,5) is *1* and (7,7) is *0*. Here the components $H_1$, $H_2$, $H_4$ and $H_5$ coalesce into one component and the matrix takes the following configuration.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | R | R |
| 2 | 1 | 1 | 1 | 1 | - | 1 | $R_3$ |
| 3 | 1 | 1 | 1 | 1 | C | 1 | - |
| 4 | 1 | 1 | 1 | 1 | C | $C_3$ | 1 |
| 5 | 1 | - | R | R | 1 | 0 | R |
| 6 | C | 1 | 1 | $R_6$ | 0 | I | I |
| 7 | C | $C_6$ | - | 1 | C | I | I |

Fig.2.2

As is clear from the above configuration any change in fragment colours has its effect on the two rows 6 and 7 and the two columns 6 and 7 and the required $I$ varies its position at the four corresponding intersecting positions. The other case when (5,5) is *0* and (7,7) is *1* can be similarly proved, $I$s taking the positions at the intersections of rows 5, 6 and columns 5, 6.

**Proposition 2.4.** *Let D be a digraph of f(D) = 2 and let for a satisfactory bicolouration of* $H_b(D)$,

*i)* $I_r \cap I_c = \phi$, *and*

*ii) A(D) contain the configuration (2).*

| | | |
|---|---|---|
| *1* | *1* | *I* |
| *R* | - | $I_c$ |
| - | *C* | $I_r$ |

......... (2)

*Then the same is true for any other bicolouration of $H_b(D)$.*

To prove the Proposition we need the following lemma:

**Lemma 2.2** *Let D be a digraph of $f(D) = 2$ with $\boldsymbol{I}_r \cap \boldsymbol{I}_c = \phi$ for a satisfactory bicolouration of $H_b(D)$ and $A(D)$ contain the configuration (2)*

*Then A(D) must contain the core matrix of N or its transpose,*

*where*

$N =$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | - | 1 | 0 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | - |
| 4 | 1 | - | 0 | 0 | - | - | - | - | - | 0 | 0 |
| 5 | 0 | 1 | 1 | 0 | 0 | - | - | - | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 1 | 0 | - | - | - | 0 | 0 | 1 |
| 7 | 0 | 0 | - | 1 | 0 | - | - | - | 0 | 0 | - |
| 8 | - | - | - | 0 | 1 | - | 0 | 0 | - | 0 | 0 |
| 9 | - | - | - | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 10 | - | - | - | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 11 | - | - | - | - | 0 | 0 | - | 1 | 0 | 0 | - |

*Fig.2.3.*

*where '-' positions are either 0 or 1 subject to the conditions that D is of $f(D) = 2$ and $\boldsymbol{I}_r \cap \boldsymbol{I}_c = \phi$.*

**Proof of lemma 2.2** Before taking up proof of the lemma we observe that for a certain combination of values *1* or *0* to the '-' positions, the digraph *D* may turn out to be of Ferrers dimension *3*. But since an interval digraph is necessarily of Ferrers dimension ≤ *2*, we simply ignore those and consider only those combination of values in the '-' positions for which D is of f(D) = 2. For instance if we consider the case when four positions (2,9), (3,11), (7,3) and (11,7) all *1* and the rest are all *0*s, then the graph $H_b(D)$ is bipartite and so $D$ is of f(D) = 2.(As a matter of fact, $H_b(D)$ in this case has seven components).

Overlooking the values of the '-' positions a possible bicolouration of $H_b(D)$ is as in the following:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $C_7$ | $I$ | $R_7$ |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | - | 1 | $R_7$ |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $C_7$ | 1 | - |
| 4 | 1 | - | $R_2$ | $R_4$ | - | - | - | - | - | 0 | 0 |
| 5 | $C_2$ | 1 | 1 | $R_3$ | 0 | - | - | - | 0 | $I_c$ | 0 |
| 6 | 1 | 1 | 1 | 1 | $C_7$ | - | - | - | $C_7$ | $C_7$ | 1 |
| 7 | $C_4$ | $C_3$ | - | 1 | $C_7$ | - | - | - | $C_7$ | 0 | - |
| 8 | - | - | - | $R_7$ | 1 | - | $R_1$ | $R_6$ | - | 0 | $R_7$ |
| 9 | - | - | - | 0 | $C_1$ | 1 | 1 | $R_5$ | 0 | $I_r$ | 0 |
| 10 | - | - | - | $R_7$ | 1 | 1 | 1 | 1 | 1 | $R_7$ | $R_7$ |
| 11 | - | - | - | - | $C_6$ | $C_5$ | - | 1 | 0 | 0 | - |

Fig.2.4.

Note that since $I_c \cap I_r = \phi$, we will not entertain those values in the '-' positions in which an $I_c$ in the above matrix also becomes an $I_r$ and an $I_r$ also becomes an $I_c$. Also note that for appropriate values, the number of components may turn out to be other than seven. As it can be observed during the proof, if it is more than seven, the additional components will not play any role, whereas as in Proposition 2.3 our arguments in the course of the proof will also hold good when it is less than seven (and some components coalesce).

With this observation, we begin our proof. We start with the configuration

| | 4 | 5 | 10 |
|---|---|---|---|
| 1 | 1 | 1 | $I$ |
| 5 | $R$ | - | $I_c$ |
| 9 | - | $C$ | $I_r$ |

We label the rows and column as above as they will turn out to be convenient and fitting in with the labelling of the matrix $N$ subsequently. The presence of $I_r$ and $I_c$ in configuration (2) require the structures

| | 8 | 10 |
|---|---|---|
| 9 | R | $I_r$ |
| 10 | 1 | R |

and

| | 1 | 10 |
|---|---|---|
| 5 | C | $I_c$ |
| 6 | 1 | C |

in the adjacency matrix. This implies that the matrix should contain the configuration

| | 1 | 4 | 5 | 8 | 10 |
|---|---|---|---|---|---|
| 1 | - | 1 | 1 | - | $I$ |
| 5 | C | R | - | - | $I_c$ |
| 6 | 1 | - | - | - | C |
| 9 | - | - | C | R | $I_r$ |
| 10 | - | - | - | 1 | R |

In the above configuration the presence of the structures

| | 4 | 10 |
|---|---|---|
| 5 | R | $I_c$ |
| 10 | - | R |

and

| | 5 | 10 |
|---|---|---|
| 6 | - | C |
| 9 | C | $I_r$ |

force us to put values *0* to (10,4) (6,5) positions as otherwise we will have $I_r \cap I_c \neq \phi$ (in either case) which will contradict our hypothesis. So the structure become

| | 1 | 4 | 5 | 8 | 10 |
|---|---|---|---|---|---|
| 1 | - | 1 | 1 | - | $I$ |
| 5 | C | R | - | - | $I_c$ |
| 6 | 1 | - | 0 | - | C |
| 9 | - | - | C | R | $I_r$ |
| 10 | - | 0 | - | 1 | R |

……… (3)

In the above configuration we can check that only column 1 and column 8 are compatible but no two rows. So we can identify column 1 and column 8 to achieve the configuration (3a).

| | 1=8 | 4 | 5 | 10 |
|---|---|---|---|---|
| 1 | - | 1 | 1 | $I$ |
| 5 | C | R | - | $I_c$ |
| 6 | 1 | - | 0 | C |
| 9 | R | - | C | $I_r$ |
| 10 | 1 | 0 | - | R |

……… (3a)

The presence of a *C* implies that it has an obstruction with an *R*. So corresponding to the *C*'s in the (9,5), (6,10) and (5,1), we consider the obstructions

| | 5 | 7 |
|---|---|---|
| 8 | 1 | R |
| 9 | C | 1 |

,

| | 10 | 11 |
|---|---|---|
| 2 | 1 | R |
| 6 | C | 1 |

and

| | 1 | 3 |
|---|---|---|
| 4 | 1 | R |
| 5 | C | 1 |

Next we draw our attention to the three *R*'s in the positions (9,8), (10,10) and (5,4). They call for the following structures in *A(D)*

| | 6 | 8 |
|---|---|---|
| 9 | 1 | R |
| 11 | C | 1 |

,

| | 9 | 10 |
|---|---|---|
| 3 | C | 1 |
| 10 | 1 | R |

and

| | 2 | 4 |
|---|---|---|
| 5 | 1 | R |
| 7 | C | 1 |

respectively, which when combined with the earlier matrix gives us

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | - | - | - | 1 | 1 | - | - | - | - | $I$ | - |
| 2 | - | - | - | - | - | - | - | - | - | 1 | R |
| 3 | - | - | - | - | - | - | - | - | C | 1 | - |
| 4 | 1 | - | R | - | - | - | - | - | - | - | - |
| 5 | C | 1 | 1 | R | - | - | - | - | - | $I_c$ | - |
| 6 | 1 | - | - | - | 0 | - | - | - | - | C | 1 |
| 7 | - | C | - | 1 | - | - | - | - | - | - | - |
| 8 | - | - | - | - | 1 | - | R | - | - | - | - |
| 9 | - | - | - | - | C | 1 | 1 | R | - | $I_r$ | - |
| 10 | - | - | - | 0 | - | - | - | 1 | 1 | R | - |
| 11 | - | - | - | - | - | C | - | 1 | - | - | - |

Note that we have arrived at the above matrix from the configuration (3) by adding 6 rows and 6 columns more to it. In the process, some rows/columns can be found compatible to one another. As in the case of lemma 2.1, it can be verified that whenever we identify any two compatible rows/columns, we either get a matrix which does not yield a satisfactory bicolouration, thereby arriving at an impossible situation, or otherwise obtain a core matrix of the matrix N.

Now the structure

| | 5 | 10 |
|---|---|---|
| 1 | 1 | $\underline{1}$ |
| 2 | - | 1 |
| 3 | - | 1 |

implies that the positions (2,5) and (3,5) are *1*.

Also the structure

| | 6 | 7 | 10 |
|---|---|---|---|
| 2 | - | - | 1 |
| 3 | - | - | 1 |
| 9 | 1 | 1 | $\underline{1}_c$ |

implies that all the positions (2,6), (2,7), (3,6) and (3,7) are *1*. Again the structure

| | 5 | 11 |
|---|---|---|
| 2 | 1 | R |
| 6 | 0 | 1 |

implies that the position (6,5) is *C*.

Next from the structures

| | 7 | 11 |
|---|---|---|
| 2 | 1 | R |
| 8 | R | - |

and

| | 5 | 11 |
|---|---|---|
| 6 | C | 1 |
| 8 | 1 | - |

we conclude that the position (8,11) is an *R*.

Also from the structures

| | 10 | 11 |
|---|---|---|
| 1 | I | - |
| 2 | 1 | R |

and

| | 5 | 11 |
|---|---|---|
| 1 | 1 | 0 |
| 6 | C | 1 |

we have the position (1,11) is an *R*.

Also the structure

| | 9 | 10 |
|---|---|---|
| 1 | - | I |
| 3 | C | 1 |

implies that the position (1,9) is *0*.

Next the structure

| | 6 | 8 | 10 |
|---|---|---|---|
| 2 | 1 | - | 1 |
| 10 | - | 1 | 1 |
| 11 | C | 1 | - |

implies that the position (2,8) must be *1*, as otherwise we have the structures $\begin{pmatrix} R & 1 \\ 1 & R \end{pmatrix}$ or $\begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix}$, none of which is possible.

Similarly from the structure

| | 6 | 8 | 10 |
|---|---|---|---|
| 3 | 1 | - | 1 |
| 10 | - | 1 | R |
| 11 | C | 1 | - |

it follows that the position (3,8) is *1*.

Also the structures

| | 7 | 10 |
|---|---|---|
| 8 | *R* | - |
| 9 | *1* | $I_r$ |

and

| | 6 | 10 |
|---|---|---|
| 9 | *1* | $I_r$ |
| 11 | *C* | - |

implies that the positions (8,10) and (11,10) are *0*.

Again the structures

| | 9 | 10 |
|---|---|---|
| 3 | *C* | *1* |
| 9 | - | $I_r$ |

,

| | 10 | 11 |
|---|---|---|
| 2 | *1* | *R* |
| 9 | $I_r$ | - |

,

| | 10 | 11 |
|---|---|---|
| 2 | *1* | *R* |
| 10 | *R* | - |

and

| | 6 | 9 |
|---|---|---|
| 3 | *1* | *C* |
| 11 | *C* | - |

implies that all the positions (9,9), (9,11), (10,11) and (11,9) are *0*.

Now the structure

| | 7 | 8 | 9 |
|---|---|---|---|
| 3 | *1* | - | *C* |
| 9 | *1* | *R* | - |
| 10 | - | *1* | *1* |

implies that the position (10,7) is *1*.

Also the structures

| | 6 | 8 | 9 |
|---|---|---|---|
| 3 | *1* | - | *C* |
| 9 | *1* | *R* | - |
| 10 | - | *1* | *1* |

and

| | 5 | 7 | 9 |
|---|---|---|---|
| 3 | *1* | - | *C* |
| 8 | *1* | *R* | - |
| 10 | - | *1* | *1* |

implies that the positions (10,6) and (10,5) are *1*.

Next the structures

| | 5 | 6 |
|---|---|---|
| 9 | *C* | *1* |
| 11 | - | *C* |

and

| | 7 | 8 |
|---|---|---|
| 8 | *R* | - |
| 9 | *1* | *R* |

imply that the positions (11,5) and (8,8) are *0*.

Now the structure

| | 4 | 10 |
|---|---|---|
| 1 | *1* | *I* |
| 2 | - | *1* |
| 3 | - | *1* |

implies that the positions (2,4) and (3,4) are *1*.

Then from

| | 4 | 9 |
|---|---|---|
| 3 | *1* | *C* |
| 10 | *0* | *1* |

we conclude that the position (10,4) is an *R*.

Now the structure

| | 4 | 5 | 6 |
|---|---|---|---|
| 1 | *1* | *1* | - |
| 9 | - | *C* | *1* |
| 10 | *R* | - | *1* |

implies that the position (1,6) is *1*.

Similarly from the structures

| | 4 | 5 | 7 |
|---|---|---|---|
| 1 | 1 | 1 | - |
| 9 | - | C | 1 |
| 10 | R | - | 1 |

| | 4 | 6 | 8 |
|---|---|---|---|
| 1 | 1 | 1 | - |
| 10 | R | - | 1 |
| 11 | - | C | 1 |

we conclude that the positions (1,7) and (1,8) are *1*.

Now the structures

| | 4 | 7 |
|---|---|---|
| 8 | - | R |
| 10 | R | 1 |

And

| | 4 | 8 |
|---|---|---|
| 9 | - | R |
| 10 | R | 1 |

implies that the positions (8,4) and (9,4) are *0*.

Substituting all these values in the configuration (4) we get the matrix

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | - | - | - | 1 | 1 | 1 | 1 | 1 | 0 | $I$ | R |
| 2 | - | - | - | 1 | 1 | 1 | 1 | 1 | - | 1 | R |
| 3 | - | - | - | 1 | 1 | 1 | 1 | 1 | C | 1 | - |
| 4 | 1 | - | R | - | - | - | - | - | - | - | - |
| 5 | C | 1 | 1 | - | - | - | - | - | - | $I_c$ | - |
| 6 | 1 | - | - | - | C | - | - | - | - | C | 1 |
| 7 | - | C | - | 1 | - | - | - | - | - | - | - |
| 8 | - | - | - | 0 | 1 | - | R | 0 | - | 0 | R |
| 9 | - | - | - | 0 | C | 1 | 1 | R | 0 | $I_r$ | 0 |
| 10 | - | - | - | R | 1 | 1 | 1 | 1 | 1 | R | 0 |
| 11 | - | - | - | - | 0 | C | - | 1 | 0 | 0 | - |

Repeating the same logic for other positions as above (required for a satisfactory bicolouration of $H(D)$ and proceeding, it can be seen that the matrix $N$ is the required matrix containing the configuration (2).

It may also be noted that, if we start from the configuration (3a) instead of the configuration (3) then as in the case of lemma 2.1, we will arrive at a core matrix of N.

**Proof of Proposition 2.4** We first suppose that the graph $H_b(D)$ has all the seven components $H_i$ ($1 \le i \le 7$) as manifest in the figure 4. We will prove that the presence of the configuration (2) is independent of the different combinations of bicolours $(R_i, C_i)$ of the components $H_i$ of the graph $H(D)$. Finally we will consider the case, when, because of the elimination of compatible rows, some components coalesce and the number of components becomes less than 7. It is very important to note in this context that after elimination of some identical rows, the property $I_r \cap I_c = \phi$ in the matrix may get lost in some case, so that according to the Proposition 2.3, the matrix $N$ contains the core matrix of $M$ as a submatrix. As in the present proposition we are interested in only those matrices for which $I_r \cap I_c = \phi$, we will simply ignore those matrices from our considerations.

As in Proposition 2.3, the proof of this proposition relies heavily on the block diagram of the matrix $N$ as given in figures 2.3 and 2.4. For convenience, we name the 9 blocks of $N$ as follows

| | *1* | *2* | *3* |
|---|---|---|---|
| *A* | $A_1$ | $A_2$ | $A_3$ |
| *B* | $B_1$ | $B_2$ | $B_3$ |
| *C* | $C_1$ | $C_2$ | $C_3$ |

Looking at the matrix $N$, we check that the 9 elements of the configuration (2) belong to the 9 different blocks, no two to the same block. The block $B_1$ contains the fragments of the components $H_2$, $H_3$ and $H_4$ excludingly, while those of $H_1$, $H_5$ and $H_6$ belong to the block $C_2$. Also we observe that $I_c$ and $I_r$ belong to the blocks $B_3$ and $C_3$ respectively.

Now we come to the question of other bicolourations by interchange of colours of the fragments in different components. We first consider the block $B_1$; A little scrutiny will reveal that for any change of colours of the components of this block there is a row which contains fragments of both colours and if by such change, the $I$ of (5,10) position loses its property of being an $I_c$, there will be another $I_c$ reappearing from amongst the $0$'s of the 10th column in the same block. This interdependence between the change of colours in block $B$ and the loss and recovery characteristic of an $I_c$ in the block $B_3$ and similarly between the blocks $C_2$ and $C_3$ for an $I_r$ is the essence of the proof and tells us all.

For instance, let $(R_2, C_2)$ be replaced by $(C_2,R_2)$. Then the position (5,10) no longer remains an $I_c$ and the configuration (2) seemingly gets lost. But from the configuration

$$\begin{array}{c|cccc} & 3 & 4 & 9 & 10 \\ \hline 2 & 1 & 1 & - & - \\ 4 & C_2 & R_4 & - & 0 \end{array}$$

it follows that

$$\begin{array}{c|c} & 9 \\ \hline 2 & 0 \\ 4 & 1 \end{array}$$

is not possible and so the $0$ of the (4,10) must be an $I$. Again the configuration

$$\begin{array}{c|cc} & 3 & 10 \\ \hline 4 & C_2 & I \\ 6 & 1 & C_7 \end{array}$$

implies that (4,10) is an $I_c$ and the configuration (2) resurfaces with the column $1$ replaced by column 3 and row 5 by row 4.

The same argument applies for every bicolouration of $H_b(D)$ and with an $I_c$ shifting its position in the column 10 of block $B_3$ and an $I_r$ in the same column of block $C_3$, we see that the configuration is manifest in every bicolouration of $H_b(D)$.

Finally we consider the case when some compatible rows are identical in the process of which some of the components of $H_i$ may coalesce. Here our arguments remain mainly the same as in the previous case; the only difference is that here the blocks lose their distinct identity and the border line between the blocks breaks down.

Let us take two examples. First consider the matrix $N'$ obtained from $N$ by identifying the rows 5 and 9. This case is quite revealing in the sense that the $I_c$ and $I_r$ of the (5,10) position and the (9,10) position are superimposed upon one another and we have an $I$ which belongs to both $\boldsymbol{I_r}$ and $\boldsymbol{I_c}$. This case is covered by Proposition 2.3 and accordingly this matrix N' should contain the core matrix of $M$ of Proposition 2.3 as a submatrix. As a matter of fact, as we can check that the matrix $M$ can be obtained from $N'$ in the following way: identify the rows 4 and 8 and again the rows 7 and 11; then consider the matrix with the rows 2, 3, 10, 6, 4, 5, 7 and the columns 1, 2, 3, 8, 9, 10, and 11; and fill in the blank positions to satisfy the requirements that $H(D)$ of the submatrix is bipartite. Since this case is outside the purview of the present Proposition we ignore this case.

For another example identifying the rows 4 and 9 in the matrix $N$, we have the matrix

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $C_7$ | $I$ | $R_7$ |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | - | 1 | $R_7$ |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $C_7$ | 1 | - |
| 5 | $C_2$ | 1 | 1 | $R_3$ | 0 | - | - | - | 0 | $I_c$ | 0 |
| 6 | 1 | 1 | 1 | 1 | $C_7$ | - | - | - | $C_7$ | $C_7$ | 1 |
| 7 | $C_4$ | $C_3$ | - | 1 | $C_7$ | - | - | - | $C_7$ | 0 | - |
| (9)4 | 1 | - | $R_2$ | $R_4$ | $C_1$ | 1 | 1 | $R_5$ | 0 | $I_r$ | 0 |
| 8 | - | - | - | $R_7$ | 1 | - | $R_1$ | $R_4$ | - | 0 | $R$ |
| 10 | - | - | - | $R_7$ | 1 | 1 | 1 | 1 | 1 | $R_7$ | $R_7$ |
| 11 | - | - | - | - | $C_4$ | $C_5$ | - | 1 | 0 | 0 | - |

We draw our attention to two particular subcases here;

(i) when either of the (6,6) and (6,7) positions are *0* and

(ii) when both of them are *1*.

In subcase (i), the components $H_2$, $H_4$ and $H_7$ coalesce into one component, say $H_7$. If we now interchange the colours of $H_7$ then the $I_c$ and $I_r$ of the (5,10) and the (4,10) are also interchanged, whereas if we change the colours of $H_7$ as well as of $H_1$, then (5,10) becomes an $I_r$ while the *0* of the (8,10) becomes an $I_c$.

In the subcase (ii), $H_2$ is distinct from $H_7$ and then interchanging the colours of $H_2$ only leads to the $I_r$ of (9,10) becoming an $I_c$ again, so that we come back to Proposition 2.3 again. (Since for this combination of colours $\mathbf{I_r \cap I_c \neq \phi}$, according to the Proposition 3 this matrix should again contain the matrix *M*, which can be verified with a careful scrutiny).

Exactly analogous to Proposition 2.4, we have the following proposition.

**Proposition 2.5** *Let D be a digraph of f(D) = 2 and let for a certain bicolouration* (**R**,*C*) *of* $H_b(D)$

*(i)* $\mathbf{I_r \cap I_c = \phi}$

*(ii) A(D) contains the configuration (4)*

| | | |
|---|---|---|
| *1* | *1* | *R* |
| *1* | *I* | $I_c$ |
| *C* | $I_r$ | - |

.......... (4)

*Then the same is true for any other bicolouration of H(D)*

The proof of the above proposition is a consequence of the following lemma.

**Lemma 2.3** *Let D be a digraph of f(D) = 2 and let* $\mathbf{I_r \cap I_c = \phi}$ *for a certain bicolouration (R,C) of H (D). If A(D) contains the configuration (4), then A(D) must contain the core matrix of the following matrix or its transpose (subject to independent permutations of rows and/or columns).*

$P =$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | - | 1 | 1 | 1 | 0 | 1 | 0 |
| 2 | 1 | 1 | 1 | 1 | - | 1 | 1 | - | 1 | 1 | 0 |
| 3 | 1 | 1 | 1 | 1 | - | 1 | 1 | 0 | 1 | 1 | - |
| 4 | 1 | 1 | 1 | 1 | - | 1 | 1 | 0 | 0 | 1 | 1 |
| 5 | - | - | - | - | - | 0 | 0 | - | 0 | 1 | 0 |
| 6 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 7 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | - | - | - | - |
| 8 | 1 | - | 0 | 0 | - | 0 | - | - | 0 | - | 0 |
| 9 | 0 | 1 | 1 | 0 | 0 | 0 | - | 0 | 0 | 0 | 0 |
| 10 | 1 | 1 | 1 | 1 | 1 | 0 | - | - | 0 | - | 0 |
| 11 | 0 | 0 | - | 1 | 0 | 0 | - | 0 | 0 | 0 | - |

Fig. 2.5

The theorem and the lemma can be proved in exactly similar way to those of Proposition 2.4 and Lemma 2.2 and so is omitted here. The possible bicolouration of the above matrix is only noted below.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | - | 1 | 1 | 1 | $R_1$ | 1 | $R_2$ |
| 2 | 1 | 1 | 1 | 1 | - | 1 | 1 | - | 1 | 1 | $R_3$ |
| 3 | 1 | 1 | 1 | 1 | - | 1 | 1 | $C_1$ | 1 | 1 | - |
| 4 | 1 | 1 | 1 | 1 | - | 1 | 1 | $C_2$ | $C_3$ | 1 | 1 |
| 5 | - | - | - | - | - | 0 | $R_8$ | - | 0 | 1 | 0 |
| 6 | 1 | 1 | 1 | 1 | 0 | $I$ | 1 | 0 | $I_c$ | $C_8$ | 0 |
| 7 | 1 | 1 | 1 | 1 | $C_7$ | 1 | 1 | - | - | - | - |
| 8 | 1 | - | $R_4$ | $R_5$ | - | 0 | - | - | 0 | - | 0 |
| 9 | $C_4$ | 1 | 1 | $R_6$ | 0 | $I_r$ | - | 0 | 0 | 0 | 0 |
| 10 | 1 | 1 | 1 | 1 | 1 | $R_7$ | - | - | 0 | - | 0 |
| 11 | $C_5$ | $C_6$ | - | 1 | 0 | 0 | - | 0 | 0 | 0 | - |

From Proposition 2.4 and 2.5 we derive the following Proposition.

**Proposition 2.6** *Let D be a digraph of F.D. 2 and let for a satisfactory bicolouration* $I_r \cap I_c = \phi$. *Then the same is true for any bicolouration.*

We see that the Proposition 2.3 and 2.6 virtually complement one another and combining the two we get the following important theorem, upon which as we will later see, our recognition algorithm for an interval digraph will heavily rely.

**Theorem 2.1** *Let D be a digraph of F.D. 2, then*

(i) *if* $I_r \cap I_c \neq \phi$ *for a certain bicolouration of* $H_b(D)$ *then the same is true for any other bicolouration of* $H_b(D)$*, and on the other hand*

(ii) *if* $I_r \cap I_c = \phi$. *for a certain bicolouration of* $H_b(D)$*, then the same is true for any other bicolouration of* $H_b(D)$

**Proof.** *(i)* Let $I_r \cap I_c \neq \phi$ for a certain bicolouration of $H_b(D)$, our proof will be complete if we can prove that $I_r \cap I_c \neq \phi$ for every satisfactory bicolouration of $H_b(D)$ (because this will imply from Proposition 2.3 that the same is true for any bicolouration of $H_b(D)$ ). Let now if possible it be not true for a certain satisfactory bicolouration of $H_b(D)$ that is $I_r \cap I_c = \phi$ for a certain satisfactory bicolouration of $H_b(D)$. Then Proposition 2.6 implies that $I_r \cap I_c = \phi$ for every bicolouration of $H_b(D)$. Contradictory to our hypothesis.

*(ii)* Follows as a consequence of *(i)*.

Combining Proposition 2.2 and Theorem 2.1, we get the following stronger version of Proposition 2.2.

**Theorem 2.2** *Let D(V,E) be a digraph of f(D) = 2. If D is an interval digraph then for any bicolouration of* $H_b(D)$, $I_r \cap I_c = \phi$.

**Theorem 2.3** *Let $D(V,E)$ be a digraph of $f(D) = 2$ and let for any bicolouration of $H_b(D)$, $I_r \cap I_c = \phi$. Then $D$ is an interval digraph iff the adjacency matrix $A(D)$ of $D$ does not contain any one of the configurations (2) and (4) of the form*

*(i)*

| | | |
|---|---|---|
| 1 | 1 | $\mathcal{I}$ |
| C | - | $\mathcal{I}_r$ |
| - | R | $\mathcal{I}_c$ |

*… … (2)*

*or (ii)*

| | | |
|---|---|---|
| 1 | 1 | R |
| 1 | $\mathcal{I}$ | $\mathcal{I}_c$ |
| C | $\mathcal{I}_r$ | - |

*… … (4)*

*or their transposes.*

**Proof.** ($\Rightarrow$) Let $A(D)$ contain a configuration of either of the given forms. First consider the simple case, when the base graph $H_b(D)$ has one component only. From the configurations, it follows that there exists an $\mathcal{I}$ such that $A(D)$ contains both the structures

| | |
|---|---|
| 1 | $\mathcal{I}$ |
| R | $\mathcal{I}_c$ |

and

| | |
|---|---|
| 1 | $\mathcal{I}$ |
| C | $\mathcal{I}_r$ |

This means that there exists an $\mathcal{I}$ which is in conflict with both the Ferrers digraphs $R \cup I_r$ and $C \cup I_c$ and so this $\mathcal{I}$ can not be included in either of $R \cup I_r$ and $C \cup I_c$ (retaining the Ferrers digraph property). Hence decomposition of $D$ into two disjoint Ferrers digraphs is not possible and so $D$ is not an interval digraph.

Next consider the case when $H_b(D)$ consist of more than one component. From Theorem 2.1 and propositions 2.4 and 2.5, it follows that the presence of the given configurations in $A(D)$ for a particular bicolouration implies the existence of the given configurations for any satisfactory bicolouration as well. Let now $(R, C)$ be any satisfactory bicolouration of $H_b(D)$.

Then as in the earlier case, there is an $\mathcal{I}$ which can not be included in the Ferrers digraphs $\boldsymbol{R}\cup\boldsymbol{I}_r$ or $\boldsymbol{C}\cup\boldsymbol{I}_c$. As this is true for any satisfactory bicolouration of $H_b(D)$, $D$ is not an interval digraph.

($\Leftarrow$). Let $D$ be not an interval digraph. Then we need to prove that $A(D)$ must contain either of the configurations (2) or (4) or their transposes. Again from propositions 2.4 and 2.5, we need to prove the result for satisfactory bicolouration only. So let $(\boldsymbol{R},\boldsymbol{C})$ be satisfactory bicolouration of $H_b(D)$ so that $G=\boldsymbol{R}\cup\boldsymbol{I}$ and $G=\boldsymbol{C}\cup\boldsymbol{I}$ are Ferrers digraphs. Also since $\boldsymbol{I}_r\cap\boldsymbol{I}_c=\phi$, $H_1=\boldsymbol{R}\cup\boldsymbol{I}_r$ and $H_2=\boldsymbol{C}\cup\boldsymbol{I}_c$ are two disjoint Ferrers digraphs.

Since $D$ is not an interval digraph, there exists an $\mathcal{I}$ say $\mathcal{I}_0$ which is conflict with an element of $\boldsymbol{R}\cup\boldsymbol{I}_r$ as well as with an element $\boldsymbol{C}\cup\boldsymbol{I}_c$. This means that $A(D)$ has the configurations

| | |
|---|---|
| - | $\mathcal{I}_0$ |
| $R/\mathcal{I}_r$ | - |

where '-'s are elements outside $\boldsymbol{R}\cup\boldsymbol{I}_r$, and

| | |
|---|---|
| - | $\mathcal{I}_0$ |
| $C/\mathcal{I}_c$ | - |

where '-'s are elements outside $\boldsymbol{C}\cup\boldsymbol{I}_c$.

Consider the configuration

| | |
|---|---|
| - | $\mathcal{I}_0$ |
| $R$ | - |

We show below that the only configuration containing this structure is of the form

| | |
|---|---|
| 1 | $\mathcal{I}_0$ |
| $R$ | $\mathcal{I}_c$ |

all other possibilities leading to contradictions. We observe that the configurations in $A(D)$

$$\boxed{\begin{array}{cc} 1 & I \\ R & 1 \end{array}} \quad \text{and} \quad \boxed{\begin{array}{cc} 1 & I \\ R & C \end{array}}$$

are impossible because $\boldsymbol{R \cup I}$ is a Ferrers digraph. Again for the same reason and because $I_0$ is in conflict with $\boldsymbol{R \cup I_r}$,

$$\boxed{\begin{array}{cc} 1 & I_0 \\ R & I \end{array}}$$

is not possible.

So the only possibility for this case is

$$\boxed{\begin{array}{cc} 1 & I_0 \\ R & I_c \end{array}}$$

Next consider the configuration

$$\boxed{\begin{array}{cc} - & I_0 \\ I_r & - \end{array}}$$

Exactly as before, it can be seen that

$$\boxed{\begin{array}{cc} 1 & I_0 \\ I_r & I_c \end{array}}$$

is the only possibility conforming to this structure. But this implies the existence of

$$\begin{array}{c|ccc} & x & y & z \\ \hline a & - & 1 & I_0 \\ b & 1 & R & - \\ c & R & I_r & I_c \end{array}$$

If the *ax*-position is *0*, then it must be *C*, but then again

$$\begin{array}{c|cc} & x & z \\ \hline a & C & I_0 \\ c & R & I_c \end{array}$$

is a contradiction because $C \cup I_c$ is a Ferrers digraph. Hence *ax*-position is *1* and the structure

$$\begin{array}{c|cc} & x & z \\ \hline a & 1 & I_0 \\ c & R & I_c \end{array}$$

is an implication of the existence of

$$\begin{array}{|cc|} \hline 1 & I_0 \\ I_r & I_c \\ \hline \end{array}$$

Similarly

$$\begin{array}{|cc|} \hline - & C \\ I_0 & - \\ \hline \end{array}$$

implies the existence of the configuration

$$\begin{array}{|cc|} \hline 1 & C \\ I_0 & I_r \\ \hline \end{array}$$

Combining Theorem 2.3 and Theorem 2.2 we state the following theorem:

**Theorem 2.4** *A digraph is an interval digraph if and only if it is of F.D. $\leq 2$ and when it is of $f(D) = 2$, for any bicolouration of $H_b(D)$,*

*(a)* $\mathbf{I}_r \cap \mathbf{I}_c = \phi$

*and (b) A(D) does not contain either of the configurations (2) and (4) and their transposes.*

In terms forbidden adjacency matrices, we state the above result in the following form.

**Corollary 2.1** *A digraph is an interval digraph if and only if it is of F.D.$\leq 2$ and A(D) does not contain the core matrix of either of the matrices M, N, P and their transposes $M^T$, $N^T$, $P^T$, where M, N, P are as given as figures 2.1, 2.3 and 2.5.*

## 2.3 Recognition Algorithm

Müller obtained a recognition algorithm of an interval digraph/bigraph in a polynomial time $O(n\, m^6 (n+m) \log n)$ , where $n$ and $m$ are the number of vertices and edges respectively of the bigraph.

In our algorithm, we first check whether Ferrers dimension of $D$ is equal to two by identifying a bipartition of $H_b(D)$. Then apply the results of the section 2 to recognition an interval digraph. Identifying whether $H_b(D)$ is bipartite generally runs in $O(n^4)$ time, where n is the number of vertices of $D$. But our procedure *bipartite* described below determines it in $O(n^3)$ time. This, intern, gives the $O(n^3)$ as the time complexity of the problem.

For a digraph $D$ of F.D. 2, consider any bicolouration $(R,C)$ of $H_b(D)$ and with reference to this bicolouration, obtain the $I_r$s and $I_c$s by the Procedure *config.* described below. If $I_r \cap I_c \neq \phi$, then $D$ is not an interval digraph. Else, check by the Procedure *config.* again, whether $H(D)$ contains the configuration (2) or (4) or their transposes. If so, the Theorem 2.3 tells us that $D$ is not an interval digraph. Otherwise $D$ is an interval digraph.

Although our algorithm takes much less time, Müller's one has an added advantage that it gives us the interval representation as well, in case $D$ turns out to be an interval bigraph.

The following Algorithm *recog.* alongwith the Procedure *config.* describes the steps for an interval digraph recognition.

**Algorithm *recog* : Interval digraph recognition**

Input: Adjacency matrix
Output: recognition of the graph $G$

1. Identify a bipartite partition of $H_b(D)$ by procedure *bipartite*.
   If no such partition is found then the graph is not an interval digraph.

2. Satisfying step(1), denote the set of all isolated vertices by $I$ and a bicolouration of $H_b(D)$ by $(R,C)$.

3. /* use procedure *config*. For Step 3 */

   for all isolated vertices $I$ in the matrix do

   begin

   (a) check if there exists a configuration of the form

   $$\begin{array}{|cc|}\hline R & I \\ 1 & R \\ \hline\end{array}$$

   (b) check if there exist a configuration of the form

   $$\begin{array}{|cc|}\hline C & I \\ 1 & C \\ \hline\end{array}$$

   (c) if an I satisfy both (a) and (b) then $G$ is not an interval digraph.

   /* $I_r \cap I_c \neq \phi$ */

   If I satisfy (a) denote it by $I_r$ else if $I$ satisfy (b) then by $I_c$.

   end;

4. . /* use procedure *config*. For Step 4 */

   for each vertex $I$ do

   begin

   check if there exist configurations of the form

   (i)

   $$\begin{array}{|cc|}\hline C & I_r \\ 1 & I \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|cc|}\hline C & 1 \\ I_r & I \\ \hline\end{array}$$

   (ii)

   $$\begin{array}{|cc|}\hline C & I_c \\ 1 & I \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|cc|}\hline R & 1 \\ I_c & I \\ \hline\end{array}$$

   if both the configurations (i) and (ii) exist then $G$ is not an interval digraph.

   end;

5. Otherwise, $G$ is an interval digraph.

**Procedure *bipartite***

Data Structure :

- For each pair of columns (i,j), j > i, maintain the two sets $L^1_{ij}$ and $L^2_{ij}$ where

$$L^1_{ij} = \{A[k, i] \mid A[k, i] = 0 \text{ and } A[k, j]=1\}$$

$$L^2_{ij} = \{A[k', j] \mid A[k', i] = 1 \text{ and } A[k', j]=0\}$$

- For each pair of columns $(i, j), j > i$, attach a tag variable $T_{ij}$, where $T_{ij}$ initially contains $0$ and is set to $1$ as the column pair is processed.
- For each i, maintain a set $S_i$ containing the column indices j for which $A[i, j]=1$
- Attach a field to each vertex indicating one colour taken from a given set of two colours. Initiate the procedure with no colour to any vertex.
- In addition, maintain a stack containing the 0's of $A(D)$, that is the vertices of $H_b(D)$. As soon as a vertex is coloured, it may be used to colour other vertices adjacent to it but still not coloured. Once a vertex is popped up from the stack, it is not pushed into the stack any more.

Step 1: Compute $L^1_{ij}$ and $L^2_{ij}$ for all $i, j$ $(j>i)$. if $L^1_{ij} = \phi$ then $L^2_{ij} = \phi$ and vice versa.

Step 2: Compute $S_i$ for all $i$.

Step 3: Find a '*0*' element in the matrix $A$ which is not already coloured.

Step 4: Assign a colour to this element and push it into the stack.

Step 5: Pop an element $A[i, j]$ from the stack.

Step 6: For all elements $k \in S_i$ do the following

if $T_{jk}$, $j<k$ $(T_{kj}, k>j)$ is not set

begin

Step 6(a): Assign the value $1$ to $T_{jk}$

Step 6(b) For each element of $L^1_{jk}$ do

if it is not already coloured, push it into the stack with the colour of $A[i, j]$

assigned to it

else if it is already coloured and of colour other than that of $A[i, j]$, then $H_b(D)$ is not bipartite.

For each element of $L^2_{jk}$ do

if it is not already coloured, push it into the stack with the colour other than

that of $A[i, j]$ assigned to it

else if it is already coloured and of the same colour of $A[i, j]$, then $H_b(D)$ is not bipartite.

end

Step 7: Repeat *Step 5* through *Step 6* until the stack is empty.

Step 8: If any 0 of $A(D)$ still remains to be coloured, repeat *Step 3* through *Step 7* else declare that the graph $H_b(D)$ is bipartite.

The following procedure *config.* describes a technique to search for a 2×2 configuration of the form

$$\begin{array}{|cc|} \alpha & \beta \\ \gamma & I \end{array}$$

in $A(D)$ which is used in steps 3 and 4 of the above algorithm.

**Procedure *config.***

Algorithms for checking the existence of the configurations of the form

$$\begin{array}{|cc|} \alpha & \beta \\ \gamma & I \end{array}$$

for all $I$ in the $n \times n$ matrix $A$.

Input: Adjacency matrix($A$)

Output: Marking the $I$'s in $A$ that form the configuration

$$\begin{array}{|cc|} \alpha & \beta \\ \gamma & I \end{array}$$

Data structure: An $n \times n$ matrix $B$ initialized by 1 to its all entries.

1. [Creation of matrix $B$]

for column $i = 1$ to $n$ do

begin

find rows $a_1, a_2, \ldots a_k$ such that $A[a_j,i]=\alpha$ for $1 \leq j \leq k$

for row $r= 1$ to $n$ do

begin

if $A[r,i]=\gamma$ then $B[r,a_j]=0$ for $1 \leq j \leq k$

end;

end;

2. For $i=1$ to $n$ do

for $j=1$ to $n$ do

begin

if $A[i,j]=I$ then

begin

for $m=1$ to $n$ do

begin

if $A[m,j]=\beta$ then

if $B[i,m]=0$ then there exists the desired configuration in the matrix for the $I$ in $[i,j]$ position.

end;

end;

end;

**Complexity Analysis:**

We first show that the complexity of Procedure *bipartite* (which determines if $H_b(D)$ is bipartite) is $O(n^3)$, where $n$ denotes the number of vertices of $D$.

The *Step 1* of the procedure can be performed in $O(n^3)$ time. The maximum number of repetition of the cycle between *Step 3* and *Step8* is $O(n^2)$. In *Step 6,* at most $n$ elements of $S_I$ are considered. If $T_{jk}$ is set, the *Step 6(a)* and *6(b)* will not be executed. Otherwise *6(a)* and *6(b)* will be repeated $O(n)$ times. In other words for each pair $(j, k)$ $O(n)$ entry in the matrix will be coloured. Hence the repetition of the cycle between *Step 3* and *Step 8* is performed in $O(n^3)$ times. Thus the complexity of the Procedure *bipartite* is $O(n^3)$.

Next it can be easily shown that the time complexity of the Procedure *Config.* is $O(n^3)$.

Now for our Algorithm *recog.* the time complexity of *Step 1* (using Procedure *bipartite* ) is $O(n^3)$ and the time complexity of *Step 3* and *4* (using Procedure *Config.* ) is $O(n^3)$. Hence the overall time complexity of our recognition algorithm is $O(n^3)$. Hence we have the following theorem.

**Theorem 2.5** *An interval digraph D can be recognised in time* $O(n^3)$*, where n is the number of vertices of D.*