MORE RESULTS ON INTERVAL DIGRAPHS, BASE INTERVAL DIGRAPHS AND CONTAINMENT DIGRAPHS

By

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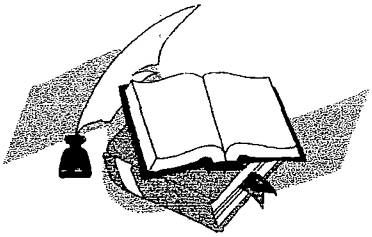


CHAPTER 1
INTRODUCTION

1.1 Basic Definitions

We review here the basic terminology about graphs, directed graphs and relations used throughout this dissertation.

Given a graph $G(V, E)$, $V= V(G)$ will denote its vertex set and $E=E(G)$ its edge set. A graph G' is a subgraph of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. A subgraph G' is a *generated subgraph* or an *induced subgraph* of G if $V(G') \subset V(G)$ and two vertices are adjacent in G' if and only if they are adjacent in G. The complement $\overline{G}(V, E)$ of a graph G has the same vertex set as G and two vertices are adjacent in $\overline{G}$ iff they are not adjacent in G.

A set of vertices of a graph G is a *stable* set (or *independent* set) if no two vertices in the set are adjacent. A *bipartite graph* is a graph $H(V, E)$ whose vertex set V can be partitioned into two stable sets, say X and Y. in this case we write $H=H(X, Y, E)$. A *complete bipartite graph* is a bipartite graph $H(X, Y, E)$ where every vertex in X is adjacent to every vertex in Y.

Let $V=\{v_1, v_2, \ldots.., v_n\}$ be the vertex set of an undirected graph G. Then *characteristic vector* of any subset $V' \subseteq V$ is the vector $x = (x_1, x_2, \ldots, x_n)$ where

$$x_i = \begin{cases} 1 & \text{if } v_i \in V' \\ 0 & \text{if } v_i \notin V' \end{cases} \quad \text{for all } i.$$

It is clear that the family of all subsets of V has one-to-one correspondence with the corners of the unit hypercube in R^n.

A directed graph (digraph) $D(V, E)$ consists of finite set V of vertices and a set E ordered pairs of vertices called edges (arcs). In this dissertation, sometimes we will write

$u \rightarrow v$ to mean $uv \in E$ and $u \not\rightarrow v$ to mean $uv \notin E$. All throughout we will assume that a digraph may have loops (i.e. edges of the form (u, u))but no multiple edges.

The *successor* set of a vertex v is the set of vertices u such that vu is an edge of the digraph $D(V, E)$. The *predecessor* set of a vertex v is the set of vertices u such that uv is an edge of the digraph $D(V, E)$.

The digraph $D(V, E)$ is an *acyclic digraph* when it contains no directed cycle.

The adjacency matrix of a di(graph) on the vertex set $V=\{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ 0, 1-matrix with 1 in the (i, j) position if only if (v_i, v_j) is an edge of the (di)graph.

The union of two digraphs $D_1=(V_1, E_1)$ and $D_2=(V_2, E_2)$ is the is the digraph $D(V, E)$ whose vertex set $V = V_1 \cup V_2$ and edge $E=E_1 \cup E_2$; similarly the intersection of two digraphs D_1 and D_2 is a digraph D consisting only of those vertices and edges that are in both D_1 and D_2.

Other definitions on graph theory will be given as and when required in the thesis. For any undefined terms one is referred to Roberts [1976], Golumbic [1980] or West [1996].

Graphs, digraphs and partial orders are all binary relations. A binary relation R on a set A is a subset of $A \times A$. A binary relation (A, R) is *reflexive* if aRa for all $a \in A$. A binary relation (A, R) is *symmetric* if $aRb \Rightarrow bRa$ for all $a, b \in A$. An undirected graph is a symmetric related set. A binary relation (A, R) is *transitive* if for all $a, b, c, \in A$, aRb and bRc implies aRc. A binary relation R on A is a partial order if R *irreflexive* and transitive. A partial order (A, R) is a linear order if for elements $a, b, \in A$, either aRb or bRa holds.. More background materials on relations can be found in Fishburn [1985].

1.2 Notations

For convenience, the most frequently used notations are listed here.

$G(V, E)$	A graph whose vertex set is V and edge set is E.
$D(V, E)$	A digraph whose vertex set is V and edge set is E.
$H(X, Y, E)$	A bipartite graph whose vertex set is partitioned into two stable sets X and Y and whose edge set is E.
xy	edge whose vertices are x and y .
$\bar{D}$	Complement of D.
$H(D)$	Associated graph of D.
$G \backslash G_1$	Subgraph of G induced by the vertex set $V(G) - V(G_1)$, for any subgraph G_1 of G.
C_n	Cycle of length n.
K_n	Complete graph with n vertices .
$K_{m, n}$	Complete bipartite graph, the size of whose stable sets are m and n.
$f(D)$	Ferrers dimension of the digraph D.
I	The set of all isolated vertices of $H(D)$.
$H_b(D)$	The graph obtained from $H(D)$ by deleting its isolated vertices.
$B(xy)$	The subgraph induced by $adj\ (x) + adj(y)$.
$\lfloor n \rfloor$	greatest integer $\leq n$
$\lceil n \rceil$	smallest integer $\geq n$.
$\mid S \mid$	Cardinality of the set S.
$A(D)$	Adjacency matrix of the digraph D.
$Adj(H)$	Biadjacency matrix of the bigraph H.

1.3 Preliminaries

The subject of intersection graph is an important area of study in graph theory. The intersection graph of a family $\mathfrak{I}$ of sets is an undirected graph $G(V, E)$ such that there is a one-to-one correspondence between $\mathfrak{I}$ and V and two vertices of V have an edge between them when the corresponding sets have a non-empty intersection. It was shown by Marczewski [1945] that all graphs are intersection graphs; because to each vertex v of G, if we associate the set of all edges incident to v, then G is the intersection graph of this family of sets. Interesting problems arise only when the families of sets have some specific topological or other properties. The families of sets most often considered in connection with intersection graphs are the families of intervals on the real line and then the corresponding intersection graph is known as *interval graph*. Detailed discussion on this topic can be found in the Golumbic [1980] and Fishburn [1985]. A recent survey by Trotter [1997] summaries a variety of recent results and open problems.

The study of directed graphs (digraphs) from the viewpoint of intersection representation began only during the last decade. Since the undirected graphs arc special types of directed graphs, the problem of finding a natural translation of the theory of intersection graphs along with its abundant literature to directed graphs is very well worth investigating. Intersection digraphs of a family of ordered pair of intervals on the real line and of arcs of a circle, respectively called *interval digraphs* and *circular arc digraphs* and related topics have been extensively studied by Sen, West and others. These results provide the background materials for this dissertation.

For a very nice treatment of the several aspects of intersection (di) graph and their applications to different fields such as, biology, computing, psychology, statistics etc. with many latest references, see the book by Mckee and McMorris [1999].

In the second chapter, first we characterize interval digraphs in terms of the interior edges with reference to a bicolouration of its associated bipartite *graph H(D)*. This gives us a complete list of forbidden configurations of an interval digraph. We then apply this characterization to obtain a recognition algorithm of an interval digraph in time which is much less than that given by Müller [1997].

Lekkerkerker and Boland [1962] used the linear order of points on the real line to characterize an interval graph which states, interalia, that any three vertices of an interval graph can be ordered in such a way that every path from the first vertex to the third vertex passes through a neighbour of the second. To obtain an analogous characterization of an interval digraph H. Müller [1997] introduced the notion of asteroidal triple of edges

(ATE) and conjectured that a digraph is an interval digraph if and only if the corresponding bipartite graph is free from ATE and other special class of bigraphs (insects).

In this context we note that an interval digraph is necessarily of Ferrers dimension ≤ 2 and its corresponding bigraph is ATE free so the question arises as to the relationship between these two classes of graphs. In chapter 3 we obtain the result that a digraph of $f(D)$ at most 2 is necessarily bichordal and ATE-free. But the converse problem is much involved. In the same chapter, we have considered this problem in the particular case, when the graph contains a strong bismplicial edge.

A base interval is an ordered pair (S, p) where S is a closed interval and p is a point of S. Previously assigning an ordered pair of base intervals to every vertex, the model of overlap base interval digraphs (Robin digraphs) was introduced and its adjacency matrix characterized.

In chapter 4 we have extended the notion of intersection to the base intervals, *base interval digraphs* and characterized the adjacency matrices of such digraphs. We have also characterized a base interval digraph in terms of intersection of two interval digraphs and then we have obtained an analogous result for undirected graph.

Finally we have considered overlap base interval digraphs where the intervals are of unit length. It has been proved in this chapter that this class of digraphs is actually the same as the class of overlap digraphs. The notion of overlap digraphs was introduced earlier [Sen, Sanyal and West, 1995].

1.4 Interval graphs

1.4.1 Interval graphs and their applications

An *interval graph* is the intersection graph of some family of intervals on the real line. That is, $G=G(V, E)$ is an interval graph if and only if there is an assignment $\boldsymbol{I}$ of a real intervals I_v to each $v \in V$ such that for all $u, v \in V$,

$$uv \in E \Leftrightarrow I_u \cap I_v \neq \varphi.$$

Interval graphs have a long and rich history. The topic is so important that a special issue of Discrete Mathematics [1985] is devoted to discuss interval graphs and related topics. They arose independently from purely mathematical consideration by Hajos [1957] in combinatorics and by the renowned molecular biologist Benzer [1959] from a problem of genetics. Benzer's problem was whether the fine structure (subelements) inside the gene were linked together in a linear arrangement. He answered this question in affirmative by studying the overlap data of mutations. In terms of graph theory a graph G can be defined by considering the substructures (mutations) to be the vertices of G with an edge between two vertices when two corresponding substructures overlap. Then Benzer's problem reduced to finding out if G was an interval graph or not.

Interval graphs are among the most useful mathematical structures for modeling real world problem. It had been used in seriation problem by Kendall [1971] and Hubert [1974], in archaeology by Skrien [1980, 1984] and in developmental psychology by Coombs and Smith [1973] to name a few. In arriving at a solution to general traffic phasing problem, stoffers [1968] and Roberts [1978, 1979] used interval graphs. Nicholson [1992] applied interval graphs to computing a protein model. More in-depth treatment and recent views in this field can be found in [Goldberg, Golumbic, Kaplan and Shamir, 1995] and [Waterman, 1995].

Recently in a relevant work [Zang, to appear] introduces the following generalization of interval graphs called a *probe interval graph* to deal with the physical mapping of DNA. A graph G is a *probe interval graph* if $V(G)$ can be partitioned into subsets P and N and each $v \in V(G)$ can be assigned an interval I_v such that $uv \in E(G)$ if and only if both $I_u \cap I_v \neq \varphi$ and at least one of u and v is in P. Interval graphs are simply probe interval graphs with $N = \varphi$.

1.4.2 Some characterizations of interval graphs

Not all graphs are interval graphs (for example, it is easy to check that C_4 is not an interval graph). First characterization of interval graphs was due to Lekkerkerker and

Boland [1962]. They first defined an asteroidal triple of vertices and characterized an interval graph in terms of this notion. A collection of three pairwise non-adjacent vertices v_1, v_2, v_3 of a graph G is an *asteroidal triple* of G if for every pair of vertices of the triple, there exists a path joining them which does not pass through a neighbour of the third.

Theorem 1.1 [Lekkerkerker and Boland, 1962]. *A graph G is an interval graph iff G is triangulated and does not contain any asteroidal triple.*

In that paper, they also provided a complete set of forbidden subgraphs for interval graphs depicted in Fig. 1.1. Thus a graph is an interval graph iff it does not contain any graph of Fig. 1.1. as induced subgraph. [Corneil, Olariu and Stewart, 1997] gives an excellent survey and synthesis of all aspects of asteroidal triple-free graphs. In that paper they showed that interval, permutation, trapezoidal, cocomparability graphs are all special classes of asteroidal triple-free graph.

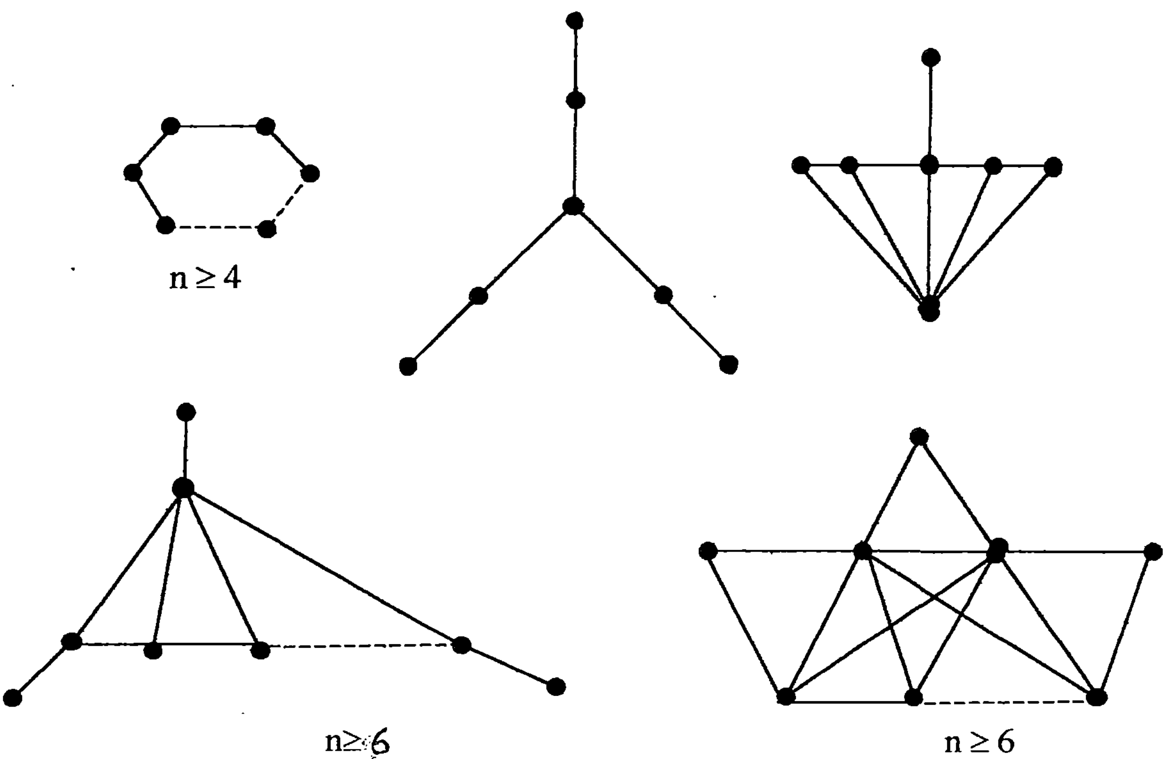


Fig 1.1 Minimal non-interval graphs

Another characterization of interval graph was given by Gilmore and Hoffman [1964] in which they had related interval digraph to what is called a *comparability* graph (or *transitive orientable* graph). A digraph is *transitive* if whenever there is an arc from *u* to v and an arc from v to w, $u \neq w$, then there is an arc from u to w. An orientation of a graph is an assignment of a direction to each of the edges of the graph to get a digraph. A graph has a transitive orientation of its edges if there is an orientation of its edges so that the resulting digraph is transitive. A graph which has a transitive orientation is called a *comparability* graph.

Theorem 1.2 [Gilmore and Hoffman, 1964]. *A graph G is an interval graph if and only if G is chordal and its complement* $\overline{G}$ *is a comparability graph.*

A maximal clique of a graph is a complete induced subgraph which is not contained in any larger such subgraph. For a graph G, its vertex-clique incident matrix $M = (m_{ij})$ is the matrix whose rows and columns correspond to the vertices and the maximal cliques respectively of the graph and

$$m_{ij} = \begin{cases} 1, & \text{if the ith vertex belongs to jth clique} \\ 0, & \text{otherwise} \end{cases}$$

A matrix is said to have consecutive one's property for rows if there is a permutation of the columns so that the 1's in each row appear consecutively.

Theorem 1.3 [Fulkerson and Gross; 1965]. *A graph G is an interval graph if and only if its vertex-clique incident matrix has a consecutive one's property for rows*

Booth and Leuker [1976] used the vertex-clique characterization to obtain a linear time algorithm for interval graph recognition. See Golumbic [1980] for an excellent exposition of the algorithm.[Simon, 1991], [Hsu, 1993] and [Corneil, Olariu, and Stewart, Proc. Ninth ACM-SIAM (SODA), 1998] contain more recent recognition algorithms.

Scheinerman [1988] in a fundamental paper introduced two equivalent models for random interval graphs. Several results about number of edges, degrees, chromatic number and other indices of almost all interval graphs were also established in that paper.

1.5 Indifference graphs

Motivated by the theory of preference and indifference in economics and psychology, indifference graphs were introduced and studied by Roberts [1969]. An undirected graph G is an *indifference graph* if given $\delta > 0$ there exists a real-valued function f on the vertices of G such that vertices u, v are adjacent if and only if $|f(u) - f(v)| \leq \delta$. Then f is an *indifference representation* of G. Roberts characterized indifference graphs and proved that they are equivalent to proper interval graphs and to unit interval graphs. A *proper interval graph* is the intersection graph of intervals in which no interval properly contains other. A *unit interval graph* is the intersection graph of intervals of unit length.

Theorem 1.4 [Roberts, 1969]. *For a graph G, the following conditions are equivalent* :

i) *G is an indifference graph ;*

ii) *G is a unit interval graph ;*

iii) *G is a proper interval graph ;*

iv) *G is an interval graph and does not contain* $K_{1,3}$ *as an induced subgraph.*

1.6 Containment Graphs and overlap Graphs

These two classes of graphs are generated by the models related to intersection. Given a collection of intervals on a real line, each pair of intervals will satisfy exactly one of the following properties, concerning the question of their intersection :

Overlap : the two intervals intersect but neither properly contains the other.

Containment : one of the two intervals properly contains the other.

Disjointness : the two intervals have empty intersection.

A graph G is an *overlap graph* if its vertices can be put into one-to-one correspondence with a collection of real intervals such that two vertices are adjacent in G if and only if their corresponding intervals overlap. The concept of overlap graph were introduced by Even and Itai [971] and were studied by Gavril [1973], Fournier [1978] and Buckingram [1980]. It turns out, nevertheless, that this class of graphs is exactly the same as the class of *circle graphs*, the intersection graph of a finite collection of chords on a circle.

Similarly, in a *containment graph* edges correspond to containment of intervals $I_v \subset I_u$ *or* $I_u \subset I_v$. Containment graphs were studied by Golumbic [1984] and also by Golumbic and Scheinerman [1985]. In [1985] they defined containment graphs in terms of partially ordered sets and showed that the class of containment graph is equivalent to the class of all comparability graphs.

1.7 Ferrers digraphs and Ferrers dimension

Ferrers digraphs play an important role in our study. This special class of digraphs was introduced independently by Guttman [1944] and Riguet [1951]. Riguet defined a *Ferrers digraph* to be a digraph $D(V, E)$ in which for all x, y, z and $t \in V$; $xy, zt \in E \Rightarrow xt \in E$ *or* $zy \in E$ (inclusive) (vertices need not be disjoint). Riguet characterized Ferrers digraphs as those whose successor sets or predecessor sets are linearly ordered by inclusion. This condition is equivalent to the transformability of the adjacency matrix $A(D)$ by independent row and column permutations to a 0, 1-matrix in which the 1's (or 0's) are clustered in the lower left in the shape of a Ferrers diagram (see Fig. 1.2).

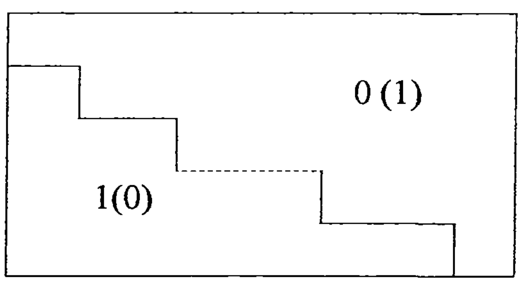


Fig. 1.2 Ferrers digraph

Riguet also showed that the above conditions are equivalent to the condition that adjacency matrix of the digraph has no 2 by 2 submatrix that is a permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are
Such a forbidden matrix is called an *obstruction* and the two zeros said to form a *couple* [Sen, Das, Roy and West 1989a]

Intersection of Ferrers digraphs was studied by Bouchet [1971] and he showed that any digraph D is the intersection of a family of Ferrers digraphs containing D. This induces one to introduce the important notion of Ferrers dimension. The *Ferrers dimension* $f(D)$ of a digraph D is the minimum number of Ferrers digraphs whose intersection is D. Bouchet [1971, 1984] also obtained several interesting result on Ferrers dimension.

The digraphs with Ferrers dimension 2 have been characterized independently by Cogis [1979] and Doignon, Ducamp and Falmagne [1984] in different contexts. In order to characterize a digraph D with $f(D)=2$, Cogis associated an undirected graph $H(D)$ with D, whose vertices correspond to the 0's of the adjacency matrix of D, and two such vertices joined by an edge if the corresponding 0's belong to an obstruction. He proved that D has Ferrers dimension at most 2 if and only if $H(D)$ is bipartite. This characterization yields a polynomial time algorithm for testing if a digraph has Ferrers dimension at most 2. Sen *et al.*, [1989a] translated the Cogis condition to an adjacency matrix condition and proved the following theorem.

Theorem 1.5 [Sen *et al*, 1989a] and [Cogis, 1979]. *For a digraph the following conditions are equivalent*

i) *D has Ferrers dimension at most 2*

ii) *The rows and columns of A(D) can be (independently) permuted so that no 0 has a 1 both below it and to its right.*

iii) *The graph H(D) of couples in D is bipartite.*

In a later work [Sen, Sanyal and West, 1995] introduced containment digraphs and showed that the class of containment digraphs is equivalent to the class of digraphs with Ferrers dimension at most 2.

Threshold graphs, a concept analogous to Ferrers digraphs were introduced by Chvatal and Hammer [1977]. The *threshold dimension* $\theta(G)$ of the graph $G = G(V, E)$ is defined to be the minimum number k of linear inequalities

$$\begin{array}{c} a_{11}x_1 + a_{12}x_3 + \dots\dots\dots + a_{1n}x_n \le t_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{k1}x_1 + a_{k2}x_2 + \dots\dots\dots + a_{kn}x_n \le t_k \end{array} \qquad (1)$$

such the $X \subseteq V(G)$ is a stable set if and only if its characteristic vector $X = \{x_1, x_2, \dots, x_n\}$ satisfies (1)

An undirected graph G whose threshold dimension $\theta(G) \le 1$ is a *threshold graph.* Equivalently $G(V, E)$ is a threshold graph if there exists a threshold assignment $[a;t]$ consisting of a labeling of the vertices by the non-negative integers and an integer threshold t such that

$$X \text{ is stable } \Leftrightarrow \sum_{x \in X} a(x) \le t \qquad (X \subseteq V(G))$$

An excellent survey of these two topics of Ferrers digraphs and threshold graphs can be found in [Cogis, 1982a]. Though these two concepts look completely different, still they are related to each other in a peculiar manner. Cogis deduced a set of known characterization of Ferrers digraphs by investigating the connection between symmetric Ferrers digraphs and threshold graph. Then in the same paper he brought about precisely the connection between symmetric Ferrers digraphs and threshold graphs. A broad overview of this topic is to be found in [Golumbic, 1980]. For a very nice and comprehensive study see [Mahadev and Peled, 1995].

The notion of dimension of a partial order was introduced by Dushnik and Miller [1941]; it is the minimum number of linear orders whose intersection is the given partial order. Bouchet showed (by proving that linear orders can equivalently be changed into Ferrers digraphs) that Ferrers dimension of a digraph is an extension of order dimension of a partial order. [Cogis, 1982 b] gave an alternative proof of this result for finite case and showed that the problem of finding the Ferrers dimension of a digraph and the problem of finding the order dimension of a partial order are polynomially equivalent. [Yannakakis, 1982] showed that the 2-dimensional posets are polynomially recognizable but the problem of designing efficient algorithms for order dimension exceeding 2 is NP-complete.

A very good summary of the notion of order dimension and analogous parameters for graphs and digraphs such as boxicity, threshold dimension and Ferrers dimension to name only a few is given in a review paper by West [1985].

1.8 Chordal bipartite graphs

The notion of chordal bipartite graph was introduced and studied by Golumbic and Goss [1978]. For an excellent survey of this topic see Golumbic [1980]. A bipartite graph $H = H(X, Y, E)$ in which every cycle of length strictly greater than 4 has a chord is a *chordal bipartite graph or bichordal graph.*

A pair of edges x_1y_1 and x_2y_2 of H is separable if there exists a set S of vertices whose removal from H causes x_1y_1 and x_2y_2 to lie in distinct connected component of H\S. The set S is an *edge separator* for x_1y_1 and x_2y_2; S is *minimal* if no proper subset of S is an edge separator for x_1y_1 and x_2y_2, with this concept the following theorem characterizes a bichordal graph.

Theorem 1.6 [Golumbic and Goss, 1978]. *A bipartite graph $H = H(X, Y, E)$ is chordal bipartite if and only if every minimal edge separator induces a complete bipartite graph.*

It can be observed that every vertex in the cycle C_{2n} $(n\geq 3)$ is the centre of a chordless path with five vertices (P_5). With this results in mind Hammer *et al.* [1989] gave a more elegant characterization of chordal bipartite graph.

Theorem 1.7 [Hammer, Maffray and Preissmann, 1989]. *Let H be a bipartite graph. Then H is chordal bipartite if and only if every induced subgraph has a vertex which is not the centre of a P_5.*

The notion of bisimplicial edge was also introduced by Golumbic and Goss [1978], and was motivated by the study of a Gaussian elimination in (0, 1)-matrices. An edge xy of a bipartite graph $H = H(X, Y, E)$ is called bisimplicial if every neighbour of x is adjacent to every neighbour of y. They consequently introduced *bipartite edge elimination scheme*, which consists in successively deleting pairs of vertices which form a bisimplicial edge until the remaining graph has no more edge. In relation to the chordal bipartite graphs they proved that every chordal bipartite graph has a bipartite edge elimination scheme but the converse is not true. Golumbic [1980] gave an example of bipartite graph which has bipartite edge elimination scheme but is not bichordal. This means that existence of bipartite edge elimination scheme is not a characteristic property of chordal bipartite graph.

Hammer *et al.* [1989] proved that every chordal bipartite graph has a vertex which is not the centre of a P_5. From this result they derived a bipartite vertex elimination scheme, which consists, in successively deleting vertices which are not the centre of a P_5 until there is no vertex. They proved that the existence of a bipartite vertex elimination scheme is a characteristic property of a chordal bipartite graph.

Perfect edge elimination ordering (a related but different notion of "bipartite edge elimination scheme") were introduced (with different names) in [Brandstadt, 1993] and [Bakonyi and Bono, 1997]. An ordering $\{e_1, \ldots, e_m\}$ of all the edges of G is *a perfect edge elimination ordering* of G if, for each $i \in \{1, \ldots, m\}$, e_i is a bisimplicial edge of the spanning subgraph of G having edge set $e_i, \ldots, e_m$. Müller [1997] and Kloks and Kratsch [1995] discuss algorithmic aspects of perfect edge elimination ordering and recognition of chordal bipartite graph.

Observe that the complement of a chordal graph can not contain an induced cycle of length greater than four. This motivates the following definition. A graph is *weakly chordal* (very often called *weakly triangulated*) if neither it nor its complement contains an induced cycle of length greater than four. Thus every chordal graph is weakly chordal. For characterization and further result on weakly chordal graph see [Sprindal and Sritharan, 1995] and [Hayward, 1996].

A chordal graph is *strongly chordal* if it has the additional property that every cycle of even length at least six has a chord that divides C into two odd length paths. Strongly chordal graphs form an intermediate family between the families of interval graphs and chordal graphs. They have been particularly important because certain graph theoretical problems have efficient computational solutions for sub-families of the family of strongly chordal graph. For structural properties of these graphs, see the fundamental paper by Faber [1983]. Roychauduri [1988] gave an algorithm for the intersection number of strongly chordal graph.

1.9 Intersection digraphs

The concept of intersection digraph was introduced independently in different contents by Beineke and Zamfirescu [1982] and Sen *et al.* [1989a]. For digraphs, the distinction between heads and tails of edges are crucial. To capture this, they assign to each vertex v of the digraph $D(V, E)$ a source set S_v and a sink set T_v. By analogy with undirected graph, a collection of ordered pairs $\{(S_v, T_v) : v \in V\}$ is an intersection representation of D when $uv \in E$ if and only if $S_u \cap T_v \neq \varphi$. Similar to the case of graphs, any digraph is an intersection digraph of some family of ordered pair of sets. This can be seen by taking the source set S_v of the vertex v to be the set of edges with v as source and the terminal set T_v to be the set of the edges with v as terminus. Naturally the problem is posed to minimize the number of elements in a universal set which will determine the intersection representation of the digraph. The intersection number $i(G)$ of an undirected graph G is the minimum size of a set U such that G is the intersection graph of the subsets of U. Erdos, Goodman and Posa [1966] showed that the intersection number of G equals the minimum number of complete subgraph needed to cover its edges. They also proved that $i(G) \leq \lfloor n^2/4 \rfloor$ for an n-vertex graph, the equality being achieved by the bipartite graph $K_{p,q}$, where $p = \lfloor n/2 \rfloor$ and $q = \lceil n/2 \rceil$. More results on this topic can be found in West [1996].

To develop analogous result for digraphs, Sen *et al.* [1989a] defined a generalized complete bipartite subdigraph (abbreviated GBS) to be a sub-digraph generated by the vertex sets X, Y, whose edges are all xy such that $x \in X$ and $y \in Y$. They said "generalized" because X, Y need not be disjoint, which means that loops may arise. The intersection number $i(D)$ of a digraph is the minimum size of U such that D is the intersection digraph of ordered pair of subsets of U. The analogue of the Erdos-Goodman-Posa result is follows :

Theorem 1.8 [Sen *et al.*,1989a]. *The intersection number of a digraph equals the minimum number of GBS's required to cover its edges and the best possible upper bound on this is n for an n-vertex digraph.*

Harary, Kabell and McMorris [1982] introduced the idea of bipartite intersection graph or intersection bigraph. Let $H = H(X, Y, E)$ be a bipartite graph with the bipartite vertex sets X and Y. If two families of sets $\{S_x; x \in X\}$ and $\{T_y : y \in Y\}$ be assigned to the vertices of X and Y respectively such that $xy \in E$ iff $S_x \cap T_y \neq \varphi$, then $H = H(X, Y, E)$ is the *intersection bigraph* of the pair of families $\{S_x\}$ and $\{T_x\}$. The two concepts of intersection bigraphs and intersection digraphs introduced independently are essentially the same; this is explained in the following manner :

Let $D(V, E)$ be an intersection digraph with the representation $\{(S_v, T_v) : v \in V\}$ Let $V=\{v_1, v_2, \ldots, v_n\}$ and let $X=\{x_1, x_2, .., x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two copies of V. Now consider a bipartite graph $H = H(X, Y, E)$ such that two vertices $x_i \in X$ and $y_j \in Y$ have an edge in H iff $v_i v_j \in E$ in D. The bipartite graph so constructed is the bipartite representation of the digraph D and is denoted by $B(D)$. It is now obvious that a digraph $D(V, E)$ is an intersection digraph of a family $\{(S_v, T_v) : v \in V\}$ iff the bipartite graph $B(D)$ is an intersection bigraph of the pair of families $\{S_x : x \in X\}$ and $\{T_y : y \in Y\}$. On the other hand $H(X, Y, E)$ is an intersection digraph iff the directed graph $D(X \cup Y, E)$, obtained from H by directing all the edges from X to Y is an intersection digraph.

Thus to study intersection bigraphs is essentially the same as to study intersection digraphs. Note that the adjacency matrix of D is the biadjacency matrix of $B(D)$ with rows and columns arranged in the obvious way and determines the bigraph completely.

1.9.1 Interval digraphs and their characterizations

By analogy with interval graphs, a digraph is an interval digraph if it has an intersection representation in which every source set and sink set is an interval on the real line. Sen, Sanyal and West [1995] proved that interval digraph is a generalized concept of interval graph. In particular, they proved the following result.

Theorem 1.9 [Sen, Sanyal and West, 1995]. *An undirected graph G is an interval graph if and only if the corresponding symmetric digraph D(G) with loops at every vertex is an interval digraph.*

Necessity of the above result is trivial. If G is an interval graph with interval I_v assigned to the vertex v, then by setting $S_v=T_v=I_v$ yields an intersection representation of $D(G)$. Conversly if $\{(S_v, T_v) : v \in V(G)\}$ is an interval representation of $D(G)$, where $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$ then they showed that $[a_v + c_v, b_v + d_v]$ yields an interval representation of G. Sen, West *et al.* [1989a, 1989b, 1996] gave several characterizations of interval digraphs. In [Sen *et al.*, 1989a] a characterization of interval digraph is obtained which is analogous to the Fulkerson and Gross characterization of interval graph. To state the result let $B\{(X_k, Y_k)\}$be a collection of GBS's whose union is the digraph D. Then define the vertex-source incident matrix for B (abbreviated V, X-matrix) to be the incident matrix between the vertices and the source sets $\{X_k\}$. Similarly, the vertex-terminal incident matrix for B (abbreviated V, Y-matrix) is the incident matrix between the vertices and the terminal sets $\{Y_k\}$. Then the following theorem gives the first characterization of interval digraphs.

Theorem 1.10 [Sen *et al.*, 1989a]. *D is an interval digraph if and only if there is a numbering of the GBS's in some covering B of D such that 1's in a row appear consecutively for both the V, X-matrix and V, Y-matrix of D.*

But more interesting characterization of interval digraph is its adjacency matrix characterization. For a given digraph D, we use $\bar{D}$ to denote the digraph whose adjacency matrix is the difference between $A(D)$ and the matrix of all ones (i.e., the complement of D). Now we state the first adjacency matrix characterization of interval digraphs.

Theorem 1.11 [Sen *et al.*, 1989a]. *The following conditions are equivalent for a digraph D.*

1) D is an interval digraph;

2) $\bar{D}$ is the union of two disjoint Ferrers digraphs ;

3) The rows and columns of the adjacency matrix of D can be permuted independently such that each 0 can be labeled with R or C in such a way that every position to the right of an R is an R and every position below a C is a C.

A matrix satisfying condition (3) above has the *partitionable zero property*, and such a matrix is a *zero partitionable matrix*.

From the definition of Ferrers digraphs, it is obvious that the complement of any Ferrers digraph is also a Ferrers digraph. Hence Ferrers dimension of D also equals the minimum number of Ferrers digraphs whose union is $\bar{D}$. From the above theorem, it is clear that an interval digraph is of Ferrers dimension at most 2. Naturally, the question arises about the converse of the statement. Sen *et al.* [1989a] gave an example of a seven vertex digraph which is of Ferrers dimension 2 but not an interval digraph. In the next chapter we have shown that the smallest such digraph has 6-vertex. Thus the class of interval digraphs form a proper subclass of the class of digraphs of Ferrers dimension at most 2.

Sen, Sanyal and West [1995] gave a less obvious characterization of Ferrers digraph using the existence of biorder representation : D is a Ferrers digraph if and only if there exist two real valued function f, g on $V(D)$ such that $uv \in E$ if and only if $f(u) \geq g(v)$. Recently, West [1998] has used this biorder representation of Ferrers digraphs to give a short proof of the above characterization of interval digraphs.

An alternative way of describing the (R, C) partition of a binary matrix is in terms of the generalized linear ones (glo) property [Sen *et al.*, 1989b]. To describe this property we need an additional concept. A *stair partition* of a matrix is a partition of its position into two subsets (L, U) by a polygonal path from the upper left to the lower right such the set L is closed under leftward or downward movement and the set U is closed under rightward or upward movement. Equivalently, U corresponds to the positions in some upper triangular matrix and L to the positions in the lower triangular matrix (see Fig 1.3).

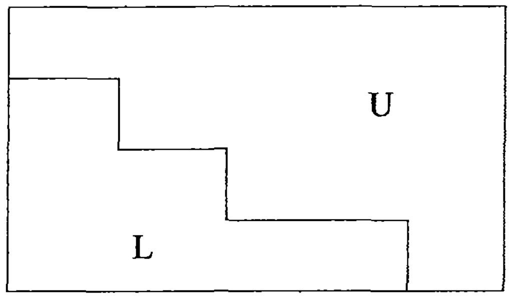


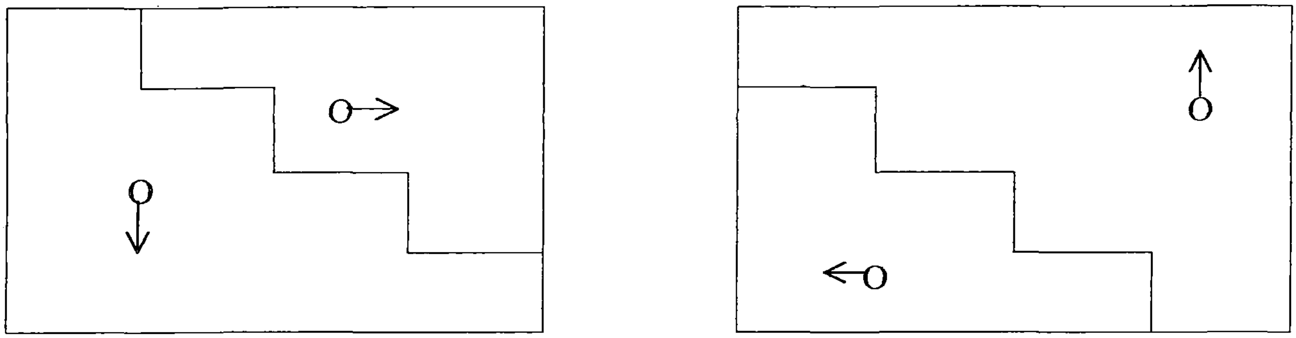
Fig. 1.3 Stair Partition

A 0, 1-matrix has the first generalized linear ones property (glop I) if it has a stair partition (L, U) such that the 1's in U are consecutive and appear leftmost in each row and the 1's in L are consecutive and appear topmost in each column. Similarly a 0, 1-matrix has the second generalized linear ones property (glop II) if it has a stair partition (L, U) such that 1's in U are consecutive and appear down most in each column and 1's in L are consecutive and appear rightmost in each row. This is illustrated in Figure 1.4. It is not difficult to see that two notions of glop I and glop II are equivalent and also that they are equivalent to the idea of (R, C) partition of a (0, 1) –matrix. So as a consequence of the previous theorem, one obtains :

Corollary 1.1 [M. Sen, P. Talukder, S. Das; Preprint]. *For a digraph D the following are equivalent :*

i) *D is an interval digraph.*

ii) *The adjacency matrix A(D) of D has the generalized linear one's property of either first kind or second kind.*



glop I glop II

Fig. 1.4 Generalized linear ones property.

Most characterization of interval graphs and interval digraphs so far involve an order of their vertices. Sanyal and Sen [1996] posed the question, "Is there any ordering among the edges of a (di)graph that characterizes an interval (di)graph?" And they answered this question in the affirmative. For this, they introduced the notion of a consistent ordering of the edges of a (di)graph.

The set of all edges of a digraph $D(V, E)$ is said to have a *consistent ordering* if E has a linear ordering (<) such that for $pq, pu, tq \in E$.

i) $pq < rs < pu \Rightarrow ps \in E \qquad (q \neq u)$

ii) $pq < rs < tq \Rightarrow rq \in E \qquad (p \neq t)$

Theorem 1.12 [Sanyal and Sen, 1996]. *A digraph $D(V, E)$ is an interval digraph if and only if its edge set has a consistent ordering*

Then by appropriate changes in the definition of consistent ordering, they obtained an analogous result for interval graphs. The set E of all edges of a graph $G(V, E)$ has a consistent ordering if E has a linear ordering (<) such that for $pq, rs, pu, tq \in E$

$$pq < rs < pu \;\Rightarrow\; ps \text{ and } pr \in E.$$

Theorem 1.13 [Sanyal and Sen, 1996]. *A graph G(V, E) is an interval graph if and only if its edge set E has a consistent ordering.*

1.9.2 Subclasses of Interval digraphs

We recall that Roberts [1969] introduced indifference graphs and proved that they are equivalent to the unit interval graphs and proper interval graphs.

Sen and Sanyal [1994] generalized the above mentioned undirected graph families by placing analogous constraints on source and sink intervals for interval digraphs. *Unit interval digraphs* are interval digraphs with interval representations such that all the source and sink intervals have unit length. *Proper interval digraphs* are interval digraphs with representation such that no source interval properly contains other source interval and no sink interval properly contains other sink interval. *Indifference digraphs* are those for which there exists an ordered pair of real valued functions f, g on the vertices such that uv is an edge if and only if $|f(u) - g(v)| \leq 1$.

Sen and Sanyal (1994) characterized the graphs in these families and proved that the families are equivalent, generalizing the results of Roberts (1969, 1976). These characterizations are generalizations of those for undirected graphs when an undirected graph is viewed as a symmetric digraph with loops. The adjacency matrix of the corresponding digraph is obtained by adding 1's on the diagonal; this is called the *augmented adjacency matrix $A^*(G)$* for an undirected graph G. A symmetric digraph with loops has an indifference representation with $f=g$, because the symmetry implies that averaging f and g will not change the resulting edges. Conversely, every indifference representation with $f=g$ yields a symmetric digraph with loops. This establishes a bijection between indifference graphs and indifference digraphs representable using $f=g$. It therefore also establishes a bijection between unit interval graph and unit interval digraphs representable by giving every vertex the same source and sink interval.

Sen *et al.* [1989a] characterized the adjacency matrix of an interval digraph in terms of *partitionable zeros property* (described earlier). Since unit interval digraphs is a subclass of interval digraphs, it is hoped that the adjacency matrix characterization for unit interval digraphs is a more restrictive version of the partitionable zeros property. *A 0*, 1-matrix has a *monotone consecutive arrangement* if there exist independent row and column permutations exhibiting the following structure : the 0's of the resulting matrix can be labeled *R* or *C* such that every position above or to the right of an *R* is an *R*, and every position below or to the left of a *C* is a *C*. The resulting matrix and the labeling is a monotone consecutive arrangement, abbreviated *MCA* (see Fig. 1.5).

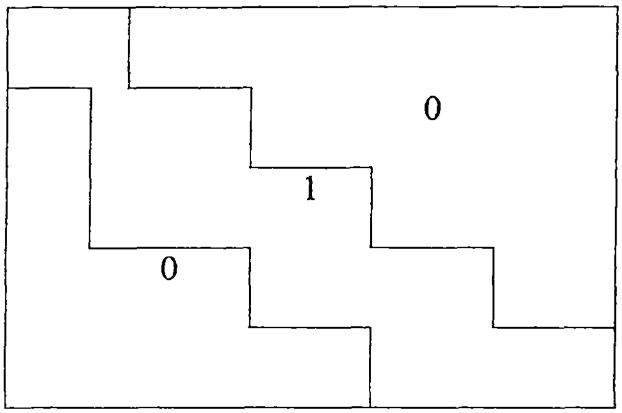


Fig. 1.5 Monotone consecutive arrangement (MCA)

Below we state the characterization of the different digraph classes by Sen and Sanyal [1994].

Theorem 1.14 [Sen and Sanyal, 1994]. *For a digraph D, the following are equivalent.*

1) *D is a unit interval digraph;*

2) *D is a proper interval digraph;*

3) *D is an indifference digraph;*

4) *The adjacency matrix of D has monotone consecutive arrangement*

Recently, West [1998] has given a short inductive proof of the equivalence of (3) and (4)

Sen and Sanyal [1994, 1996] gave several other characterizations, but they did not give a forbidden submatrix characterization for adjacency matrix of unit interval digraphs. Motivated by Roberts [1969] that an interval graph is an unit interval graph if and only if it does not contain the bipartite graph $K_{1,3}$ as an induced subgraph, Lin, Sen and West [1995, 1997] characterized the interval digraph that are unit interval digraphs. Particularly, they prove the following :

Theorem 1.15 [Lin & West; 1995 and Lin, Sen & West; 1997]. *A zero partitionable matrix has a MCA if and only if it does not contain any of the three 3 by 4 matrices listed below and their transposes.*

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Steiner [1996] gave a nice algorithm to re;cognize an indifference digraph (bigraph) in linear time. His recognition algrorithim uses permutation graph. A graph $G(V, E)$ on $V = \{v_1, v_2, \ldots, v_n\}$ is a permutation graph if there is a labeling $\ell: V \to \{1,2, \ldots, n\}$ and a permutation π of $\{1, 2, \ldots, n\}$ such that $v_i v_j \in E$ if an only if $\ell(v_i) < \ell(v_j)$ and $\ell(v_j)$ precedes $\ell(v_i)$ in π. The function ℓ is called the permutation labeling of G and π is the defining permutation.

Theorem 1.16 [Steiner, 1996]. *A digraph $D(V, E)$ is an indifference digraph if and only if its bipartite graph $B(D)$ is a permutation graph*

Since a bipartite permutation graph can be recognized in linear time [Spinrad, Brandstadt; 1987], he concluded that :

Theorem 1.17 [Steiner, 1996]. *An indifference digraph can be recognized in linear time.*

As a subclass of interval digraphs, *interval nest digraphs* were introduced by Prisner (1994). It has a representation $\{(S_v, T_v): v \in V\}$ where each T_v is contained in S_v. He gave some application of interval nest digraph model to the real world situation. In the same paper, he had shown that some parameters for interval nest digraphs can be computed in polynomial time.

Prisner had also studied reflexive interval digraphs. Using the adjacency matrix characterization of interval digraph (Theorem 1.11) it easily follows that a digraph is a reflexive interval digraph iff its complement is (edge) disjoint union of two loopless Ferrers digraphs. He also showed that loopless Ferrers digraphs are precisely interval order. A poset $P(V,<)$ is an interval order if there is some family of intervals $\{I_v : v \in V\}$ such that $x<y$ if and only if I_x lies completely to the left of I_y.

Theorem 1.18 [Prinser, Preprint]. *A digraph D is a reflexive interval digraph if and only if D is the disjoint union of two interval orders.*

The underlying graph $U(D)$ of a digraph $D(V,E)$ has V as vertex set and two vertices x and y are adjacent in $U(D)$ whenever $xy \in E$ or $yx \in E$. The characterization of an interval digraph that it is the intersection of two Ferrers digraphs whose union is complete has no analogue for reflexive interval digraphs, because the complement of an interval order is no longer an interval order. Nevertheless, the underlying graphs of reflexive interval digraphs can be expressed by the intersection of two interval orders.

A *trapezoid graph* is the intersection graph of a family of trapezoids whose parallel sides all lie on a single pair of parallel lines. The complements of trapezoid graphs are exactly the comparability graphs of posets of interval dimension at most 2, that is intersection of two interval orders. So we have the following theorem due to Prisner.

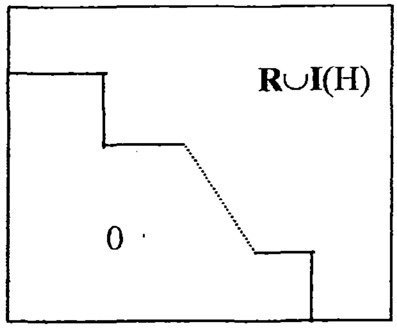
Theorem 1.19 [Prisner, Preprint]. *The underlying graph of a reflexive interval graph is a trapezoid graph.*

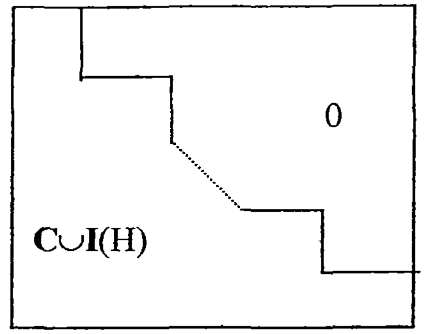
Since trapezoid graphs are weakly triangulated [Dagnan, Golumbic, and Pinter; 1988] it follows that the underlying graph of a reflexive interval graph is weakly triangulated. Prisner then uses the result to compute several parameters of this digraph more efficiently.

1.9.3 Interior edges

The notion of interior edges was introduced by Das and Sen [1993]. In may be recalled from section 1.3 that to characterize a digraph D of $f(D) \leq 2$, Cogis [1982] associated an undirected graph $H(D)$ with D and proved that D is of $f(D) \leq 2$ if and only if $H(D)$ is bipartite. Also it is known that for a given digraph D with $f(D) = 2$, the complement $\bar{D}$ is the union of two Ferrers digraphs (not necessarily disjoint). These two Ferrers digraphs are realizations of $\bar{D}$. And obviously realizations of $\bar{D}$ is not unique. Now we note that $H(D)$ may have more than one connected components; besides it may have one or more isolated vertices (corresponding to the 0's of $A(D)$ which do not belong to any obstruction). The graph obtained by deleting the isolated vertices from H(D) denoted by $H_b(D)$, is the *bare* graph associated with D [Doignon *et al.*, 1984]; and the set of all isolated vertices is denoted by $\boldsymbol{I(H)}$ or $\boldsymbol{I}$.

For a digraph D with $f(D) = 2$ any bicolouration of $H_b(D)$ does not lead to a covering of $\bar{D}$ by two Ferrers digraphs. But to prove the characterization of a digraph of $f(D)$ at most 2, Cogis adopted a constructive method to show that there always exists a suitable bicoluration of $H_b(D)$ that yields a realization of $\bar{D}$ as the union of two Ferrers digraphs. As a matter of fact, he obtained the particular bicoluration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ in such a way that adjoining all the edges of $\boldsymbol{I(H)}$ to each $\boldsymbol{R}$ and $\boldsymbol{C}$ yielded the required Ferrers digraphs realization G_1 and G_2 so that $\bar{D} = G_1 \cup G_2$, $G_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$. Such a bicoluration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ was termed a satisfactory bicoluration by Das and Sen [1993]. Let $(\boldsymbol{R}, \boldsymbol{C})$ be a satisfactory bicolouration of $H_b(D)$ leading to a realization of $\bar{D} = G_1(V, E_1) \cup G_2(V, E_2)$ where $E_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $E_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$. Let the rows and columns of $A(G_1)$ be so arranged that all the ones are clustered in the upper right. Similarly, the rows and columns of $A(G_2)$ are so arranged that all the ones are clustered in the lower left.

$A(G_1) =$ 

$A(G_2) =$ 

Then an edge corresponding to an $I \in G_1$ is said to be an *interior edge* of G_1, denoted by I_r, if there exists a configuration of the form

$$\begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$$

in $A(G_1)$; Similarly, an $I \in G_2$ is said to be an *interrior edge* of G_2, denoted by I_c, if there exists a configuration of the form

$$\begin{pmatrix} C & 0 \\ I & C \end{pmatrix}$$

in $A(G_2)$.The set of all interior edges of G_1 is denoted by $\boldsymbol{I_r}$ (G_1) or $\boldsymbol{I_r}$, and the set of all interior edges of G_2 is denoted by $\boldsymbol{I_c}$. Note that the sets $\boldsymbol{I_r}$ and $\boldsymbol{I_c}$ are identified with reference to a particular realization of $\bar{D}$ and will change if the realization changes.

With this notion of interior edges, Das and Sen [1993] proved that if a digraph D of $f(D) = 2$ is an interval digraph then for any satisfactory bicolouration of $H_b(D)$, $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \varphi$.

But the converse is not true. They gave an example of an eight vertex digraph with $f(D) = 2$ which is not an interval digraph and for which $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \varphi$. In chapter 2 of the present thesis, we continue from their work, and have been able to solve the problem of finding all possible configuration of a digraph having $f(D)=2$ but which is not an interval digraph. Incidentally, this leads us to a recognition algorithm of an interval digraph in a more efficient way.

1.9.4 Recognition Algorithm

Recognition of interval digraphs is a major open problem is this area. Recently Müller [1997] has found a polynomial algorithm to recognize an interval digraph. He has used a dynamic programming method to recognize an interval bigraph and accordingly interval digraph. A similar approach was used earlier in [Bodlaender, Kloks and Kratsch; 1995] and [Deogun, Kloks and Kratsch and Müller; 1994] to compute the tree-width and the vertex ranking number respectively, of permutation graphs. To describe the method we require a number of new ideas and definition.

Analogous to the result on interval graph that an interval graph is chordal, it can be easily seen that an interval bigraph is bichordal. For pair *a, b* of non-adjacent vertices of $G(V,E)$, a set $S \subset V$ is a *minimal a-b separator* if

i) a and b belong to different connected components of the subgraph $G \setminus S$, and

ii) no proper subset of S has the above property.

A minimal *a-b* separator is also called a minimal separator. Muller has then used the notion of a complete separator and introduced an *anchored segment*. A set of vertices S is a *complete separator* if

i) either $S = \varphi$ or S is a separator such that the subgraph $G(S)$ induced by S is a complete bipartite graph and

ii) S is either a minimal separator or at most the union of two minimal separators of G.

Let now L and R be two complete separators of G. Then clearly $G\setminus(L \cup R)$ is a disconnected graph. If C is a connected component of $G\setminus L \cup R$, then the subgraph H' induced by the vertices of C and the vertices in $L \cup R$ is called a segment of G and the triples (L, H', R) *anchored segment* of G. The algorithm to recognize an interval bigraph is briefly described below.

Assume that a chordal bipartite graph is given. Find out all the complete separators and the edges of G.

If a segment forms a complete bipartite subgraph of G, then the unique, short interval representation of this segment is just a point only. Müller defined length of a bipartite interval representation of an interval bigraph as the sum of the lengths of its intervals.

Then he called an interval representation *short* if all the intervals are closed, have integer end points and have minimum length. Müller showed how to combine two or more segments with short interval representation to obtain a large one and this is the most crucial step in the algorithm. It turns out that the important parts of the bipartite interval representation are the complete bipartite subgraphs bounding it on both sides. If for a subgraph H' the anchored segment (L, H', R) is not unique, the different bipartite interval representation of H exist. An anchored segment obtained by suitably enlarging two GBS's of a segment is a *realization*. In the same paper he has shown that a chordal bipartite graph is an interval bigraph iff the anchored segment (φ, G, φ) is realizable. He had also shown that the algorithm takes time of order O $(nm^6(m+n) \log n)$, where n is the number of vertices and m is the number of edges of the graph.

In the present thesis we have obtained a recognition algorithm from a different viewpoint. An interval digraph is first characterized in terms of interior edges with reference to a bicolouration of its associated bipartite graph $H(D)$. From this, a complete list of forbidden configurations of an interval digraph is obtained. Then this characterization is applied to obtain a recognition algorithm of an interval digraph in time $O(n^3)$.

1.10 Containment digraphs and overlap digraphs

Sen, Sanyal and West [1995] extended the containment and overlap model to representations of digraphs. The containment digraph of a family $\mathcal{J} = \{(S_v, T_v) : v \in V\}$ is the digraph with vertex set V in which there is an edge from u to v if and only if S_u property contains T_v. It is easy to represent any digraph as a containment digraph; given vertex set $V = \{v_i\}$, let $T_{v_i} = \{i\}$, and let S_{v_i} be the element - i together with $\{j : v_j \in N^+(v_i)\}$, where $N^+ (v_i)$ is the set of successor (out-neighbour) of v_i. Naturally, one has to restrict the pair of sets in $\mathcal{J}$ to obtain interesting class of digraphs. If $\mathcal{J}$ is a family of order pairs of intervals, then the resulting containment digraph is an *interval containment digraph*. The characterization of interval containment digraph uses Ferrers digraphs.

Just as interval digraphs are closely related to interval graphs, so interval containment digraphs are closely related to interval containment posets. A containment representation of a poset assigns to each element $x \in p$ a set S_x such that $x < y$ it and only if $S_x \subset S_y$. It is well known that interval containment posets are precisely the posets of dimension 2 [Dushnik and Miller, 1941] and [Madej and West, 1991]. Furthermore, the Ferrers dimension of the comparability digraph of a poset (a digraph that is irreflexive, and transitive) equals the order dimension the poset [Bouchet, 1971 and Doignon *et al.*, 1984]. Hence it is not surprising that the interval containment digraphs are precisely the digraphs of Ferrers dimension 2, which was proved in [Sen, Sanyal & West, 1995].

To study overlap digraphs represented by intervals they [1995] have used a more restrictive model than the direct analogue with undirected graphs. *A right overlap digraph* (ROI-digraph) is a digraph represented by a family $\mathcal{J}$ of ordered pairs of intervals such that there is an edge from u to v if and only if (i) S_u and T_v overlap (no containment) and (ii) $\inf S_u < \inf T_v$. To characterize the adjacency matrix of these digraphs they gave the following definition : A 0, 1-matrix has a *P, R-partition* if its rows and columns can be permuted independently so that its 0's can be labeled P or R such that (1) the position to the right *and* the positions above any R are also 0's labeled R, and (2) positions to the left *or* the positions below any P are also 0's labeled P (Fig. 1.6).

As illustrated in the Fig. 1.6, it is easy to see that the R's constitute a Ferrers digraph and the P's constitute the union of two Ferrers digraphs. Hence any digraph whose adjacency matrix has a P-R partition is a digraph of Ferrers dimension at most 3, and they precisely proved that a digraph is a ROI-digraph if and only if its adjacency matrix has a P-R partition. They also observed that the digraphs having right-overlap interval representations in which all the intervals have unit length, are same as the unit interval digraphs or indifference digraphs.

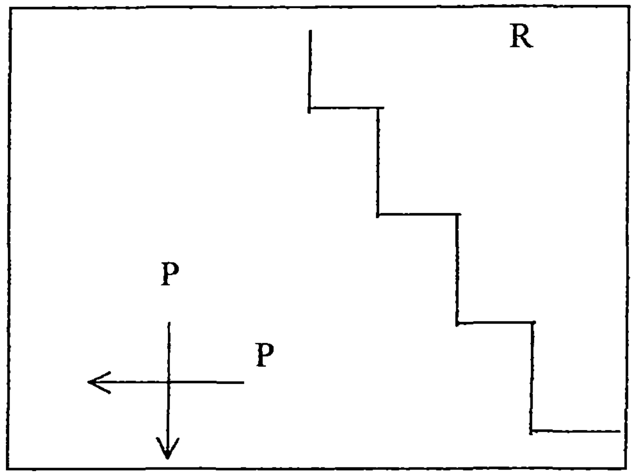


Fig 1.6

In this context we can similarly introduce left overlap interval digraph (LOI-digraph) replacing the condition (ii) of the definition of a ROI by the condition (ii)$'$ inf T_v < inf S_u. Then we observe that if we take the mirror image of intervals with respect to any point on the real line, an ROI-digraph becomes an LOI-digraph and vice versa. Thus the class of ROI-digraphs coincides with the class of LOI –digraph.

1.11 Robin digraphs

Sanyal [1994] posed the question of characterizing a binary matrix having independent row and column permutations where 0's (in the matrix) are such that a 0 has all positions 0 throughout any one of the four directions, *viz*, left, right, above and below (see fig 1.7). such a matrix is a *4-directable binary matrix*.

Also a square matrix having consecutive 1's property for rows was characterized by Sen *et al.* (1989a) in terms of an interval-point digraph; it is represented by a family $\mathfrak{I} = \{(S_v, p_v) : v \in V\}$ of ordered pairs where S_v is a closed interval and p_v is any arbitrary point and $uv \in E$ if and only if $p_v \in S_u$. Such an ordered pair (S_v, p_v) was called a *pointed interval*. Now it can be easily seen that a four directable matrix is the adjacency matrix of the intersection of two digraphs D_1 and D_2, where $A(D_1)$ has consecutive 1's property for rows and $A(D_2)$ has consecutive 1's property for columns. So it is easy to infer that adjacency matrix of a digraph D is four directable iff $D = D_1 \cap D_2'$ where D_1 and D_2 are two interval-point digraphs and D_2' is the converse of D_2.

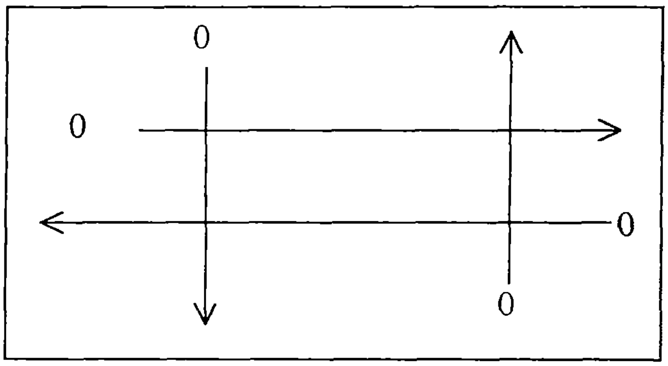


Fig 1.7 4 – directable matrix

To give another characterization of the 4-directable matrix, he considered p_u as a member of the closed interval S_u and called it a base point of S_u. Such an ordered pair (S_v, p_v) is called a *base interval*. Then a generalized version of an overlap digraph was introduced in the following way : Let $\mathfrak{I} = \{((S_v, p_v), (T_v, q_v)) : v \in V\}$ be a family of ordered pairs of base *intervals* (S_v, p_v) and (T_v, q_v) then a digraph $D(V, E)$ is a *right overlap base interval digraph (Robin digraph)* when $uv \in E$ if and only if

i) S_u and T_v overlap (no containment)

ii) $\text{Inf}\, S_u < \text{Inf}\, T_v$ and

iii) $p_u, q_v \in S_u \cap T_v$.

Then he proved that a digraph has a overlap base interval representation (right or left) iff its adjacency matrix has 4-directable property.

The idea of representation of a digraph by the intersection model of the ordered pair of base interval was given by Sanyal [1994]. In the Chapter 4 of the present thesis, we characterize the adjacency matrix of these digraphs (called the *base interval digraphs*) by showing that it is a particular form of 4-directable binary matrix. This is a modified version of the characterization given by Sanyal in his thesis [1994]. Then we characterize a base interval digraph in terms of intersection of two interval digraphs and overlap base interval digraph in terms of intersection of two containment digraphs. Lastly, we consider the particular case when all the intervals of Robin digraph are of unit length and prove that Robin digraphs with unit length interval are precisely the ROI – digraph.

1.12 Miscellaneous

Lin *et al.* [1999] has shown that every digraph is the intersection digraph of a family of pairs of subtrees of a star. For any digraph, we can let each sink set T_v consist of a distinct leaf of the star and let each source set S_v consist of the centre and the leaves assigned to the out-neighbors (i.e. successors) of v. Hence for an arbitrary digraph D, it makes sense to define the parameter $L(D)$, the leafage of D, to be the minimum number of leaves in the host tree in any subtree representation of D. They also proved that the interval digraphs are precisely the digraphs with leafage 2. They also defined catch representation of a digraph D in the following manner : In a subtree representation of a digraph if every sink set is restricted to be a singleton vertex, then this is a *catch representation* of D, and D is a *catch-tree digraph*. The *catch leafage* of a digraph D is the minimum number of leaves in the host tree in any catch representation of D. then they studied upper and lower bounds for the leafage and catch leafage and showed that the bounds are best possible, but can be arbitrarily weak.

It is well known that a representation of an interval digraph is not unique and an interval (di)graph may have many representations differing not only in length, but more importantly, in the relative positions of the intervals. To study the relative positions of the intervals in interval representation of an interval (di)graph is the problem of chronological orderings of an interval (di)graph. Skrien [1980, 1984] studied the problem for undirected graph. In a recent paper Sen *et al.* (Preprint) studied the corresponding problem for directed graph. Actually their results generalize the corresponding results on interval graph by Skrien and describes how, given an interval digraph, the order of intervals of one representation differs from another. In other words it describes the various interrelations between the end points of the intervals for a digraph to be an interval digraph.

CHAPTER 2*

FORBIDDEN CONFIGURATIONS AND RECOGNITION ALGORITHM OF INTERVAL DIGRAPH/BIGRAPH

2.1 Introduction

An *interval digraph* is a directed graph $D(V, E)$ for which every vertex $v \in V$ is assigned a pair of closed intervals (S_v, T_v) such that uv is an edge (arc) iff S_u and T_v have a non-empty intersection. An *interval bigraph* is a basically equivalent concept of an interval digraph. It is a bipartite graph $B(U, V, E)$ having bipartite sets U and V, for which there are two families of intervals $\{S_u : u \in U\}$ and $\{T_v : v \in V\}$ such that $uv \in E$ iff $S_u \cap T_v \neq \phi$.

An interval bigraph was introduced in [Harray,Kabel, McMorris,1982] while an interval digraph was introduced in [Sen *et al.*,1989a]. That the two concepts are equivalent can be seen from the following .

Given a digraph $D(V, E)$, consider the bipartite graph $B=B(D)$ whose partite sets are two disjoint copies U and V of the set V of D and let two vertices u and v in $B(D)$ be adjacent iff $uv \in E$. Then it is not difficult to show that D is an interval digraph iff $B(D)$ is an interval bigraph. On the other hand, $B(U, V, E)$ is an interval bigraph iff the directed graph $D(U \cup V, E)$, obtained from B by directing all the edges from U to V is an interval digraph.

Several characterizations of an interval digraph / bigraph are known [Müller,1997; Sanyal&Sen,1996; Sen *et al.*, 1989a]. In [Sen *et al.*,1989a] it was characterized in terms of its adjacency matrix and in terms of Ferrers digraphs. We recall the following theorem that characterizes an interval digraph.

Theorem [Sen *et. al.*, 1989a] *The following conditions are equivalent.*

(A) D is an interval digraph.

(B) The rows and columns of the adjacency matrix of D can be (independently) permuted so that each 0 can be replaced by one of {R, C} in such a way that every R has only R's to

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* This Chapter has been communicated to Discrete Appl. Math

its right and every C has only C's below it.

(C) D is intersection of two Ferrers digraphs whose union is complete.

From the above theorem it follows that the Ferrers dimension $f(D)$ of an interval digraph D is at most 2. It was also shown by them that the converse is not true and in fact there exists a digraph of $f(D) = 2$which is not an interval digraph. The Ferrers dimension of a digraph D will also be referred to as the Ferrers dimension of its corresponding bigraph $B(D)$. A digraph with $f(D) = 2$ was characterized independently by Cogis [1979] and also in [Doignon, Ducamp, Falmagne,1984;Sen *et al.*,1989a; Sen, Sanyal and West ,1995] in different contexts.

Cogis [1979] introduced the concept of the *associated graph* $H(D)$ corresponding to a digraph D. It is the graph whose vertices correspond to the 0's of the adjacency matrix $A(D)$ of D with two such vertices are joined by an edge in $H(D)$ when the corresponding 0's form the permutation matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $A(D)$.The 0's are then said to form an *obstruction.* Alternatively $H(D)$ can be defined in following manner: let $D = (V, E)$ be a digraph, i.e., $E \subseteq V \times V$. Then $H(D)$ is an undirected graph with vertex set $(V \times V) \setminus E$ and two non-edges (u, v) and (x, y) of D are adjacent in $H(D)$ if and only if $(x, v) \in E$ and $(u, y) \in E$.

Cogis [1979] proved that $f(D)$ of a digraph D is at most 2 iff $H(D)$ is bipartite. Then he used this result to obtain a recognition algorithm for a digraph of $f(D) = 2$ in a polynomial time.

Müller [1997] obtained a dynamic programming algorithm to recognize an interval bigraph in a polynomial time. He first observed that an interval bigraph is chordal bipartite. It is easily observed that a cycle of length at least 6 is of Ferrers dimension 3. So a bigraph which contains an induced cycle of length ≥ 6 is necessarily of $f(D) \geq 3$. Since an interval digraph (bigraph) is of $f(D)$ at most 2, it follows that it must be bichordal. In order to obtain his algorithm, Müller relies on the theorem by Golumbic and Goss [6] that a bipartite graph is chordal bipartite iff each minimal vertex separator induces a complete bipartite subgraph. He then recursively constructs a bipartite interval representation of a graph from interval representations of its proper subgraphs.

Das and Sen [1993] tried to characterize an interval digraph in terms of forbidden configurations of its adjacency matrix. As a matter of fact, Müller had also made an attempt to solve this problem. In section 2.2, we continue from the paper by Das and Sen [1993] and obtain a complete list of forbidden configurations of the adjacency matrix of an interval digraph. In the process we obtain in section 2.3, a recognition algorithm of an interval digraph in a polynomial time $O(n^3)$.

2.2. Forbidden Configurations of Interval Digraphs/Bigraphs

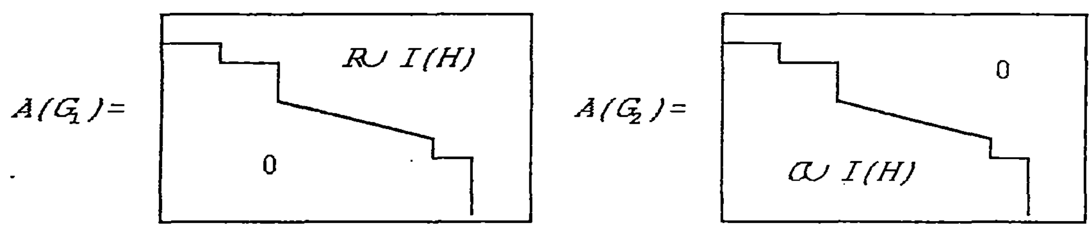
As noted in the introduction, an interval digraph is of Ferrers dimension at most *2*, but the converse is not true. In [Das and Sen, 1993], an effort was made to find out the forbidden configurations of an interval digraph from the perspective of its relations with the associated bipartite graph *H(D)* of *D*. The present paper is, in effect, a continuation of that paper. So it may be worth recalling the main results contained therein for the sake of motivation. Cogis [1979] proved that $f(D)$ of a directed graph D is at most *2* iff *H(D)* is bipartite. The graph *H(D)* may have more than one connected component; besides it may have one or more isolated vertices (corresponding to the *0*'s which do not belong to any obstruction). The graph obtained by deleting the isolated vertices from *H(D)* is denoted by $H_b(D)$ and is called *the bare graph associated with D* [Doignon *et al.*, 1984].

Let D be a digraph of f(D) = 2 so that *H(D)* is bipartite. The set of all isolated vertices of *H(D)* is denoted by $\boldsymbol{I(H)}$ or $\boldsymbol{I}$ and a bicolouration of *H(D)* by $(\boldsymbol{R}, \boldsymbol{C})$. Recall that a colouration of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colours. Naturally, a bicolourable graph uses two colours only. If $H_b(D)$ has more than one connected components $H_1, H_2, ..., H_p$, a bicolouration of H_i will be denoted by $(\boldsymbol{R_i}, \boldsymbol{C_i})$. It is evident that $\boldsymbol{R}=\cup\boldsymbol{R}_i$ and $\boldsymbol{C}=\cup\boldsymbol{C}_i$ for any labelling of the bicolouration $(\boldsymbol{R_i}, \boldsymbol{C_i})$ of H_i. Also the elements of the set $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{R_i}$, $\boldsymbol{C_i}$, or $\boldsymbol{I}$ are denoted by the corresponding capital letters R, C, R_i, C_i, or I respectively. The stable sets R_i and C_i are called the *fragments* of $H_b(D)$. While proving his result, Cogis obtained the particular bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ in such a way that adjoining all the edges of $\boldsymbol{I(H)}$ to each of $\boldsymbol{R}$ and $\boldsymbol{C}$ yielded the required Ferrers digraph realization G_1 and G_2 where $G_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$. Such a bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ for which $G_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$ are Ferrers digraphs, is called a *satisfactory bicolouration.* Clearly if $H(D)$ has no isolated vertex then D is an interval digraph.

While the recognition of a digraph of f(D) = 2 requires the realization of its complement as the union of two Ferrers digraph G_1 and G_2, not necessarily disjoint, such that $D^c = G_1 \cup G_2$, the problem for an interval digraph recognition, however is to cover its complement by two Ferrers digraph which should necessarily be disjoint, $D^c = H_1 \cup H_2$, $H_1 \cap H_2 = \phi$. This is equivalent to adjoining every edge of $\boldsymbol{I}(H)$ into only one of two digraphs $G_1(V, \boldsymbol{R})$ and $G_2(V, \boldsymbol{C})$ so that they become two disjoint Ferrers digraphs.

To this end, the notion of interior edges was introduced in the same paper.

Let $(\boldsymbol{R}, \boldsymbol{C})$ be a satisfactory bicolouration of $H_b(D)$ leading to a realization of $D^c = G_1(V, E_1) \cup G_2(V, E_2)$ where $E_1 = \boldsymbol{R} \cup \boldsymbol{I}(\boldsymbol{H})$ and $E_2 = \boldsymbol{C} \cup \boldsymbol{I}(\boldsymbol{H})$. Let the rows and columns of $A(G_1)$ be so arranged that all the ones are clustered in the upper right. Similarly, the rows and columns of $A(G_2)$ are so arranged that all the ones are clustered in the lower left.



An edge $I \in \boldsymbol{I}$ is said to be an interior edge of G_1, denoted by I_r, if there exists a configuration of the form

$$\begin{array}{|cc|} \hline R & I \\ 0 & R \\ \hline \end{array}$$

in $A(G_1)$; similarly, an $I \in \boldsymbol{I}$ is said to be an interior edge of G_2 denoted by I_c, if there exist a configuration of the form

$$\boxed{\begin{matrix} C & 0 \\ I & C \end{matrix}}$$

in $A(G_2)$. With reference to a particular realization of D^c as the union of G_1 and G_2, $D^c = G_1 \cup G_2$, the set of all interior edges of G_1 is called interior of G_1 and is denoted by $\boldsymbol{I_r}(G_1)$ or $\boldsymbol{I_r}$ and all interior edges of G_2 is called interior of G_2 and is denoted by $\boldsymbol{I_c}(G_2)$ or $\boldsymbol{I_c}$. Note that the sets $\boldsymbol{I_r}$ and $\boldsymbol{I_c}$ are identified with reference to a particular realization of D^c and will change if the realization changes. With these notions, it was proved in the same paper that for a digraph of f(D) = 2, the property $\boldsymbol{I_r \cap I_c \neq \phi}$ is invariant under any satisfactory bicolouration of $H_b(D)$. This means that if $\boldsymbol{I_r \cap I_c \neq \phi}$ for a certain satisfactory bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ of a digraph D of f(D) $=2$ then the same is true for any satisfactory bicolouration of $H_b(D)$. As a matter of fact, the following proposition was proved in [Das and Sen, 1993].

Proposition 2.1[Das and Sen, 1993]. *Let D be a digraph of f(D) = 2. If $\boldsymbol{I_r \cap I_c \neq \phi}$ for a certain satisfactory bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$, then the same is true for any satisfactory bicolouration of $H_b(D)$.*

Lastly the paper concluded with the following proposition.

Proposition 2.2 [Das and Sen ,1993]). *Let D be a digraph of f(D) = 2. If D is an interval digraph, then for any satisfactory bicolouration of $H_b(D)$, $\boldsymbol{I_r \cap I_c = \phi}$; but the converse is not true.*

In the Theorems 2.1 and 2.2 of the present chapter we do away with the restriction of a satisfactory bicolouration and prove the same result for any bicolouration $H_b(D)$. Thus the Theorems 2.1 and 2.2 of this chapter are improvements upon the previous one. This generalization, as we will later see, will have a lasting effect when we come to the question of recognition algorithm.

For this generalization, we require extending the definition of I_r and I_c for any bicolouration of $H_b(D)$. With reference to *any* bicolouration $(\mathbf{R},C)$ of $H_b(D)$, an $\mathcal{I}_0 \in \boldsymbol{I}$ will be termed $\mathcal{I}_r$ or $\mathcal{I}_c$ if $A(D)$ contains a configuration

$$\boxed{\begin{array}{cc} R & \mathcal{I}_0 \\ 1 & R \end{array}} \quad \text{or} \quad \boxed{\begin{array}{cc} C & \mathcal{I}_0 \\ 1 & C \end{array}}$$

respectively. We also need the following notions of a core matrix of a matrix and of the compatibility between two rows/columns in a matrix.

Let two rows (columns) of the adjacency matrix of a digraph D be identical. It means that the out-neighbours (in-neighbours) of the two vertices are the same. Alternatively the two vertices of the bipartite graph obtained by vertex splitting operations [Müller, 1997] are copies of one another. Since an interval digraph (bigraph) property is a hereditary property, we are not interested in such identical row or columns (copies). Deleting those rows or columns of a matrix A which are identically equal to a previous row (or column) the resulting matrix will be called the core matrix of A and the corresponding digraph, the core digraph of D.

In a $(0,1)$ matrix, we will frequently use a '-' in any position to indicate that it is either 0 or 1. The rows (or columns) of a binary matrix are compatible, if for some combination of values of the '-' positions they become identical; otherwise they are incompatible. For example in the matrix M below of Fig. 2.1, the rows 2 and 3 are compatible, because they become identical but putting the values 0 to the positions (2,5) and (3,7); but since (1,6) and (2,6) positions have values 0 and 1 respectively, the rows 1 and 2 are incompatible.

By a *configuration of an adjacency matrix A*, we shall mean a sub-matrix of A obtained by any (independent) permutation of rows and of columns.

Proposition 2.3 *Let D be a digraph of Ferrers dimension 2 and let $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \phi$ for a satisfactory bicolouration $(\mathbf{R},C)$ of $H_b(D)$. Then the same is true for any other bicolouration of $H_b(D)$.*

The proof of the proposition relies heavily on the following lemma.

Lemma 2.1 *Let D be a digraph of $f(D) = 2$ and let $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \phi$ for a satisfactory bicolouration of $H_b(D)$. Then the adjacency matrix $A(D)$ of D must contain the core matrix of the matrix*

M or its transpose M^T (subject to independent permutations of rows and/or columns) where

$$M = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & - & 1 & 0 \\ 3 & 1 & 1 & 1 & 1 & 0 & 1 & - \\ 4 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 5 & 1 & - & 0 & 0 & - & 0 & 0 \\ 6 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & - & 1 & 0 & 0 & - \\ \hline \end{array}$$

Fig. 2.1

Proof of lemma 2.1. We first make some observations on the matrix *M*. If the values of the '-' positions are all *0*'s, then the column 2 becomes identical with the column 3 and so also the rows 2 and 3. Then the core matrix of *M* with a bicolouration of the vertices of *H(D)* is of the form

$$\begin{array}{|cccccc|} \hline 1 & 1 & 1 & 1 & R_1 & R_2 \\ 1 & 1 & 1 & C_1 & 1 & R_3 \\ 1 & 1 & 1 & C_2 & C_3 & 1 \\ 1 & R_4 & R_5 & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ C_4 & 1 & R_6 & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ C_5 & C_6 & 1 & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ \hline \end{array}$$

But if the values of the (5,5) and (7,7) positions are both 1 then all the components coalesce into one component. Now we begin the proof of lemma 2.1. Since $I_r \cap I_c \neq \phi$, there is an $\mathcal{I} \in$

$I_r \cap I_c$ for which the configurations

$$\boxed{\begin{array}{cc} 1 & R \\ R & I \end{array}} \quad \text{and} \quad \boxed{\begin{array}{cc} 1 & C \\ C & I \end{array}}$$

must be present in the adjacency matrix $A(D)$ of D.

So the adjacency matrix $A(D)$ must have configuration

$$\begin{array}{c|ccc} & 1 & 4 & 6 \\ \hline 1 & - & 1 & R \\ 4 & 1 & - & C \\ 6 & C & R & I \end{array} \qquad \ldots\ldots\ldots (1)$$

We label the rows and columns of the configuration conveniently so that they will ultimately coincide with those of the matrix M. Now every R and C in the above configuration must be in obstruction with some C and R respectively in the required matrix $A(D)$. We pay our attention to them.

The R and C of (1,6) and (4,6) position require the structure

$$\begin{array}{c|cc} & - & 6 \\ \hline 1 & 1 & R \\ - & C & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & 6 & - \\ \hline - & 1 & R \\ 4 & C & 1 \end{array}$$

respectively. Note that we have not labelled the new rows and columns in the above two structures. Several possibilities may occur; we can give them different labels or we can identify two rows and/or two columns, whenever we find them compatible. Our aim is now to explore all possibilities and find out the forbidden configurations.

First we consider the case when two rows and columns in the above structure are given different, as in the following

$$\begin{array}{c|cc} & 5 & 6 \\ \hline 1 & 1 & R \\ 3 & C & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & 6 & 7 \\ \hline 2 & 1 & R \\ 4 & C & 1 \end{array}$$

In this case the configuration gets the form

$$\begin{array}{c|ccccc} & 1 & 4 & 5 & 6 & 7 \\ \hline 1 & - & 1 & 1 & R & - \\ 2 & - & - & - & 1 & R \\ 3 & - & - & C & 1 & - \\ 4 & 1 & - & - & C & 1 \\ 6 & C & R & - & I & - \end{array} \qquad \ldots\ldots\ldots\ 1(a)$$

Next we explore other possibilities as regards the shape of the matrix when some rows/columns in the above configuration coincide.

For example, we first consider the case by identifying rows 2 and 3 in the configuration 1(a). Then the configuration is

$$\begin{array}{c|ccccc} & 1 & 4 & 5 & 6 & 7 \\ \hline 1 & - & 1 & 1 & R & - \\ 2=3 & - & - & C & 1 & R \\ 4 & 1 & - & - & C & 1 \\ 6 & C & R & - & I & - \end{array} \qquad \ldots\ldots\ldots\ 1(b)$$

Note that all the four rows in 1(b) are incompatible to one another and so we try to identify the possible compatible columns. As an example, let us see what happens when column 1 becomes identical with column 7 and also column 4 with column 6. Then the configuration becomes

$$\begin{array}{c|ccc} & 1(7) & 2(5) & 6 \\ \hline 1 & - & 1 & R \\ 4 & 1 & - & C \\ 6 & C & R & I \\ 2(3) & R & C & 1 \end{array}$$

In this case we look at the structure

$$\begin{array}{c|cc} & 4(5) & 6 \\ \hline 6 & R & I \\ 2(3) & C & 1 \end{array}$$

Here C and I are in obstruction in the matrix $\boldsymbol{C \cup I}$, which is contradictory to our hypothesis, because $\boldsymbol{C \cup I}$ is a Ferrers digraph in a satisfactory bicolouration. So this possibilities is ruled out.

By similar reasoning, we can check that in whatever way we identify the columns either in the configuration 1(b) or in 1(a) we will reach an impossible situation. Therefore we are now left to search for the matrix coming up from the configurations 1(a) and 1(b) only.

First we consider the configuration 1(a). In that configuration the structure

$$\begin{array}{c|cc} & 6 & 7 \\ \hline 1 & R & - \\ 2 & 1 & R \\ 6 & I & - \end{array}$$

implies that positions (1,7) and (6,7) must be 0 (otherwise it would contradict the property that $\boldsymbol{R \cup I}$ is a Ferrers digraph for a satisfactory bicolouration). Again since $\boldsymbol{C \cup I}$ is a Ferrers digraph, the structure

	5	6
3	C	1
4	-	C
6	-	I

implies that the positions (4,5) and (6,5) are *0*.

Now by the same logic,

	1	6
1	-	R
4	1	C

implies that (1,1) position cannot *C* or *I* Also

	1	4
1	-	1
6	C	R

means that (1,1) position cannot be *R* or *I* So (1,1) position must be 1. Similarly (4,4) position is *1*. The configuration thus takes the shape

	1	4	5	6	7
1	1	1	1	R	0
2	-	-	-	1	R
3	-	-	C	1	-
4	1	1	0	C	1
6	C	R	0	I	0

The structure

	5	7
1	1	0
4	0	1

means that the two *0*'s are of two distinct colours and for a satisfactory bicolouration the position (1,7) must be *R* and position (4, 5) is *C*.

Next the structure

$$\begin{array}{c|cc} & 1 & 6 \\ \hline 1 & 1 & R \\ 2 & - & 1 \\ 4 & 1 & C \end{array}$$

implies that the position (2,1) must be *1*. On similar grounds the positions (3,1), (2,4) and (3,4) are all *1*.Thus the configuration gets the form

$$\begin{array}{c|ccccc} & 1 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 1 & R & R \\ 2 & 1 & 1 & - & 1 & R \\ 3 & 1 & 1 & C & 1 & - \\ 4 & 1 & 1 & 0 & C & 1 \\ 6 & C & R & 0 & I & 0 \end{array}$$

Now we consider the obstruction corresponding to *C* and *R* in the positions (6,1) and (6,4) respectively. For them we have the structures

$$\begin{array}{c|cc} & 1 & - \\ \hline - & 1 & R \\ 6 & C & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} & - & 4 \\ \hline 6 & 1 & R \\ - & C & 1 \end{array}$$

Arguing similarly as before we arrive at the matrix

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & R & R \\ 2 & 1 & 1 & 1 & 1 & - & 1 & R \\ 3 & 1 & 1 & 1 & 1 & C & 1 & - \\ 4 & 1 & 1 & 1 & 1 & C & C & 1 \\ 5 & 1 & - & R & R & - & 0 & 0 \\ 6 & C & 1 & 1 & R & 0 & I & 0 \\ 7 & C & C & - & 1 & 0 & 0 & - \end{array}$$

We note that rows 2 and 3 are the only compatible rows and the columns 2 and 3 are the only compatible columns in the above matrix. We also note that the rows 5 and 7 constructed in the way can not be merged with any other row. Replacing *R*'s and *C*'s by *0*'s we arrive at the required matrix *M*.

If we now start with the configuration 1(b), and consider the obstructions of C and R in the positions (6,1) and (6,4) respectively , then exactly as before we will arrive only at the matrix M where with rows 2 and 3 identifying together. ■

Proof of Proposition 2.3. We divide the proof into three cases according to the values of the positions (5,5) and (7,7) in the above matrix *M*.

(i) When both of them are *1*;

(ii) When both of them are *0*; and

(iii) When one of them is *1* and the other is *0*.

In case (i), the graph $H_b(D)$ is connected and we have nothing to prove.

In case (ii), the number of components of graph $H_b(D)$ is 6 and a possible bicolouration is given by

	1	2	3	4	5	6	7
1	1	1	1	1	1	R_1	R_2
2	1	1	1	1	-	1	R_3
3	1	1	1	1	C_1	1	-
4	1	1	1	1	C_2	C_3	1
5	1	-	R_4	R_5	I	I	I
6	C_4	1	1	R_6	I	I	I
7	C_5	C_6	-	1	I	I	I

The above matrix has the interesting feature that if the matrix is divided into four blocks as in the figure, the upper left (UL) block has all its elements equal to *1*, while all the *I*s comprise the lower right (LR) block. The upper right (UR) block has the fragments of the components H_1, H_2 and H_3 whereas those of H_4, H_5 and H_6 all have their places in the lower left (LL) block.

We now come to the question of interchange of colours. Suppose, as a case in point, we interchange the colours of H_1 only without changing the colours of the other components. Here we observe that the column 6 has two fragments of the same colour and so the I of the (6,6) position loses the property that it is both an I_r and I_c. But then the column 5 has fragments of two colours and so the I in the (6,5) position becomes both I_r and I_c. If we interchange the fragment of colours of any number of components in the UR-block, keeping the colours of the components in the other block unchanged, we find that at least one column in the UR-block must have fragments of two different colours and one of the Is in the 6th row becomes both an I_r and an I_c.

Similar things happen when we interchange the fragment colours of the components in the LL-block, this time the Is of the 6th column coming to satisfy the requirement.

This loss and gain property of an $I \in \mathbf{I}_r \cap \mathbf{I}_c$ is clearly manifest in the block diagram of the matrix and remains the motivating spirit behind our assertion.

Considering the other possibility, if we interchange the fragment colours in both the blocks, then all because of the in-built pattern of the different blocks in the matrix, one column in the UR-block and one row in the LL-block have fragments of both the colours and the corresponding row and the column intersect at the required I

For an instance, suppose the colours of components H_2, H_5 and H_6 are interchanged, the first lying in the UR-block and the other two in the LL-block. Then the two blocks UR and LL take the look

$$\begin{array}{c|ccc}
 & 5 & 6 & 7 \\
\hline
1 & 1 & R_1 & C_2 \\
2 & - & 1 & R_3 \\
3 & C_1 & 1 & - \\
4 & R_2 & C_3 & 1
\end{array}$$

UR-block

and

$$\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
5 & 1 & - & R_4 & C_5 \\
6 & C_4 & 1 & 1 & C_6 \\
7 & R_5 & R_6 & - & 1
\end{array}$$

LL-block

Here we see that all the columns in the UR-block and the 5th row in the LL-block have the features of containing fragments of both colours and the corresponding Is get the required criterion.

Lastly we come to case (iii) when of the two positions (5,5) and (7,7) one has the value *1* and other *0*. Specifically, suppose (5,5) is *1* and (7,7) is *0*. Here the components H_1, H_2, H_4 and H_5 coalesce into one component and the matrix takes the following configuration.

	1	2	3	4	5	6	7
1	1	1	1	1	1	R	R
2	1	1	1	1	-	1	R_3
3	1	1	1	1	C	1	-
4	1	1	1	1	C	C_3	1
5	1	-	R	R	1	0	R
6	C	1	1	R_6	0	I	I
7	C	C_6	-	1	C	I	I

Fig.2.2

As is clear from the above configuration any change in fragment colours has its effect on the two rows 6 and 7 and the two columns 6 and 7 and the required I varies its position at the four corresponding intersecting positions. The other case when (5,5) is *0* and (7,7) is *1* can be similarly proved, Is taking the positions at the intersections of rows 5, 6 and columns 5, 6.

Proposition 2.4. *Let D be a digraph of f(D) = 2 and let for a satisfactory bicolouration of $H_b(D)$,*

i) $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \phi$, *and*

ii) A(D) contain the configuration (2).

$$\begin{array}{|ccc|}\hline 1 & 1 & I \\ R & - & I_c \\ - & C & I_r \\ \hline\end{array} \qquad \cdots\cdots\cdots \qquad (2)$$

Then the same is true for any other bicolouration of $H_b(D)$.

To prove the Proposition we need the following lemma:

Lemma 2.2 *Let D be a digraph of $f(D) = 2$ with $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \phi$ for a satisfactory bicolouration of $H_b(D)$ and $A(D)$ contain the configuration (2)*

Then $A(D)$ must contain the core matrix of N or its transpose,

where

$$N = \begin{array}{c|cccc|cccc|ccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & - \\ \hline 4 & 1 & - & 0 & 0 & - & - & - & - & - & 0 & 0 \\ 5 & 0 & 1 & 1 & 0 & 0 & - & - & - & 0 & 0 & 0 \\ 6 & 1 & 1 & 1 & 1 & 0 & - & - & - & 0 & 0 & 1 \\ 7 & 0 & 0 & - & 1 & 0 & - & - & - & 0 & 0 & - \\ \hline 8 & - & - & - & 0 & 1 & - & 0 & 0 & - & 0 & 0 \\ 9 & - & - & - & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 10 & - & - & - & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 11 & - & - & - & - & 0 & 0 & - & 1 & 0 & 0 & - \\ \hline \end{array}$$

Fig.2.3.

where '-' positions are either 0 or 1 subject to the conditions that D is of $f(D) = 2$ and $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \phi$.

Proof of lemma 2.2 Before taking up proof of the lemma we observe that for a certain combination of values *1* or *0* to the '-' positions, the digraph *D* may turn out to be of Ferrers dimension *3*. But since an interval digraph is necessarily of Ferrers dimension ≤ 2, we simply ignore those and consider only those combination of values in the '-' positions for which D is of f(D) = 2. For instance if we consider the case when four positions (2,9), (3,11), (7,3) and (11,7) all *1* and the rest are all *0*s, then the graph $H_b(D)$ is bipartite and so D is of f(D) = 2.(As a matter of fact, $H_b(D)$ in this case has seven components).

Overlooking the values of the '-' positions a possible bicolouration of $H_b(D)$ is as in the following:

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	C_7	$\mathcal{I}$	R_7
2	1	1	1	1	1	1	1	1	-	1	R_7
3	1	1	1	1	1	1	1	1	C_7	1	-
4	1	-	R_2	R_4	-	-	-	-	-	0	0
5	C_2	1	1	R_3	0	-	-	-	0	$\mathcal{I}_c$	0
6	1	1	1	1	C_7	-	-	-	C_7	C_7	1
7	C_4	C_3	-	1	C_7	-	-	-	C_7	0	-
8	-	-	-	R_7	1	-	R_1	R_6	-	0	R_7
9	-	-	-	0	C_1	1	1	R_5	0	$\mathcal{I}_r$	0
10	-	-	-	R_7	1	1	1	1	1	R_7	R_7
11	-	-	-	-	C_6	C_5	-	1	0	0	-

Fig.2.4.

Note that since $I_c \cap I_r = \phi$, we will not entertain those values in the '-' positions in which an $\mathcal{I}_c$ in the above matrix also becomes an $\mathcal{I}_r$ and an $\mathcal{I}_r$ also becomes an $\mathcal{I}_c$. Also note that for appropriate values, the number of components may turn out to be other than seven. As it can be observed during the proof, if it is more than seven, the additional components will not play any role, whereas as in Proposition 2.3 our arguments in the course of the proof will also hold good when it is less than seven (and some components coalesce).

With this observation, we begin our proof. We start with the configuration

	4	5	10
1	1	1	$\mathcal{I}$
5	R	-	$\mathcal{I}_c$
9	-	C	$\mathcal{I}_r$

We label the rows and column as above as they will turn out to be convenient and fitting in with the labelling of the matrix N subsequently. The presence of I_r and I_c in configuration (2) require the structures

$$\begin{array}{c|cc} & 8 & 10 \\ \hline 9 & R & I_r \\ 10 & 1 & R \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 1 & 10 \\ \hline 5 & C & I_c \\ 6 & 1 & C \end{array}$$

in the adjacency matrix. This implies that the matrix should contain the configuration

$$\begin{array}{c|ccccc} & 1 & 4 & 5 & 8 & 10 \\ \hline 1 & - & 1 & 1 & - & I \\ 5 & C & R & - & - & I_c \\ 6 & 1 & - & - & - & C \\ 9 & - & - & C & R & I_r \\ 10 & - & - & - & 1 & R \end{array}$$

In the above configuration the presence of the structures

$$\begin{array}{c|cc} & 4 & 10 \\ \hline 5 & R & I_c \\ 10 & - & R \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 5 & 10 \\ \hline 6 & - & C \\ 9 & C & I_r \end{array}$$

force us to put values 0 to (10,4) (6,5) positions as otherwise we will have $I_r \cap I_c \neq \phi$ (in either case) which will contradict our hypothesis. So the structure become

$$\begin{array}{c|ccccc} & 1 & 4 & 5 & 8 & 10 \\ \hline 1 & - & 1 & 1 & - & I \\ 5 & C & R & - & - & I_c \\ 6 & 1 & - & 0 & - & C \\ 9 & - & - & C & R & I_r \\ 10 & - & 0 & - & 1 & R \end{array} \quad \ldots\ldots\ldots \ (3)$$

In the above configuration we can check that only column 1 and column 8 are compatible but no two rows. So we can identify column 1 and column 8 to achieve the configuration (3a).

$$\begin{array}{c|cccc}
 & 1{=}8 & 4 & 5 & 10 \\
\hline
1 & - & 1 & 1 & I \\
5 & C & R & - & I_c \\
6 & 1 & - & 0 & C \\
9 & R & - & C & I_r \\
10 & 1 & 0 & - & R
\end{array} \quad \ldots\ldots\ldots \text{(3a)}$$

The presence of a *C* implies that it has an obstruction with an *R*. So corresponding to the *C*'s in the (9,5), (6,10) and (5,1), we consider the obstructions

$$\begin{array}{c|cc}
 & 5 & 7 \\
\hline
8 & 1 & R \\
9 & C & 1
\end{array}\ , \qquad
\begin{array}{c|cc}
 & 10 & 11 \\
\hline
2 & 1 & R \\
6 & C & 1
\end{array} \qquad \text{and} \qquad
\begin{array}{c|cc}
 & 1 & 3 \\
\hline
4 & 1 & R \\
5 & C & 1
\end{array}$$

Next we draw our attention to the three *R*'s in the positions (9,8), (10,10) and (5,4). They call for the following structures in *A(D)*

$$\begin{array}{c|cc}
 & 6 & 8 \\
\hline
9 & 1 & R \\
11 & C & 1
\end{array}\ , \qquad
\begin{array}{c|cc}
 & 9 & 10 \\
\hline
3 & C & 1 \\
10 & 1 & R
\end{array} \qquad \text{and} \qquad
\begin{array}{c|cc}
 & 2 & 4 \\
\hline
5 & 1 & R \\
7 & C & 1
\end{array}$$

respectively, which when combined with the earlier matrix gives us

$$\begin{array}{c|cccc|cccc|ccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
1 & - & - & - & 1 & 1 & - & - & - & - & I & - \\
2 & - & - & - & - & - & - & - & - & - & 1 & R \\
3 & - & - & - & - & - & - & - & - & C & 1 & - \\
\hline
4 & 1 & - & R & - & - & - & - & - & - & - & - \\
5 & C & 1 & 1 & R & - & - & - & - & - & I_c & - \\
6 & 1 & - & - & - & 0 & - & - & - & - & C & 1 \\
7 & - & C & - & 1 & - & - & - & - & - & - & - \\
\hline
8 & - & - & - & - & 1 & - & R & - & - & - & - \\
9 & - & - & - & - & C & 1 & 1 & R & - & I_r & - \\
10 & - & - & - & 0 & - & - & - & 1 & 1 & R & - \\
11 & - & - & - & - & - & C & - & 1 & - & - & - \\
\hline
\end{array}$$

Note that we have arrived at the above matrix from the configuration (3) by adding 6 rows and 6 columns more to it. In the process, some rows/columns can be found compatible to one another. As in the case of lemma 2.1, it can be verified that whenever we identify any two compatible rows/columns, we either get a matrix which does not yield a satisfactory bicolouration, thereby arriving at an impossible situation, or otherwise obtain a core matrix of the matrix N.

Now the structure

$$\begin{array}{c|cc} & 5 & 10 \\ \hline 1 & 1 & I \\ 2 & - & 1 \\ 3 & - & 1 \end{array}$$

implies that the positions (2,5) and (3,5) are *1*.

Also the structure

$$\begin{array}{c|ccc} & 6 & 7 & 10 \\ \hline 2 & - & - & 1 \\ 3 & - & - & 1 \\ 9 & 1 & 1 & I_c \end{array}$$

implies that all the positions (2,6), (2,7), (3,6) and (3,7) are *1*. Again the structure

$$\begin{array}{c|cc} & 5 & 11 \\ \hline 2 & 1 & R \\ 6 & 0 & 1 \end{array}$$

implies that the position (6,5) is *C*.

Next from the structures

$$\begin{array}{c|cc} & 7 & 11 \\ \hline 2 & 1 & R \\ 8 & R & - \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & 5 & 11 \\ \hline 6 & C & 1 \\ 8 & 1 & - \end{array}$$

we conclude that the position (8,11) is an R.

Also from the structures

$$\begin{array}{c|cc} & 10 & 11 \\ \hline 1 & I & - \\ 2 & 1 & R \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & 5 & 11 \\ \hline 1 & 1 & 0 \\ 6 & C & 1 \end{array}$$

we have the position (1,11) is an R.

Also the structure

$$\begin{array}{c|cc} & 9 & 10 \\ \hline 1 & - & I \\ 3 & C & 1 \end{array}$$

implies that the position (1,9) is 0.

Next the structure

$$\begin{array}{c|ccc} & 6 & 8 & 10 \\ \hline 2 & 1 & - & 1 \\ 10 & - & 1 & 1 \\ 11 & C & 1 & - \end{array}$$

implies that the position (2,8) must be 1, as otherwise we have the structures $\begin{pmatrix} R & 1 \\ 1 & R \end{pmatrix}$ or $\begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix}$, none of which is possible.

Similarly from the structure

$$\begin{array}{c|ccc} & 6 & 8 & 10 \\ \hline 3 & 1 & - & 1 \\ 10 & - & 1 & R \\ 11 & C & 1 & - \end{array}$$

it follows that the position (3,8) is *1*.

Also the structures

$$\begin{array}{c|cc} & 7 & 10 \\ \hline 8 & R & - \\ 9 & 1 & I_r \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 6 & 10 \\ \hline 9 & 1 & I_r \\ 11 & C & - \end{array}$$

implies that the positions (8,10) and (11,10) are *0*.

Again the structures

$$\begin{array}{c|cc} & 9 & 10 \\ \hline 3 & C & 1 \\ 9 & - & I_r \end{array}, \quad \begin{array}{c|cc} & 10 & 11 \\ \hline 2 & 1 & R \\ 9 & I_r & - \end{array},$$

$$\begin{array}{c|cc} & 10 & 11 \\ \hline 2 & 1 & R \\ 10 & R & - \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 6 & 9 \\ \hline 3 & 1 & C \\ 11 & C & - \end{array}$$

implies that all the positions (9,9), (9,11), (10,11) and (11,9) are *0*.

Now the structure

$$\begin{array}{c|ccc} & 7 & 8 & 9 \\ \hline 3 & 1 & - & C \\ 9 & 1 & R & - \\ 10 & - & 1 & 1 \end{array}$$

implies that the position (10,7) is *1*.

Also the structures

$$\begin{array}{c|ccc} & 6 & 8 & 9 \\ \hline 3 & 1 & - & C \\ 9 & 1 & R & - \\ 10 & - & 1 & 1 \end{array}$$

and

$$\begin{array}{c|ccc} & 5 & 7 & 9 \\ \hline 3 & 1 & - & C \\ 8 & 1 & R & - \\ 10 & - & 1 & 1 \end{array}$$

implies that the positions (10,6) and (10,5) are *1*.

Next the structures

$$\begin{array}{c|cc} & 5 & 6 \\ \hline 9 & C & 1 \\ 11 & - & C \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & 7 & 8 \\ \hline 8 & R & - \\ 9 & 1 & R \end{array}$$

imply that the positions (11,5) and (8,8) are *0*.

Now the structure

$$\begin{array}{c|cc} & 4 & 10 \\ \hline 1 & 1 & 1 \\ 2 & - & 1 \\ 3 & - & 1 \end{array}$$

implies that the positions (2,4) and (3,4) are *1*.

Then from

$$\begin{array}{c|cc} & 4 & 9 \\ \hline 3 & 1 & C \\ 10 & 0 & 1 \end{array}$$

we conclude that the position (10,4) is an *R*.

Now the structure

$$\begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & - \\ 9 & - & C & 1 \\ 10 & R & - & 1 \end{array}$$

implies that the position (1,6) is *1*.

Similarly from the structures

	4	5	7
1	1	1	-
9	-	C	1
10	R	-	1

	4	6	8
1	1	1	-
10	R	-	1
11	-	C	1

we conclude that the positions (1,7) and (1,8) are *1*.

Now the structures

	4	7
8	-	R
10	R	1

And

	4	8
9	-	R
10	R	1

implies that the positions (8,4) and (9,4) are *0*.

Substituting all these values in the configuration (4) we get the matrix

	1	2	3	4	5	6	7	8	9	10	11
1	-	-	-	1	1	1	1	1	0	I	R
2	-	-	-	1	1	1	1	1	-	1	R
3	-	-	-	1	1	1	1	1	C	1	-
4	1	-	R	-	-	-	-	-	-	-	-
5	C	1	1	-	-	-	-	-	-	I_c	-
6	1	-	-	-	C	-	-	-	-	C	1
7	-	C	-	1	-	-	-	-	-	-	-
8	-	-	-	0	1	-	R	0	-	0	R
9	-	-	-	0	C	1	1	R	0	I_r	0
10	-	-	-	R	1	1	1	1	1	R	0
11	-	-	-	-	0	C	-	1	0	0	-

Repeating the same logic for other positions as above (required for a satisfactory bicolouration of $H(D)$ and proceeding, it can be seen that the matrix N is the required matrix containing the configuration (2).

It may also be noted that, if we start from the configuration (3a) instead of the configuration (3) then as in the case of lemma 2.1, we will arrive at a core matrix of N.

Proof of Proposition 2.4 We first suppose that the graph $H_b(D)$ has all the seven components H_i ($1 \le i \le 7$) as manifest in the figure 4. We will prove that the presence of the configuration (2) is independent of the different combinations of bicolours (R_i, C_i) of the components H_i of the graph $H(D)$. Finally we will consider the case, when, because of the elimination of compatible rows, some components coalesce and the number of components becomes less than 7. It is very important to note in this context that after elimination of some identical rows, the property $\boldsymbol{I_r \cap I_c = \phi}$ in the matrix may get lost in some case, so that according to the Proposition 2.3, the matrix N contains the core matrix of M as a submatrix. As in the present proposition we are interested in only those matrices for which $\boldsymbol{I_r \cap I_c = \phi}$, we will simply ignore those matrices from our considerations.

As in Proposition 2.3, the proof of this proposition relies heavily on the block diagram of the matrix N as given in figures 2.3 and 2.4. For convenience, we name the 9 blocks of N as follows

	1	*2*	*3*
A	A_1	A_2	A_3
B	B_1	B_2	B_3
C	C_1	C_2	C_3

Looking at the matrix N, we check that the 9 elements of the configuration (2) belong to the 9 different blocks, no two to the same block. The block B_1 contains the fragments of the components H_2, H_3 and H_4 excludingly, while those of H_1, H_5 and H_6 belong to the block C_2. Also we observe that I_c and I_r belong to the blocks B_3 and C_3 respectively.

Now we come to the question of other bicolourations by interchange of colours of the fragments in different components. We first consider the block B_1; A little scrutiny will reveal that for any change of colours of the components of this block there is a row which contains fragments of both colours and if by such change, the $\mathcal{I}$ of (5,10) position loses its property of being an $\mathcal{I}_c$, there will be another $\mathcal{I}_c$ reappearing from amongst the 0's of the 10th column in the same block. This interdependence between the change of colours in block B and the loss and recovery characteristic of an $\mathcal{I}_c$ in the block B_3 and similarly between the blocks C_2 and C_3 for an $\mathcal{I}_r$ is the essence of the proof and tells us all.

For instance, let (R_2, C_2) be replaced by (C_2, R_2). Then the position (5,10) no longer remains an $\mathcal{I}_c$ and the configuration (2) seemingly gets lost. But from the configuration

$$\begin{array}{c|cccc} & 3 & 4 & 9 & 10 \\ \hline 2 & 1 & 1 & - & - \\ 4 & C_2 & R_4 & - & 0 \end{array}$$

it follows that

$$\begin{array}{c|c} & 9 \\ \hline 2 & 0 \\ 4 & 1 \end{array}$$

is not possible and so the 0 of the (4,10) must be an $\mathcal{I}$ Again the configuration

$$\begin{array}{c|cc} & 3 & 10 \\ \hline 4 & C_2 & \mathcal{I} \\ 6 & 1 & C_7 \end{array}$$

implies that (4,10) is an $\mathcal{I}_c$ and the configuration (2) resurfaces with the column 1 replaced by column 3 and row 5 by row 4.

The same argument applies for every bicolouration of $H_b(D)$ and with an $\mathcal{I}_c$ shifting its position in the column 10 of block B_3 and an $\mathcal{I}_r$ in the same column of block C_3, we see that the configuration is manifest in every bicolouration of $H_b(D)$.

Finally we consider the case when some compatible rows are identical in the process of which some of the components of H_i may coalesce. Here our arguments remain mainly the same as in the previous case; the only difference is that here the blocks lose their distinct identity and the border line between the blocks breaks down.

Let us take two examples. First consider the matrix N' obtained from N by identifying the rows 5 and 9. This case is quite revealing in the sense that the I_c and I_r of the (5,10) position and the (9,10) position are superimposed upon one another and we have an I which belongs to both $\boldsymbol{I_r}$ and $\boldsymbol{I_c}$. This case is covered by Proposition 2.3 and accordingly this matrix N' should contain the core matrix of M of Proposition 2.3 as a submatrix. As a matter of fact, as we can check that the matrix M can be obtained from N' in the following way: identify the rows 4 and 8 and again the rows 7 and 11; then consider the matrix with the rows 2, 3, 10, 6, 4, 5, 7 and the columns 1, 2, 3, 8, 9, 10, and 11; and fill in the blank positions to satisfy the requirements that $H(D)$ of the submatrix is bipartite. Since this case is outside the purview of the present Proposition we ignore this case.

For another example identifying the rows 4 and 9 in the matrix N, we have the matrix

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	C_7	I	R_7
2	1	1	1	1	1	1	1	1	-	1	R_7
3	1	1	1	1	1	1	1	1	C_7	1	-
5	C_2	1	1	R_3	0	-	-	-	0	I_c	0
6	1	1	1	1	C_7	-	-	-	C_7	C_7	1
7	C_4	C_3	-	1	C_7	-	-	-	C_7	0	-
(9)4	1	-	R_2	R_4	C_1	1	1	R_5	0	I_r	0
8	-	-	-	R_7	1	-	R_1	R_4	-	0	R
10	-	-	-	R_7	1	1	1	1	1	R_7	R_7
11	-	-	-	-	C_4	C_5	-	1	0	0	-

We draw our attention to two particular subcases here;

(i) when either of the (6,6) and (6,7) positions are 0 and

(ii) when both of them are 1.

In subcase (i), the components H_2, H_4 and H_7 coalesce into one component, say H_7. If we now interchange the colours of H_7 then the I_c and I_r of the (5,10) and the (4,10) are also interchanged, whereas if we change the colours of H_7 as well as of H_1, then (5,10) becomes an I_r while the 0 of the (8,10) becomes an I_c.

In the subcase (ii), H_2 is distinct from H_7 and then interchanging the colours of H_2 only leads to the I_r of (9,10) becoming an I_c again, so that we come back to Proposition 2.3 again. (Since for this combination of colours $\boldsymbol{I_r \cap I_c \neq \phi}$, according to the Proposition 3 this matrix should again contain the matrix M, which can be verified with a careful scrutiny).

Exactly analogous to Proposition 2.4, we have the following proposition.

Proposition 2.5 *Let D be a digraph of f(D) = 2 and let for a certain bicolouration* ***(R,C)*** *of* $H_b(D)$

(i) $\boldsymbol{I_r \cap I_c = \phi}$

(ii) A(D) contains the configuration (4)

$$\begin{array}{|lll} \hline 1 & 1 & R \\ 1 & I & I_c \\ C & I_r & - \end{array} \qquad \cdots\cdots\cdots \quad (4)$$

Then the same is true for any other bicolouration of H(D)

The proof of the above proposition is a consequence of the following lemma.

Lemma 2.3 *Let D be a digraph of f(D) = 2 and let* $\boldsymbol{I_r \cap I_c = \phi}$ *for a certain bicolouration (R,C) of H (D). If A(D) contains the configuration (4), then A(D) must contain the core matrix of the following matrix or its transpose (subject to independent permutations of rows and/or columns).*

P =

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	-	1	1	1	0	1	0
2	1	1	1	1	-	1	1	-	1	1	0
3	1	1	1	1	-	1	1	0	1	1	-
4	1	1	1	1	-	1	1	0	0	1	1
5	-	-	-	-	-	0	0	-	0	1	0
6	1	1	1	1	0	0	1	0	0	0	0
7	1	1	1	1	0	1	1	-	-	-	-
8	1	-	0	0	-	0	-	-	0	-	0
9	0	1	1	0	0	0	-	0	0	0	0
10	1	1	1	1	1	0	-	-	0	-	0
11	0	0	-	1	0	0	-	0	0	0	-

Fig. 2.5

The theorem and the lemma can be proved in exactly similar way to those of Proposition 2.4 and Lemma 2.2 and so is omitted here. The possible bicolouration of the above matrix is only noted below.

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	-	1	1	1	R_1	1	R_2
2	1	1	1	1	-	1	1	-	1	1	R_3
3	1	1	1	1	-	1	1	C_1	1	1	-
4	1	1	1	1	-	1	1	C_2	C_3	1	1
5	-	-	-	-	-	0	R_8	-	0	1	0
6	1	1	1	1	0	I	1	0	I_c	C_8	0
7	1	1	1	1	C_7	1	1	-	-	-	-
8	1	-	R_4	R_5	-	0	-	-	0	-	0
9	C_4	1	1	R_6	0	I_r	-	0	0	0	0
10	1	1	1	1	1	R_7	-	-	0	-	0
11	C_5	C_6	-	1	0	0	-	0	0	0	-

From Proposition 2.4 and 2.5 we derive the following Proposition.

Proposition 2.6 *Let D be a digraph of F.D. 2 and let for a satisfactory bicolouration* $I_r \cap I_c = \phi$. *Then the same is true for any bicolouration.*

We see that the Proposition 2.3 and 2.6 virtually complement one another and combining the two we get the following important theorem, upon which as we will later see, our recognition algorithm for an interval digraph will heavily rely.

Theorem 2.1 *Let D be a digraph of F.D. 2, then*

(i) *if* $I_r \cap I_c \neq \phi$ *for a certain bicolouration of* $H_b(D)$ *then the same is true for any other bicolouration of* $H_b(D)$*, and on the other hand*

(ii) *if* $I_r \cap I_c = \phi$. *for a certain bicolouration of* $H_b(D)$*, then the same is true for any other bicolouration of* $H_b(D)$

Proof. *(i)* Let $I_r \cap I_c \neq \phi$ for a certain bicolouration of $H_b(D)$, our proof will be complete if we can prove that $I_r \cap I_c \neq \phi$ for every satisfactory bicolouration of $H_b(D)$ (because this will imply from Proposition 2.3 that the same is true for any bicolouration of $H_b(D)$). Let now if possible it be not true for a certain satisfactory bicolouration of $H_b(D)$ that is $I_r \cap I_c = \phi$ for a certain satisfactory bicolouration of $H_b(D)$. Then Proposition 2.6 implies that $I_r \cap I_c = \phi$ for every bicolouration of $H_b(D)$. Contradictory to our hypothesis.

(ii) Follows as a consequence of *(i)*.

Combining Proposition 2.2 and Theorem 2.1, we get the following stronger version of Proposition 2.2.

Theorem 2.2 *Let D(V,E) be a digraph of* $f(D) = 2$. *If D is an interval digraph then for any bicolouration of* $H_b(D)$, $I_r \cap I_c = \phi$.

Theorem 2.3 *Let $D(V,E)$ be a digraph of $f(D) = 2$ and let for any bicolouration of $H_b(D)$, $I_r \cap I_c = \phi$. Then D is an interval digraph iff the adjacency matrix $A(D)$ of D does not contain any one of the configurations (2) and (4) of the form*

(i)

$$\begin{array}{|ccc|} \hline 1 & 1 & \mathcal{I} \\ C & - & \mathcal{I}_r \\ - & R & \mathcal{I}_c \\ \hline \end{array} \quad \ldots\ldots(2)$$

or (ii)

$$\begin{array}{|ccc|} \hline 1 & 1 & R \\ 1 & \mathcal{I} & \mathcal{I}_c \\ C & \mathcal{I}_r & - \\ \hline \end{array} \quad \ldots\ldots(4)$$

or their transposes.

Proof. ($\Rightarrow$) Let $A(D)$ contain a configuration of either of the given forms. First consider the simple case, when the base graph $H_b(D)$ has one component only. From the configurations, it follows that there exists an $\mathcal{I}$ such that $A(D)$ contains both the structures

$$\begin{array}{|cc|} \hline 1 & \mathcal{I} \\ R & \mathcal{I}_c \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|cc|} \hline 1 & \mathcal{I} \\ C & \mathcal{I}_r \\ \hline \end{array}$$

This means that there exists an $\mathcal{I}$ which is in conflict with both the Ferrers digraphs $\boldsymbol{R \cup I_r}$ and $\boldsymbol{C \cup I_c}$ and so this $\mathcal{I}$ can not be included in either of $\boldsymbol{R \cup I_r}$ and $\boldsymbol{C \cup I_c}$ (retaining the Ferrers digraph property). Hence decomposition of D into two disjoint Ferrers digraphs is not possible and so D is not an interval digraph.

Next consider the case when $H_b(D)$ consist of more than one component. From Theorem 2.1 and propositions 2.4 and 2.5, it follows that the presence of the given configurations in $A(D)$ for a particular bicolouration implies the existence of the given configurations for any satisfactory bicolouration as well. Let now $(\boldsymbol{R}, \boldsymbol{C})$ be any satisfactory bicolouration of $H_b(D)$.

Then as in the earlier case, there is an $\mathcal{I}$ which can not be included in the Ferrers digraphs $\boldsymbol{R \cup I_r}$ or $\boldsymbol{C \cup I_c}$. As this is true for any satisfactory bicolouration of $H_b(D)$, D is not an interval digraph.

($\Leftarrow$). Let D be not an interval digraph. Then we need to prove that $A(D)$ must contain either of the configurations (2) or (4) or their transposes. Again from propositions 2.4 and 2.5, we need to prove the result for satisfactory bicolouration only. So let $(\boldsymbol{R},C)$ be satisfactory bicolouration of $H_b(D)$ so that $G=\boldsymbol{R \cup I}$ and $G=\boldsymbol{C \cup I}$ are Ferrers digraphs. Also since $\boldsymbol{I_r \cap I_c} = \phi$, $H_1=\boldsymbol{R \cup I_r}$ and $H_2=\boldsymbol{C \cup I_c}$ are two disjoint Ferrers digraphs.

Since D is not an interval digraph, there exists an $\mathcal{I}$ say $\mathcal{I}_0$ which is conflict with an element of $\boldsymbol{R \cup I_r}$ as well as with an element $\boldsymbol{C \cup I_c}$. This means that $A(D)$ has the configurations

$$\begin{array}{|cc|}\hline - & \mathcal{I}_0 \\ R/\mathcal{I}_r & - \\ \hline\end{array}$$

where '-'s are elements outside $\boldsymbol{R \cup I_r}$, and

$$\begin{array}{|cc|}\hline - & \mathcal{I}_0 \\ C/\mathcal{I}_c & - \\ \hline\end{array}$$

where '-'s are elements outside $\boldsymbol{C \cup I_c}$.

Consider the configuration

$$\begin{array}{|cc|}\hline - & \mathcal{I}_0 \\ R & - \\ \hline\end{array}$$

We show below that the only configuration containing this structure is of the form

$$\begin{array}{|cc|}\hline 1 & \mathcal{I}_0 \\ R & \mathcal{I}_c \\ \hline\end{array}$$

all other possibilities leading to contradictions. We observe that the configurations in $A(D)$

$$\boxed{\begin{array}{cc} 1 & I \\ R & 1 \end{array}} \quad \text{and} \quad \boxed{\begin{array}{cc} 1 & I \\ R & C \end{array}}$$

are impossible because $\boldsymbol{R \cup I}$ is a Ferrers digraph. Again for the same reason and because I_0 is in conflict with $\boldsymbol{R \cup I_r}$,

$$\boxed{\begin{array}{cc} 1 & I_0 \\ R & I \end{array}}$$

is not possible.

So the only possibility for this case is

$$\boxed{\begin{array}{cc} 1 & I_0 \\ R & I_c \end{array}}$$

Next consider the configuration

$$\boxed{\begin{array}{cc} - & I_0 \\ I_r & - \end{array}}$$

Exactly as before, it can be seen that

$$\boxed{\begin{array}{cc} 1 & I_0 \\ I_r & I_c \end{array}}$$

is the only possibility conforming to this structure. But this implies the existence of

$$\begin{array}{c|ccc} & x & y & z \\ \hline a & - & 1 & I_0 \\ b & 1 & R & - \\ c & R & I_r & I_c \end{array}$$

If the *ax*-position is 0, then it must be C, but then again

$$\begin{array}{c|cc} & x & z \\ \hline a & C & I_0 \\ c & R & I_c \end{array}$$

is a contradiction because $C \cup I_c$ is a Ferrers digraph. Hence *ax*-position is *1* and the structure

$$\begin{array}{c|cc} & x & z \\ \hline a & 1 & I_0 \\ c & R & I_c \end{array}$$

is an implication of the existence of

$$\boxed{\begin{array}{cc} 1 & I_0 \\ I_r & I_c \end{array}}$$

Similarly

$$\boxed{\begin{array}{cc} - & C \\ I_0 & - \end{array}}$$

implies the existence of the configuration

$$\boxed{\begin{array}{cc} 1 & C \\ I_0 & I_r \end{array}}$$

Combining Theorem 2.3 and Theorem 2.2 we state the following theorem:

Theorem 2.4 *A digraph is an interval digraph if and only if it is of F.D. ≤ 2 and when it is of $f(D) = 2$, for any bicolouration of $H_b(D)$,*

(a) $I_r \cap I_c = \phi$

and (b) A(D) does not contain either of the configurations (2) and (4) and their transposes.

In terms forbidden adjacency matrices, we state the above result in the following form.

Corollary 2.1 *A digraph is an interval digraph if and only if it is of F.D.≤ 2 and A(D) does not contain the core matrix of either of the matrices M, N, P and their transposes M^T, N^T, P^T, where M, N, P are as given as figures 2.1, 2.3 and 2.5.*

2.3 Recognition Algorithm

Müller obtained a recognition algorithm of an interval digraph/bigraph in a polynomial time $O(n\, m^6(n+m)\log n)$, where n and m are the number of vertices and edges respectively of the bigraph.

In our algorithm, we first check whether Ferrers dimension of D is equal to two by identifying a bipartition of $H_b(D)$. Then apply the results of the section 2 to recognition an interval digraph. Identifying whether $H_b(D)$ is bipartite generally runs in $O(n^4)$ time, where n is the number of vertices of D. But our procedure *bipartite* described below determines it in $O(n^3)$ time. This, intern, gives the $O(n^3)$ as the time complexity of the problem.

For a digraph D of F.D. *2*, consider any bicolouration *(R,C)* of $H_b(D)$ and with reference to this bicolouration, obtain the I_rs and I_cs by the Procedure *config.* described below. If $I_r \cap I_c \neq \phi$, then D is not an interval digraph. Else, check by the Procedure *config.* again, whether $H(D)$ contains the configuration (2) or (4) or their transposes. If so, the Theorem 2.3 tells us that D is not an interval digraph. Otherwise D is an interval digraph.

Although our algorithm takes much less time, Müller's one has an added advantage that it gives us the interval representation as well, in case D turns out to be an interval bigraph.

The following Algorithm *recog.* alongwith the Procedure *config.* describes the steps for an interval digraph recognition.

Algorithm *recog* : Interval digraph recognition

Input: Adjacency matrix
Output: recognition of the graph G

1. Identify a bipartite partition of $H_b(D)$ by procedure *bipartite*.
   If no such partition is found then the graph is not an interval digraph.

2. Satisfying step(1), denote the set of all isolated vertices by I and a bicolouration of $H_b(D)$ by ***(R,C)***.

3. /* use procedure *config*. For Step 3 */

for all isolated vertices I in the matrix do

begin

(a) check if there exists a configuration of the form

$$\boxed{\begin{array}{cc} R & I \\ 1 & R \end{array}}$$

(b) check if there exist a configuration of the form

$$\boxed{\begin{array}{cc} C & I \\ 1 & C \end{array}}$$

(c) if an I satisfy both (a) and (b) then G is not an interval digraph.

/* $I_r \cap I_c \neq \phi$ */

If I satisfy (a) denote it by I_r else if I satisfy (b) then by I_c.

end;

4. . /* use procedure *config*. For Step 4 */

for each vertex I do

begin

check if there exist configurations of the form

(i)

$$\boxed{\begin{array}{cc} C & I_r \\ 1 & I \end{array}} \quad \text{or} \quad \boxed{\begin{array}{cc} C & 1 \\ I_r & I \end{array}}$$

(ii)

$$\boxed{\begin{array}{cc} C & I_c \\ 1 & I \end{array}} \quad \text{or} \quad \boxed{\begin{array}{cc} R & 1 \\ I_c & I \end{array}}$$

if both the configurations (i) and (ii) exist then G is not an interval digraph.

end;

5. Otherwise, G is an interval digraph.

Procedure *bipartite*

Data Structure :

- For each pair of columns (i,j), j > i, maintain the two sets L^1_{ij} and L^2_{ij} where

$$L^1_{ij} = \{A[k, i] \mid A[k, i] = 0 \text{ and } A[k, j]=1\}$$
$$L^2_{ij} = \{A[k', j] \mid A[k', i] = 1 \text{ and } A[k', j]=0\}$$

- For each pair of columns $(i, j), j > i$, attach a tag variable T_{ij}, where T_{ij} initially contains 0 and is set to 1 as the column pair is processed.

- For each i, maintain a set S_i containing the column indices j for which $A[i, j]=1$

- Attach a field to each vertex indicating one colour taken from a given set of two colours. Initiate the procedure with no colour to any vertex.

- In addition, maintain a stack containing the 0's of $A(D)$, that is the vertices of $H_b(D)$. As soon as a vertex is coloured, it may be used to colour other vertices adjacent to it but still not coloured. Once a vertex is popped up from the stack, it is not pushed into the stack any more.

Step 1: Compute L^1_{ij} and L^2_{ij} for all i, j $(j>i)$. if $L^1_{ij}=\phi$ then $L^2_{ij}=\phi$ and vice versa.

Step 2: Compute S_i for all i.

Step 3: Find a '*0*' element in the matrix A which is not already coloured.

Step 4: Assign a colour to this element and push it into the stack.

Step 5: Pop an element $A[i, j]$ from the stack.

Step 6: For all elements $k \in S_i$ do the following

if T_{jk}, $j<k$ $(T_{kj}$, $k>j)$ is not set

begin

Step 6(a): Assign the value 1 to T_{jk}

Step 6(b) For each element of L^1_{jk} do

if it is not already coloured, push it into the stack with the colour of $A[i, j]$

assigned to it

else if it is already coloured and of colour other than that of $A[i, j]$, then $H_b(D)$ is not bipartite.

For each element of L^2_{jk} do

if it is not already coloured, push it into the stack with the colour other than

that of $A[i, j]$ assigned to it

else if it is already coloured and of the same colour of $A[i, j]$, then $H_b(D)$

is not bipartite.

end

Step 7: Repeat *Step 5* through *Step 6* until the stack is empty.

Step 8: If any 0 of $A(D)$ still remains to be coloured, repeat *Step 3* through *Step 7*

else declare that the graph $H_b(D)$ is bipartite.

The following procedure *config.* describes a technique to search for a 2×2 configuration of the form

$$\boxed{\begin{matrix}\alpha & \beta \\ \gamma & I\end{matrix}}$$

in $A(D)$ which is used in steps 3 and 4 of the above algorithm.

Procedure *config.*

Algorithms for checking the existence of the configurations of the form

$$\boxed{\begin{matrix}\alpha & \beta \\ \gamma & I\end{matrix}}$$

for all I in the $n \times n$ matrix A.

Input: Adjacency matrix(A)

Output: Marking the I's in A that form the configuration

$$\boxed{\begin{matrix}\alpha & \beta \\ \gamma & I\end{matrix}}$$

Data structure: An $n \times n$ matrix B initialized by 1 to its all entries.

1. [Creation of matrix B]

for column $i = 1$ to n do

begin

```
      find rows a1, a2, ... ak such that A[aj,i]=α  for 1 ≤j ≤k
      for row r= 1 to n do
         begin
           if A[r,i]=γ then B[r,aj]=0 for 1 ≤j ≤k
         end;
    end;
2. For i=1 to n do
   for j=1 to n do
    begin
      if A[i,j]=I then
       begin
         for m=1 to n do
          begin
             if A[m,j]=β then
                 if B[i,m]=0 then there exists the desired configuration  in the matrix for the  I
in     [i,j] position.
          end;
      end;
   end;
```

Complexity Analysis:

We first show that the complexity of Procedure *bipartite* (which determines if $H_b(D)$ is bipartite) is $O(n^3)$, where n denotes the number of vertices of D.

The *Step 1* of the procedure can be performed in $O(n^3)$ time. The maximum number of repetition of the cycle between *Step 3* and *Step8* is $O(n^2)$. In *Step 6,* at most n elements of S_I are considered. If T_{jk} is set, the *Step 6(a)* and *6(b)* will not be executed. Otherwise *6(a)* and *6(b)* will be repeated $O(n)$ times. In other words for each pair (j, k) $O(n)$ entry in the matrix will be coloured. Hence the repetition of the cycle between *Step 3* and *Step 8* is performed in $O(n^3)$ times. Thus the complexity of the Procedure *bipartite* is $O(n^3)$.

Next it can be easily shown that the time complexity of the Procedure *Config.* is $O(n^3)$.

Now for our Algorithm *recog.* the time complexity of *Step 1* (using Procedure *bipartite*) is $O(n^3)$ and the time complexity of *Step 3* and *4* (using Procedure *Config.*) is $O(n^3)$. Hence the overall time complexity of our recognition algorithm is $O(n^3)$. Hence we have the following theorem.

Theorem 2.5 *An interval digraph D can be recognised in time $O(n^3)$, where n is the number of vertices of D.*

CHAPTER 3

BIGRAPHS/DIGRAPHS OF FERRERS DIMENSION 2 AND ASTEROIDAL TRIPLE OF EDGES (ATE)

3.1 Introduction

Previously, we have noted that the concepts of interval digraphs and interval bigraphs are closely related and, in fact, basically equivalent. Analogous to the notion of containment graph, a containment digraph was introduced and studied in [Sen, Sanyal and West, 1995]. Equivalently, a *containment bigraph* is a bipartite graph $B(U, V, E)$ for which there are two families of intervals $\{S_u : u \in U\}$ and $\{T_v : v \in V\}$ such that $u \in U$ and $v \in V$ are adjacent if and only if $S_u \supset T_v$. In this chapter the digraph D and the corresponding bigraph $B=B(D)$ (obtained from D by vertex splitting operation [Müller, 1997]) will be used interchangeably. The adjacency matrix of D is the *biadjacency matrix* of B. The intervals corresponding to the members of U and V and the *source intervals* and *sink intervals* respectively.

A pair of edges x_1y_1 & x_2y_2 of a bipartite graph $H(X,Y,E)$ is separable [Golumbic, 1980] if the corresponding vertices induce the subgraph $2K_2$ in H; in this case its biadjacency matrix contains a 2×2 permutation submatrix. A bigraph containing at least a pair of separable edges is *a separable bigraph.* Otherwise it is *non-separable*.

It is clear that the bipartite analogue of a Ferrers digraph is a non-separable bigraph and that of a digraph D of higher $f(D)$ is separable. The Ferrers dimension of a digraph will also be called the *Ferrers dimension of the corresponding bigraph.* It was proved [Sen *et al.*, 1989a] that a bigraph is an interval bigraph if and only if it is the intersection of two non-separable bigraphs whose union is complete. It was also proved that a bigraph is an interval bigraph if and only if the rows and columns of its biadjacency matrix can be permuted independently so that each 0 can be replaced by one of $\{R, C\}$ in such a way that every R has only R's to its right and every C has only C's below it. The matrix is said to have an *$\{R, C\}$ partition of zeros* or *zero partitionable property.*

While an interval bigraph is necessarily of Ferrers dimension at most 2, a containment bigraph is exactly what characterizes a bigraph of *f(D)* at most 2. The following theorem is a characterization of a containment bigraph translated from its digraph version.

Theorem 3.1 [Sen *et al.*, 1989a] and [Sen, Sanyal & West, 1995]. *The following conditions are equivalent :*

i) *B is a containment bigraph;*

ii) *B is of Ferrers dimension at most 2;*

iii) *The rows and columns of the adjacency matrix of B have an independent permutation so that no 0 has a 1 both to its right and below it.*

The rearranged matrix with this permutation of rows and columns is referred to as F_2 *matrix*. Also we shall refer to this property as F_2 *property* for the rearranged matrix.

It is clear that the class of interval bigraphs form a proper sub-class of the class of containment bigraphs. In fact, the bigraph in Fig. 3.1. is an example of a containment bigraph which is not an interval bigraph.

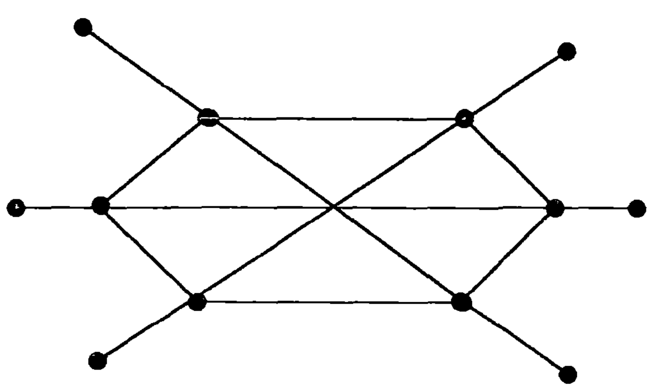


Fig. 3.1

Its adjacency matrix has the characterizations of Ferrers dimension 2 but does not have $\{R, C\}$ –partition of zeros.

Motivated by the study of Gaussian elimination in (*0, 1*)–matrices, Golumbic and Goss [1978] introduced the notion of chordal bipartite graph, an analogue of chordal graph. It was characterizd by Hammer *et al.*in [1989].

A bigraph is called *chordal bipartite* or simply *bichordal,* if every cycle of length ≥ 6 has a chord. Müller [1997] showed that an interval bigraph is a bichordal graph. It is now relevant to observe that every (even) cycle of length ≥ 6 is of Ferrers dimension ≥ 3; consequently it follows that a bigraph B having Ferrers dimension at most 2 does not contain a cycle of length ≥ 6 and equivalently a containment bigraph is necessarily bichordal.

Lekkerkerker and Boland [1962] used the notion of asteroidal triple of vertices to characterize an interval graph and then obtained a complete list of forbidden subgraphs of an interval graph. Analogously in the present chapter, we define an *asteroidal triple of edges* (*ATE*), as follows : Three mutually separable edges e_1, e_2, e_3 of a graph G are said to form an *asteroidal triple of edges,* if for any two of them, there is a path from the vertex set of one to the vertex set of the other that avoids the neighbours of the third edge. This definition is a slightly modified version of the definition given by Müller [1997]. Here we consider the three edges to be mutually separable.

Analogous to the characterization of an interval graph in terms of asteroidal triples, Müller in the same paper conjectured that a chordal bipartite graph is an interval digraph iff it is ATE-free and also free from a class of bigraphs, he termed insects. In section 3.2 we give a counter example to show that the conjecture is not true. In the next section we study the notion of strong and weak bisimplicial edges in a bigraph. Then we try to obtain the significance of an ATE in a bigraph. In section 3.4 we first show that a bigraph B having $f(B) = 2$ is also ATE-free which strengthens the result by Müller that an interval bigraph is ATE -free. Next we address the problem of characterizing an ATE-free bigraph and in this endeavor, we first give a counter-example (Fig. 3.5) to show that an ATE-free bichordal graph is not necessarily of Ferrers dimension 2. Finally in this section, we show that when a bighraph contains a strong bisimplicial edge, the bigraph of example 3.1 (Fig. 3.5) is the only forbidden subgraph of a bichordal, ATE-free bigraph having Ferrers dimension 2 and this is the main result of this chapter.

One final remark : of the two equivalent directed graph and bipartite graph models, the latter one seems preferable. Because, as we have seen, the adjacency matrix and the permutation of its rows and columns play an important role in their characterization; but the (independent) permutation of rows and columns for the adjacency matrix of a bigraph gives us the (same) graph up to isomorphism, where as this is not the same for the digraph case. Moreover, results on the well studied notion of chordal bipartite graphs fits in very well and we rely heavily on them in our present study.

3.2 Müller's conjecture : a counterexample

Müller considered two bigraphs B and B^* whose biadjacency matrices are

1	1	1	1	0	0
1	1	1	0	1	0
1	1	1	0	0	1
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0

Adj(B)

1	1	1	1	0	0
1	1	1	0	1	0
1	1	1	0	0	1
1	0	0	1	1	1
0	1	0	1	1	1
0	0	1	1	1	1

Adj(B)*

and then defined an insect to be a bigraph G such that $B \subseteq G \subseteq B^*$. He showed that an interval bigraph is ATE-free and also insect-free. He then conjectured that a bichordal graph is an interval bigraph iff it is ATE-free and insect-free. The following example shows that the conjecture is not true.

Consider the bigraph of Figure 3.2.

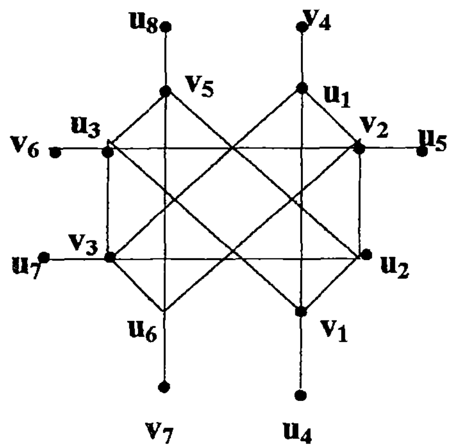


Fig. 3.2

The biadjacency matrix in terms of a bicolouring of $H_b(D)$ of the above graph is given by

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1	1	1	1	1	R	R	R
u_2	1	1	1	C	1	I	R
u_3	1	1	1	C	1	1	R
u_4	1	R	R	I	R	I_r	R
u_5	C	1	R	I_c	R	I_r	I_r
u_6	C	1	1	C	1	C	1
u_7	C	C	1	I_c	R	I_c	I_r
u_8	C	C	C	C	1	I_c	I

It is clear that for this bigraph $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \phi$ and it contains the configuration

	v_1	v_5	v_6
u_2	1	1	I
u_5	C	R	I_r
u_7	C	R	I_c

Hence from Theorem 2.4 it immediately follows that D is not an interval digraph and thereby the bigraph is not an interval bigraph.

It is to be noted that in [Das and Sen, 1993] the digraph in example 1 was shown to be an example of a digraph which is not an interval digraph. The bigraph in Figure 3.2 that we have considered in present section is a slight variant of that example and can be obtained by deleting the column 4 of the adjacency matrix.

3.3 Bisimplical edges : Strong & Weak

Throughout this chapter, we will assume that no vertex of *H(X, Y, E)* is a copy of one another. Let $e = xy$ be an edge of a bipartite graph *H(X,Y,E)*. Also let $B(e) = B(xy)$ denote the subgraph induced by $adj(x) + adj(y)$. An edge $e = xy$ of the bipartite graph H is called *bisimplicial* if $B(e)$ or $B(xy)$ is complete. Analogous to the notion of strong and weak simplicial vertices, a bisimplicial edge $e=xy$ of H is said to be *strong* if $H \backslash B(e)$ is connected; other wise it is *weak*. Note that a graph may contain strong bigimplicial edges as well as weak bisimplical edges. (For example in the graph of example 3.1, the edge xy is a strong bisimplical edge where as the edge x_3y_2 is a weak bisimplical edge). We begin with the following theorems which guarantee the existence of bisimplicial edges in a bichordal graph.

Theorem 3.2 [Golumbic, 1980] *Let H be a connected bichordal graph. If H is separable, then it has at least two separable bisimplical edges.*

Theorem 3.3 [Golumbic and Goss, 1978] *Let H be a connected bichordal graph. If H is non-separable then every vertex of H is incident with some bisimplicial edge of H.*

Regarding Theorem 3.3, we have the following interesting observation to make. It has been seen in the introduction that a non-separable bigraph is equivalent to a Ferrers digraph and so its adjacency matrix can be arranged in the form of a Ferrers diagram with 1's blocked in the upper right corner as in the following Figure.

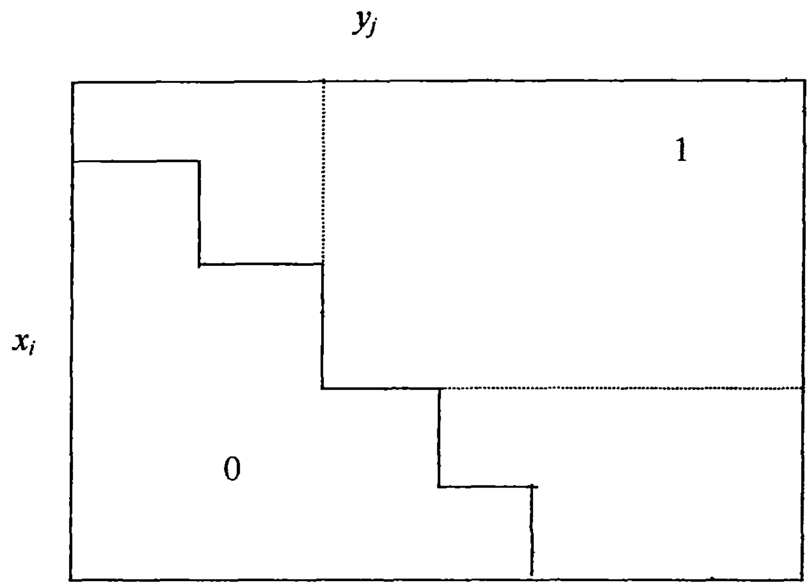


Fig. 3.3

So given any vertex say x_i, of the bigraph $H(X, Y, E)$, there is a permutation of the members of its X-set; so that x_i becomes the last member in its equivalent class. Now consider the vertex y_j in $Adj(x_i)$, such that y_j is the first member in its equivalent class. Then we can easily see the $Adj(x_i)$ and $Adj(y_j)$ form a rectangular block of 1's in the diagram, which induces a complete bipartite subgraph and by the way leave the vertices of its complement totally isolated. Consequently it follows that the bisimplicial edges of a non-separable connected bigraph are all weak.

The converse, however, is not true, as can be seen from the following graph (Fig 3.4)

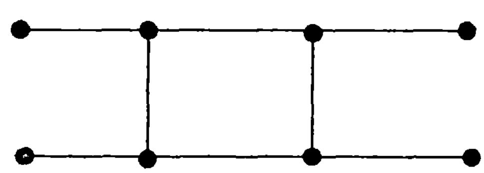


Fig. 3.4

The above graph is separable with no strong bisimplicial edge.

Proposition 3.1 *If a chordal bipartite graph G has exactly two bisimplicial edges, then they must be strong.*

Proof. Let $e=xy$ be a weak bisimplicial edge of H and let $C_1, C_2, .., C_k(k\geq 2)$ be the components of $H\backslash B(e)$. Now consider the subgraph H_1 induced by the vertices of $B(e)$ and C_1. If C_1 consists of a single vertex v, it is a leaf of H and so the edge incident to it is a bisimplicial edge of H. If C_1 is non-trivial component, then H_1 is separable and so has at least two bisimplicial edges; the bisimplicial edges other than e must belong entirely to C_1 and since C_i's are all disconnected, they are also bisimplicial edges of H. So H has more than k (at least $k+1$, $k\geq 2$) bisimplicial edges, which contradicts the hypothesis. ∎

Let H (X, Y, E) be a containment bigraph. In its containment representation let δ_v $(v=x$ or $y)$ be the interval corresponding to the vertex v and $r(v)$, $l(v)$ be its right and left end point respectively.we call δ_v as end interval if

i) $r(v') > l(v)$

or ii) $l(v') < r(v)$ for all vertex v' belongs to the same partite set as v. In case (i) δ_v is the *left end interval* and in case (ii) δ_v is the *right end interval.*

3.4 ATE-free bigraphs and bigraphs of Ferrers dimension 2

In this section, first we prove that a bigraph having Ferrers dimension at most 2 is necessarily bichordal and ATE-free.

Proposition 3.2 A bigraph of Ferrers dimension ≤ 2 is bichordal and ATE-free.

Proof. We have already observed in the introduction that a cycle of length ≥ 6 is of Ferrers dimension 3 and accordingly a bigraph H with $f(H) \leq 2$ is necessarily bichordal. To show that a bigraph with $f(H) \leq 2$ is ATE-free, we rely on the characterization of a bigraph with Ferrers dimension ≤ 2 which states that H is of $f(H) \leq 2$ iff after suitable arrangement of rows and columns, its (bi)adjacency matrix becomes such that for any 0 in the matrix either every position to its right is a 0 or every position below it is 0.

Let $e_i = x_iy_i$ $(i=1, 2, 3)$ be three mutually separable edges of H. From the characterization of a containment bigraph as stated above, the rearranged F_2-matrix has the submatrix in the following form

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \end{array}$$

(because otherwise, any different permutation will violate its F_2 matrix characterization).

Now we consider a path P joining one end of e_1 to one end of e_3 say, $u_1, \ldots\, x_iy_p \ldots x_jy_q \ldots u_3$, where $u_1 = x_1$ or y_1 and $u_3 = x_3$ or y_3. To reach e_3 from e_1, a close look into the rearranged matrix show that the path has a subpath from one edge x_iy_p to x_jy_q which calls for either of the two possibilities :

1) there must exists two vertices, say x_i and x_j on P such that x_2 lie between x_i and x_j in the rearranged matrix.

2) there must exists two vertices, say, y_p and y_q on P such that y_2 lie between y_p and y_q.

Then the sequence of vertices of the subpath is either $x_iy_px_jy_q$ or $y_px_iy_qx_j$. In the first case y_p precedes y_2 and so the position x_2y_p cannot be 0 (because in that case this 0 will leave 1 both to its right and below). So x_2y_p position must be 1. This means that x_2 is adjacent to the path. Similarly for the second case we see that y_2 is adjacent to the path. This proves that H is ATE-free. ■

Note that if a bichordal graph has three strong bisimplicial edges, then they must form an ATE. Since a containment bigraph is ATE-free, it follows that a containment bigraph can not contain more than 2 strong bisimplical edges. In this context, we have the following interesting result concerning the relative position of the intervals representing a strong bisimplicial edge in a containment bigraph.

Proposition 3.3 *Let $e=xy$ be a strong bisimplicial edge of a containment bigraph $H(X,Y,E)$. Then for any containment model of H, the intervals representing x and y are the end intervals (on the same end).*

Proof. Let $e = xy$ be a strong bisimplicial edge of H, so that $H\backslash B(e)$ is connected. Then in any representation of H

$\delta_y \subset \delta_x'$ for the vertices x' of $B(e)$ only,

and $\delta_x \supset \delta_y'$ for the vertices y' of $B(e)$ only,

Let, if possible, there is a representation of H for which δ_y is not an end (say, left) interval. Then there exists vertices y_1, y_2 of H such that

$l(y) > r(y_1)$ and $r(y) < r(y_2)$.

Since no vertices of H is copy of another, there exist vertices x_1, x_2 of $H\backslash B(e)$ for which $x_1y_1 \in E$ and $x_2y_2 \in E$.

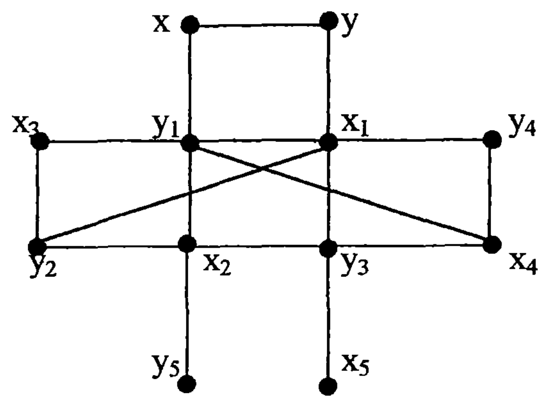
Again since $x_1, x_2 \in H\backslash B(e)$,

$r(x_1) < r(y)$ and $l(y) < l(x_2)$;

because otherwise x_1 or x_2 would belong to $B(e)$. but then the edges x_1y_1 and x_2y_2 can not be connected by any path in $H\backslash B(e)$, meaning that $H\backslash B(e)$ is disconnected. This contradiction shows that δ_y is an end interval. It follows that δ_x along with other source intervals of $B(e)$ are the end intervals (on the same end as δ_y). ∎

Now we are in a position to address the converse problem to Proposition 3.2. Our question is whether a bichordal and ATE-free bigraph is of Ferrers dimension at most 2. The answer is, infact, negative. Below we give an example of bigraph which is bichordal, ATE free but of Ferrers dimension 3. The process in which we have obtained this example is however a matter of long deliberation and will be clear when we come to the next theorem.

Example 3.1 The following bigraph (Figure 3.5) is bichordal, ATE-free but of Ferrers dimension 3.



The bigraph H_0

Fig. 3.5

It can be seen that the associated graph $H(H_0)$ of the above bigraph H_0 has an odd cycle and so from Cogis [1979], it follows that $f(H_0) > 2$. If, however, we delete any vertex from H_0, its Ferrers dimension becomes equal to 2. So clearly $f(H_0) = 3$.

Below we state the central result of this chapter in Theorem 3.4.

Let $e=xy$ be a bisimplicial edge of the bipartite graph $H=H(X,Y,E)$.

We write $H_1(X',Y',E') = H \setminus \{x,y\}$, $B_1 = B_1(X_1,Y_1,E_1) = B(xy)\setminus\{x,y\}$,

$H_2(X_2, Y_2, E_2) = H\setminus B(xy) = H_1\setminus B_1$.

$N(e)$ = vertex set of $B(e)$ i.e., the set of neighbours of x and y.

X'_2 be the set of those members of X_2 which are adjacent to some member of B_1 and $X''_2 = X_2 - X'_2$. So $X_2 = X'_2 \cup X''_2$. Similarly we can define Y'_2 and Y''_2 so that $Y_2 = Y'_2 \cup Y''_2$.

Next we denote the subgraphs induced by the vertices $X_1 \cup Y'_2$ and $X'_2 \cup Y_1$ by P and Q respectively.

Theorem 3.4. *Let a bipartite graph $H(X,Y,E)$ be bichordal and ATE-free and contains a strong bisimplicial edge. Then either $f(H) = 2$ or H contains the bigraph H_O of Fig. 3.5 as an induced subgraph.*

The proof of the theorem is actually very long and requires a very careful and meticulous reading.

To prove the above theorem we first prove the following lemma.

Lemma 3.1 *Let H be a bichordal and ATE-free bigraph and let e=xy be a strong bisimplicial edge of H. then the induced subgraphs P and Q as defined above are non-separable.*

Proof of the lemma. If possible, let the induced subgraph P have separable edges x_iy_i and x_jy_j where $x_i,\ x_j \in X_1$, and $y_i,\ y_j \in Y'_2$. Since H_2 is connected, it must contain a path between y_i and y_j. Now if this path is of length 2, say $y_ix'y_j$, $x' \in X_1$, then we have a 6-cycle $x_iyx_jy_jx'y_ix_i$ in H. On the other hand if the path is of length > 2, say $y_ix' \ldots x''y_j$, then three edges $e=xy$, $e' = x'y_i$ and $e''=x''y_j$ are mutually separable and constitute an ATE of H; for we have paths yx_iy_i between e and e' and yx_jy_j between e and e'' which avoid $N(e'')$ and $N(e')$ respectively and $N(e)$ does not contain any vertex of the component H_2.

Similarly it can be shown that the subgraph Q is also non-separable.

Proof of Theorem 3.4. We recall that no vertex of H is a copy of one another. By the above lemma, the subgraphs P and Q are non-separable (i.e. their biadjacency matrices are Ferrers bigraphs) so we can order the vertices of Y'_2 and X'_2 such that, $Adj(H_1)$, the biadjaceney matrix of H_1 has the following configuration (Fig. 3.6).

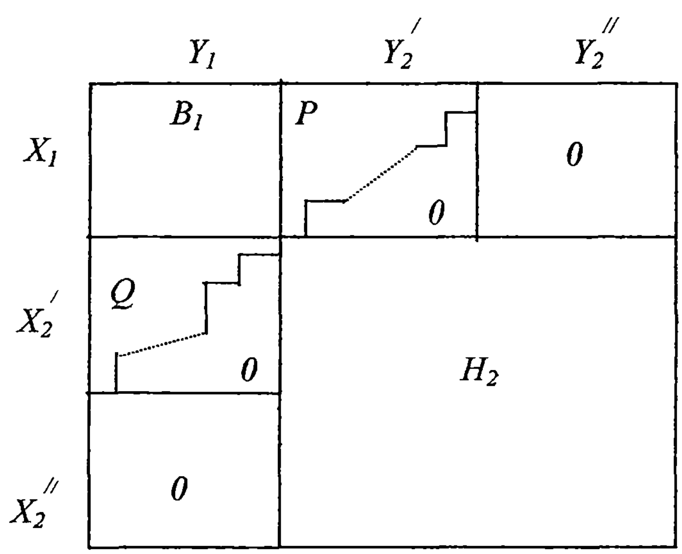


Fig. 3.6 Biadjacency matrix of H_1

We recall that the vertices $X_2 = X'_2 \cup X''_2$ and $Y_2 = Y'_2 \cup Y''_2$ induce the subgraph H_2. Consequently $Adj(H_2)$, the biadjacency matrix of H_2 is a submatrix of the Fig 3.6, where the row and column arrangement are the same as in $Adj(H_1)$ (i.e., of Fig 3.6). Also it is to be noted that in the $Adj(H_2)$ we can permute the rows (columns) of X''_2 (Y''_2), and the rows (columns) of X'_2 (Y'_2) which belong to the same partitioned class, without changing the structure of Fig 3.6. These permutations will be referred to as the *permissible permutations* in this chapter.

We will show that when a bigraph H satisfies the given conditions of the theorem and free from H_0, $Adj(H_1)$ with its rows and columns arranged as in Fig 3.6 will exhibit the characteristics of a bigraph of Ferrers dimension 2 (Theorem 3.1) and once this is established, we place the x-row and y-column of the bisimplical edge $e=xy$ to the top and extreme left of $Adj(H_1)$ and this will prove the theorem. This will be established if we can show that $Adj(H_2)$ has the property that no 0 has a 1 both to its right and below it. For this we will show below that, if $Adj(H_2)$ for any permissible permutation of its rows and columns, contains a configuration of the form

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array}$$

Where '-' positions is either 0 or 1, then the bigraph H contains either an ATE or a 6-cycle or the forbidden graph H_0 as induced subgraph.

To complete the proof of the theorem, we need to consider following cases :

case 1. Neither P nor Q is complete bipartite graph;

case 2. P is complete but Q is not ;

case 3. Q is complete but P is not;

case 4. Both P and Q are complete.

We will observe that while a 6-cycle or an ATE will be present in all the four cases, the forbidden graph H_0 will occur only in case 4, when P and Q are both complete bipartite graphs.

Case 1. Suppose for any permissible permutation of its rows and columns, $Adj(H_2)$ contains the configuration

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array}$$

This case is to be divided again into four subcases subject to whether the vertices x_i, x_j and the vertices y_i, y_j belong to the same partitioned class or to distinct partitioned classes.

Subcase 1a. x_i and x_j belong to two distinct partitioned classes of X_2 and so do y_i and y_j belong to two distinct classes of Y_2.

Subcase 1b. y_i and y_j belong to the same partitioned class, whereas x_i and x_j belong to two distinct classes.

Subcase 1c. x_i and x_j belong to the same partitioned class of X_2 whereas y_i and y_j belong to two distinct classes of Y_2.

Suncase 1d. x_i and x_j belong to the same partitioned class and y_i and y_j belong to the same partitioned class.

Subcase 1a. In this case clearly $Adj(H_1)$ contains a configuration

$$\begin{array}{c|ccc} & y_1 & y_i & y_j \\ \hline x_1 & 1 & 1 & 0 \\ x_i & 1 & 0 & 1 \\ x_j & 0 & 1 & - \end{array}$$

Where $x_1, y_1 \in V(B_1) = X_1 \cup Y_1$.

Now if the '-' position is 1, then the above configuration is a 6-cycle. So we suppose that '-' position is a 0. Then the three edges $e=xy$, $e_i=x_iy_j$, $e_2=x_jy_i$ of H are mutually separable. Also we have path xy_1x_i between e and e_1 and path yx_1y_i between e and e_2 which avoid respectively $N(e_2)$ and $N(e_1)$. Also there exist path between e_1 and e_2 which avoids $N(e)$ (since e_1 and e_2 are two edges of the connected component H_2 and no vertex of it is adjacent to e). So $\{e, e_1, e_2\}$ constitute an ATE of H.

Sub case 1b. In this case $Adj(H_1)$ contains a configuration

$$\begin{array}{c|ccc} & y_1 & y_i & y_j \\ \hline x_1 & 1 & - & - \\ x_i & 1 & 0 & 1 \\ x_j & 0 & 1 & - \end{array}$$

where $x_1 \in X_1$, $y_1 \in Y_1$ and x_1y_i, x_1y_j positions are both 1 or both 0.

It is possible that by permuting the vertices y_i an y_j, we can get a F_2-matrix, except of course when we are confronted with a vertex x_k belonging to still another partitioned class of X_2 and having the following configuration in $Adj(H_1)$.

$$\begin{array}{c|cccc} & y_1 & y_2 & y_i & y_j \\ \hline x_1 & 1 & 1 & - & - \\ x_i & 1 & 1 & 0 & 1 \\ x_j & 1 & 0 & 1 & 0 \\ x_k & 0 & 0 & 0 & 1 \end{array}$$

where $x_1 \in X_1$ and $y_1, y_2 \in Y_1$.

Here also we find three mutually separable edges $e=xy$, $e_1=x_jy_i$ and $e_2 = x_ky_j$. And these three edges constitute an ATE of H; for the paths xy_1x_j, $xy_2x_iy_j$ between e, e_1 and e, e_2 avoid the neighbour of e_2 and e_1 respectively and because H_2 is connected the path between e_1 and e_2 avoids neighbours of e.

Sub case 1c. This case is similar to case 1b and so is omitted.

Subcase 1d First suppose that x_i, $x_j \in X'_2$ and y_i, $y_j \in Y'_2$. Now one possibility is that we can permute x_i, x_j and/or y_i, y_j and get $Adj(H_1)$ as F_2-matrix straightway without facing any obstruction elsewhere. Otherwise, the four configurations are the instances in $Adj(H_1)$, where there are vertices x_k, belonging to a class other than that of x_i, x_j and y_k, belonging to a class other than that of y_i and y_j, when we fail to derive the F_2-matrix directly.

	y_1	y_i	y_j	y_k
x_1	1	1	1	0
x_i	1	0	1	0
x_j	1	1	0	1
x_k	0	0	1	-

Fig 3.7 (i)

	y_1	y_k	y_i	y_j
x_1	1	1	0	0
x_k	1	-	1	0
x_i	0	1	0	1
x_j	0	0	1	-

Fig 3.7 (ii)

	y_1	y_i	y_j	y_k
x_1	1	1	1	0
x_k	1	1	0	-
x_i	0	0	1	0
x_j	0	1	0	1

Fig 3.7 (iii)

	y_1	y_k	y_i	y_j
x_1	1	1	0	0
x_i	1	1	0	1
x_j	1	0	1	0
x_k	0	-	0	1

Fig 3.7 (iv)

where $x_1 \in X_1$ and $y_1 \in Y_1$.

In these cases we can see after a careful scrutiny that H contains either an ATE or 6-cycle. (For example, if $x_j\, y_j$ position is 0 then $\{x_jy_i,\ x_iy_j,\ xy\}$ is an ATE of the graph that contains Fig (ii) or (iii) as a submatrix in its adjacency matrix.

The cases when x_i, $x_j \in X''_2$; y_i, $y_j \in Y'_2$ or when x_i, $x_j \in X'_2$; y_i, $y_j \in Y''_2$ are similar to the previous case so are omitted.

Finally to complete the subcase 1d, we suppose that x_i, $x_j \in X''_2$; y_i, $y_j \in Y''_2$. As earlier, two following configurations occur when we cannot get $Adj(H_1)$ as F_2-matrix directly by permuting x_i, x_j and y_i, y_j.

		Y_1	Y'_2 y'	Y''_2 $y_i\ y_j$
X_1			1	0
X'_2	x'	1	-	1 0
X''_2	x_i	0	1	0 1
	x_j		0	1 -

Fig 3.8 (i)

		Y_1 Y_1	Y'_2 y'	Y''_2 $y_i\ y_j$
X_1			1	0
X'_2	x'	1	-	0 1
X''_2	x_i	0	0	0 1
	x_j		1	1 -

Fig. 3.8 (ii)

Note that here $x' \in X'_2$ and $y' \in Y'_2$.

Now in each cases for any value in the '-' position the presence of ATE and 6-cycle can be easily checked.

Case 2. & 3. Proof in these cases are similar to the case 1 (i.e. if, for any permissible permutation of its rows and columns, $Adj(H_2)$ contains the configuration

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

then the bigraph H contains either an ATE or a 6-cycle) and so are omitted.

Case 4. Here both P and Q are complete bipartite graphs. Since no vertex of H is a copy of one another, it is clear that X_1 and Y_1 are singleton sets, let $X_1 = \{x_1\}$ and $Y_1 = \{y_1\}$. So $Adj(H)$ has the following structure :

$$\begin{array}{c|c|c|c|} & y & y_1 & Y_2 \\ \hline x & 1 & 1 & 0 - - - - - - - 0 \\ \hline x_1 & 1 & 1 & \\ \hline X_2 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & H_2 \\ \hline \end{array}$$

Fig. 3.9

Suppose if possible $Adj(H_2)$ contains a configuration (1). Here we will show that H_o is the only graph which is of Ferrers dimension > 2 but does not contain a 6-cycle or an ATE.

We observe that the structure

	y	y_1	y_I	y_j
x	1	1	0	0
x_1	1	1	1	0
x_i	0	1	0	1
x_j	0	0	1	-

of *Adj(H)* with *xy* as strong bisimplicial edge implics that *H* must contain an ATE or C_6 according as '-' position is a *0* or *1*. So if *Adj*(H_2) contains the configuration (1), (and since *Adj(H)* is symmetric) *Adj(H)* must have one of the following structures :

	y	Y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	-	-
x_i	0	1	0	1
x_j	0	1	1	-

and

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	-	-
x_i	0	1	0	1
x_j	0	0	1	-

Where the positions x_1y_i and x_1y_j are both *0* are both *1*.

Here we suppose that both the positions $x_1\, y_i$ and $x_1\, y_j$ are 1. (We are not considering the possibility that x_1y_i and x_1y_j are *0*, since later we add all possible row and / or column to the above matrices).

Not that the x_jy_j position in either of the matrices is *0* or *1*. To facilitate the mater, we replace the x_j row by two rows x_{j_1} and x_{j_2} to the matrices, one row taking the value '*0*' and the other taking the value '*1*' is the corresponding y_j column. So we get the matrices

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	1	1
x_i	0	1	0	1
x_{j_1}	0	1	1	0
x_{j_2}	0	1	1	1

Fig 3.10 (i)

and

	y	y_1	y_i	y_j
x	1	1	0	0
x_1	1	1	1	1
x_i	0	1	0	1
x_{j_1}	0	0	1	0
x_{j_2}	0	0	1	1

Fig 3.10 (ii)

We label the vertices x_i, x_{j_1}, x_{j_2} by x_3, x_4 and x_2 and the vertices y_i, y_j *by* y_3, y_2 respectively in both the Figures for the sake of convenience.

Clearly, by permuting the x_3, x_4 and x_2 row and y_3, y_2 column of Fig 3.10 (i), *Adj(H)* gets the following F_2 –matrix structure (Fig 3.11(i)). Also permuting the rows and columns of the matrix in Fig. 3.10(ii) we get F_2-matrix of Fig 3.11(ii)

	y	y_1	y_2	y_3
x	1	1	0	0
x_1	1	1	1	1
x_2	0	1	1	1
x_3	0	1	1	0
x_4	0	1	0	1

Fig 3.11 (i)

and

	y	y_1	y_2	y_3
x	1	1	0	0
x_1	1	1	1	1
x_3	0	1	1	0
x_2	0	0	1	1
x_4	0	0	0	1

Fig 3.11 (ii)

Naturally, the question arises : is it possible that by adding a row / column to the rearranged matrices of the above Figures we will get a matrix which forbids its F_2-representation? And we have to address this important question every time, when ever we come across a matrix having F_2-characteristics.

We answer this question through a very long and exhaustive searching process, where we will show that

(a) in the case of Fig 3.11(ii) these new additions (to forbid its being an F_2-matrix) will always lead us to either a 6-cycle or an ATE, where as

(b) in the case of Fig. 3.11(i), these attempts yield in addition to a 6-cycle or an ATE, the only one forbidden graph H_o (Fig 3.5).

In the present chapter we will prove our point for graph of Fig. 3.11(i) through a detailed study. The proof for the graph of 3.11(ii) is of similar nature and so will be omitted.

We note that the bigraph H_o is a singularly important bigraph in the present chapter, in the sense that for the case when H contains a strong bisimplicial edge it will prove to be the only smallest bichordal ATE free bigraph with Ferrers dimension >2.

Because of this importance, we first take into account the particular means of adding suitable rows and columns to Fig 11(i) (to forbid F_2-matrix) that yields the graph H_o.

To the matrix of Fig. 3.11 (i), there are several alternatives for adding rows. Among these, we consider the particular row (name it x_5)

	y	y_1	y_2	y_3
x_5	0	0	0	1

Then the new matrix gets the followings F_2-representation.

	y	y_1	y_2	y_3
x	1	1	0	0
x_1	1	1	1	1
x_2	0	1	1	1
x_3	0	1	1	0
x_4	0	1	0	1
x_5	0	0	0	1

Fig 3.1 2

To this, we add two new columns, say y_4 and y_5 to get the matrix

	y	y_1	y_2	y_3	y_4	y_5
x	1	1	0	0	0	0
x_1	1	1	1	1	1	0
x_2	0	1	1	1	-	-
x_3	0	1	1	0	0	0
x_4	0	1	0	1	-	-
x_5	0	0	0	1	-	-

Fig 3.13

Now it is a matter of verification that when y_4 and y_5 column of the above matrix have the structure as in Fig 3.14 then we get the matrix which is the biadjacency matrix of the crucially important bigraph H_o. For the other structures of y_4 and y_5 columns, when Fig. 3.13 contain the submatrix

$$\begin{array}{c|cc} & y_4 & y_5 \\ \hline x_i & 0 & 1 \\ x_j & 1 & - \end{array}$$

where x_i, x_j are any two among the x_2, x_3, x_4 rows, then it can be checked that the graph H must contain either an ATE or a 6-cycle.

	y	y_1	y_2	y_3	y_4	y_5
x	1	1	0	0	0	0
x_1	1	1	1	1	1	0
x_2	0	1	1	1	0	1
x_3	0	1	1	0	0	0
x_4	0	1	0	1	1	0
x_5	0	0	0	1	0	0

Fig 3.14 biadjacency matrix of H_0

Now we come to the detailed and arduous task of considering all possible ways of adding different rows and columns to Fig 3.11(i) for forbidding F_2-characteristics.

We continue from Fig 3.12 obtained from Fig 11(i) by adding the row x_5

$$\begin{array}{c|cccc} & y & y_1 & y_2 & y_3 \\ \hline x_5 & 0 & 0 & 0 & 1 \end{array}$$

We bypass the question of adding other possible rows to it (which will be taken care of later), and consider adding columns to Fig. 3.12.

To this matrix if we add a new column, say, y_i with $x_1y_i = 0$ and $x_3y_i=1$, it can be seen that $\{xy, x_3y_i, x_5y_2\}$ is an ATE or $x_1y_2\,x_3y_i\,x_5y_3x_1$ is a 6-cycle in H according as $x_5y_i = 0$ or 1. So in our search for new graphs other than a 6-cycle or an ATE, we are to consider the following possibilities :

(A) $\begin{array}{c|c} & y_i \\ \hline x & 0 \\ x_1 & 1 \\ x_3 & 0 \end{array}$ (B) $\begin{array}{c|c} & y_i \\ \hline x & 0 \\ x_1 & 0 \\ x_3 & 0 \end{array}$ (C) $\begin{array}{c|c} & y_i \\ \hline x & 0 \\ x_1 & 1 \\ x_3 & 1 \end{array}$

Note that for each of the cases x_2y_i, x_4y_i and x_5y_i may take either of values *0* or *1* and so we have to consider 2^3 possibilities for each of the cases.

A little deliberation will reveal that we have actually reached the graph H_o from Fig. 3.12 by adding the two columns y_4 and y_5 having configurations of cases (A) and (B) respectively with particular values to the other positions.

Now for the detailed study, we first add a column y_6 having the configurations of case (C) to get the following matrix.

	y	y_1	y_2	y_6	y_3
x	1	1	0	0	0
x_1	1	1	1	1	1
x_3	0	1	1	1	0
x_2	0	1	1	-	1
x_4	0	1	0	-	1
x_5	0	0	0	-	1

Fig 3.15

In order that the graph of the above matrix retains its bichordality we observe that if any '-' position in the y_6 columns is *0* then all positions below it are also *0* (since otherwise, the graph has a C_6 as an induced subgraph).

It is very important to note here that for an exhaustive study of the matrix obtained by adding all possible combinations of columns to Fig 3.12, it is imperative that we should consider possibilities, when two or more of such new columns have the same conFigurations from either of the cases (A), (B) and (C).

Below we add two columns y_j and y_k to Fig. 3.15 where any column (or both the columns) has (have) the configurations of either (A) or (B), and get the matrix.

	y	y_1	y_2	y_6	y_3	y_i	y_j
x	1	1	0	0	0	0	0
x_1	1	1	1	1	1	-	-
x_3	0	1	1	1	0	0	0
x_2	0	1	1	-	1	-	-
x_4	0	1	0	-	1	-	-
x_5	0	0	0	-	1	-	-

Fig 3.16

Now we can easily verify (as in the matrices of Fig 3.13 and Fig 3.15) that when '-' positions in the above matrix takes either the values *0* or *1*, then either it is a F_2-matrxix or other wise it contains a 6-cycle or an ATE or the graph H_o as an induced subgraph.

With the above deliberations we have addressed the problem of additions of extra columns to Fig 3.15 completely. Now we come back to the question of adding new possible rows to Fig 3.15 (which was left earlier when we added only the x_5 row to Fig 3.11 (i)). It can be seen that the following possible rows remain to be added to Fig. 3.15.

	y	y_1	y_2	y_6	y_3
1. x_i	0	1	0	0	0
2. x_j	0	1	1	0	0
3. x_k	0	1	0	1	0
4. x_l	0	0	0	1	0
5. x_m	0	0	0	0	0
6. x_n	0	0	1	-	-

Note that in the last case we have an ATE $\{xy, x_5y_3, x_ny_2\}$ or a 6-cycle $x_3y_1x_4y_3x_ny_2x_3$ in the graph *H* according as x_ny_3 is *0* or *1*. So we omit this row from our consideration.

We already observed that when x_5y_6 position is 1, then x_2y_6 and x_4y_6 positions are also 1. So if x_5y_6 position is 0 and we add x_l row to the Fig 3.15, then the positions x_2y_6, x_4y_6 are 1. (since otherwise we have an ATE $\{x_5y_2, x_ly_6, xy\}$in the graph). Thus we distinguish two subcases :

i) when we add the above mentioned rows to the Fig 3.15 together with x_l row.

ii) When we add above rows to the Fig 3.15 but not the x_l row.

In the case (i) we have the matrix of Fig 3.17 (i) (or its submatrix), where as in the case (ii) we have the matrix of Fig 3.17 (ii) (or its submatrix).

Note that matrix of Fig 3.17(i) is a F_2-matrix. Also the matrix of Fig. 3.17(ii) is a F_2-matrix when its bigraph is free from ATE and 6-cycle.

At this point we make an important observation. To the matrix of Fig 3.11(i) if we add a row other than x_5 and then as before add the column y_6 and all other possible rows then we will arrive at the same matrices as Fig. 17(i) and 17(ii).

	y	y_1	y_3	y_6	y_2
x	1	1	0	0	0
x_l	1	1	1	1	1
x_2	0	1	1	1	1
x_3	0	1	1	1	0
x_j	0	1	1	0	0
x_k	0	1	0	1	0
x_4	0	1	0	1	1
x_i	0	1	0	0	0
x_l	0	0	0	1	0
x_5	0	0	0	-	1
x_m	0	0	0	0	0

Fig 3.17 (i)

	y	y_1	y_6	y_3	y_2
x	1	1	0	0	0
x_l	1	1	1	1	1
x_3	0	1	1	1	0
x_k	0	1	1	0	0
x_2	0	1	-	1	1
x_j	0	1	0	1	0
x_4	0	1	-	0	1
x_i	0	1	0	0	0
x_5	0	0	-	0	1
x_m	0	0	0	0	0

Fig 3.17 (ii)

It is clear that Fig 17(i) and 17(ii) exhausts all possibilities of adding rows to Fig. 15. So we are in the final stage, when we will address the problem of adding all possible columns to Fig 17(i) and 17(ii), wherefrom the final solution to the problem will emerge.

Our purpose is now to add various possible columns which will forbid its F_2-characteristics and arrive at possible new graphs other then a 6-cycle, an ATE or H_o.

Note that for such forbidding, a matrix obtained by adding new possible columns to Fig. 17(i) or 17(ii) must contain a submatrix

$$\begin{array}{c|cc} & y' & y'' \\ \hline x' & 0 & 1 \\ x'' & 1 & - \end{array}$$

where one at least of the two columns y' and y'' is a new one.

We observe that if x' and x'' rows are taken from the rows x_2, x_3 , x_4 and x_5 and all the other positions in the y' and y'' column are 0, we arrive at a situation similar to Fig. 3.16. So by similar arguments as therein, we conclude that either the graph is of Ferrers dimension 2 or otherwise it contains a 6-cycle, or an ATE or H_o. Thus we are left to the case when every added column has a 1 in either of the x_i, x_j, x_k, x_l, x_m-rows. We divide all the columns to be added to Fig 3.17(i) (or 3.17(ii)) into two types :

Type I. In an added column there is a1 in at least one of the three rows x_i, x_j and x_k. Denote the corresponding columns by y_i, y_j and y_k respectively. This means that x_iy_i position in the y_i column is 1 and so are x_jy_j and x_ky_k. Note that other position of these columns may have any value. So any two or all the three columns y_i, y_j and y_k may in some case become identical and the proof will follow the same course in such case.

Type II. All the positions (of the added columns) in x_i, x_j and x_k row are 0.

First we consider the case, when all the added columns are of Type–I. In this case we will prove the assertion that either $f(H) = 2$ or the bigraph H contains a 6-cycle or an ATE.

As mentioned earlier we consider the added columns in the matrix of Fig. 3.17(i). The other case for Fig. 3.17(ii) is similar and so will be omitted. Here the matrix is

	y	y_1	y_2	y_6	y_3	y_i	y_j	y_k
x	1	1	0	0	0	0	0	0
x_1	1	1	1	1	1	-	-	-
x_3	0	1	1	1	0	-	-	-
x_2	0	1	1	1	1	-	-	-
x_j	0	1	1	0	0	-	1	-
x_k	0	1	0	1	0	-	-	1
x_4	0	1	0	1	1	-	-	-
x_i	0	1	0	0	0	1	-	-
x_l	0	0	0	1	0	-	-	-
x_5	0	0	0	-	1	-	-	-
x_m	0	0	0	0	0	-	-	-

In the above matrix we first observe that all the positions x_1y_i, x_1y_j, x_1y_k must be 1. For example, if $x_1y_j = 0$ then we have a 6-cycle $x_1y_1x_jy_jx_5y_3x_1$ or an ATE $\{xy, x_5y_3, x_jy_j\}$ in the graph H according as $x_5y_j = 1$ or 0.

Now we permute the rows and columns of the above matrix and obtain the following

	y	y_1	y_i	y_j	y_2	y_6	y_k	y_3
x	1	1	0	0	0	0	0	0
x_1	1	1	1	1	1	1	1	1
x_i	0	1	1	-	0	0	-	0
x_j	0	1	-	1	1	0	-	0
x_2	0	1	-	-	1	1	-	1
x_3	0	1	-	-	1	1	-	0
x_k	0	1	-	-	0	1	1	0
x_4	0	1	-	-	0	1	-	1
x_l	0	0	-	-	0	1	-	0
x_5	0	0	-	-	0	-	-	1
x_m	0	0	-	-	0	0	-	0

Fig. 3.18

Consider a submatrix of Fig. 3.18, when only one column y_i (of Type I) is added to Fig. 3.17 (i). Note that if the graph of the submatrix is to be of Ferrers dimension >2, there must be a *1* below a *0* in the y_i column (and the positions of the two rows can not be interchanged). In this case we can verify the existence of a 6-cycle or an ATE. As an

if $x_j y_i = 1$, so $x_j y_i = 0$. Now we have a C_6 $x_i y_j x_3 y_6 x_l y_i x_i$

example, let $x_3 y_i = 0$ and $x_k y_i = 1$. In this case we have a 6-cycle $x_j y_i x_k y_6 x_3 y_2 x_j$ or an ATE $\{x_i y_i, x_j y_2, x_l y_6\}$ according as $x_l y_i$ is ... *1* or 0.

Similar proof will follow, when instead of y_i, only one column y_j or y_k is added to Fig 3.17(i) to form a submatrix of Fig. 3.18.

Now we consider the case when we require adding not one column, but two columns, say y_i and y_j to Fig. 3.17(i) making the bigraph of Ferrers dimension >2. In this case the matrix must contain the forbidden submatrix.

$$\begin{array}{c|cc} & y_i & y_j \\ \hline x' & 0 & 1 \\ x'' & 1 & 0 \end{array}$$

In this case also, some deliberation will reveal the existence of a 6-cycle or an ATE involving both the vertices y_i and y_j in it. We consider an example. Suppose $x''=x_5$ and $x'=x_l$ Now in this submatrix (of Fig 3.18) if $x_5 y_6=0$ then we observe that $\{xy, x_5 y_3, x_l y_6)$ is an ATE of the bigraph. So $x_5 y_6$ position must be *1*. Next $x_i y_j$ position must be *0*, since otherwise we have a 6-cycle $x_i y_i x_l y_6 x_5 y_j x_i$ in the bigraph. Now we verify that $x_j y_i x_l y_6 x_5 y_j x_j$ is a 6-cycle or $\{xy, x_5 y_l, x_l y_j\}$ is an ATE of the bigraph according as $x_j y_i$ is a *1* or *0* respectively.

Next we consider the matrix of Fig. 3.18. If its bigraph H is of $f(H) >2$ then the cases just stated above will arise again and similarly we can verify that H must contain either an ATE or a 6-cycle.

Now we consider the case when addition of columns of type II only to Fig 3.17(i) or 3.17 (ii) leads us to a bigraph of Ferrers dimension >2. It is a matter of routine deliberation to verity our assertion in this case also.

The same assertion can be verified for the case when we have to add columns of both the type I and II. It will be very interesting to observe in this context, that for this bigraph to be of Ferrers dimension >2, the columns of type I has no role to play and as a consequence the ATE or 6-cycle present in the graph can be seen to be independent of the vertices y_i, y_j and y_k (columns of Type I).

One last case still remains to be considered. Note that during the process of expansion of Fig 17(i) or 17(ii), by adding new columns to them, two or more rows of the same form in Fig 17(i) or 17(ii) may be repeated with '-' s in the corresponding position of the added columns. Also two or more columns of the same form may get a repetition, with the result that the new bigraph turns out to be of Ferrers dimension >2. We just mention here that the exhaustive search in this case also will lead us to the same old story and no new graph. ∎

Combining proposition 3.3 and theorem 3.4 we sum up the main result of this chapter in the following form :

Theorem 3.5. *A bi(di) graph H of Ferrers dimension ≤ 2 is bichordal and ATE free. On the other hand, in case when a bichordal and ATE free bigraph contains a strong bisimplicial edge them the graph H_0 of Fig 3.5 is the only forbidden subgraph for a bigraph of Ferrers dimension at most 2.*

CHAPTER 4*

DIGRAPHS REPRESENTED BY INTERVALS HAVING BASE POINTS

4.1 Introduction

We recall from chapter 1, that intersection, overlap and containment model for digraphs were introduced and characterized by Sen *et al.* In [Sen *et al.*, 1989a] it was shown that a digraph D is an interval-point digraph iff its adjacency matrix $A(D)$ has consecutive ones property for rows. Again Sen and Sanyal [1994] has shown that a digraph is an indifference digraph iff it's $A(D)$ has monotone consecutive arrangement property. By this we mean that there exists independent row and column permutations exhibiting the following structure of $A(D)$: the 0's of the resulting matrix can be labeled R or C such that every position above or to the right of an R is an R, and every position below or to the left of a C is a C.

Since an indifference digraph has also consecutive one's property for rows, a question immediately arises as to under what conditions an interval-point digraph reduces to an indifference digraph. In section 4.2. We answer this question and show that the class of indifference digraphs is the same as the class of interval-point digraphs, where the source intervals are of unit length.

The notion of 'base interval' was introduced by Sanyal [1994]. If S_v is a closed interval and p_v is a point of S_v, then the ordered pair (S_v, p_v) is called a *base interval.* Replacing a pair of intervals by a pair of base intervals $\{(S_v, p_v), (T_v, q_u) : v \in V\}$, a digraph was obtained in the following manner : $uv \in E$ *iff*

(i) S_u and T_v overlap (no containment),

(ii) $\inf S_u < \inf T_v$ and

(iii) $p_u, q_v \in S_u \cap T_v$. Such a digraph was termed *right overlap base interval (Robin)* digraph and was characterized by Sanyal interms of its adjacency matrix as follows : A

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digraph is a *Robin digraph* iff its adjacency matrix has a 4-directable property (defined earlier in chapter 1, fig. 1.7). Replacing the condition of overlapping of two intervals S_u and T_v in Robin digraph, by the conditions of intersection, Sanyal introduced the idea of base interval digraphs. Also the problem of characterization of these digraphs was initiated by him. In the section 4.3 we characterize these digraphs in terms of their adjacency matrices. As a particular case of 4-directable matrix we consider a binary matrix where 0's have a partition into two classes, say X and Y; a binary matrix will be said to have an *X-Y partition* if its rows and columns can be labeled either X or Y such that (i) the positions to the right or the position above any X are 0's labeled X, (ii) the positions to the left or positions below any Y are also zeros labeled Y, and moreover (iii) if any column contains both X and Y which have all X's and Y's to their right and left respectively then the row corresponding to Y must occur below the row corresponding to X. Similarly if any row contains both X and Y which have all X's and Y's to the above and below respectively then the column corresponding to Y must occur to the left of the column corresponding to X (fig. 4.1). It may be noted that this definition is a modified form of the definition of X-Y partition given by Sanyal [1994].

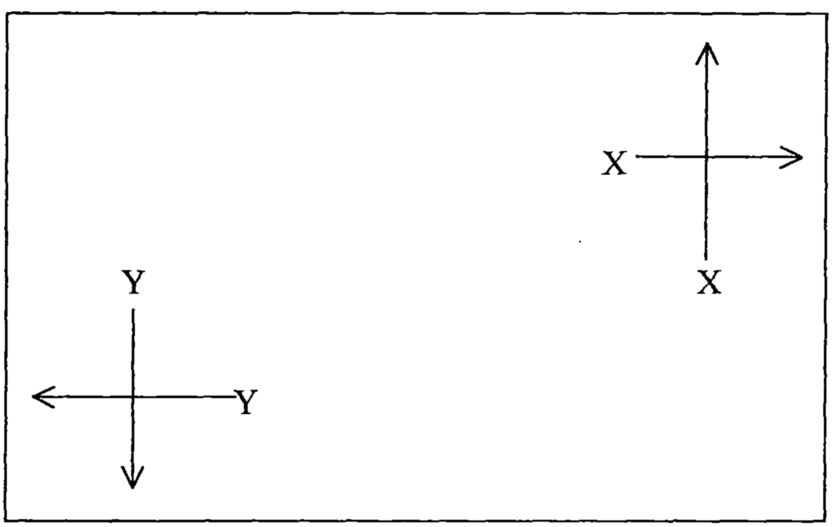


Fig 4.1 X-Y Partition

In section 4.3 we will show that these binary matrices characterize the adjacency matrices of base interval digraphs.

Now we draw our attention to an X-Y partitioned binary matrix. If we form a binary matrix $A(D_1)$ consist only of those 0's which correspond to the X's occurring consecutively in a row and the Y's occurring consecutively in a column, we note that since $X \cap Y = \varphi$, this new matrix is actually the adjacency matrix of an interval digraph. Similarly considering the X's occurring in a column and the Y's occurring in a row we get another interval digraph. Thus we see that a base interval digraph is practically the intersection of two interval digraphs. In the same section we present detailed proof of the characterization. In the same section we also introduce the concept of base interval in an undirected graph and obtain an analogous characterization of what has been termed a base interval graph.

In section 4.4, we deal with an overlap base interval digraph and show how closely it is related to the notion of an interval containment digraph. The interval containment digraph $D(V, E)$ of a family $\mathfrak{I} = \{(S_v, T_v) : v \in V\}$ of an ordered pairs of intervals is the digraph with vertex set V in which $uv \in E$ iff S_u contains T_v. These digraphs were characterized by Sen, Sanyal and West [1995] and it was shown that this class is equivalent to the class of digraphs of Ferrers dimension 2. Also it was proved by Sanyal [1994] that a digraph is a Robin digraph iff its adjacency matrix is 4-directable. Noting that this digraph is of Ferrers dimension 4 and that a digraph of Ferrers dimension 2 is nothing but an interval containment digraph, it immediately follows that a Robin digraph must be the intersection of two interval containment digraphs. Here we probe into the necessary and sufficient condition for a Robin digraph to be the intersection of two particular interval containment digraphs.

In the last section we deal with Robin digraph where the intervals are of unit length. For this we first observe that an overlap digraph though of Ferrers dimension 3 is such that its adjacency matrix is 4-directable. So, it should be expressed in terms of a Robin digraph. Probing this question we indeed get the result that an overlap digraph is actually a Robin digraph with unit length intervals.

In this chapter sometime we use the symbol $u \rightarrow v$ to mean that uv is an edge of the digraph $D(V, E)$.

4.2 Interval-point digraph and indifference digraph

An interval-point digraph is an interval digraph where the sink interval reduces to a point. In other word every vertex v is assigned a pair (S_v, p_v) where S_v is an interval and p_v is a point and $uv \in E$ iff $p_v \in S_v$. Adjacency matrix of the interval-point digraphs were characterized by Sen *et al* [1989a] in the following way : adjacency matrix of the interval point digraph has consecutive ones property for rows and conversely. Since adjacency matrix of an indifference digraph has also consecutive ones property for rows, the immediate question that arises is under what condition an interval point digraph reduces to an indifference digraph.

For this, we first observe that if the source intervals $[a_i, b_i]$ corresponding to the vertex v_i are such that

$$a_i \leq a_{i+1}, \qquad bi \leq b_{i+1}, \qquad i = 1, 2, \ldots, n-1$$

then arranging the columns in increasing order of c_i's [the terminal points corresponding to v_i's] and the rows in the increasing order of a_i's (or b_i's) the adjacency matrix $A(D)$ of D exhibits an MCA and interval-point digraph becomes an indifference digraph.

That the converse of the above statement is also true, as follows from the proof of Theorem 2 of [Sen *et al,* 1989a]. As a matter of fact repeating the arguments as it is there, the assignments

$$a(v_k) = \min \{ i, v_i v_k \in E\}$$

and $$b(v_k) = \max \{i, v_i u_k \in E\}$$

and the condition of MCA guarantees that the indifference digraph is an interval-point digraph with the above property. Hence we have the following proposition :

Proposition 4.1 *An interval point digraph $D(V, E)$ where v_i corresponds to the pair $([a_i, b_i], c_i)$ is an indifference digraph iff the intervals are such that*

$$a_i \leq a_{i+1}, \qquad bi \leq b_{i+i}, \qquad i = 1, 2, \ldots, n-1$$

We define a *unit interval-point* digraph as one where all the source intervals are of unit length and a *proper interval-point* digraph is one there a source interval does not contain properly another source interval.

Now we use the above proposition to prove the following :

Theorem 4.1 *For a digraph D(V, E) the following conditions are equivalent :*

1) *D is an indifference digraph;*

2) *D is a unit interval-point digraph ;*

3) *D is a proper interval-point digraph.*

Proof. 1) $\Rightarrow$ 2) Let $\{f(v), g(v)) : v \in V\}$ be an indifference representation of a digraph $D(V, E)$. Corresponding to the vertex v of D, construct a unit interval-point representation $(S_v, g(v)/2)$ where

$$S_v = \left[\frac{f(v)}{2} - \frac{1}{2}, \frac{f(v)}{2} + \frac{1}{2}\right]$$

Now $uv \in E \Leftrightarrow |f(u) - g(v)| \leq 1 \Leftrightarrow \left|\frac{f(u)}{2} - \frac{g(v)}{2}\right| \leq \frac{1}{2} \Leftrightarrow \frac{g(v)}{2} \in S_u \Leftrightarrow uv$ is an edge of

of the unit interval-point digraph.

2) $\Rightarrow$ 3) follows obviously.

3) $\Rightarrow$ 1) follows from the proposition 1.

As a matter of fact we note that if $\{([a_v, b_v], c_v) : v \in V\}$ is a unit interval-point representation of $D(V, E)$ then $f(v) = a_v + b_v$, $g(v) = 2c_v$ will be an indifference representation of $D(V,E)$.

4.3 Base interval digraph

A digraph $D(V, E)$ is a base interval digraph if its vertex set V has one-to-one correspondence with a family of ordered pairs of base intervals $\{S_u, p_v), (T_v, q_v) : v \in V\}$, $p_v \in S_v$, $q_v \in T_v$ and $u \to v$ if and only if $p_u, q_v \in S_u \cap T_v (\neq \varphi)$. The following theorem characterizes the adjacency matrix of a base interval digraph.

Theorem 4.2 *A necessary and sufficient condition that a digraph is a base interval digraph is that its adjacency matrix has an X-Y partition.*

Proof (necessary) Let $\{(S_v, p_v), (T_v, q_v) : v \in V\}$ be a base interval representation of a digraph $D(V, E)$ where $S_v = [a_v, b_v]$, $p_v \in S_v$ and $T_v = [C_v, d_v]$, $q_v \in T_v$; then we have $u \to v$ if and only if $(p_v, q_v) \in S_u \cap T_v$. So $uv \notin E$ if and only if one of the four inequalities (i) $b_u < q_v$, (ii) $p_u < c_v$, (iii) $q_v < a_u$, and (iv) $d_v < p_u$ holds.

From section 4.1 it is clear that D is of Ferrers dimension 4; that is D is the union of four Ferrers digraphs. These four Ferrers digraphs are obtained from the pairs (u, v) satisfying the above four inequalities. Arranging the rows of the adjacency matrix in increasing order of the values of p_u and its column in the increasing order of the values of q_v we see that the matrix exhibits 4-directable property.

Denote a 0 in the adjacency matrix by X if it corresponds to a position uv which satisfies the inequalities (i) or (ii), and otherwise by Y. It is to be noted that a position uv may satisfy both the inequalities (i) and (ii); that is, they are not mutually exclusive. Similarly for (iii) and (iv). Now it is a matter of verification that the sets X and Y satisfy the conditions of the X-Y partition and that $X \cap Y = \varphi$. This gives the required X-Y partition of the adjacency matrix.

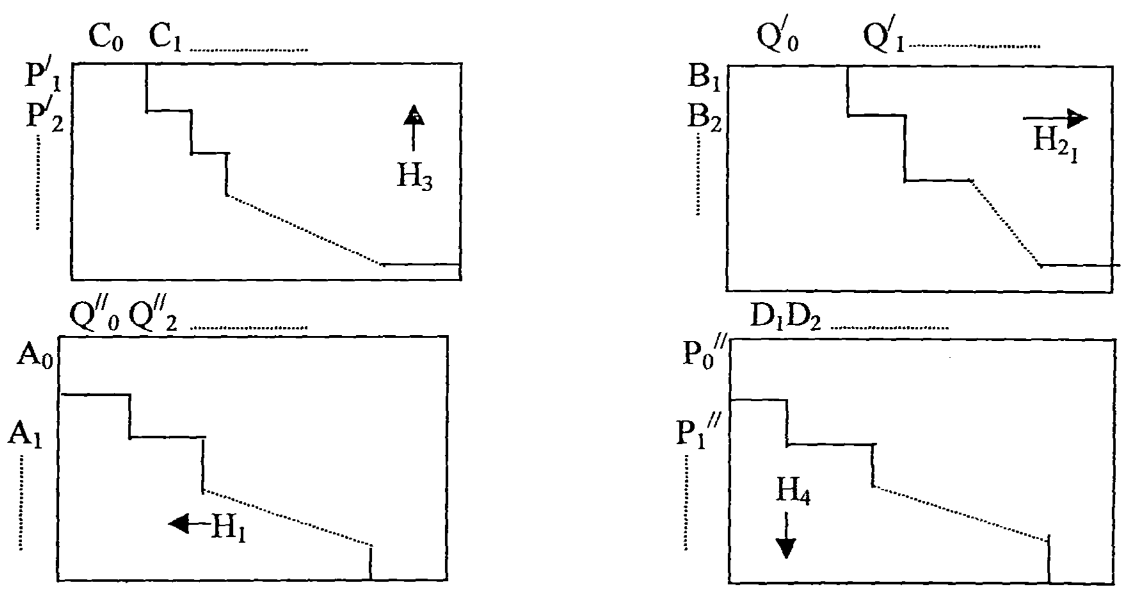


Fig. 4.2 decomposition of a matrix having X-Y partition.

For sufficiency, consider a permutation of the rows and columns of the adjacency matrix $A(D)$ that exhibits an X-Y partition. As observed earlier, D is the union of four Ferrers digraphs, which we view as sets of positions in the adjacency matrix given by

i) H_1 consisting of the Y's in the $A(D)$ that have only Y's to their left.

ii) H_2 consisting of the X's in $A(D)$ that have only X's to their right.

iii) H_3 consisting of the X's in $A(D)$ that have only X's above them.

iv) H_4 consisting of the Y's in $A(D)$ that have only Y's below them.

Now we want to construct base intervals (S_v, p_v) and (T_v, q_v) where $p_v \in S_v = [a_v, b_v]$ and $q_v \in T_v = [c_v, d_v]$ for all $v \in V$, such that uv is outside of H_i, $i = 1, 2, 3, 4$ if and only if $a_u < q_v < b_u$ and $c_v < p_u < d_v$ which gives the base interval representation of the digraph D. The values for the end points of the intervals and the base points within them will come from a topological ordering of an auxiliary acyclic digraph. We shall use H_1, H_2, H_3, H_4 to define eight partitions of V. Because the successor sets of a Ferrers digraph is ordered by inclusion, we can define a natural partition of the rows of the adjacency matrix with two rows in the same block if and only if the successor sets of the two corresponding vertices are identical. Furthermore, the blocks of the partition are indexed naturally by the inclusion ordering on the successor sets. The same is true of the predecessor sets and the columns of the adjacency matrix.

For H_1 we can permute the rows to achieve Ferrers diagram in the lower left. Also for H_2, we can permute the rows to achieve this in the upper right. So the natural terminal partitions of H_1 and H_2 have the columns in the same order. Similarly for H_3 and H_4 we can permute the columns to active Ferrers diagram in the upper right and lower left respectively. Here again the natural source partitions of H_3 and H_4 have rows in the same order. This is illustrated in the figure 4.2, where we have given the names to the blocks of the partitions.

Let $A = \{A_i\}$, $B = \{B_i\}$, $C = \{C_i\}$ and $D = \{D_i\}$. Since the rows of H_3, H_4 and columns of H_1, H_2 are in the same order, we can define additional partitions P_0, P_1, ..., P_s and Q_0, Q_1,

…., Q_t that maintain the order of the rows, where each block P_i is the intersection of one P'_j and one P''_k, and each Q_i is the intersection of one Q'_j and one Q''_l. In other words, the partition $P = \{P_i\}$ is the common refinement of $\{P'_j\}$ and $\{P''_k\}$ with fewest blocks, indexed by the shared order on the rows, and similarly for $Q = \{Q_i\}$. Note that the indexing of the various types of P's agrees with the row order for H_3 and H_4, and the indexing for the Q's agrees with the column order for H_1 and H_2.

We construct an auxiliary digraph $Z=Z_1 \cup Z_2$ with vertices $A \cup B \cup C \cup D \cup P \cup Q$, which we call nodes to distinguish them from the vertices of the original digraph. We will assign distinct integers to these nodes via a map f Each $v \in A_i$ will receive $f(A_i)$ as the value of of a_v; similarly b_v, c_v, d_v, p_v and q_v are set from the values of f on B, C, D, P and Q respectively. We put an edge in Z from one node to another when we want the number assigned to the first node to be less than the number assigned to the second, and then f will be chosen to increase along every edge. Since we want the p-values and q-values to be increasing in rows and columns in accordance with the discussion of the X-Y partition above, we put $Q_i \to Q_j$ in Z if $i<j$, and similarly $P_i \to P_j$ if $i<j$.

First we construct the digraph Z_1 in the following way :

If $u \in A_i$ and $v \in Q''_j$ with $i \geq j$ then $uv \notin E(D)$ $(uv \in H_1)$ and we want $q_v < a_u$; on the other hand if $i<j$ then possibly $u \to v$ and we need to allow this by $q_v > a_u$. Hence for the pair Ai, Q_l with $Q_l \subseteq Q''_j$ we put $A_i \to Q_l$ if $i<j$ but $Q_l \to A_i$ if $i \geq j$. This defines a linear ordering on $A \cup Q$. Similarly for the pair B_i, Q_l with $Q_l \subseteq Q'_j$ we put $B_i \to Q_l$ if $i \leq j$ but $Q_l \to B_i$ if $i > j$. This again defines a linear ordering on $B \cup Q$.

But the interaction between this two ordering, we first observe that in the ordering $A \cup Q$ (or $B \cup Q$) between two Q node we have at most one A (or B) node. Combining these two ordering $A \cup Q$ and $B \cup Q$ we form a linear ordering on $A \cup B \cup Q$ such that between two Q node if there exist two node A_i and B_j with $A_i \cap B_j \neq \varphi$ then we place B_j succeeding A_i, but if $A_i \cap B_j = \varphi$ then we place them in any order.

Now f-values of the nodes are increasing from left to right in the linear ordering $A \cup B \cup Q$; so if B_k precede A_i we must have $A_i \cap B_k = \varphi$, (since a_v's and b_v's are end points of real interval). Thus we must require $f(A_i) < f(B_k)$ if $v \in A_i \cap B_k$. To check this let in the linear ordering $A \cup B \cup Q$, B_k precede A_i and $u \in A_i \cap B_k$. Also let Q_j be a node between A_i and B_k and $v \in Q_j$. Now $B_k \to Q_j$ with $u \in B_k$ and $v \in Q_j$ implies $uv \in H_2$ i.e. uv is an X. Again $Q_j \to A_i$ with $u \in A_i$ and $v \in Q_j$ implies $uv \in H_1$ that is uv is an Y; so $X \cap Y \neq \varphi$ which is impossible.

Similarly we construct the linear ordering Z_2 with notes $C \cup D \cup P$.

For the interaction between Z_1 and Z_2 so as to obtain $Z = Z_1 \cup Z_2$, we note that the value of base point of a source interval must lies within it, so we must require $f(A_i) \leq f(P_l) \leq f(B_j)$ if there is a vertex $v \in A_i \cap P_l \cap B_j$. We represent this by placing edges from A_i to P_l and P_l to B_j. Similarly for base point of terminal interval we require $f(C_i) \leq f(Q_k) \leq f(D_j)$ if $v \in C_i \cap Q_k \cap D_j$. And we represent this by placing edges from C_i to Q_k and Q_k to D_j.

Our problem now is to show that $Z = Z_1 \cup Z_2$ is acyclic. If it be so, consider a numbering $f: V(Z) \to R$ such that $XY \in E(Z)$ implies $f(X) < f(Y)$. Then using the values of f to determine a_v, b_v, c_v, d_v, p_v, q_v as described above, we have created base intervals (S_v, p_v) and (T_v, q_v) where $S_v = [a_v, b_v]$ $T_v = [c_v, d_v]$, $a_v < p_v < b_v$ and $c_v < q_v < d_v$ such that $uv \in E(D)$ if and only if $a_u < q_v < b_u$ and $c_v < p_u < d_v$.

Here we shall show that the auxiliary digraph Z has no cycle. We note that in each of the matrix $H_i (i = 1, 2, 3, 4)$, the indices of the blocks as we go down the row or go to the right along the column are in increasing order. Also in each ordering, the indices on a particular type of node appear in increasing order. For example the block. P_i occures above the block P_j in the matrix H_3 (or H_4) if $i < j$; and a block C_i occures earlier than the block C_j in the matrix H_3 if $i < j$.

Now we claim that the directed graph Z with nodes A, B, C, D, P, Q has no cycle. Let if possible the digraph Z has a cycle. Below we consider the following possibilities in which a cycle may occur.

Case 1 Let $B_j\, A_i\, P_k\, P_m$ be a cycle (fig 4.3(i)).Then we must have $k<m$. If there is no node Q_r between B_j and A_i, then without loss of generality we can interchange the node B_j and A_i and get rid of the cycle (Fig. 4.3(ii)) . So let there be a node Q_r between B_j and A_i (Fig. 4.3(iii)). The edge $B_j \to Q_r$ implies that the edges uw

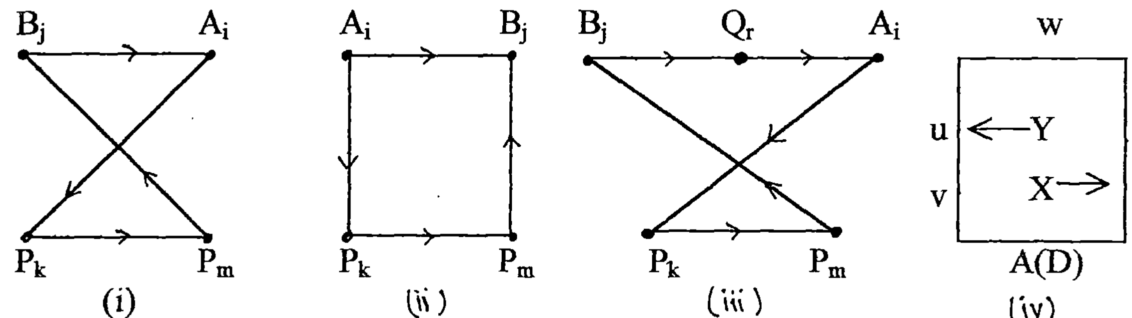


Fig 4.3

for which $u \in B_j$ and $w \in Q_r$ all belongs to H_2. This means that the position (u, w) in the matrix $A(D)$ is X and positions to its right are all X. Similarly $Q_r \to A_i$ implies that for $w \in Q_r$ and $v \in A_i$ the position vw in the matrix $A(D)$ is Y and the positions to the left in the row are all Y (see fig 4.3(iv)). Again $A_i \to P_k$ implies that $v \in P_k$ and $P_m \to B_j$ implies that $u \in P_m$, and since $k<m$, the u-row occur below the v-row which violates the condition (iii) of the X-Y partition.

Case 2. Let there be a cycle of the form B_i ....., $Q_j\, D_r$.... $P_s\, B_i$, (Fig. 4.4). Now $P_s \to B_i$ implies that $u \in P_s \cap B_i$ and $Q_j \to D_r$ implies $v \in Q_j \cap D_r$. Again $B_i \to Q_j$ with $u \in B_i$ and $v \in Q_j$ implies $uv \in H_2$ i.e. uv is an X. Similarly $D_r \to P_s$ with $u \in P_s$ and $v \in D_r$ implies $uv \in H_4$ i.e. uv is an Y. Therefore $uv \in X \cap Y$, which is impossible since $X \cap Y = \varphi$.

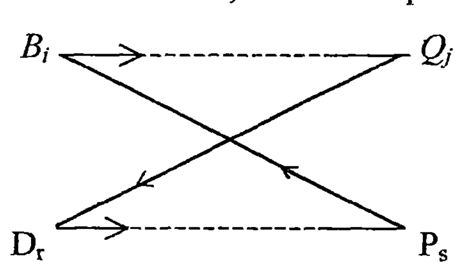


Fig. 4.4

Similarly we can arrive at a contradiction from the other cases. ∎

Example 4.1. Here we illustrate the above method of construction of base intervals from an *X-Y* partitionable binary matrix.

Consider the *X-Y* partitionable binary matrix M.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	1	1	1	0	1	0	0
v_2	1	1	1	0	0	0	0
v_3	0	0	1	1	0	0	0
v_4	0	0	1	0	1	0	0
v_5	0	0	0	0	1	1	0
v_6	0	1	1	0	1	0	0
v_7	0	0	0	0	1	1	0

Here the four Ferrers digraphs H_i $(i = 1,2,3,4)$ are as follows :

Q''

$H_1 =$ A

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1							
v_2							
v_6	0						
v_3	0	0					
v_4	0	0					
v_5	0	0	0	0			
v_7	0	0	0	0			

Q'

$H_2 =$ B

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_2				0	0	0	0
v_3					0	0	0
v_1						0	0
v_4						0	0
v_6						0	0
v_5							0
v_7							0

C

$H_3 =$ P'

	v_1	v_2	v_3	v_5	v_4	v_6	v_7
v_1					0	0	0
v_2					0	0	0
v_3						0	0
v_4						0	0
v_5							0
v_6							0
v_7							0

D

$H_4 =$ P''

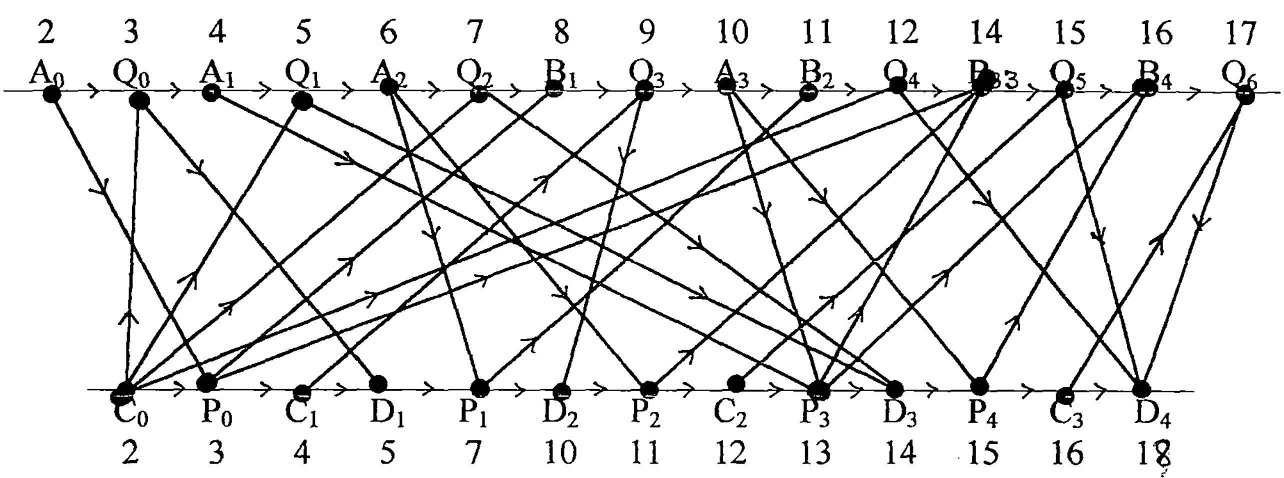
	v_1	v_4	v_2	v_3	v_5	v_6	v_7
v_1							
v_2							
v_3	0						
v_4	0	0					
v_5	0	0					
v_6	0	0					
v_7	0	0	0	0			

The vertex partitions from H_1 & H_2 are given by

i	0	1	2	3	4	5	6
A_I	v_1, v_2	v_6	v_3, v_4	v_5, v_7			
B_I		v_2	v_3	v_1, v_4, v_6	v_5, v_7		
Q'_i	v_1, v_2, v_3	v_4	v_5	v_6	v_7		
Q''_i		v_1	v_2	v_3, v_4	v_5, v_6, v_7		
$Q_i = Q'_i \cap Q''_i$	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Again the vertex partitions from H_3 & H_4 are given by

i	*0*	*1*	*2*	*3*	*4*
C_i	v_1, v_2, v_3, v_4	v_4	v_6	v_7	
D_i		$v_1,$	v_4	v_2, v_3	v_5, v_6, v_7
P'_i		v_1, v_2	v_3, v_4	v_5, v_6, v_7	
P_i''	v_1, v_2	v_3	v_4, v_5, v_6	v_7	
$P_i = P'_i \cap P''_i$	v_1, v_2	v_3	v_4	v_5, v_6	v_7



The auxiliary Digraph Z

Fig 4.5

Then the resulting topological ordering yields the following sequences

i	0	1	2	3	4	5	6
Ai	2	4	6	10			
Bi		8	11	14	16		
Ci	2	4	12	16			
Di		5	10	14	18		
Pi	3	7	11	13	15		
Qi	3	5	7	9	12	15	17

Now picking out *a(v), b(v), c(v), d(v), p(v)* and *q(v)* for each vertex v, we have the following base interval representation for M.

i	1	2	3	4	5	6	7
(S_{v_i}, p_{v_i})	[2, 14], 3	[2, 8], 3	[6, 11], 7	[6, 14], 11	[10, 16], 13	[4, 14], 13	[10, 16], 15
(T_{v_i}, q_{v_i})	[2, 5], 3	[2, 14], 5	[2, 14], 7	[4, 10], 9	[2, 18], 12	[12, 18], 15	[16, 18], 17

So for we have characterized the adjacency matrix of a base interval digraph in terms of an *X-Y* partition of the matrix. Below we take a look into the *X-Y* partition of the matrix again to obtain yet another characterization of a base interval digraph.

Let R_1 denote the zeros of *A(D)* where a zero has all positions zero to its right and let R_2 denote the zeros of *A(D)* where a zero has all positions zero to its left. Similarly let $C_1(C_2)$ denote the zeros where zero has all positions zero above (below) it. Note that $R_1 \cap C_1$ and $R_2 \cap C_2$ are not necessarily empty but $R_1 \cap C_2 = \varphi$ and $R_2 \cap C_1 = \varphi$ (Fig 4.6).

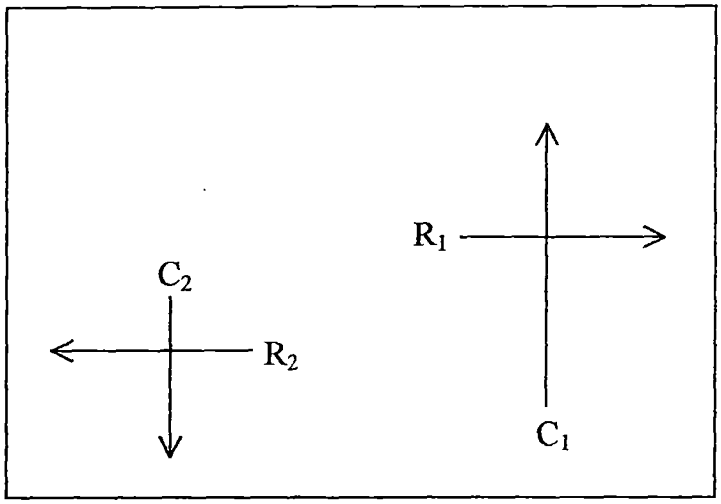


Fig. 4.6

Now construct a binary matrix from *A(D)* which has zeros corresponding to R_1 and R_2 only. This matrix has consecutive ones property for rows and accordingly the corresponding digraph is an interval-point digraph. Similarly construct a binary matrix from *A(D)* which has zeros corresponding to C_1 and C_2 only. This matrix has consecutive ones property for columns and so converse of this matrix is the adjacency matrix of an interval-point digraph. Accordingly *D* is the intersection of these two digraphs.

Again construct a binary matrix from *A(D)* which has zeros corresponding to R_1 and C_2 only. This matrix has partitionable zeros property and so the corresponding digraph is an interval digraph. Similarly construct a binary matrix from *A(D)* which has zeros corresponding to C_1 and R_2 only. Reversing the order of rows and columns of this matrix we observe that it has again partitionable zeros property and the corresponding digraph is an interval digraph. Thus *D* is the intersection of two interval digraphs. These observations motivate us to prove the following.

Theorem 4.3. *For a digraph D(V, E) the following conditions are equivalent :*

1) *D is a base interval digraph with base interval representation* $\{([a_v, b_v], p_v), ([c_v, d_v], q_v) : v \in V\}$.

2) $D = D_1 \cap D'_2$ *where* D_1 *is the interval-point digraph* $\{([a_v, b_v], q_v) : v \in V\}$ *and* D_2 *is the interval-point digraph* $\{ ([c_v, d_v], p_v) : v \in V\}$ *with* $p_v \in [a_v, b_v]$ *and* $q_v \in [c_v, d_v]$ *and* D'_2 *is the converse of* D_2.

3) $D = F_1 \cap F_2$ *where* F_1 *and* F_2 *are two interval digraphs, the pairs of intervals corresponding to a vertex v in* F_1 *and* F_2 *being of the form* $\{([a_v, p_v], [c_v, q_v]) : v \in V\}$ *and* $\{([p_v, b_v], [q_v, d_v]) : v \in V\}$ *respectively.*

Proof. 1) $\Rightarrow$ 2). Let $D(V, E)$ be a base interval digraph

Then $uv \in E \Rightarrow p_u, q_v \in S_u \cap T_v$ where $S_u = [a_u, b_u]$, $T_v = [c_v, d_v]$

$\Rightarrow p_u \in [c_v, d_v]$ and $q_v \in [a_u, b_u]$

$\Rightarrow uv \in D'_2$ and $uv \in D_1$

$\Rightarrow uv \in D_1 \cap D'_2$

where D_1 and D_2 are two interval-point digraphs $\{([a_v, b_v], q_v) : v \in V\}$ and $\{([c_v, d_v], p_v) : v \in V\}$ respectively.

Again when $uv \notin E$ then easily we see that $uv \notin D_1$ or $uv \notin D'_2$ or both i.e. $uv \notin D_1 \cap D'_2$. Thus if D has a base interval respective on then D is intersection of interval point digraph.

2) $\Rightarrow$ 3) Let $D = D_1 \cap D'_2$ where D_1 and D_2 are two interval-point digraphs with the given representation.

Consider two interval digraphs F_1 and F_2 where the pairs of intervals corresponding to the vertex v are $([a_v, p_v], [c_v, q_v])$ and $([p_v, b_v], [q_v, d_v])$ respectively for F_1 and F_2. Let $uv \in E$ and let $p_u < q_v$.

Then uv is an edge of D'_2

$\Rightarrow p_u \in [c_v, d_v]$

$\Rightarrow p_u \in [c_v, q_v]$

$\Rightarrow [a_u, p_u] \cap [c_v, q_v] \neq \varphi$

$\Rightarrow uv \in F_1$.

Again uv is an edge of D_1

$$\Rightarrow q_v \in [a_u, b_u]$$

$$\Rightarrow q_v \in [p_u, b_u]$$

$$\Rightarrow [p_u, b_u] \cap [q_v, d_v] \neq \varphi$$

$$\Rightarrow uv \in F_2$$

Thus $uv \in F_1 \cap F_2$

Similarly, when $p_u \geq q_v$ we can prove $uv \in F_1 \cap F_2$. When $uv \notin E$, then it is easy to see that either $uv \notin F_1$ or $uv \notin F_2$ or both; that is $uv \notin F_1 \cap F_2$.

So $\quad D = F_1 \cap F_2$..

3) $\Rightarrow$ 1) Let $D = F_1 \cap F_2$ where the pairs of interval in F_1 and F_2 corresponding to a vertex v are of the form $([a_v, p_v], [c_v, q_v])$ and $([p_v, b_v], [q_v, d_v])$ respectively. Corresponding to a vertex v of D consider a base interval representation $([a_v, b_v], p_v)$, $([c_v, d_v], q_v)$ where $p_v \in [a_v, b_v]$ and $q_v \in [c_v, d_v]$.

Now $uv \in F_1 \cap F_2 \Rightarrow uv \in F_1$ and $uv \in F_2$

$$\Rightarrow [a_u, p_u] \cap [c_v, q_v] \neq \varphi \text{ and } [p_u, b_u] \cap [q_v, d_v] \neq \varphi$$

$$\Rightarrow p_u \in [c_v, q_v] \text{ or } q_v \in [a_u, p_u] \text{ and } p_u \in [q_v, d_v] \text{ or } q_v \in [p_u, b_u]$$

First and last cases are simultaneously possible and second and third cases are simultaneously possible. So in either case $[a_u, b_u]$ and $[c_v, d_v]$ intersect and $p_u, q_v \in [a_u, b_u] \cap [c_v, d_v]$.

Also if $uv \notin D$ then either $uv \notin F_1$ or F_2 or both. Let $uv \notin F_1$ then with $p_u \notin [c_v, d_v]$ or $q_v \notin [a_u, b_u]$. So uv is not an edge of the base interval digraph. ■

Unit base interval digraphs are those base interval digraphs in which all the source base intervals and sink base intervals are of unit length. As a immediate consequence of

Theorem 4.3, we have the following corollary which shows that such digraphs are the intersection of two indifference digraphs.

Corollary 4.1. *$D(V, E)$ is a unit base interval digraph with unit base interval representation $\{(S_v, p_v), (T_v, q_v) : v \in V\}$ iff $D = D_1 \cap D'_2$ where D_1 is the interval-point digraph $\{(S_v, q_v) : v \in V)$ and D'_2 is the interval point digraph $\{(T_v, p_v) : v \in V\}$ with $p_v \in S_v$ and $q_v \in T_v$.*

4.3.1. Base interval graph

We introduce the notion of base interval graph for an undirected graph and then characterize its adjacency matrix.

Let $\mathfrak{I}=\{(S_v, p_v) : v \in V\}$ be a family of base intervals with $p_v \in S_v$. We say that a graph $G(V, E)$ is a base interval graph when $uv \in E \Leftrightarrow p_u \in S_v$ and $p_v \in S_u$.

If G is a base interval graph then the corresponding symmetric digraph with loops at each vertex is a base interval digraph, as can be seen by taking

$$(T_v, q_v) = (S_v, p_v), \ v \in V.$$

Consequently its adjacency matrix has an X-Y partition. While characterizing a base interval graph we will observe in the next theorem how the symmetry of the adjacency matrix fits in with the X-Y partition. In fact we will see that an (i, j) entry in the adjacency matrix of G is an X iff (j, i) entry is an Y.

Theorem 4.4 *The following statements are equivalent for a graph $G(V, E)$:*

1) G is a base interval graph

2) There exists a simultaneous permutation of the rows and columns of the augmented adjacency matrix $A(G)$ of G such that if an entry in the upper triangular matrix is zero, then all entries to the right of it or above it are zeros and if an entry in the lower triangular matrix is zero then all entry to the left of it or below it are zeros.

Proof. As indicated above, the part 1) $\Rightarrow$ 2) follows from the *X-Y* partition of the corresponding digraph with loops at every vertex and the symmetry of the adjacency matrix. So below we prove the other part only. Let after a suitable simultaneous permutation of rows and columns, $A(G)$ has the stated properties. From $A(G)$ we form a matrix $A(D_I)$ which has a consecutive ones property for rows by converting all the 0 entry to 1 which lie between the first one and last one in any row. Then D_I is an interval-point digraph. We now show that if G^* denotes the symmetric digraph with loop at every vertex, corresponding to the graph G, then

$$G^* = D_I \cap D'_I$$

We note that while forming D_I from G^* we have not deleted any edge from G^* and so $G^* \subset D_I$. Again since G^* is a symmetric digraph, $G^* \subset D'_I$. Thus $G^* \subset D_I \cap D'_I$.

On the other hand, if some entry (i, j) is a 1 in $A(D_I)$ which is a zero in $A(G)$ then by the construction of $A(D_I)$ there is at least a 1 to the left or to the right of this position in $A(G)$ (according as this entry is in the lower triangle or in the upper triangle). Consequently by the hypothesis all the entries below it or above it in $A(D)$ must be 0's. Hence all the positions to the right or left of (j, i) entry in $A(G)$ are 0's. So the entry 0 in the (j, i) position does not come in the way of consecutive one's property of D_I and hence the position remains 0 in $A(D_I)$.

This means that (i, j) entry in $A(D'_I)$ is 0 which implies in turn that the (i, j) entry in $D_I \cap D'_I$ is a zero. Thus $D_I \cap D'_I \subset G^*$.

Now from the consecutive one's arrangement of the rows of D_I construct an interval-point digraph $\{(S_v, p_v) : v \in V\}$ and since every element in the main diagonal of D_I is 1 we have $p_v \in S_v$ for all $v \in V$. This is the base interval representation for G. ∎

Again proceeding along the same line as in Theorem 4.3, we can prove the following theorem.

Theorem 4.5. *The following statements are equivalent* :

1) *$G(V, E)$ is a base interval graph where a vertex $v \in V$ is assigned a pair $([a_v, b_v], p_v)$, $p_v \in [a_v, b_v]$.*

2) *If G^* denotes the symmetric digraph corresponding to G with a loop at each vertex then $G^* = D_1 \cap D'_1$. Where D_1 is the interval-point digraph, the vertex v being assigned the same pair $([a_v, b_v], p_v)$, $p_v \in [a_v, b_v]$.*

3) *$G(V, E)$ is the intersection of two interval graphs G_1 and G_2 where the intervals corresponding to a vertex v in G_1 and G_2 are $[a_v, p_v]$ and $[p_v, b_v]$ respectively.*

4.4 Robin digraph and interval containment digraph

Motivated by the facts that a Robin digraph is of Ferrers dimension 4 and an interval containment digraph is of Ferrers dimension 2, we show below that how a Robin digraph can be characterized in terms of the intersection of two interval containment digraphs.

Let $D(V,E)$ be a Robin digraph with Robin representation $\{(S_v, p_v), (T_v, q_v)) : v \in V\}$ where $S_v = [a_v, b_v]$, $p_v \in S_v$ and $T_v = [c_v, d_v]$, $q_v \in T_v$

Then $uv \in E \Rightarrow a_u < c_v < b_u < d_v$ and $c_v \leq p_u$, $q_v \leq b_u$

$$\Rightarrow [a_u, b_u] \supset [c_v, q_v] \text{ and } [p_u, b_u] \subset [c_v, d_v]$$

$$\Rightarrow uv \in D_1 \text{ and } uv \in D'_2.$$

$$\Rightarrow uv \in D_1 \cap D'_2.$$

where D_1 and D_2 are two interval containment digraphs with representation $\{([a_v, b_v], [c_v, q_v]) : v \in v\}$ and $\{([c_v, d_v], [p_v, b_v]) : v \in V\}$ respectively.

Conversely, let

$$uv \in D_1 \cap D'_2.$$

$$\Rightarrow uv \in D_1 \text{ and } uv \in D'_2.$$

Corresponding to the vertex v construct the pair of intervals $([a_v, b_v], [c_v, d_v])$, then $a_u < c_v$, $b_u < d_v$ and $p_u \in S_u$, $q_v \in T_v$; also $q_v \in S_u$, $p_u \in T_v$.

Therefore $[a_u, b_u]$ and $[c_v, d_v]$ overlap with inf $S_u <$ inf T_v and p_u, $q_v \in S_u \cap T_v$.

Thus D is a Robin digraph with the above representation. This proves the following :

Theorem 4.6 *A digraph $D(V, E)$ is a Robin digraph with Robin representation $\{(S_v, p_v), (T_v, q_v) : v \in V\}$ if and only if $D = D_1 \cap D'_2$ where D_1 and D_2 are two interval containment digraphs with representations $\{([a_v, b_v], [c_v, q_v] : v \in V\}$ and $\{([c_v, d_v], [p_v, b_v]) : v \in V\}$ respectively.*

4.5 Unit Robin digraph

In this section we study a Robin digraph where the intervals are of unit length. In this case we will observe that Ferrers dimension of its adjacency matrix reduces to 3 and moreover this class becomes equivalent to the class of overlap digraph.

Theorem 4.7 *If D is a digraph then the following conditions are equivalent :*

1) D is a Robin digraph with intervals of unit length.

2) The rows and columns of the adjacency matrix of D can be permuted independently so that its 0's can be labeled R or P such that (i) the positions to the right and positions above any R are also 0's labeled R and (ii) the positions to the left or positions below any P are also 0's labeled P.

3) *D is a right overlap interval digraph.*

Proof. 1) $\Rightarrow$ 2). Let $\{((S_v, p_v), (T_v, q_v)) : v \in V\}$ be a right overlap base interval representation of a digraph D, where $S_v = [a_v, b_v]$, $p_v \in S_v$, $T_v = [c_v, d_v]$, $q_v \in T_v$ and $|S_v| = 1$, $|T_v| = 1$. Let m_v and n_v be the midpoints of S_v and T_v respectively. If $uv \notin E$, then $m_u \geq n_v$ or $p_u < c_v$ *or* $q_v > b_u$. The last two possibilities are not mutually exclusive. We label the u-v position of the adjacency matrix by R if $m_u \geq n_v$ and by P if $p_u < c_v$ *or* $q_v < b_u$. Now we arrange the rows of the matrix in decreasing order of values m_u and columns in decreasing order of values n_v. If $(u, v) \in R$; then every position to the right and every position above (u, v) is also R. If $(u, v) \in P$ and if $p_u < c_v$ then every position to the left of (u, v) is P and if $(u, v) \in P$ with $q_v > b_u$ then every position below (u, v) is P.

2) $\Rightarrow$ 3) See [Sen, Sanyal and West, 1995] for proof.

3) $\Rightarrow$ 1) We observed in the section 1.10 that the class of ROI digraph and of LOI-digraph are the same. So we may consider a LOI-digraph. Let $\{(S_v, T_v) : v \in V\}$ be a given LOI representation of D, where $S_v = [a_v, b_v]$, $T_v = [c_v, d_v]$.

We want to construct a Robin representation (S'_v, p_v), (T'_v, q_v) where $S'_v = [a'_v, b'_v]$, $T'_v = [c'_v, d'_v]$, $p_v \in S'_v$ and $q'_v \in T'_v$ and S'_v, T'_v are of the same length. From the LOI-representation of D, we have for $u \to v$ the following inequality holds :

$$c_v < a_u < d_v < b_u$$

Choose a number $l > \max_{v \in V} \{d_v - c_v, b_v - a_v\}$, that is, l is greater than the length of any interval of the LOI-representation.

Now setting

$$b'_u = a_u, \ a'_u = b'_u - l$$

$$d'_u = d_u, \ c'_u = d'_u - l$$

$$q_u = c_u, \ p_u = b_u - l,$$

we easily verify that all the intervals S'_u, T'_u are all of the same length l, $p_u \in S'_u$, $q_u \in T'_u$ and $u \to v$ iff (i) $\inf S'_u < \inf T'_v$ and (ii) $p_u, q_v \in S_u \cap T_v \neq \varphi$. ∎

Example 4.2 To illustrate the above method of construction of Robin digraph of same length interval from a LOI-digraph, consider the following LOI-representation of a digraph

$v_1 \to ([11, 18], [10, 14])$

$v_2 \to ([15, 21], [10, 12])$

$v_3 \to ([13, 18], [14, 20])$

$v_4 \to ([19, 25], [10, 16])$

$v_5 \to ([21, 23], [20, 22])$

$v_6 \to ([17, 21], [18, 24])$

$v_7 \to ([17, 21], [14, 20])$

Here we may take $l = 10$. And using the formula described in the proof we have the following Robin representation of the digraph with interval of the same length.

$v_1 \to S_1 = [1, 11], \quad p_1 = 8, \qquad T_1 = [4, 14], \quad q_1 = 10$

$v_2 \to S_2 = [5, 15], \quad p_2 = 11, \qquad T_2 = [2, 12], \quad q_2 = 10$

$v_3 \to S_3 = [3, 13], \quad p_3 = 8, \qquad T_3 = [10, 20], \quad q_3 = 14$

$v_4 \to S_4 = [9, 19], \quad p_4 = 15, \qquad T_4 = [6, 16], \quad q_4 = 10$

$v_5 \to S_5 = [11, 21], \quad p_5 = 13, \qquad T_5 = [12, 22], \quad q_5 = 20$

$v_6 \to S_6 = [7, 17], \quad p_6 = 11, \qquad T_6 = [14, 24], \quad q_6 = 18$

$v_7 \to S_7 = [7, 17], \quad p_7 = 11, \qquad T_7 = [10, 20], \quad q_7 = 14$

CONCLUSION

In this dissertation, we have provided a characterization of interval digraph (bigraph) by a list of forbidden matrices (configurations). These configurations, in turn, help us to find a recognition algorithm of an interval digraph in O (n^3) time. Also we have attempted to provide a characterization of bigraphs of Ferrers dimension 2 in terms of asteroidal triple of edges (ATE). Lastly we have characterized base interval (di) graph and Robin digraph with unit length interval and established their interrelation with the class of interval, overlap or containment digraphs. But still a good number of related problems remain unsolved in this area. We list below some of these which arise immediately as a consequence of our work and further research work may be carried out in future.

1. Sen *et al.* [1995] showed that interval digraph can be viewed as a generalization of interval graph. But does such relationship exist between circular arc digraph and circular arc graph ? Between interval containment digraph and interval containment graph ? Or between overlap digraph and overlap graph ?

2. Sanyal and Sen [1996] introduced the concept of consistent ordering of the edges of a digraph (graph) and used it to obtain a new characterization of an interval digraph (graph). One can study if this characterization can be used to obtain a recognition algorithm of an interval digraph (graph) more efficiently.

3. In chapter 3, we have made an attempt to characterize a bigraph (digraph) of Ferrers dimension 2 in terms of asteroidal triple of edges and could solve the problem in only one case when the graph contains a strong bisimplicial edge. But the other case, when all the bisimplicial edges of the graph are weak remains unresolved.

4. Müller [1997] gave some example of ATE's in a bigraph. But the complete list of ATE in a bigraph remains to be found out.

5. An algorithm to recognize the existence of ATE in a bigraph remains to be found out.

6. One may study the bigraphs corresponding to its forbidden configurations of interval bigraphs and find out the correlation between these bigraphs and ATE; consequently one may find out forbidden bigraph characterization of interval bigraphs.

7. Using biorder representation of Ferrers digraph, West [1998] gave a short inductive proof of the interval digraph characterization. Also one may find out short inductive prof of interval digraph characterization in terms of (R, C) partition.

8. Sanyal [1994] characterized the adjacency matrices of Robin digraphs. In chapter 4 we have characterized the adjacency matrix of base interval (di) graph. Also in the same chapter we have characterized the overlap base interval digraphs and base interval graphs in terms of intersection of two digraphs having other model represented by intervals. But one may introduce the notion of 'base circular arc' in the similar fashion of base interval and investigate the corresponding problems for the case when the base interval is replaced by base circular arc.

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