Chapter III

RUSSELL'S PARADOX IN RELATION TO THE IDEA OF LOGICAL SYSTEMS

Logical systems have been very useful tools for Philosophers, Logicians and Mathematicians.

First, logic has been useful in clarifying long standing problems in Philosophy. Secondly, logic has been useful in accessing the validity of Philosophical arguments. Thirdly, logic has been useful in developing new theories - new Philosophical oppositions. And fourth, the recent advances in logic have shed great light on the nature of logic itself, such as the nature of mathematics and the relationship of logic to mathematics.

It has been appreciated that the predication of properties to properties leads to a difficult known as syntactical paradoxes. We may illustrate the point in the following manner. Let us call any property which can be predicted of itself, a **predicable property**, any property which can not be predicate of itself an **impredicable property**. We can say that the property of being **common** is a predicable property, since it seems true being common is a common property. But can we say that the property of being rare is a rare property?

What about the property of being impredicable? Can this property be truly predicated of itself? The unfortunate answer seems to be that if the property of being impredicable is predicated of itself that it is **not** predicated of itself. And if is not **not** predicated of itself, then it is predicated of itself. Hence the paradox. To make this clear let us symbolize the property of being predicable as P and the property of being impredicables as P'. Let us bear in mind that to say the given property F is P is to say that FF, and to say that the given property F is P' is to say that $\sim$FF.

To start with P' is itself P' or else P' is P. Take the first possibility namely the P' is P'. If P' is P' then P' is predicated of itself and hence P' is P. So if P' is P' then P' is P.

It follows that is P' is P, then it is P, and if P' is P, then it is P'. What we have shown is that if the property of being impredicable is **impredicable**, then it is predicated of itself, and hence it is predicable. And if the property of being impredicable is **predicable**, then it is not predicated of itself and hence it is **impredicable**.

This contradictory result can be made by writing down the definition of P' and then constructing a simple argument as follows.

1. $\bar{P}F = df \sim FF$

That is to say a property F is impredicable is to say that it is not the case that F is F. From which it follows that given any property F, F is P' if and only if it is not the case that F is F.

2. $(F)\ (\bar{P}F \equiv \sim FF)$

Hence by substituting $\bar{P}$ for F, by universal instantiation (U.I.).

3. $\bar{P}\bar{P} \equiv \sim \bar{P}\bar{P}$

From which an explicit contradiction can be determined.

We may show in this context the logical structure of the paradox of the liar. It was first posed by ancient Greek. A modernized version of the paradox may be stated as under :

It seems reasonable to support that every declarative sentence is either **true** or **false**.

But consider the sentence (1), sentence (1) is false.

Is sentence (1) true or is it false? The unfortunate answer seems to be that if sentence (1) is true then it is false, and if it is false, then it is true.

Let us take the first possibility namely that sentence (1) is true. If (1) is true and (1) says that (1) is false, then it follows that (1) is false. So if (1) is true then (1) is false.

Now suppose (1) is false. If (1) is false and (1) says that (1) is false, then it follows that it is false that (1) is false, and therefore it follows that (1) is true. So if (1) is false, then (1) is true. Either way we have a contradiction and hence a paradox.

It is often thought that the liar paradox can be solved by ruling out any sentence which refers to itself. The logician divided among themselves as to whether, the liar paradox is a paradox of self reference. We shall take up the matter for discussion in a latter chapter.

Russell's paradox is a logical paradox that has very serious repercussion in the theory of classes and thus also in the foundation of mathematics. Consider the class of all classes that do not belong to themselves. Does it belong to itself? Answering either "yes" or "no" results is a contradiction. From this Russell drew the conclusion that no such class exists, but it is not easy to justify this conclusion.

The field of set theory was pioneered by Cantor and further developed by Russell and Zermalo. The original motivation that led to the formulation of the set theory derived from problems arising out of investigation into certain type of infinite set and recognition of a need for a theory of infinite. Russell showed that most of received mathematics is deducible in set theory. Burali-Forti paradox and Cantor's paradox called into question, the size and existence of certain sets, and Rüssell's paradox showed that the unrestricted conception of sets as being determined by all predicates or conditions was inconsistence. Sets are conceived of either being "build" from their members or as being defined by a condition. For instance the condition 'is red' defines the set of red things. Russell's paradox indicate not every condition defines a set. He showed the unrestricted conception of red leads to contradiction.

We shall therefore consider Russell's paradox in the following order

(a) We shall first consider the Russell's conception of set and how sets are built.

(b) How are sets determined by a condition and then we shall be in a position to appreciate what has come to be known as Russell's paradox.