Chapter II

THE CONCEPT OF MATHEMATICAL LOGIC AND SOME RELATED ISSUE : THE IDEA OF PARADOX

To express thinking in mathematical terms and to subject it to computing is a very old idea. Pythagoras thought of numbers as being the essence of things and consequently the relation between numbers were thought to be the relation between things. Accordingly ideas and numbers, thinking an operation with numbers had the same meaning.

At the end of the last century mathematical logic received its precise profile as a result of the researches of Peano, the Italian mathematician and Frege, the German mathematician. Along the lines of Frege, Bertrand Russell argue to understand a logic as a mathematical theory and such a conception found its best expression in **Principia Mathematica**, (Hence forth PM), written together with A.Whitehead. Therefore since 1910 mathematical logic has come to exist and any subsequent development are but compilations and attempts to find solutions to certain difficulties which arise from the very nature of the formal construction of this theory. Symbolic logic has been conceived by mathematicians to explain logical processes in mathematics.

A very important part in the development of mathematical logic has been played by the contradiction of the set theory. These are known as paradoxes. Any one who intends to construct a formal system has to ensure that the formation rules, the derivation rules that he puts into his system should not, between them, generate antinomies. He wants to exclude the paradoxes, to guarantee that none of the known ones will arise within his system, and if possible that no as yet unknown ones will break out their either. But it is a well known fact that paradoxes do arise and they constitute a radical challenge to the rationality of human thinking. Paradoxes are items about which it is difficult to say anything comprehensive without ourself falling into contradiction. If we are unwilling to adopt a general skepticism about reason, we must take up the challenge. But this does not mean that the paradoxes can be excluded from formal system. The challenge consists in showing how we can comment on them, wherever they arise, without being committed to contradicting ourself. For the constructor of formal system a solution needs only be an exclusion device. But a philosophically inclined thinker may do something different. He may try to show that these are only apparent antinomies. For example the so called paradox of material implication is said to be only aparrent. The case that a true proposition can be derived either from a true or a false premise is a consequence of defining implication in terms of negation and disjunction.

It is possible to take the following stands in respect of the paradoxes.

A. We may say that the paradoxes are unimportant and that they arise from the misuses of languages.
B. A paradox may be important in one context but not in other contexts.
C. A paradox can be solved or removed by valid proofs.
D. A paradox compels us to revise our naive ideas.
E. A paradox compels us to distinguish sharply between levels of language.
F. A paradox show that our natural habits of thought and speech are in coherent and require revision, and so on.

Let us now ask what is a paradox and what it would be to solve it. Typically a paradox is a apparently sound proof of an unacceptable conclusion, in most of the cases the conclusion is unacceptable because it is self contradictory. It is possible to distinguish between the branches of the reasoning within the paradox, and the conclusion of one branch contradicts that of the other[1]. To illustrate the case we may point to the liar paradox namely, an Indian says that all Indians are liars. In any version of the simple liar we seem to be able to argue that if the utterance is true then it is false, so it must be false, but also that if it is false then it is true, so it must be true that is, it must be both true and false. J.L. Mackie (**Truth Probability and Paradox**, p.238). This paradox is an ancient one. But over the last seventy years a paradox like that has cropped up within the attempts to formalize set theory and the foundation of arithmetic and to describe exact language structure stating precise rules for determine truth and falsehood. It is easy to see how antinomies threaten such projects. If the formation rules for a formal system permit the construction of items which the other rules of the system than require to be characterized in incompatible ways, then as long as the system recognises the argument form "p, Not-p, therefore q", any well-formed formula becomes provable within the system. Proof within the system ceases to discriminate and it will be impossible to use any interpretation of the system for any ordinary purpose.

Let us take two examples of paradoxes. We may think of a card such that a on one side of which is written, "The statement of the other side of the card is false." While on the other side is written, "The statement of the other side of the card is true". Usually such paradoxes are classified as a paradoxes of self-reference. There is an opinion to the effect that the example of the card does not contain any self reference. In order to have a more self-referring paradox we may think of a liar telling us "what I am now saying is true." Russell's a class paradox has been described as a paradox of self-reference. Is the class of all and only those classes that are members of themselves is a member of itself or not? If the class of self-membered classes is not a member of itself then it is not a member of itself.

And similarly if the class in question is a member of itself then it is a member of itself. What is paradoxical is that we can not decide whether this class is a member of itself not.

We may now mention some of the most important paradoxes of logical or mathematical type.

1. C. Burali-Forti was an Italian mathematician. He published his famous paper: **Una-questions Sui numeri transfiniti** in 1897. In this paper Burali-Forti exposed the paradox of Peano's formal structure. We may try to express the paradox as follows :

In the set theory it is demonstrated that

A. Any set of ordinal numbers defines ordinal numbers.

B. This ordinal number exceeds by one the greatest number in the relevant series.

C. The series of ordinals is well-ordered.

Let us now consider the series of all ordinal numbers. This series defines an ordinal number we shall call it Ω, which is the greatest of all ordinal numbers. If this be the case that the series of all ordinals contains the ordinal number Ω, defined by the series and so the ordinal number it defines is $\Omega+1$ and not Ω. The contradiction is self evident. The greatest ordinal number is not the greatest.

2. Cantor discovered this antinomy almost at the same time when Burali-Forti published his paper referred to above. Cantor's paradox was brought to public knowledge only in 1927 by Zermelo, in his complete edition of Cantor's works. Cantor's paradox may be stated as follows.

If M is the set of all sets and Mc its cardinal number, then Nc its greatest possible cardinal number. On the other hand a theorem in the set theory says that the cardinal number of all the subset of a set M is greatest than the cardinal number of the set M. So the greatest cardinal number is not the greatest.

There are other important logical mathematical paradoxes. These may be briefly mentioned.

3. Richard's paradox (1905) - This paradox is concerned with the construction of the definite decimal numbers.

4. Zermalo-König's paradox (1905) is concerned with the construction of well-ordered sets.

5. Berry's paradox (1906) is similar to Richard's paradox.

6. Skolen's paradox (1923) refers to formalization or axiomatization of set theory.

7. Gödel's paradox is very similar to liar paradox.

We propose to undertake a critical statement of Russell's paradox in the following chapter. But before we do that a brief account of Russell's paradox may be offered by way of introduction.

The logic of classes was well constructed by Peano before PM Russell and Whitehead took over from Peano the whole calculus of classes. In the third chapter of PM they have enumerated the properties of classes. A class, they say must be something wholly determined by its membership, for example the class "Even primes" is identical with a class "numbers identical with 2." When the class is given the membership it is determined. Further the null class is just as good as any other. It is also to be noted that two different sets of object can not form the same class. Again in the same sense in which there are classes there must also be classes of classes. For example, "The combination of **n** things taken **m** at a time", where the n things form a class and each such a class is a member of the specified set of combinations, which set is therefore another class whose members are classes. The class of unit classes or of couples is absolutely indispensable. The former is the number 1 and the latter the number 2. Thus without classes arithmetic becomes impossible.

It must therefore be the case that under all circumstances it is meaningless to suppose a class identical with one of its member. If such a supposition have any meaning then "α is a member of α" and so would α is not a member of α. If we call this class A, then

$$\alpha \in K \equiv_{\alpha} \alpha \sim \in K$$

If we call this class K, then we have

$$\alpha \in K. \equiv_{\alpha} .\alpha \sim \in K$$

But this is a contradiction. Hence "α is a member of α" and "α is not a member of α" must always be meaningless.

For Russell there is nothing surprising about this conclusion and he thinks we must take special notice to consequences of it. In the first place a class consisting only one member must not be identical with that one member. To think so would absolutely be meaningless not simply false. In the second place if the class of all class were a Class, then we must distinguish between two occurrences of the same symbol on the left and right side of membership. Peano had put the case as follows -

$$\text{Cls} \in \text{cls}$$

The first Cls need to have the different meaning from the second.

It was the intension of the authors of PM to construct the whole of mathematics in symbolic manner, in course of the reconstruction there appear the logical paradoxes in the very formalism. In order to solve the paradox Russell was compelled to introduce the well known theory of Types, so that the paradoxes could be avoided. In the following chapters we shall go over to a detail enquiry into that.

References

1. *Principia Mathmatica* : Russell and Whitehead OUP.

2. *Truth and the Liar Paradox* : ED. Robert Martin OUP.

3. *Truth Probability and Paradox*:J.L. Mackie, OUP.

4. *Development of Logic*: Kneale Kneale, OUP.