Chapter I

INTRODUCING RUSSELL'S MATHEMATICAL PHILOSOPHY

The object of Russell's mathematical Philosophy is to show that mathematics can be shown to be a logical development of certain basic ideas; and that mathematics can be reduced to logic.

The number of a class is the class of all those classes which are similar to it. Classes are similar when their members can be put into a one-to-one relation with each other.

An infinite cardinal number satisfies the equation, n equals a n plus 1.

By distinguishing between types of entities it is possible to avoid paradoxes which have perplexed philosophers for centuries.

Mathematical truths are a priori and have nothing to do with facts about the world, they are logical tautologies.

Of Russell's three books on mathematical philosophy one is **Introduction to mathematical philosophy**. This work predates the famous **Principia Mathematica,** and I propose to take it as a representative work. It is expressed almost wholly in specially devised symbols and containing formal proofs showing that mathematics can be "reduced" to logic. Since these proofs are somewhat formidable for one not versed in mathematics and not having an aptitude for mathematical symbolism.

The general thesis of the book is that we can start with the familiar portions of mathematics, say such a statement as "2 plus 2 equals 4", and go either "up" into higher mathematics, leading to the consideration of fractions, real numbers, complex numbers, infinite numbers, integration and differentiations, or "down" into lower mathematics, leading through analysis to notions of greater abstractness and greater logical simplicity. The latter route, which is the approach adopted primarily in recent mathematics, and consequently less familiar to non-mathematicians, asks not what can be defined and deduced when we assume that "2 plus 2 equals 4" but what more general ideas and principles can be found in terms of which "2 plus 2 equals 4" can be defined and deduced. In other words, the most obvious and easy things in mathematics do not come logically at the beginning, but somewhere in the middle, just as the bodies which are easiest to see are neither those which are very near nor those which are very far, but those which are at a "moderate" distance. The easiest conceptions to grasp in mathematics are neither the complex and intricate ideas not the logically simple and abstract ideas, but the common-sense notions involved in the whole numbers.

The reason why such a study can be called "mathematical philosophy" is to be found in the fact that although many of the notions considered in this type of investigation- **numbers**, **class**, **relations**, **order**, **continuity**, **infinity** have been traditionally examined by philosophers, without very much success, interesting results can be obtained when the methods of speculative philosophers are replaced by the more refined and precise methods of the mathematicians and logicians. In order to stress his point Russell frequently argues that these newer conceptions render the traditional philosophical problems insoluble, or perhaps meaningless. Perhaps the best examples of Russell's mathematical philosophy is his definition, following Giuseppe Peano, of the notion of the natural numbers. One could hardly imagine a concept which would seem clearer to the ordinary man than that exemplified by the series:

0,1,2,3,...... n, n+1, ... Yet Peano shows that though this notion is familiar it is not understood. It can be reduced, in fact, to three primitive ideas and five primitive propositions. The notions involved in the use of "primitive" must first be explained.

Since all terms that are defined only by means of other terms, we must accept some terms as initially clear in order to have a starting point for our definitions. This procedure does not require that these latter terms be incapable of definition, for we might have stopped too soon and we might be able to define them if we go further. On the other hand, there may be certain terms which are logically simple in the sense that they cannot be analyzed into any other terms. The decision between these two possibilities is not important for logic; all that is needed is the knowledge that since human powers are finite, definitions must always begin with some terms which are at least undefined at the moment, though perhaps not permanently.

Primitive propositions have the same status. Whenever we prove propositions to be true, we do so by reducing them to other propositions, which must themselves be proved by reducing them to still other propositions. Ultimate proof, therefore, presumably can not be achieved unless we assume certain propositions to be self-evident. But mathematicians have quite generally abandoned the notions of "self-evidence" since it seems to rest on psychological rather than on logical considerations and permits truth to vary from individual to individual. However, they have granted that there must be in any formal system certain propositions, usually called "postulates", which are unproved **within the** system, though they may be provable by going to still more basic postulates outside the system.

Since Peano's axioms (we shall deal extensively with Peano's system in a later chapter) do not guarantee that there will be anything in the world which exemplifies them, and since we want our numbers to be such as can be used for the purpose of counting common objects, we should supplement Peano's work by making it into a theory of arithmetic. This was done by another mathematician, named Gottlob Frege. It requires the introduction of the notion of "class". A class may be defined in two ways : by enumerating its members - say, Brown, Jones, and Robinson - or by mentioning a defining property as when we speak of "inhabitants of London". The former is called "extensional", the latter "intentional". The latter is more fundamental, since extension can always be reduced to intention, but intention often cannot, even theoretically, be reduced to extension. This is important for the definition of numbers, for numbers themselves form an infinite collection and cannot be enumerated. Furthermore, it is probably true that there is an infinite numbers of collections in the world — for example, an infinite numbers of pairs. Finally we wish to define "number" in such a way as to permit the existence of infinite numbers as well as finite ones, and this requires that we be able to speak of the terms in an infinite collection by means of a property which is common to all its members and peculiar to them.

Proceeding in this way, Russell shows how it is possible to demonstrate when two classes "have the same number" — that is, exhibit a property in terms of which, their numbers could be defined. This can be done by showing that the classes are "similar", where "similarity" is defined in terms of having a one-to-one relation to each other.

For example, in countries where neither polygamy nor polyandry is permitted, the relation "spouse of" constitutes a relation on the basis of which the class of married men can be shown to be similar to the class of married women. The use of this criterion does not require that we be able to **count** either class, and we can know that the numbers of married men is the same as the number of married women without knowing the number either. The notion of similarity is therefore logically more simple that the notion of counting, though not necessarily more familiar. If we now make a bundle of all pairs, of all trios, and of all quarters and then extend this to a bundle of all classes that have only one member (unit class), and to a bundle of all classes that have no members (null classes), we could then go on to say that by "2" we mean the property which is common to all pairs, by "3" we mean the property which is common to all trios, and so on.

However, Russell does not choose to do so because he is afraid that if we suppose some property in nature which we call "twoness" we may be unconsciously creating a metaphysical entity whose existence is debatable. Of the class of couples we can be sure, but of the metaphysical 2 we cannot. Therefore, he defines the number "2" simply as the class of all couples, not as the property which they all possess. And more generally, the number of a class is the class of all those classes which are similar to it. This definition sounds odd, but it is precise and indubitable and can be shown to apply to the world in such a way as to make arithmetic possible.

The notion of "infinity", which has puzzled philosophers since the days of the Greeks, can easily be defined. There are many different kinds and levels of infinite numbers, and only the simplest, the infinite cardinal numbers, are examined. Russell points out that what he had previously called the "natural" numbers can also be called the "inductive" numbers, such usage indicating merely that we are naming the numbers in terms of Peano's fifth postulate rather than in terms of something else. The principle of mathematical induction can be crudely stated in the form, "what can be inferred from next to next can be inferred from first to last".

Suppose that we now take under consideration the collection of the inductive number themselves. This collection cannot itself have as its number one of the inductive numbers, for if we suppose **n** to be any inductive number, the inclusion of zero in our collection compels us to say that the number of such a collection will be **n** plus **1**. Hence, the number of inductive numbers is a new number, which is different from all of them and is not possessed to all inductive properties. This number is called an "infinite cardinal number" and its unusual character is shown in the fact that it satisfies the equation, **n** equals **n** plus **1**. A class possessing such characteristics is called a "reflexive" class and number of such a class is a reflexive cardinal number. A still more surprising characteristic of an infinite cardinal number is that it satisfies the equation, **n** equals **2n**. For example, the number of even inductive numbers is the same as the number of all inductive numbers, both odd and even. Leibniz used this fact to prove that infinite numbers are impossible, but modern mathematical logicians use it only to show that the commonly accepted belief that the whole is greater than one of its parts is really not true and is based on an unperceived vagueness in some of its terms.

Granting the existence of infinite numbers an interesting question arises : Does there exist in the world a class the number of whose members is infinite? An affirmative answer to this question appears to be demonstrable. Assume that the number of individuals (the meaning of the term "individual" is left undefined for the moment) is some finite number, say **n**. Now there are mathematical truths which informs us, first, that given a class of **n** members there are 2^n ways of selecting some of its members and, second, that 2^n is always greater than n. If we now start with a class of **n** members, than add to this the class of classes that may be formed from **n**, namely, $\mathbf{2^n}$, then add the class of classes of classes that may be formed from this, namely, $\mathbf{2^m}$ and so on, we shall have a total whose number is the infinite cardinal. Hence, the number of "things" in the world is infinite.

Russell confesses that he formally believed this to be a valid proof. But he now rejects it because it involves what has come to be called the "confusion of types". This fallacy consists in the formation of "impure" classes. If there are **n** individuals in the world, and **2**^{n} classes of individuals, we can not form a new class whose number is **n** plus **2**^{n}. Classes are logical constructions, not things, and the two "types" cannot be combined. Plato argued that since the number 1 has being, but is not identical with being, 1 plus being equals 2; then 1 plus 2 equals 3; and so on; hence the world is infinite. Two mathematicians, Bernard Bolzano and Richard Dedekind, argued that because ideas of things are "things", and because there is an idea of every thing, the class of "things" is a reflexive class (since it is similar to a part of itself) and its number must be infinite. Russell tries to show not only that these "proofs" have an air of hocus-pocus about them, but also that unless we prevent this confusion of types we shall be able to prove all sorts of self-contradictory statements - for example, that if a class is a member of itself, it is not a member of itself. One way to avoid both the feeling of uneasiness and the paradoxes is to define the word "individual" as referring to an entity of a certain "type", the word "class" as referring to an entity of another type, the word "relation" as referring to an entity of still another type, and so on.

The problem which Russell here formulates has given rise in recent literature to the distinction between languages and meta-languages, a meta language being a language which talks about a language. It then becomes important not to confuse these two "types" of language because absurdities and paradoxes may develop if we do.

In conclusion, Russell returns to the general question concerning the nature of mathematical philosophy and its relations to logic and empirical knowledge. Mathematics, formally defined as the science of quantity, can no longer be so defined. Many branches of geometry have nothing to do with quantity, and even arithmetic, which is commonly thought to deal with numbers, concerns itself rather with the more basic ideas of one-to-one relations and similarity between classes. The generalization of the notion of order also means that mathematics is no longer particularly concerned with the number series.

What, then, is this new study which starts with mathematics and ends with discussions of classes, relations and series? It may be called indifferently either "logic" or "mathematics", the choice of name is not important. But its characteristics should be clarified.

We can say that the "form" of a proposition is that which remains unchanged when every constituent of the proposition "Socrates is earlier than Aristotle" has the same form as "Napoleon is greater than Wellington". In every language there are certain words whose function is merely to indicate form. These are called "logical constants". For example, the word "is" in the proposition "Socrates is human" is a logical constant. In pure mathematics the only constants which occur are logical constants. In these sense, therefore, logic and mathematics are one.