Chapter IX

CONCLUDING REMARKS

In this part of the dissertation we shall be chiefly concerned with making a couple of critical comments on the basic concepts that have come up in course our discussion of Russell's paradox.

I. The concept of 'propositional function' has played a significant role in Russell's formulation of set. It is with the help of the notion of propositional function and its satisfaction that Russell defines the notion of a set or class. Yet a logician and mathematician of later days does not seem to be happy with Russell's concept of a propositional function. Alfred Tarski for example appears to have some reservation concerning the concept of propositional function.

He contends that the 'x is an integer' is not a sentence, because the variable x does not have a meaning by itself. Tarski adduces the following reasons. The question - Is x is an interger ? Cannot answered meaningfully. Tarski deplores that in text books of elementary mathematics such an impression is conveyed that it is possible to attribute an independent meaning to variables. The symbols "x" or "y" is taken to denote the so called "variable numbers". Tarski thinks that the statement of these kinds have their source in gross misunderstanding. The "variable number" x could not possibly have any specified property. It could be neither positive nor negative nor equal to zero. The property of such a number would change from case to case. The number would sometimes be positive, sometimes negative and sometimes equal to zero. We do not find entities of such a kind in our world at all. Their existence would contradict the fundamental laws of thought. The point for Tarski is that the classification of symbols into constants and variables does not have any analogue in the form of a similar classification for the numbers such as "constant number" and "variable numbers".

The phrase "x is an integer" is not a sentence, but it does have the grammatical form of a sentence. It does not express the difinite assertion and therefore can be neither conformed nor refuted. We can only obtain a sentence from "x is an integer" when we replace "x" by a constant denoting a definite number. If "x" is replaced by the symbol "1" the result is a true sentence, where as a false sentence arises on replacing "x" by "1/2". The point is summarized by Tarski in the following manner :

An expression, which contains variable, when replaced by constants becomes a sentence and it is what Tarski calls a sentential function. It may be noted now that Tarski's **sentential function** is what Russell's called a **propositional function**. Tarski's "sentence" is Russell's proposition.

In mathematics however the concept of function is a little different. A mathematical function such as "x+y=5" is composed entirely by mathematical symbol and not of word of language. Mathematician would call it a formula. We may therefore say that the Russell's "propositional function" and Tarski's "sentential function" are properly speaking concepts of logic rather than of mathematics.

II. Let us now turn to Russell. His theory of classes and of class membership follows closely in the footsteps of his immediate predecessors. It is in terms of classes Russell Proposes to define natural numbers, and through that definition all the fundamental notions of arithmetic. Arithmeticians like Peano had already maintained that all other numbers could be defined in terms of the natural numbers. If Russell can define the natural numbers in terms of classes then he may be taken to have proved that mathematics has no need of **numericals**, as distinct from nearly logical **constants**.

We have seen that Russell defines a cardinal number as the number of a class, as the class of all classes similar to the given class. A class has six members, if it belongs to the class to which all classes belong. 'Similar' has a special technical sense. It means that **having the same number as**. It has been argued that Russell's definition of cardinal number is circular. Russell has been said to have defined the number of a class as that class to which all classes with the same number belong. Let us see what Russell can say in his own defense. He can define 'similarity' or 'having the same number' in non-numerical terms. Two classes may be said to have the same number when they can be correlated one to one. Russell maintains that he does not need the number one to establish a 'one-to-one' correlation. A relation is one-to-one if x stands in relation to y and so does x^1, then x and x^1 are identical. And if x has this relation to y and y^1 then y and y^1 are identical. We may put the case in terms of concrete example : there is a one-to-one correlation between legal wives to legal husbands in a monogamous community. This means that if x is the legal husband of y and x^1 is the legal husband of y, then x and x^1 are identical. If x is the legal husband of y and y and y^1 are identical. Thus Russell maintains that his definition of numbers in terms of similar classes involves no circularity.

III. Russell's definition of number illustrates the central technique of Russell's philosophical method. This is what he calls the **principle of abstraction**. Normally a **number** is picked out by abstraction from a set of group which possesses a common numerical property. But Russell points out that there is only such property. Russell objects that there is no way of showing that there is only one property. One we have picked out. We are in search of a single property but there is a class of properties. Russell claims that the principle of abstraction avoids this difficulties.

It defines by referring to a class consisting of all the classes which have unique relation namely one-to-one correspondence to each other there may be a property common to all the members of these classes but we need not make that presumption. Russell's philosophical motivation is to reduce the number of entities of properties which are to be presumed to exist inorder to give the complete account of the world.

IV. We may look at the notion of a propositional function a little more closely. A definition of the variable presupposes the notion of any. The variable represent any one of a set or class. A variable is a symbol which denotes any one of class of element. Frege had suggested that the variable keeps an **empty place** which must be filled in by an element of the class represented by a variable in order that the expression in which the variable occurs may be completed. Frege's statement is some what ambiguious. It does not take into account the fact that variables represent any one but not a determinant one, of a set of element, set being defined by the specific functional relation. It is for this reason that Russell brings in the specific form of function called a propositional function.

It is by no means easy to detemine precisely the nature of a propositional function. From the PM We gather that a propositional function is something which contains a variable x and expresses as a **proposition** as soon as a value is assigned to x. It differs from a proposition by the fact that it is ambiguious, that is, it contains variable of which the value is unassigned. Russell also points out that a propositional function agrees with the ordinary function of mathematics in the fact of containing an unassigned variable; where it differs is in the fact that the values of the function are the proposition. It is to be noted that Russell lays stress upon the ambiguity of the propositional function and since that this ambiguity constitutes the sole difference between the propositional function and a proposition. By 'ambiguity' Russell means **indeterminateness** which is the characteristic of the variable. But it may be questioned whether Russell's way of putting the matter is clear enough. What is important is that there is a determinant correspondence specified by the functional relation holding between the variables. This is the characteristic of the mathematical function. Russell further says that propositional functions are the fundamental kind from which the more usual kind of functions such as 'sin x' or 'log x' or' the father of x' are derrived. Such a function is called descriptive function, that is, it means that the term having such and such relation to x so that the function describes the value of x which satisfies the function. We may therefore say that in the case of a propositional function there ~~are~~ seems to be nothing corresponding to the value of the function, since there is nothing corresponding to the term described by the mathematical descriptive function. The two kinds of functions :

Mathematical and propositional functions are not exactly analogous. The propositional function does not have anything corresponding to the dependent variable, for example, in the expression 'y = log x'.

We may briefly indicate the symbolism of PM in respect of propositional function. The large Greek letters, ϕ, ψ, X are used to express variable functions involving one variable, e.g. "$\phi(x)$", "$\psi(x)$". It is usually found convenient to omit the brackets round the variable and thus we write "$\phi(x)$" as "ϕx". Propositional functions involving two, or more, variables may be written "$\phi(x,y)$", "$\phi(x,y,z)$". But it is usually more convenient to express a function involving two variables by the symbols for relations, as follows : "XRY".

V. Wittgenstein has been one of the early critics of Russell's paradox, which he believes could be disposed of. In the **Tractatus Logico-Philosophicus** (3.331) Wittgenstein remarks that Russell does not mention the meaning of signs when establishing the rules for them. Wittgenstein argues that an useless sign is meaningless. The meaning of a sign is no part of logical syntax. He further says that no proposition can make a statement about itself, because a propositional sign cannot be contained in itself. A function therefore is not contained itself, and therefore a function cannot be its own argument. If, Wittgenstein says, the function F(x) is its own argument, then in the proposition 'F'(F(fx))' the outer and the inner F's cannot have the same meanings. The inner function has the form $\phi(fx)$, while the outer one has $\psi(\phi(fx))$. F is common to both the functions, but it itself signifies nothing. Wittgenstein then argues that instead of 'F(Fu)' if we write '$(\exists\phi) : F(\phi u).\phi u = Fu$' then the paradox is disposed of.

Whether Wittgenstein's argument really disposes of Russell's paradox is a matter we need not decide now at once. what is intriguing is the fact Russell who had written the "Introduction" to the **Tractatus** did not take notice of Wittgenstein's argument. Wittgenstein's objections rests on his theory of symbolism. According to him logic is a syntax showing how to use the symbols correctly. He distinguishes between a **sign** and a **symbol**. A sign becomes a symbol only when it has significant use. We can build up a logical grammar of signs in disregard of their symbolic meanings. Based on this theory of symbolism Wittgenstein thinks Russell's theory of types erroneous. He emphasizes the actual impossibility of a formal sign speaking of its content. If the problem is to constitute a syntax of signs devoid of any content, then their possible meanings cannot be hinted at, since otherwise their formal nature be altered.

VI. In recent times Quine has dealt extensively with set theory and its logic. Quine's work is basically a ramification of Russell's work. Quine's book **Set theory and its Logic**, is dedicated to Russell, whose ideas loomed large in this subject and whose writings inspired Quine's interest. We propose to have a look at Quine's statement and critical points made in the context of that classical body of the set theory which comes down from Cantor to Russell.

We should note that Quine-defines set theory as the mathematical theory of classes. He agrees with Zermalo's view, namely "set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions of 'number', 'order', and 'function' taking them in their simple form and to develop them by the logical foundation of all arithmetic and analysis".[1] [Quoted by Quine in **Set theory**, Introduction page 4]

As Quine conceives the problem of set theory, the question is; what classes there are, or, a major concern of set theory is to decide what open sentences (or what Russell called 'propositional functions') to view as determining clauses.

Quine adds also that because of Russell's paradox and other antinomies much of set theory has to be persued self-consciously. Quine has given full credit to cantor for "the discoverty or creation" of set theory. Quine states that Cantorian set theory consists of the following :

(a) The general assumption of the existence of classes and general laws concerning them.

(b) The derivation of the theory of relations from (a), and particularly a theory of functions.

(c) Then the integers are defined and the real numbers and the arithmatical laws that govern the integer of real numbers are derived.

(d) Finally, one gets infinite numbers and the theory of relative sizes of infinite classes. Quine knows that (d) constitutes the most characteristic business of Set theory.

Quine has followed in distinguishing between **Abstract set theory** and point-set theory. Quine thinks that his own work is an abstract set theory. He also says that abstract set theory is more concerned about its logic.

VII. We may now state Russell's contribution to philosophy of mathematics in a summary form. His theory of classes and of class-membership at first follows in the footsteps of his immediate predecessors. He proposes to define natural member in terms of classes, and through that definition all the fundamental notions of arithmetic. Peano had maintained that all other members could be defined in terms of the natural numbers. Russell's claim as to the definability of natural members in terms of classes implies that mathematics has no need of **numericals**, as distinct from merely **logical**, constants.

We have seen that Russell defines a cardinal number, which is always the number of a class, as "the class of all classes similar to the given class". A class has six members, if it belongs to the class to which all classes similar to it belong. 'Similar' has the special technical sense of 'having the same number as'. Is Russell's definition circular ? He has to meet this objection. Does he not define the number of a class as that class to which all classes with the same number belong ? Russell would reply that he defines 'similarity' or 'having the same member' in non-numerical terms. Two classes have the same member if they are correlated one-to-one. One does not even need the number one to establish 'one-to-one' correlation. Russell says that a relation is one-to-one if **x** stands in this relation to **y**, and so does $\mathbf{x}^1$, then **x** and $\mathbf{x}^1$ are identical. If **x** has this relation to **y** and to $\mathbf{y}^1$, then **y** and $\mathbf{y}^1$ are identical. To say that there is a one-to-one correlation between legal wives and legal husbands in a monogamous society is to assert that **x** is a legal husband of **y** and $\mathbf{x}^1$ is the legal husband of **y**, then **x** and $\mathbf{x}^1$ are identical. And **x** is the legal husband of **y** and $\mathbf{y}^1$, then **y** and $\mathbf{y}^1$ are identicl. As per the argument Russell would remark that his definition is not circular.

Russell's definition of number is significant enough in illustrating his philosophical method. This is known, and is called by Russell, 'the principle of abstraction'. This principle, as it has often been suggested, may also be called the principle of **dispensing** with abstractions. Let us see how this principle works. A 'number' is picked out, by abstraction, from a set of groups which possess a common numerical property - this is what we ordinarily do. But Russell says that there is no way of showing that there is only one such property - the one we have picked out. Abstrction leaves us, indeed, with a class of properties instead of a single property. The 'principle of abstraction avoids this difficulty. The principle defines by reference to a class consisting of all the classes which have a unique relation, i.e., one-to-one correspondence, to each other. Such a definition does not rule out the possibility that there is a property common to all the members of these classes. But it does not need to make that presumption. At this point emerges a principal driving force behind Russell's philosophy - the desire to reduce the number of entities and properties which must be presumed to exist in order to give a 'complete account of the world'.

The definition of number in terms of classes is not itself paradoxical. But, as Russell has noted, it threatens to produce a paradox. The notion of 'a class of all classes' is particularly difficult. The class of all classes is itself a class. It follows that it is itself a member of the class of all classes. It means that it includes itself as a member. The case is generalisable. The class of things which are not men is itself something which is not a man. But it is also true that there are classes which do not include themselves. For example, the class of things which **are** men is **not** itself a man.

We must then admit that classes can be either of two types : (a) those which are members of themselves, and (b) those which are not members of themselves. Let us consider the class which consists of all the classes which are not members of themselves. Is this class a member of itself or not ? If it is a member of itself, then it is not one of the classes which are not members of themselves. And yet to be a member of itself, it **must** be one of those classes. Here, then,there is a manifest contradiction. Equally if it is not a member of itself, then it is not one of those classes which are not members of themselves. This again is a contradiction, which leads us to an antinomy. Either alternative implies a contradiction.

This contradiction has come to be known as Russell's paradox. There have been paradoxes earlier in Philosophy, for example, we have referred to the famous liar's paradox in the first chapter, as an instance of self-referencial paradox. It is noteworthy that Russell has restated the liar's paradox and has given it a great importance. Suppose a man says "I am lying". In this case if what the man says is true he is lying that is what he says not true. Again if what he says is not true, then also he is lying that is what he says is true. Does this paradox stand on the same footing with the paradox of the class of all classes ? The liar paradox could be passed by as a mere ingenuity. But the paradox of all class of all classes could not be similarly regarded.

We have considered, in an earlier chapter, Frege's awareness of the difficulties of the logical system by which he attempted to establish logical bases of arthmetic. Frege had written to a correspondence : "I found it hard to decide to accept the classes or the extension of concept, because it does not appear to say enough to me, and this indeed has proved to be so ... I am now sure that it was a mistake to venture so far in this initial idea of mine"[1]. Russell was aware of Frege's work and sent him his paradox. Frege was greatly purterbed. He wrote in the appendix to his **Fundamental Laws of Arithmetic** that hardly anything more is unfortunate can affect a scientific writer than to have one of the foundations of his edifice, shaken after the work is finished. In short Frege thought that Russell's paradox did shake the foundation.

Frege's reaction to Russell's paradox may be put as follows. If the logistic construction of arithmetic is to be carried through we must be able to pass from a properly constitute a concept to its extension. So that we ought to be able to talk without contradiction about the members of the property constituted class of classes which are not members of themselves. Yet this is just what Rusell's paradox seem to rule out. It may be noted in this context that Frege attempted the solution of the difficulty. He modified his account of equal extension so as to exclude the extention of a concept from the class of object which fall under it. In doing so it will no longer will be permissible to say that the class of things which are not meant (the extension of the concept of **not-men**) itself not a man. In other words, the class of classes which are not member of themselves is a member of itself. Frege believed that by addition of limiting condition to his proof he would be able to avoid Russell's paradoxes.

We have seen in a preceeding chapters that Russell introduced the theory of types as a solution of his paradoxes, though he was never wholly satisfied with it. Russell describes his theory of types as chaotic and unfinished, even though it has had important effects on the development of contemporary philosophy.

Let us now look back to Russell's own solution to his paradoxes. Russell says that all paradoxes arise out of a certain kind of vicious circle. Whenever we suppose that a collection of objects may contain members which can only be defined by means of the collection as a whole,the vicious circle raises its head in Mathematics and in Logic. Let us take a case. Suppose we say that, 'All propositions have the property ϕ.' Now this is itself a proposition. The class of propositions has among its member the proposition that all propositions have the property ϕ. This is a contradiction because the class is both completed or not completed. It then brings out the fact that there is **no such class**.

What shall we do now ? Russell says that statements about all propositions are meangingless. The only way of developing a theory of propositions is to break up the pseudo-totality into sets of propositions. Each set has to be a genuine totality and a separate account must be given of each such set. This breaking up is the object of the theory of types, but we should also note that the theory of types applies to propositional functions rather than a proposition. Russell thought that it is not propositions but propositional function that is more important for mathematics.

The theory of types has two forms - The simple and the ramified. It is sometimes said that there are two theories of types. In the earlier chapters we have dwelt upon the simple theory of types, and had left the ramified theory unconsidered. However we shall say something more about it.

The simple theory depends upon the conception of a range of significance. In the propositional function '$\underline{x}$ is mortal', $\underline{x}$ can be replaced by a certain constants, in such a way as to form a true proposition. But in certain cases the resulting proposition may be neither true nor false but meaningless. The trichotomy, true, false and meaningless is nothing new as Russell himself has pointed out. In Mill's **System of Logic** one can come across it, traces of the tricotomy may be found in Frege as well. The constants which when substituted for x are said to constitute the range of significance or type of the function. In the case of '$\underline{x}$ is mortal' the range of significance is restricted to particular entities. Mortality can be sensibly asserted of any particular thing. But it is without meaning to describe "class of men" or "humanity" as being mortal. The general principle is that a function must always be of a higher type than its argument. In the proposition 'socates is mortal', 'Socates is the argument" of "mortal" which is the function. It is now clear that mortality or 'mortal' cannot be meaningfully predicated of 'class men'. It is so because a thing can be a member of a class, and a class cannot be a member of anything less than a class of classes. An individual can be a member of a club but club cannot be a member of anything less than a association of class. In the paradox of the class which is a membr of itself this rule is ignored when it is presumed that all classes are of a single type and that any class could be a member of another class a vicious circle arises. 'The class of all classes' becomes a class additional to all the classes of which it is the class. Russell has tried to show that if the distinction between the types is formally maintained then one would not say of a class either that it is or that it is not a member of itself. In this way the paradox would vanish. It is true that distinction between types are respected in everyday speech. No one says that 'humanity is not a man'. But the problem lies elsewhere. The difference in type between humanity and a man is obvious. But such fundamental notions of Logic as truth, falsity, fuction, property class etc. have no fixed definite type. As a result different kinds of truth, for example,

a) The first order truth, (x is y)

b) The second order truth, (x is y) is true

c) The thrid order truth, (((x is y) is true) is true) are simple talked about as truth. Paradoxes inevitably arise as we are led to imagine that proposition about truth are about themselves. In point of fact (b) & (c) are really higher order truths about the first order truth. The only way out of the paradox is explicitly to mention what **order** of truths or classes or functions we are talking about.

There hangs an uneasiness about the concept of type. How is it possible to say that **Socrates** and **mankind** are of different types, unless there is some single general sense of type. To ascribe a single function **are of different types** to arguments of different types is to sin against the different types. Several commentators, notably, Rudolf Carnap, have suggested a linguistic interpretation of the theory of types. Russell has been inclined to say that it is a mistake to speak of **entities** as being of this or that type. It is expressions which differ in type. It is the **words** 'Socrates' and 'humanity' have different **syntactical** functions. This means that the theory of types has its application in the language of the second order language.

It has been suggested that Russell has grouped together paradoxes that are different in character e.g. the paradox about classes and the paradox of the liar. The paradox about classes arise within the attempt to construct a logical system. But the paradox of the liar is linguistic and arises only when we try to **talk about** that system. The simple theory of types aims at resolving paradoxes within a logical system. And the logician needs not concern himself with the paradoxes of the linguistic sort. These can be removed by clearing up ambiguities.

References :

1.Wittgenstein, L, ***Tractatus Logico-Philosophiens***, tr. D.F.Pears and B.F.McGuinnes, London, 1961

2. Quoted by Anton Dumitriu in ***History of Logic***, Vol.4, p.63, Abacus Press, Turnbridge Wells, Kent, 1977.

3. Russell's "Reply to his critics" in ***The Philosophy of Bertrand Russell***, ed.

4. Smart, J.J. "Whitehead and Russell's Theory of Types", ***Analysis***, 1950.

5. Weiss, P., "The Theory of Types", ***Mind***, 1928.