Chapter VIII

RUSSELL'S PARADOX AND ITS CRITICS

We shall now consider a couple of critical points concerning the set theoretical problems.

W.O.Quine has dealt with the problem concerning the theory of proof in class theory vis-a-vis mathematics. He remarks that the general method of proof in the theory of classes is deduction of theorem from axioms. In mathematics the axioms give laws for the special predicate "=". In the theory of classes the starting point consists of axioms giving laws for the special predicate "∈". According to Quine the most notable point is the principle of construction namely every movadic predicate has a class as extension which may be formulated as follows

$(\exists\alpha)(x)(x \in \alpha. \equiv Fx)$

There is an α, and for every value of x, x is a member of α equals that x has the predicate or property F.

We have already noted that the reduction of the theory of natural numbers to the theory of classes and relation was done by Frege. The execution of the whole reduction program with full detail came with Russell's PM. Hence it is a remarkable fact that the concept of all the branches of pure mathematics can be defined within the notion of the theory of classes. So it may be said that the classical mathematics reduces to logic but this entails a problem.

Kurt Gödel has argued for the incompletability of the number theory. From

Gödel's conclusion there follows the corresponding conclusion regarding the theory of classes. Gödel's discovery came as a shock to preconceptions. Common sense had been on the side of supposing that complete method of proof for number theory and for mathematics in general, not merely were possible and principle, but were even already at hand. Where does the mathematical truth lie if not in the possibility of truth. Since the number theory is translatable into the notation of the theory of classes, the incompletability of the number theory implies the incompletability of the theory of classes. Any effort towards a complete deductive theory of classes and of classical mathematics is doomed to failure. In view of the Gödel's result our knowledge about the class of numbers is subjected to unexpected limitations.

Gödel's proof was published in 1931, and it was the proof of the existence of formally undecidable propositions in any formal system of arithmetic. His **first incompleteness theorem** states that in any formal system S of arithmetic, there will be a sentence P of the language of S such that if S is consistent, neither P nor its negation can be proved in S. The technique used in proving this result is to translate the syntax of the language of S into arithmetic, thus making S capable of representing its own syntax. This makes it possible to show that there must be a sentence P of S which can be interpreted, very roughly, as saying 'I am not provable'. It is shown that if S is consistent, this sentence is not provable, and hence, it is sometimes argued, P must be true.

Gödel's results bring a new branch of mathematical theory to maturity. It is known as Metamathematics or proof theory. The subject matter of this branch is a mathematical theory itself.

Russell's paradox of set theory implies that if there were such a set as designated by an open sentence such as

~ (x is not a member of x),

then it determines no set at all. In that case it would have to be a member of itself if and only if not a member of itself . There have been attempts to get over the logical uneasiness generated by Russell's paradox. Nobody wishes to assert that the class of man is a man.

It is so because the class does not belong to itself. When we say that something belongs to a class we mean that it falls under the concept whose extension the class is. Now, let us try to understand the concept, namely the class does not belong to itself. The extension of this concept is a class of classes that does not belong to themselves. Let us call it class K. Does it belong to itself ? Let us suppose that it does. If anything belong to a class it falls under the concept whose extension the class is. Thus if our class renounced itself it is a class that does not belong to itself. This supposition leads to self-contradiction. Again if we suppose that our class K does not belong to itself, then it falls under the concept whose extension it itself is. Therefore it belongs to itself. In this case we get a contradiction.

This is a very unsatisfactory situation. Can we say that there might be concepts if no corresponding classes exist. If we may be permitted to suggest that the concept of a class which is not a member of itself is such a concept with no corresponding classes, then Russell's paradox may disappear. This is so thought because in the conjecture there is no class of all classes which are not members of themselves. In point of fact the conjecture is known as Frege's way out by Quine[1]. But Frege did not reconcile himself to this revolutionary idea.

Russell in course of time came to suggest that there could be some propositional functions which did not determine genuine classes. In that case the propositional function is non-predicated. But the contradiction arises because if classes are admitted incautiously and they turn out to be self productive. In short non-predicative functions determine self producting classes.

It has also been suggested that such classes could be excluded by special limitation on the use of the class notation. But this suggestion is also not very satisfactory.

It may be worked out while to ask about, William and Martha Kneale say, the sole cause of difficulty, in Russell's account, they put forward the view that Russell's "no classes" theory itself discovers a paradox. Could we not talk in some other way in order to avoid paradox ? Can we not talk of the class of ...? The following suggestions may be considered. In step of the class which is supposed to contain all the classes that are not members of themselves, let us consider the property of being a property which does not exemplify itself. If this property exemplified itself then it cannot exemplify itself; and if it does not exemplify itself then it must exemplify itself. So it is clear that even if there is no talk of classes the nature of the trouble is the same as is the original paradox.

It may be noted that the French mathematician Henry, Poincaré had suggested that Russell's paradox resulted from the wrong conception of mathematics. Poincare' accepted Russell's distinction of predicative and non-predicative functions and pointed out that some phrases which looks like definition do not define anything.

Russell believed that the paradoxes all had a common root in their violation of a definitely valid rule which he called '**the vicious circle principle**.

Formulation of this principle

"If provided a certain collection had a total, it would have members only definible check in terms of that total, then the said collection has no total."

In Poincare's view the vicious circle which gives rise to the paradoxes of the theory of sets is connected with the attempt to treat of a definite set as a completed one.

It is interesting to know Russell's reaction to Poincare's criticism. Russell agrees that all paradoxes as due to vicious circle. But such circles can be avoided by following a principle, namely whatever involves all of a collection must not be one of a collection. Russell says further that Poincare′ did not appreciate the difficulties of applying the principles.

We have already taken into account how close Russell's paradox is to the theory of paradox. In point of fact, Russell's paradox arises as a consequence of not heeding to the type distinction. In course of time it has been found that all paradoxes are not of the same stock. A distinction has come to be made between logical paradoxes and epistomological paradoxes. This distinction was made by F.P.Ramsay in his paper titled : "Foundation of Mathematics". Russell's paradox is a logical paradox, and so were the paradoxes of Burali-Forti and Cantor, but it has epistomological overtones.

An epistomological paradox is sometimes called semantical paradox. An example of semantical paradox may be illustrated by words which designate property but is not exemplified, by itself. For example, the word 'French' is not a French word. Such word is called 'heterological'. The contradiction will arise if and when we ask whether or not, the word 'heterological' is heterological. If it is heterological then it designates a property which is not exemplified, and since it designates heterological it is not heterological. But if it is not heterological then the property which it designates is exemplified by itself, so that it is heterological. The contradiction therefore is this: - If it is, then it is not and if it is not, then it is heterological. Let us symbolise the property of heterologicality by '**Het**'. Then the contradiction may be stated as follows

Het ('Het') $\equiv \sim$ Het ('Het')

Now the problem is how shall one eliminate the contradiction. It has been suggested that an elimination is possible by advocating a more complicated version of the theory of logical types, known as the Ramified theory of logical types. According to simple theory of logical types all entities are divided into different logical types of which is lowest contents all individuals, the next consists of all properties of individuals designated by functions of individuals. The next type will consist of all properties of properties of individuals and so on. This is in short the simple theory of logical types. The Ramified theory divides each type, excepting the type of individuals into a further hierarchy. Thus all functions which may be significantly predicated of individuals are divided into different orders into different orders.

The difference between the two theory of types may be stated as under. The simple theory of types proceeds from speaking about of all functions or property. We are permitted to speak only about all functions of functions of individuals. The Ramified theory of types refrains from speaking about of all functions and properties of a given type. We are permitted to speak only about all first order functions of a given type or all second order functions of a given type.

It should also be noted that the Ramified theory of logical types divides propositions into a hierarchy of propositions of different orders. Any propositions is said to be a first order proposition if it made no reference to any totality of propositions.

It has been noticed that the Ramified theory of logical types prevents the paradox of the liar namely that an Indian says that all Indians are liars. This theory, including both the hierarchy of types and the hierarchy of orders, was recommended by Russell for its ability to solve certain contradictions.

Notes

I. It is well known that Quine has built a logical system meant to replace the system of PM, but which maintains all the theories of this system. The theory of types is replaced in his system by the "theory of stratification". Quine has shown that his system excludes the antinomies of Burali Forti and that of Russell, but it is however not certain that it excludes all possible antinomies. This is another story,. we simply note the fact, we are not directly concerned with the matter.

II. Gödel's contributions in mathematical logic are important and spectacular. Of these the following are worth mentioning for our purpose.

(a) The arithmetisation of meta-languages. The problem was also treated by Leibniz.

(b) The proof of limitation of the logical formalisms.

(c) The corollary of this proof, according to which the consistency of a system cannot be demonstrated in the system itself.

(d) The impossibility of demonstrating the consistency of the real members starting from the consistency of the whole numbers.

III. It is claimed that the paradoxes in the set theory find their solution in the intuitionistic conception. Paradoxes do invalidate the principle of the excluded middle, they offer propositions which cannot be declared either true or false. In order to find a way out, the Dutch mathematician L.E.J. Brouwer set up a new logical philosophical doctrine as intuitionism. His ideas began to be expounded about 1907.

Brouwer's initial principle is that any proposition which has any contents must indicate one or more states of affairs which are well defined and accessible to our experience. The consequence of this principle is that, according to Brouwer, in the field of infinite collections it is meaningless to say that a certain element belongs to a set E. without being able to point this out. How can we then claim that a collection has an infinity of elements, if we cannot point out every member of the collection? Thus a proposition has a certain content when our immediate intuition relates it to certain states of facts. In the following proposition a, "every element in the set K has the property P", if the set K is infinite, then the negation of this proposition, "a is false" does not satisfy the principle, as it is not possible to prove for an infinity of elements, these states of facts which impede them having the property P.Brouwer's conclusion is that - without any justification - mathematicians, imitating what happens in philosophy, extrapolate logical truths, considering them as being "ideal" and admitting them as valid even where a direct verification is not possible, namely the field of infinite. The belief in the unlimited efficiency of the principle of the excluded middle for the study of natural laws involves the belief in the finite character and atomic structure of the world. In other words, the principle of the excluded middle is not valid and originates only by the projection of mathematics on a finite system in natural sciences. In Brouwer's opinion, this is why paradoxes have appeared.

Only in the range of the finite does a negation of a proposition have a precise meaning. The principle of the excluded middle cannot be applied in a general form. The proposition "the object A has the property P", if it is affirmed that it is absurd that this proposition is false, if does follow that it is true. The absurdity of absurdity does not imply truth, though the converse is true. The absurdity of absurdity gives a particular modality to propositions, which differs from the simple notions of true and false. By straightening the domain of validity of the principle of the excluded middle, Brouwer believes he can eliminate the paradoxes of the infinite.

In intuitionistic mathematics neither logic nor philosophy is presumed. Mathematics is made independent of any hypothesis outside itself, and the whole mathematics is reduced to arithmetic, the fundamental concepts of which are offered by intuition.

IV. Rudolf Carnap has a point against Russell's formulation of the paradox. Carnap does not think that Russell's introduction of the distinction between types is really avoid the so-called antinomies. Russell's antinomy centres on the concept of those properties which do not apply to themselves. So long as no distinction is made between predicates of different levels, it will appear meaningful to say of a property F that either it applies to itself or it does not. Thus we might make some such definition as the following : a property **is impredicable** in case it does not apply to itself. Symbolically,

'Impr (F) $\equiv \sim$ F (F)'

Substituing for the free variable 'F' of this definitional formula the defined predicate 'Impr' itself, we obtain

'Impr (Impr) $\equiv \sim$ Impr (Impr)'

But this sentence, in Carnap's idiom, like every sentence of the form

'p$\equiv \sim$ p' is L-false. By 'L-false' Carnap means that the range of the sentence is the total range, hence provided it is true in every possible case. Every L-true sentence is true, for since it holds in every possible case. Hence the range of an L-false sentence is the complement of the L-true sentence.

As the definition leads to a contradiction, it is antinomous. Carnap now makes the point that if the distinction of types is introduced, then the expression

'F(F)' is not an admissible substantial formula because a predicate must always be of higher level than its argument expression. In other words, the definition above cannot be set up, and the antinomy above cannot be set up, and the antinomy vanishes with it.

Carnap follows Ramsey in remarking that Russell originally undertook a further subdivision of the types, which led to the so-called ramified system of types. In connection with this ramified system certain fresh difficulties arise, for whose elimination Russell required the axiom of reducibility. According to this axiom, for any function of an arbitrary order there exists another function of the first order formally equivalent with it. The introduction of this axiom, required by the foundations of mathematics, has been criticized, mainly because it could lead to new paradoxes. Ramsey has shown that a further sub-division of types is unnecessary, and that the simple system of types is sufficient. Thus the axiom reducibility becomes superfluous. As a consequence of this criticism, especially that of L.Chwistck Russell recognized the difficulties of the ramified theory of types and the axiom of reducibility, and he has pointed it out in the "Introduction" to the second edition of **Principia**.

References

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