Chapter V

RUSSELL'S FORERUNNERS : PEANO AND AND CANTOR

Historically speaking the mathematics of number is a late development. We say this because a mathematics of number has not traditionally been organised in an axiomatic form. Arithmetic, school Algebra, the Differential and Integral Calculus have customarily been presented as collections of rules of calculation, rather than in the form of axiomatized systems of laws. The modern mathematics of numbers had originated in the mathematics of Babylonians, Hindus and Arabs than in that of Greeks. The Greeks gave geometrical interpretation to numbers.

The Babylonians, Hindus and Arabs developed symbols and rules of calculations and made it possible to deal with numerical problems more abstractly than could the Greeks. But eastern mathematics did not concern themselves

(a) with giving proofs, and

(b) organising their knowledge of numbers into axiometric form.

So it remains that Geometry has been handed down to us in an axiomatized form which Euclid had given. The development of the mathematics of numbers has given rise to Philosophical puzzlement. The whole numbers are not disturbing, fractions have been regarded as quotients of whole numbers. But when the Babylonians face a difficulty in referring to the result of substracting a number from itself, a symbol for zero was introduced and it began to be treated as though it were a genuine number. Another source of uneasiness must have been encountered in the case of negative numbers. Sometimes they have been said to be Ghost of number and so on.

Philosophical puzzlements about various kinds of numbers came to be tackled by nineteenth centry mathematicians by developing a **unified theory of numbers**. This was a very important achievement. These mathematician tried to show.

(a) The mathematical theories concerning sophisticated kinds of numbers can be reduced to or constructed from a theory concerning only the basic kinds of numbers.

(b) How each of the more sophisticated kinds of numbers (together with operations such as addition and multiplication) performable on numbers of that kind can be **defined** in terms of the whole numbers and the operations performable upon them.

(c) They also showed that these can be done in such a way that the laws which govern more sophisticated kinds of numbers can be **deduced** from the laws that govern the whole numbers. This development is called Arithmetization of analysis. That is to say the unified theory of numbers is concerned with showing that analytical mathematics can be reduced to elementary number theory of arithmetic.

We have tried to show that unified theory of numbers enables us to regard the various kinds of numbers as belonging to a single family. We shall take a look at the way higher types of numbers can be reduced to the basic kinds of numbers.

The basic kinds of numbers are called natural numbers (unfortunately there are mathematicians who do not like to include zero among natural numbers but there are others who count it in). By natural numbers are meant all those numbers each of which can be reached by starting from zero and adding one as often as necessary. The Italian mathematician Peano was the first to organise the fundamental laws of rational numbers in axiomatic form.

We propose to give a brief outline of Peano's achievement in the axiomatization of the arithmetical system. This is in **Arithmetices Principia**.

The axiomatic theory of numbers set out in **Arithmetices Principia** (The Principles of Arithmetic) consists of the exclusive logical presentation of the axiomatic structure of the system of natural numbers. Peano's axiomatization is in essence with the following. He has shown that the theory of natural numbers can be finally constructed starting from 3 primitive ideas and 5 initial propositions, besides the purely logical ones. The importance of this achievement was remarkable. Bertrand Russell underlines the importance of Peano's axiomatization (Introduction to Mathematical Philosophy, Chap.1, London, 1919). He affirmed that the three ideas and five propositions had become somehow the guarantees of the entire tradition in pure mathematics.

The primitive ideas in Peano's final aximoatization are the following Primitive ideas.

(1) Zero

(2) Number (Numerous - interger positions)

(3) Successor (Sequens)

These ideas are not defined, but only intuitively accepted. It is accepted that it is known what "zero" is. By "number" Peano understands the class of "natural numbers" and to say "2 is a number" means "2 belongs to the class of natural numbers". By "successor" he indicates the immediately following number to a given one. 1 is the successor of 0,2 is the successor of 1 and so on Peano's axioms :

(1) 0 (zero) is a number.

(2) The successor of a number is a number.

(3) Two numbers cannot have the same successor.

(4) 0 (zero) is no successor to any number.

(5) Any property of 0 (zero) as well as of any successor to a number which has this property is common to all numbers.

With these 3 primitive ideas and 5 axioms, Peano succeeds in reconstructing the arithmetic of natural numbers, and later the whole of arithmetic.

Though Peano's axiomatization is the best logical achievement in the field of arithmetic, it still suffers from a drawback which has been pointed out by Russell[1]. He says that Peano's primitive ideas, namely 0 (zero), number and successor, are compatible with an infinite number of different interpretations which all satisfy the conditions set by the 5 axioms. For instance, supposing that "0" means 100, and supposing also that the expression "number" stands for all numbers starting with 100 in the natural sequence of numbers, then the axioms are satisfied, even the fourth axiom, because, although 100 is the successor of 99, this quantity 99 is not a "number" in the sense given to this word in this instance. Let us now give to 0 (zero) its usual meaning, says Russell further, and let us designate by "number" that which we usually mean by even numbers; the "successor" will result if we add the number "2". So, the figure 1 will represent the quantity 2, the figure 2, the quantity 4 etc. The sequence of natural numbers will then be:

zero, two, four, six, eight, ...

It is clear that Peano's axioms are satisfied.

Let us suppose now that "0" stands for the quantity "one" and that we call "numbers" the group

1, 1/2, 1/4, 1/8, 1/16, ...

admitting that "successor" means "one half". The 5 axioms of Peano are valid for the above group.

An infinite number of examples can be thus given : any progression satisfies Peano's axioms and could consequently be taken as "the sequence of natural numbers". In Peano's system, says Russell, there is no possibility of distinguishing the various interpretations of the primitive ideas. In other words, what can be reproached to Peano's axiomatization is the fact that it is not characteristic of natural numbers (hence of the whole arithmetic), and consequently neither for the whole of mathematics.

It may be noticed that peano's axioms contained three undefined terms : zero, successor and natural numbers. It should also be noticed that the axioms by themselves do not show us what these terms are supposed to mean, nor do they give us any hint that the terms refer to anything real. Now if we are to accept the axioms are true then we must supply meaning and evidence from our sides. As the three terms are used in the axioms, we are to passively assume that "zero" refer to some one definite entity among numbers, and that there is just one entity among numbers that is its immediate successor. It follows from the axioms that nothing holds in a natural numbers which is not covered by the axioms. Also follows that since zero is not a successor to any natural numbers the series cannot circle back to its starting point. Another important point to know is that the series of natural numbers cannot stop. On the basis of Peano's axioms we can introduce the names of the numbers. For example we may say, and in fact we do, "one" is by definition the name of immediate successor of zero or "two" by definition is the name of the immediate successor of one.

Peano's axioms do not provide us with a complete theory of the natural numbers. Why? There appear to be two reasons for these

(1) If we limit ourselves to Peano's three primitive terms and his five axioms, it becomes impossible for us to define addition and multiplication for natural numbers and cannot also express them **within the system**. Can we prove within the system such laws as the sum of natural number x and y is always the same number as the sum of y and x ? Answer is no.

If we are to reduce higher kind of numbers to natural numbers we need two important terms, namely set and ordered pair. These are not included among Peano's primitives.

How are we to understand the terms set and ordered pair ? These terms are understood in a rarefied way. A set is a class, collection or group of things. The things belonging to a set may be of any kind, concrete (e.g. set of books) or abstract (e.g. set of theories). They may or may not be closely similar or closely connected with one another. But the essential point is that a set has to be thought of as a single entity, and has to be distinguished from the things that are members of it. The set is a very different thing from its member. Let us take a concrete example. Let there be a set of mathematicians. The members of this set will include every mathematician but nothing else. Each member is a mathematician but a set of all mathematicians is not itself a mathematician. The set is numerous that is it has many members but no mathematician is numerous.

Two sets are sometimes said to be identical when they have exactly the same members. For example, the set of equilateral triangles is identical to the set of equi-angular triangles. Supposing we are permitted to say that there can be sets having no members, interesting results follow. According to the criterian of identity all empty sets are identical. This means there can be only one empty set. For example the set of golden mountains is identical to the set of square circles. These sets have exactly the same members - none.

Let us now turn to the relationships between the sets. One set is a subset of another when all the members of the former are members of the later. Thus the set of mathematicians is a subset of the set of human beings. we must be very cautious to distinguish between a subset of and a member. Copernicus is a member but not the subset of a set of mathematicians. The set of mathematicians is of course a subset but not a member of the set of human beings.

Let us turn to the concept of ordered pair. This term is also to be understood in the rarerified sets. Let us say that an ordered pair consists of two things of any kind whatever considering a certain order. The things may be concrete or abstract similar or dissimilar. Ordered pairs can also be identical. One ordered pair (x,y)can be said to be identical with another ordered pair (z, w) provided that the two first items, x and z are identical and the two second items y and w are identical. On this basis it should be possible to define ordered pairs as a kind of set of sets.

Let us now see how higher kinds of numbers can be defined. The process of defining the higher kinds of numbers by reducing them to the rational numbers. For example a theory of rational numbers, sets and ordered pairs. This is the first step in defining the higher kinds of numbers. The second step would consist in developing the theory of real numbers, again basing it upon our theory of the rational numbers. The theory of complex number is also developed in the same way. The process of developing the higher kinds of numbers may be formulated as follows. At each step we take for granted that it is understood what the preceeding kinds of numbers are and what it means to add and multiply them. On this basis we define what the next kinds of number are, and we define what it shall mean to add and to multiply them.

The most interesting points to note in this connection is that with this higher kind of numbers name "one" has several meanings. It appears first as the name of a natural numbers i.e. the immediate successor of zero. Then it appears as the name of a rational number. In this case the rational number one is the set containing the ordered pair (1,1), (1,2), (1,3). Again "one" appears as the name of real number. A real number is a set of rationals. The real number one is the set of rational numbers smaller than the rational number one. The point is that we must distinguish between the natural number one, the rational number one, the real number one and so on. The same numeral is used to stand for all of this but they are essentially distinct mathematical entities. The German mathematician Kronecker significantly remarked, "Dear God made the whole number, all the others are human works"[2].

It may well have been noticed that the definitions of higher kinds of numbers we have already made use of the term set. The idea of developing a theory of sets goes back to the German mathematician Cantor in the late nineteen centuty. Cantor's particular contribution was his theory of infinite set and of transfinite numbers. Let us see what these notions are.

Let us consider the passengers in a bus. If the conductor does not allow more passengers to get into the bus, then every seat is occupied by one passenger or every passenger occupies one seat. We can now say that the set of passenger and the set of seats are in one-to-one correlation. This example may be taken to illustrate the notion of one-to-one correspondence. Cantor's theory employs this important notion. Formally speaking we may say that let there be two sets S_1 and S_2. Now the members of the set S_1 standing one-to-one correspondence with the members of the set S_2 only if the members of the one set are associated with those of the other, such that with each member of S_1 exactly one member of S_2 is associated and if each member of S_2 is exactly one member of S_1 is associated. Let us now go back to our example of the bus. The set of passengers has the same number of members as has the set of seats. But if every seat is occupied by a passenger and some passengers do not have seats then a set of passengers shall be said to be larger than the set of seats.

The matter is not simple as it looks like. Questions and doubt arise in this connection.

(a) What do we really should mean when we say that two sets are related to each other by one-to-one correspondence ? Shall we mean that there are two numerically different sets and that they have the same members and that there are the same number of members ? In that case may we say that the two sets in question come within the domain of the axiom of extentionality, which defines the notion of set identity.

(b) If the point made in (a) is unexceptional then what is the guarantee that there are two numerically different sets and not one set. We say this because one-to-one correspondence may be viewed as obtaining between the members of the same set, of course reflexively. Let us illustarate the point. Suppose that there is a set of six unambiguous words. In that case the words and the meaning will be related in a reflexive manner.

Difficulties such as these will necessarily prevail in a conceptual discipline such as mathematics. We do not venture to suggest a solution but at times the awareness of the existence of the problem could be more rewarding that an actual solutions of it.

Let us now go back to Cantor. In our example of the bus we considered two sets of finite size, for buses can not be of an infinitely large size. Cantor had subjected that the members of infinitely large sets can stand in one-to-one correlation. His idea was that when sets contain infinitely many members we can compare the sizes of sets. Cantor also held that two infinite sets are to have the same size if and only if their members can be correlated to one-to-one. Further an infinite set is said to be larger than another if and only if when all the members of the latter are associated with members of the former, and some members of the former are left over.

Let us give some concrete examples of the correlated sets. We all know that numbers are either odd or even. In that case the set of odd numbers having such members as 1,3,5,7,9... is of the same size as the set of even numbers because we can correlate their members in such a way that each odd number is associated with its immediate successor. That is to say the members of the two sets are related to one-to-one and it is on the basis of correspondence of the members of two sets that the sets are said to be of the same size.

Let us vary the example to find more surprising consequences. Let us take two sets one of odd numbers and the other consisting of natural numbers as members. Can we say that these two sets are of the same size. Cantor thinks that by associating the first odd numbers with the first natural numbers and the **n**th odd number with the **n**th natural number, we can show that the members of these two sets are correlated to one-to-one.

Odd numbers

1,3,5,7,9,11, ...

Natural numbers

0,1,2,3,4,5,6, ... (pace Peano)

Looking at the table above it may be remarked that unless one agrees to bring in a point above **n**th numbers both odd and natural that the two sets can not perhaps be said to have the same size.

Cantor thinks that the set of natural numbers is of the same size of the set of rational numbers. But how can that be ? We all know that the set of rational numbers are imagined to be much larger. Let us see how does Cantor show that what he does is the following. Let us arrange the rational numbers in a series so that every rational number has its definite place in the series and also that it is within a finite number of steps from the beginning of the series.

$$
\begin{array}{ccccc}
\frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \ldots\ldots\ldots \\[1ex]
\frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \ldots\ldots \\[1ex]
\frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \ldots\ldots\ldots \\[1ex]
. & . & . & . & \\
. & . & . & . & \\
. & . & . & . & \ldots\ldots\ldots
\end{array}
$$

We may now be able to correlate the first rational with the first natural number and Cantor adds that we may do so, in general the **n**th rational with **n**th natural number. In the series we have expressed each rational as a fraction.

It may be noted that the array is open to the right and down i.e. the series in both the direction does not stop. Every rational number must occur somewhere either of the right or downwards.

It is also to be noticed that the array is two-dimensional but we can arrange all its members in a linear series, if we start with the upper left corner and weave diagonally through the array. So we get

0/1	→	1/1		2/1	→	3/1		....
	↙		↗		↙		↗	
0/2		1/2		2/2		3/2		....
↓	↗		↙		↗			
0/3		1/3		2/3		3/3		....
	↙		↗					
0/4		1/4		2/4		3/4		....
↓	↗							
0/5		1/5		2/5		3/5		....
0/6		1/6		2/6		3/6		....
⋮		⋮		⋮		⋮		

The linear series we get

0/1, 1/1, 0/2. 0/3, 1/2, 2/1, 3/1, 2/2, 1/3,

0/4, 0/5, 1/4, 2/3, 3/2, ...

Cantor wanted to have a series in which every rational numbers occurs only once. And each member of the series with a finite number of steps of the beginning :

Rational numbers : $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}$ ....

Natural numbers : 0, 1, 2, 3, 4, 5 ....

It is in this way that a one-to-one correlation between the members of the set of rational numbers and the members of the set of natural numbers can be shown. Therefore the two sets are of the same size.

Some surprising results follow from Cantor's philosophy of number.

(a) The odd numbers form a subset of the natural numbers. They are just as numerous as natural numbers. And it is also a fact that the natural numbers are as numerous as the rational numbers. Considering the point it might strike some one that the whole thing goes against Euclid's axiom that the whole is greater than the any of its parts. The reason for this surprising state of affairs is that Euclid thought only of finite wholes and never about the Cantorian infinite whole. We are simply noting the point and it is not easy to reach a clear-cut conclusion.

(b) Cantor may be taken to have suggested that all infinite sets are of the same size. But this is not the case. Let us consider the real numbers that are greater than zero but not greater than one. Cantor has maintained that there are more of these real numbers than there are natural numbers.

Now the set of these real numbers may be supposed to be of the same size as the set of all natural numbers. That would mean that these real numbers could somehow all be arranged in a series, for example (r_1, r_2, r_3, ..., r_n, ...) so that the first real number is correlated with the real natural in the series of natural numbers and the **n**th real number could be correlated with that **n**th natural number.

Real natural number can be represented in decimal notation as a non-terminating decimal. A non-terminating decimal means that it never reaches a place after which all the digits are "0".

Examples of non-terminating decimal

$$1/3 = .3333.....$$

Terminating decimal can also be put into non-terminating form 0.303 can be expressed as non-terminating form as 0.3029999 ...

But if we consider the real number (r_0) represented by the following non-terminating decimal its first digit is to be "5" if the first digit of r, is not "5" and is to be "6" otherwise, its second digit is to be "5" if the second digit of r_2 is not "5" and is to be "6". Similar explanation will be imposed in the case of **n**th digit.

This non-terminating decimal must represent a real number greater than 0 and less than 1, yet this real number r_0 is so defined that it can not be identical with any real number in the series $r_1, r_2, r_3, \ldots, r_n, \ldots$

Thus the real number r_0 has not been correlated with any natural number. This contradicts our supposition that a one-to-one correlation between these real numbers and the natural numbers was possible. Therefore such a one-to-one correlation is not possible, and there are more of real numbers than natural numbers. Hence we conclude that the set of real numbers is larger than the set of rational numbers.

Cantor's main contribution lies in the direction of development a theory of transfinite cardinal numbers. We may explain the notions as follows.

"A cardinal number measures the size of a set, finite or infinity; transfinite cardinals measure the sizes of infinite set."[3]

(a) It follows that a set of natural numbers will have the smallest transfinite cardinal number.

(b) The set of real numbers will have a larger transfinite cardinal number.

(c) The set of all subsets of the set of real numbers will have an even larger transfinite cardinal number.

How did Cantor arrive at the conclusion mark (c)? The question can be answered in the following manner.

A non-empty set may be either finite or infinite. But in either case a non-empty set has more subsets than it has members. This means that the cardinal numbers of the set of subsets of a given non-empty set must always be larger than the cardinal number of the given set. We find that however large a cardinal number is, there may be other cardinal numbers larger than it. So it follows that there are infinite number of cardinal numbers and they can be arranged in an ascending order.

We may now round up our discussion of Cantor's theory of transfinite cardinal number. We have noticed that Cantor's conclusion have been surprising enough to make us feel uneasy. But they are quite natural from the mathematical point of view. His results are to be looked upon as theorems of a system whose axioms express the basic laws of a natural numbers and of sets and ordered pair.

It may be pointed out in passing that not all mathematicians agree with Cantor's theory of transfinites. For example the intuitionist school of mathematics takes a radial line. According to them the numbers are the creations of the mind.

Definitions construct the entity defined and therefore a set cannot be regarded as existing unless it has previously being constructed by our deciding what its members are.

There is another school of mathematics represented by Russell and Whitehead. According to them the law of the mathematics of number are derivable from or can be reduced to logic alone. This is known as the logistic thesis. Acccording to the logistic thesis the laws of Arthmetic and the rest of the mathematics of the number are related to those of logic in the same way as the theorems of Geometry are related to its axioms.

"Ordered pair" and the laws governing sets of ordered pair are counted as belonging to logic rather than to mathematics. They defined the natural numbers as certain kinds of sets of sets. "0" is defined as the set of all empty set. "One" is defined as the set of all non-empty sets, each of which is such that anything belonging to it are identical and so on. One set of sets is said to be immediate successor of another when one member is removed from any set belonging to the former, then the diminished-set belongs to the latter. All the natural numbers have been defined in the logistic thesis as anything belonging to every set to which zero belongs and to which belongs the immediate successor of anything that belongs.

Refferences

1. ***Principles of Mathematics***, Chap. I
2. Quoted in ***Introduction to the Foundations of Mathematics*** by Raymond Wilder, New York, 1962.
3. Stephen Berger, ***Philosophy of Mathematics***, p.66.