Chapter IV

RUSSELL'S CONCEPT OF CLASS

In this chapter we shall be concerned with Russell's derivation of the concept of 'class'. Usually the concept of class is taken as a primitive concept. But Russell's does not take it as a primitive notion. On the contrary he seeks to **derive** the notion and it will be our purpose to see how does he do that.

Russell appears to derive the notion of a class from the notion of propositional function.

For Russell a proposition is a form of words which expresses what is either true or false. He further says that the word 'proposition' consists of symbols as may give expression to truth and falsity. Let us see what Russell means by the statement. The bare formula

'$(a + b)^2 = a^2 + 2ab + b^2$'

is not a proposition. That is to say the bare formula

'$(a + b)^2 = a^2 + 2ab + b^2$'

does not say anything definite unless we further add or suppose that a and b are to have such and such values and it is only then, that the formula would become a proposition.

When no assumptions are made about the values of the variables the mathematical formula would be a propositional function. As Russell explains it, a propositional function is an expression containing 'undetermined constituent'. When values are assigned to this constituent, the expression becomes a proposition. The bare formula is a function whose values are propositions.

Russell distinguishes a propositional function from a descriptive function. Lĕt us consider the expression "the most difficult theorem in geometry" is a descriptive function but not a propositional function although its values are propositions. A descriptive function describes a proposition which propositional function does not.

The expression 'x is human' is a propositional function, since x is undetermined, i.e. it is neither true nor false. But when a values is assigned to x, it becomes a true or false proposition.

Further, Russell adds that any mathematical equation is a propositional function, so long as its variables have no definite values. A propositional function is a mere schema, 'an empty receptacle for meaning not something already significant'[1].

Russell think that the propositional functions involved in the theory of classes and relations.

We have already mentioned that Russell does not treat class as a primitive idea. In the calculus of classes the symbols for classes do not represent object called the classes. According to him classes are logical fiction or incomplete symbols. Let us ask why does Russell say that 'class' is a logical fiction or incomplete symbol. He does not regard classes as part of the 'ultimate furniture' of the world. This means classes differ from the particulars or individuals. He says that we cannot take classes in the pure extensional way. And this would make impossible for us to understand how there can be such a class as a null class.

It is by no means easy to determine precisely the nature of a propositional function. In the **Principia Mathematica** Russell gives the following explanation.

A propositional function differs from a proposition solely by the fact that it is ambiguous. It is the ambiguity of a propositional function that constitutes the difference. Critics of Russell find the ambiguity some what unfortunate. By 'ambiguity' Russell seems to mean indeterminateness which is characteristic of the variable. To say that an expression containing a variable is ambiguous expression involves a misleading use of language. What is important is that there is a determinate correspondence specified by the functional relation holding between the variables. This is the characteristic of mathematical function.

Russell further distinguishes between the propositional function and an undeterminate value of the function. To mark this distinction he writes '$\hat{x}$ is hurt' to express the propositional function and 'x is hurt' to express the undeterminate value of the function. To this distinction we shall return later.

Let us now consider in what respect a propositional function resembles, and in what respect it differs from an ordinary mathematical function. Russell points out that they agree in containing unassigned variables and differing in that the values of a propositional functions are propositions. This does not seem a sufficiently precise account of the matter. Russell says that propositional functions are the fundamental kind from which other kinds of functions, such as 'sin x', 'log x' or 'The father of x' are derived. Such functions are called **descriptive** function. They mean the term having such as such relation to x, so that the function describes the value of x which satisfies the function. Thus 'log x' means 'the logarithm of x'. If we substitute the determinate value 3 for x in the expression 'log x'. We get the 'logarithm of 3'. Now the 'logarithm of 3' **describes** the number 0.4771. Just as 'the father of Rama' describes Dasharatha.

It is important to distinguish the **number describing** (0.4771) from the **description of it** (log 3), since the same number may have many different descriptions just as the same man may have many different descriptions applicable to him. For example the same number 2 may be described by 'the only even prime', 'the square root of 4' or 'the sum of 1 and 1'. In the mathematical function y=log x, y is the value of the function and if 3 be is substituted for x, then the value of the function is 0.4771.

Frege had supposed that there is a analogy between a propositional function and a mathematical function. We may now consider his view.

Frege thought that every proposition was a description which describes either they are **true** or they are **false**. For example, take "$2^2 = 4$", it describesthe truth. Again "3<2" describes the false. The functional expression $X^2 = 4$, in Frege's view has two values. The true for one argument i.e. true. But false for all other arguments, namely $1^2, 3^2, 4^2$ ... etc. If Frege's view is not mistaken, if his view were correct then propositional function is analogous to mathematical functions. Russell's has discussed Frege's view in **Principles of Mathematics**. If Frege's view were correct we could write "y = x is hurt" and then y would be the dependent variable just as in "y = log x". But there is nothing in the propositional function standing to "x is hurt" as y stands to "log x", in the mathematical function, "y = log x". Thus the two kinds of function are not exactly analogous. In the case of a propositional function there seem to be nothing corresponding to the value of the function. Since there is nothing corresponding to the term described by the mathematical descriptive function, we can not then regard the propositional function as having anything exactly corresponding to the dependent variable in the expression "y = log x".

Derivation of the Concept of the Class from Propositional Function

We have already indicated that Russell derives the concept of class from the concept of propositional function. But before we consider Russell's derivation. We should try to understand the notion of a class from a common sense point of view.

The notion of a **class** is just as familiar as the notion of an **individual**. Accordingly there are class propositions (e.g. All men are mortals) and singular proposition (e.g. Russell is a mathematician). No one supposes, that a class is a object of the same kind and type of an individual. Therefore we must ask what a class is. We shall do it in a two fold way :

(a) By considering how we come to recognize a class.

(b) How we use class symbols.

Let us consider the case of **the people who are learned**. These constitute a class and they are referred to by the terms scholars. So we can say, **scholars who are all the individuals who are learned**. We have now a set of individuals distinguished from other sets of individuals in that each individuals of the set possesses the property of being learned. There are two ways of selecting the individuals who form a class. One way is to enumerate the individuals one after the other. The order of enumeration may be indifferent. The second way is to select a certain property which may belong to many individuals.

Class names of geometrical figure are not derived by an enumeration of all the figures instantiating the class-forming property. It is a set determined by a conjunction of properties belonging to each individuals in the set and to no other individuals. Such a set is a class.

There is an important difference between an enumerative set and a class. The later is determind by a property or conjunctions of properties. The former is selected by an enumeration. The enumerative selection of a class is possible only in the case of finite classes i.e. classess consisting of finite number of members. An infinite class is not denumerable so that such a class must be determined by a conjunction of properties, by means of which the class is selected.

(b) The most remarkable statement that Russell makes concerning classes or the concept of a class is that to be concerned with classes is to be concerned with **the** in the plural, for example 'the inhabitants of Calcutta'. He has also shown that a cardinal number is to be defined as a class of classes and that the number 1 is to be defined as the class of all unit classes i.e. of all that have just one member. When the number 1 is defined as the class of all unit classes we must be clear about the notion of unit class. A class **a** is said to be a 'unit' class if the propositional function x is an **a** is not always false. This gives us a definition of unit class if we already know what a class in general is.

Russell also says the symbol of classes are near conveniences and classes are in fact 'logical fictions'. Russell has a feeling that no definition of class is finally satisfactory. Therefore he says that the classes cannot be regarded as part of the ultimate furniture of the world. Classes cannot be taken in a pure extensional way as simply heaps, because in that case the null class cannot be regarded as a heap. And secondly, a class which has only one member would be identical of that one member.

Therefore Russell is led on to define classes by propositional function which are true of the member of the class and false of other things. Now it should be carefully noted that classess are not to be taken as identical with the propositional functions that may be true of its member. So we have to decide that

(a) the classess cannot be things of the same sort as their members;

(b) that they cannot be just heaps or aggregates; and also

(c) that they cannot identified by propositional functions.

It still remains very difficult to see what classes actually are. Are they something more than the symbolic fictions or they are just symbolic fictions.

What after all is Russell's intension in saying that classess are symbolic fictions and how does the view help us?

From the consideration made above the following conclusion should follow.

(i) Every propositional function must determine a class, consisting of those arguments for which the function is true. Russell's final position is that the class is rendered determined by a propositional function and that every propositional function determines an appropriate class.

(ii) Two formally equally propositional functions should determine the same class. Two propositional functions which are not formally equivalent must determine different classess. That is, a class is determined by its membership and no two different classess can have the same membership. If a class is determined by a function $\phi(x)$, we may say that **a** is a member of the class if $\phi(a)$ is true.

(iii) Russell then proceeds to formulate the notion of classess of classess. **In the Introduction to Mathematical Philosophy** Russell gives the following definition of class of classes.

"The number of a class is the class of all those classes that are similar to it"[2]. The most crucial notion in this definition is that of similarity. Two classes are said to be similar where there is a one-one relation which co-relates the terms of the one class each with one term of the other class. One class is said to be similar to another when there is a one-one relation of which one class is the domain while the other is the converse domain. Two finite classes have the same number of terms if they are similar. The act of counting consists in establishing a one-one correlation between the set of objects counted and the natural numbers that are used up in the process. There are as many objects in the set to be counted as there are numbers upto the last number used in the counting.

It then follows that number is any collection which is the number of one of its member. It may be seen that Russell defines numbers of a given class. The class of fathers will be all those who are somebody's father.

(iv) The most puzzling conclusion what Russell seeks to derive from what has gone before is a distinction between meaninglessness and falsity. There may be certain propositions about classes that may be meaningless but not false. For example, to suppose a class as a member of itself or not a member of itself; this statement results from a contradiction of which we propose to deal with in what to follows:

Russell introduces his concept of the contradiction of the greatest cardinal. The number of classes contained in a given class is always greater than the number of the members of the class, and hence Cantor's paradox shows that there is no greatest cardinal number. But if we add together with one class the individuals, the classes of classes of individuals etc., we obtain a class of which its own subclasses would be members. The class consisting of all objects that can be counted must have a cardinal number which is the greatest possible. Since all its sub-classes will be members of it, there cannot be more of them than there are members. Hence we arrive at a contradiction. Russell's now seeks to apply the proof of Cantor's paradox to a supposed class of all imaginable objects and thereby derive a fallacy which he calls confusion of types.

Confusion of Types :

Russell begins with the idea of the contradiction of the greatest cardinal. The number of classes in a given class is always greater than the number of members of the class. From this he is to infer that there is no greatest cardinal number. If we would add together with one class that individual, the classes of individuals etc., we obtain a class of which one such class will be a member. "The class consisting of all objects that can be counted, of whatever sort, must if there be such a class, have a cardinal number which is the greatest possible"[3]. Since all its subclasses will be members of it, there cannot be more of them than there are members. Hence we arrive at a contradiction.

Russell discovers this contradiction in Cantor's proof that there is no greatest cardinal. Before we proceed to consider Russell's discovered contradiction, we should have a brief statement of Cantor's proof. Let us put Cantor's paradox in the following manner :

Let C be the set of all sets. Then every subset of C is also a member of C is a subset of c, i.e.

$$2^c \subset C$$

But $2^c \subset C$ implies that

$\# (2^c) < \# (c)$

However, according to Cantor's theorem,

$\# (C) < \# (2^c)$

thus the concept of the set of all sets leads to a contradiction. [Notations are explained in the Appendix]

The comprehensive class if it is to embrace every thing then it must embrace itself as one of its members.

As Russell says "if there is such a thing as "everything" then "everything" is something, and is a member of the class "everything"[4].

But we know that the class is not a member of itself. Mankind is not a man. The assemblage of all classes which are not members of themselves is a class. And now the question is, is it a member of itself or not?

If it is one of those classes that are not members of themselves or it is not a member of itself. If it is not the case, then it is not one of those classes that are not members of themselves that is it is a member of itself. There are two hypotheses, that it is and that it is not.Each implies its contradictory. This is a contradiction.

References :

1. Mathematical Philosophy, p.157

2. Ibid, p.18.

3. Ibid, p.136.

4. Ibid, p.136.