CHAPTER - IV

CIRCULAR-ARC DIGRAPHS

4.1. Introduction

A digraph is a circular-arc digraph if it is the intersection digraph of a family of ordered pairs of arcs on a circle. In chapter -II, we gave several characterizations of interval digraphs. The purpose of this chapter is to give corresponding characterizations of circular-arc digraphs. We will also give another characterization of interval digraphs.

A 0,1-matrix has the consecutive ones property for rows [columns] if its columns [rows] can be permuted so that the 1's in each row [column] are consecutive. We can replace "consecutive" by "circular" in the definition to obtain the circular ones property ; by 1's being "circular", we mean their positions are consecutive when we view the positions as a cycle, with the first row (or column) following the last. Due to Helly property for intervals on the real line (a pairwise intersecting family of intervals has a common point), a graph is an interval graph if and only if the incidence matrix between its vertices and maximal cliques has the consecutive ones property for columns. The analogous circular ones property for columns of the incidence matrix

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is a sufficient but not a necessary condition for a circular-arc graph. In particular, deleting a triangle from a complete graph on 6 vertices yields a circular-arc graph for which this condition fails. For circular-arc digraphs, the situation is similar, we get a sufficient but not necessary condition from the circular ones property for the adjacency matrix, as seen in Sec.4.2.

Nevertheless, Tucker's characterization of circular-arc graphs has an analogue for circular-arc digraphs. He introduced a concept called the <u>quasi-circular ones property</u> for a 0,1-matrix. This property uses consecutive appearances of 1's in both rows and columns. Given a 0-1,matrix A, let V_i [W_j] be the 1's appearing consecutively in row i [column j], starting with the first 1 on or after the main diagonal and wrapping around (if possible) until the first 0 is reached. Then A has the <u>quasi-circular ones property</u> if the union of the sets V_i and W_j cover all the ones in A. In particular, let the *augmented adjacency matrix* $A^*(G)$ of a graph be the adjacency matrix (0,1-matrix with $a_{ij} = 1$ if $v_i v_j \in E$), together with $a_{ii} = 1$. For $A^*(G)$, the sets V_i and W_j all start at the main diagonal. Tucker [1971] proved that G is a circular-arc graph if and only if its vertices can be indexed so that $A^*(G)$ has the quasi-circular ones property.

To obtain an analogous characterization of circular-arc digraphs, we need some additional concepts. A <u>stair partition</u> of a matrix is a partition of its positions into two sets (L,U) by a polygonal path from the upper left to the lower right, such that the set L is closed under leftward or downward movement, and the set U is closed under rightward or upward movement. Equivalently, U corresponds to the non-zero positions in some upper triangular matrix and L to the non-zero positions in some lower triangular matrix. The polygonal path will be called a <u>stair of the matrix</u>.

A 0,1-matrix A has the <u>generalized linear ones property</u> if it has a stair partition (L,U) such that the 1's in U are consecutive and appear leftmost in each row and 1's in L are consecutive and appear topmost in each column. We will prove in Sec.4.3 that D is an interval digraph if and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting matrix has the generalized linear ones property. We prove this theorem by verifying that the generalized linear ones property is equivalent to one of the conditions given in Theorem 2.5 in chapter -II. In particular, a 0,1-matrix has the *partitionable zeros property* if each 0 can be replaced by one of {R,C} in such a way that every R has only R's to its right and every C has only C's below it.

This characterization of interval digraphs does extend to circular-arc digraphs in the manner one would hope. Given a 0,1-matrix A and a stair partition (L,U), let V_i [W_j] be the 1's in row i that begin at the stair and continue rightward [downward] (around if possible) until the first 0 is reached. Then A has the <u>generalized circular ones property</u> if it has a stair partition (L,U) such that the V_i's and W_j's together cover all 1's of A. In Sec.4.3, we prove that a digraph D is a circular arc digraph if and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting matrix has the generalized circular ones property. Note how this corresponds to Tucker's characterization ; an augmented adjacency matrix with the quasi-circular ones property has the generalized circular ones property with the stair partition in which U consists of the main diagonal and the positions above it, and the doubly -directed digraph of a circular-arc graph with a loop at each vertex has a circular-arc representation by putting both S_v and T_v equal to the arc assigned to v in the original circular-arc representation.

In section 4.4, we discuss the relationship between Ferrers dimension and circular-arc digraphs. The characterization of circular-arc digraphs described above implies that the complement of every intersection of two Ferrers digraphs is a circular-arc digraph. Also, there are circular-arc digraphs of arbitrarily large Ferrers dimension, but there are complementary digraphs of Ferrers dimension 3 that are not circular-arc digraphs.

4.2. Elementary characterizations

Recall that a (0,1) matrix M is said to have the circular ones property for rows (or columns) if the columns (or rows) of M can be permuted so that the ones in each row (or column) are circularly consecutive. In sec.2.3, we gave an elementary sufficient condition for interval digraphs in terms of the adjacency matrix. In fact, D has an intersection representation by intervals in which all terminal sets are single points if and only if its adjacency matrix has the consecutive ones property for rows. Analogously, we have a sufficient condition for circular-arc digraphs that characterizes those with similarly restricted representations :

Theorem 4.1. *A digraph* D (V,E) *has an intersection representation by a circular -arcs in which all terminal arcs are single points, if and only if its adjacency matrix has the circular ones property for rows (or columns).*

Proof. D has such a representation if and only if there are arcs S(v) and points t(v) in [0,1) such that uv ∈ E if and only if t(v) ∈ S(v), where we may assume the t(v)'s are distinct. These exist if and only if numbering the vertices in circular order of t(v) exhibits the circular ones property for rows of the adjacency matrix. ■

This condition is not necessary in general, as shown by the following example.

Example 4.1. Consider the digraph D(V,E) given in *Fig.* 4.1, consisting of a two -directional path along vertices 1,2,3,4,5 plus a loop at 3.

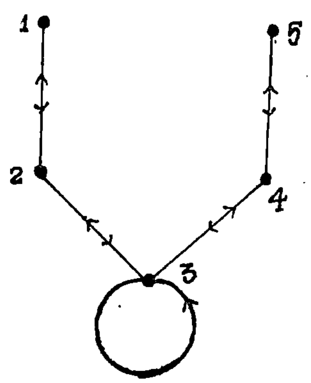


Fig. 4.1

Its adjacency matrix A(D) appears below.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix does not have circular ones property for rows (or columns), but D has the following circular arc representation :

i	1	2	3	4	5
S_i	{-2}	{1}	[-2,0]	{2}	{-1}
T_i	{1}	{-2}	[0,2]	{-1}	{2}

From the representation, we see that this is not only a circular-arc digraph, but in fact an interval digraph.

To obtain an elementary characterization of interval digraphs we defined generalized complete bipartite sub-digraph (GBS) of a digraph D to be a sub digraph generated by vertex sets X, Y whose edges are all xy such that $x \in X$, $y \in Y$. (X,Y are not necessarily disjoint) If $\mathcal{B} = \{(X_k, Y_k)\}$ is a collection of GBS's whose union is D, then the <u>vertex-source incidence matrix</u> for $\mathcal{B}$ (V,X-matrix) is the incidence matrix between the vertices (indexing the rows) and the source sets $\{X_k\}$ (indexing the columns). Similarly, the <u>vertex-terminus incidence matrix</u> for $\mathcal{B}$ (V,Y-matrix) is the incidence matrix between the vertices and the terminal sets $\{Y_k\}$.

Our first characterization of circular-arc digraphs uses these incidence matrices. Its proof is analogous to the proof of the corresponding characterization of interval digraphs (sec.2.3), but we include it here for completeness.

Theorem 4.2. *A digraph D(V,E) is a circular arc -digraph iff there is a numbering of the GBS's in some covering* **B** *of D such that the ones in rows appear on a cycle for both the V,X-matrix and V,Y-matrix of D.*

Proof. For sufficiency, consider such a **B** whose union is D, and let (X_k, Y_k) be a common numbering of the columns of the V,X- and V,Y-matrices that exhibits the circular ones property for both. Assign $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$, where a_v, b_v, c_v, d_v are defined by $v \in X_k$ if and only if $a_v \leq k \leq b_v$ (circularly) and $v \in Y_k$ if and only if $c_v \leq k \leq d_v$ (circularly). Then $S_u \cap T_v \neq \phi$ if and only if $u \in X_k$ and $v \in Y_k$ for some k.

For necessity, consider a representation of D by a family $\{(S_v, T_v)\}$ of ordered pairs of arcs. We may assume they are closed and have endpoints indexed circularly by integers, with $S_v=[a_v, b_v]$ and $T_v=[c_v, d_v]$. For each integer k belonging to any of these intervals, define a GBS $B_k = (X_k, Y_k)$ of the intersection digraph of the interval pairs, by setting $X_k = \{v : k \in S_v\}$ and $Y_k = \{v : k \in T_v\}$. Then $S_u \cap T_v \neq \phi$ if and only if $u \in X_k$ and $v \in Y_k$ for some k, so the intersection digraph of the interval pairs is in fact the union of the specified GBS's. Furthermore, by construction the resulting V,X- and V,Y- matrices have the simultaneous circular ones property for rows. ■

4.3. Adjacency matrix characterizations

In order to phrase our adjacency matrix characterization for interval digraphs in a fashion useful for circular-arc digraphs, we prove the following lemma.

Lemma 4.1. *The partitionable zeros property is equivalent to the generalized linear ones property.*

Proof. The generalized linear ones property implies the partitionable zeros property by changing each 0 in the upper part of the stair partition to R and each 0 in the lower part to C. For the converse, consider an assignment of all R's and C's to 0's that exhibits the partitionable zeros property. Note that no C appears above an R, and no R appears to the left of a C. Therefore, if we let U' consist of all R's and positions above them and L' consist of all

C's and positions to their left, then U' and L' are disjoint and can be extended arbitrarily to a stair-partition U,L with U' ⊆ U and L' ⊆ L that demonstrates the generalized linear ones property. ■

Theorem 4.3 *A digraph D is an interval digraph if and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting matrix has the generalized linear ones property.*

Proof. Theorem 2.5 of chapter-II states that D is an interval digraph if and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting matrix has the partitionable zeros property. ■

We note that we could very well prove the above theorem of characterization of interval digraphs starting from scratch in a very similar manner to the proof given in chapter -II for characterization of interval digraphs in terms of intersection of Ferrers digraphs. As we will see, the proof of our characterization of circular -arc digraphs will also prove Theorem 4.3 directly, so that we need not depend on the proof in Chapter-II, and indeed it provides a shorter proof than that of the characterization of interval digraphs in terms of the partitionable zeros property. We proceed now to the main result.

Theorem 4.4. *A digraph D is a circular-arc digraph if and only if the rows and columns of its adjacency matrix can be permuted independently so that the resulting natrix has the generalized circular ones property*

Proof.

Necessity. Suppose D has a circular-arc representation $\{S_v, T_v\}$, with S_v = [a(v),b(v)] and T_v = [c(v),d(v)], where a(v),b(v),c(v),d(v) $\in$ [0,1) and arcs extend clockwise from their initial point to their final point. Label the rows and columns of A(D) in increasing order of a(v)'s and c(v)'s respectively, with ties broken arbitrarily. Let these two orderings be $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ respectively. Cover D by two subdigraphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$, assigning edges by

$$uv \in E_1 \text{ if and only if } c(v) \in [a(u), b(u)]$$

$$uv \in E_2 \text{ if and only if } a(u) \in (c(v), d(v)]$$

Note that $uv \in H_1 \cup H_2$ if and only if $S_u \cap T_v \neq \phi$, so $D = H_1 \cup H_2$. They need not be disjoint,; $uv \in H_1 \cap H_2$ if and only if $S_u \cup T_v$ is the entire circle. It suffices to obtain a stair-partition such that the ones corresponding to H_1 belong to U, and those corresponding to H_2 belong to L, and the partition exhibits the generalized circular ones property.

We construct such a partition from the orderings of u and v.

We may assume $a(u_1) = 0$, by shifting the origin, if necessary. Let $a_{n+1} = c_{n+1} = 1$, $c_0 = 0$, and otherwise $a_i = a(u_i)$ and $c_j = c(v_j)$. Partition the indices into disjoint classes $C_1,\ldots,C_n$ of consecutive indices such that $C_r = \{j : a_r \leq c_j < a_{r+1}\}$. Similarly, partition the indices into disjoint consecutive classes $A_0,\ldots,A_n$ such that $A_s = \{ i : c_s \leq a_i < c_{s+1}\}$. By construction, $1 \in A_0$, but otherwise A_r or C_s may be empty.

Define a stair-partition of A by putting position i,j in U if $i \in A_r$, $j \in C_s$ with $j \geq i$, otherwise position i,j is in L. To verify that this stair-partition of A establishes the generalized circular ones property, suppose $A_{ij} = 1$ with $i \in A_r$, $j \in C_s$. If $u_i v_j \in E(H_1)$, then S_{u_i} extends clockwise from a_i as far as c_j (it may pass 0). This means S_{u_i} contains $c_{j'}$ for $j' < j$ in C_s and if $s \neq r$, for all $j' \in C_{s-1} \cup C_{s-2} \cup \ldots \cup C_r$, circularly. Similarly, if $u_i v_j \in E(H_2)$, then T_{v_j} extends clockwise from c_j as far as a_i (it may pass 0). This means T_{v_j} contain $a_{i'}$ for $i' < i$ in A_r and if $r \neq s-1$, for all $i' \in C_{r-1} \cup C_{r-2} \cup \ldots \cup C_{s-1}$, circularly. In particular, $A_{i'j} = 1$ for these i'.

Sufficiency. Suppose that A(D) (appropriately permuted) has a stair-partition U,L with the generalized circular ones property. By a circular shift of the rows, if necessary, we may assume $(1,1) \in U$. Let $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ be numberings of the vertices according to the rows and columns of A. We will construct a circular-arc representation of D by reversing the procedure above. First, we define $a_1,\ldots,a_n$, $c_1,\ldots,c_n$ from the stair-partition; then we define $b_1,\ldots,b_n$ from the ones in the rows of A and $d_1,\ldots,d_n$ from the ones in the columns of A.

The positions in the two parts of a stair-partition are separated by a polygonal path from the upper left corner to lower right corner that takes 2n steps, crossing n rows and n columns. Having assumed that position 1,1 is in U, the first step crosses row 1. The sequence of rows and columns crossed defines a natural ordering on $\{a_i\} \cup \{c_j\}$. Define a_i or c_j to be the value of its index in this ordering; note $a_1 = 1$.

These values run from 1 to 2n. We will also assign $\{b_i\} \cup \{d_j\}$ from these values and then put $S_{u_i} = [(a_i-1)/2n, (b_i-1)/2n]$ and $T_{v_j} = [(c_j-1)/2n, (d_j-1)/2n]$ to obtain a circular arcs in [0,1). To define b_i, consider row i of A. If the leftmost position in row i in U is zero, set $b_i = a_i$. Otherwise, the ones in row i extend circularly rightward until some column j, just before hitting the first zero. In this case, set $b_i = c_j$. Similarly, in order to define d_j, consider column j of A. If the highest position in column j in L is zero, set $d_j = c_j$. Otherwise, the ones in column j extend circularly downward until some row i just before hitting the first zero. In this case, set $d_j = a_i$.

We claim that $\{ S_{u_i} \}$ and $\{ T_{v_j} \}$ give a circular-arc representation of D. This is true if $A_{ij} = 1$ if and only if $c_j \in [a_i, b_i]$ or $a_i \in [c_j, d_j]$. If $c_j \in [a_i, b_i]$, then $c_j \neq a_i$ implies $b_i \neq a_i$. Therefore, there are ones in row i of U, and $c_j \in [a_i, b_i]$ implies the ones extend circularly to the right as far as column j. Hence $A_{ij} = 1$. By similar reasoning $a_i \in [c_j, d_j]$ also implies $A_{ij} = 1$.

If $S_{u_i} \cap T_{v_j} = \phi$, first suppose $a_i < c_j$. This means that row i is crossed before column j by the separating path in the stair-partition, so $(i,j) \in U$. Also $a_i < c_j$ and $S_{u_i} \cap T_{v_j} = \phi$ imply $b_i < c_j$, which means the ones in row i do not extend far enough circularly rightward to reach column j. Also, $a_i \notin [c_j, d_j]$ means the ones in column j do not extend far enough circularly downward to reach row i. Since A has the generalized circular ones property, this implies $A_{ij} = 0$. When $a_i > c_j$, in which case $(i, j) \in L$, similar reasoning also implies $A_{ij} = 0$. ■

Note that the sufficient argument above in fact is a short proof that if A(D) has the generalized linear ones property, then D is an interval digraph. Thus the lengthy proof of Theorem 2.5 involving topological orderings to construct an interval representation of a digraph whose adjacency matrix has the partitionable zeros property can be replaced to obtain a short proof of the following characterization.

Theorem 4.5. The following are equivalent :

(A) D *is an interval digraph.*

(B) *The adjacency matrix* A(D) *has independent row and column permutations that exhibit the partitionable zeros property.*

(C) *The adjacency matrix* A(D) *has independent row and column permutations that exhibit the generalized linear ones property.*

Proof. (A) implies (B) using row and column orderings constructed as in the preceding proof. Indeed, the method of proof of Theorem 4.4 shows that (A) and (C) are equivalent, and Lemma 4.1 is the equivalence of (B) and (C). ■

To illustrate we consider the following two examples - one for interval representation and the other for circular-arc representation.

Example 4.2. Consider the digraph D(V,E) whose adjacency matrix after a rearrangement is

	v_1	v_8	v_3	v_9	v_{10}	v_6	v_7	v_2	v_4	v_5
v_1	1	1	1	1	1	0	0	0	0	0
v_2	1	1	1	1	1	0	0	0	0	0
v_3	0	1	1	1	1	0	0	0	0	0
v_4	0	1	1	1	1	1	1	0	0	0
v_5	0	0	1	1	1	1	1	1	1	1
v_6	0	0	1	1	1	1	1	0	0	0
v_7	0	0	1	1	1	0	0	0	0	0
v_8	0	0	0	0	1	0	1	1	1	1
v_9	0	0	0	0	1	0	1	1	1	0
v_{10}	0	0	0	0	1	0	0	1	0	1

A(D)

For the given stair S of the rearranged matrix, the two digraphs H_1 and H_2 are given by

	v_1	v_8	v_3	v_9	v_{10}	v_6	v_7	v_2	v_4	v_5
v_1	1	1	1	1	1					
v_2				1	1					
v_3					1					
v_4					1	1	1			
v_5					1	1	1	1	1	
v_6						1	1			
v_7							0			
v_8									1	1
v_9										0
v_{10}										1

H_1

	v_1	v_8	v_3	v_9	v_{10}	v_6	v_7	v_2	v_4	v_5
v_1										
v_2	1	1	1							
v_3		1	1	1						
v_4		1	1	1						
v_5			1	1						
v_6			1	1	1					
v_7			1	1	1	0				
v_8					1	0	1	1		
v_9					1	0	1	1	1	
v_{10}					1	0	0	1	0	

H_2

The linear ordering of $\{a_i\} \cup \{c_i\}$ is as follows:

a_1	c_1	c_8	c_3	a_2	c_9	a_3	a_4	a_5	c_{10}	a_6	c_6	a_7	c_7	c_2	a_8	c_4	a_9	a_{10}	c_5
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

The values of b_i's and d_i's are equal to one or other of a_j's or c_j's. To find, say d_9, consider the column 4 in H_2 that corresponds to the vertex v_9. Since for this column the first zero appears just after the row 7 corresponding to the vertex v_7, so $d_9 = a_7$.

To determine d_6 (say) consider the column 6 in H_2 corresponding to the vertex v_6. Since the highest position in L for this column is 0, so $d_6 = c_6$.

The values of $a_i b_i c_i d_i$ thus obtained are given in the following table:

i	1	2	3	4	5	6	7	8	9	10
a_i	1	5	7	8	9	11	13	16	18	19
b_i	10	10	10	14	20	14	13	20	18	20
c_i	2	15	4	17	20	12	14	3	6	10
d_i	5	19	13	18	20	12	18	8	13	19

Accordingly, the interval representations for D are given by

i	1	2	3	4	5	6	7	8	9	10
S_i	[1,10]	[5,10]	[7,10]	[8,14]	[9,20]	[11,14]	[13]	[16,20]	[18]	[19,20]
T_i	[2,5]	[15,19]	[4,13]	[17,18]	[20]	[12]	[14,18]	[3,8]	[6,13]	[10,19]

contnd.

Example 4.3. Consider the adjacency matrix alongwith the stair given by

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1	1	1	1	1	1	0	0	1	0	1
v_2	1	1	1	1	1	0	0	1	0	1
v_3	0	1	1	1	1	0	0	0	0	1
v_4	0	1	1	1	1	1	1	0	0	1
v_5	1	0	1	1	1	1	1	1	1	1
v_6	0	0	1	1	1	0	1	0	0	0
v_7	0	0	1	1	1	0	0	0	0	0
v_8	1	1	1	1	1	0	1	1	1	1
v_9	0	0	0	0	1	0	1	1	1	0
v_{10}	1	1	1	1	0	0	0	1	0	1

For the stair drawn as above the two digraphs H_1 and H_2 are respectively given by

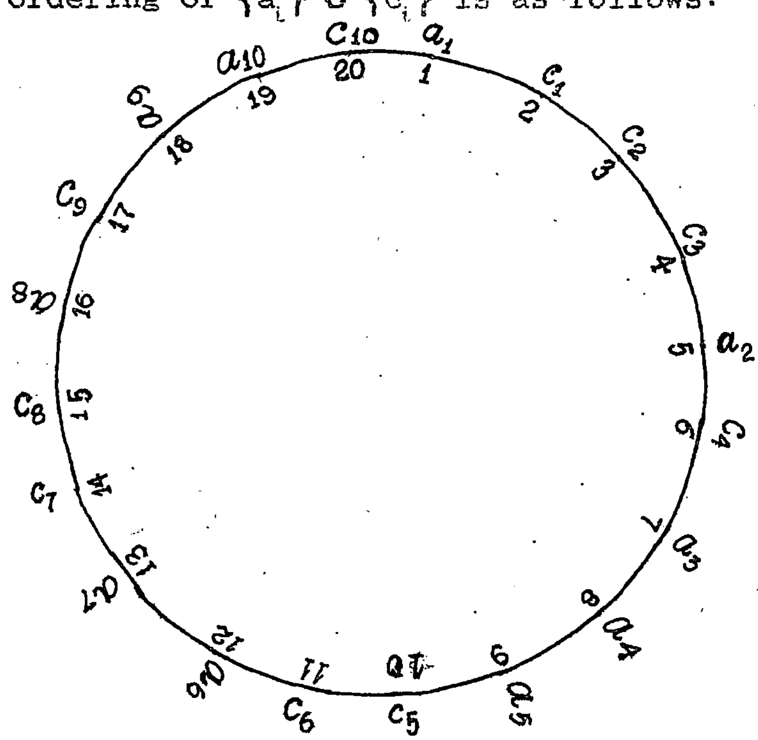
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1	1	1	1	1	1					
v_2				1	1					
v_3					1					
v_4					1	1	1			
v_5	1				1	1	1	1	1	1
v_6							1			
v_7							0			
v_8	1	1	1	1	1				1	1
v_9										0
v_{10}	1	1	1	1						1

H_1

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1							1			1
v_2	1	1	1				1			1
v_3		1	1	1						1
v_4		1	1	1						1
v_5			1	1						1
v_6			1	1	1	0				
v_7			1	1	1					
v_8			1	1	1		1	1		
v_9					1		1	1	1	
v_{10}								1	0	

H_2

The circular ordering of $\{a_i\} \cup \{c_i\}$ is as follows:



b_i's and d_i's are equal to one or other of a_j's or c_j's. To find, say b_4, consider row 4 in H_1 that corresponds to the vertex v_4. Since for this row, the first zero appears just after the column 7 corresponding to vertex v_7, so $b_4 = c_7$. To determine b_7 (say), consider row 7 in H_1 corresponding to the vertex v_7. Since the left-most position in U for this row is 0, so $b_7 = a_7$.

The hour-markers of a_i b_i c_i d_i on the 20-hour clock thus obtained are given in the following table:

i	1	2	3	4	5	6	7	8	9	10
a_i	1	5	7	8	9	12	13	16	18	19
b_i	10	10	10	14	2	14	13	10	18	6
c_i	2	3	4	6	10	11	14	15	17	20
d_i	5	8	16	16	18	11	18	5	18	9

So the circular -arc representation of D is

i	1	2	3	4	5	6	7
S_i	[1,10]	[5,10]	[7,10]	[8,14]	[9,2]	[12,14]	[13]
T_i	[2,5]	[3,8]	[4,16]	[6,16]	[10,18]	[11]	[14,18]

cont.

i	8	9	10
S_i	[16,10]	[18]	[19,6]
T_i	[15,5]	[17,18]	[20,9]

4.4. Circular-arc digraphs and Ferrers dimension.

In chapter - II, we considered the relationship between interval digraphs and Ferrers digraphs. In the terminology we have been using in this thesis, a Ferrers digraph is a digraph whose adjacency matrix has independent row and column permutations such that the resulting matrix has a stair partition (L,U) in which L consists entirely of 1's and U consists entirely of 0's. Equivalently, the successor sets are linearly ordered by inclusion, as are the predecessor sets. Equivalently, the adjacency matrix has no 2X2 submatrix that is a permutation matrix.

The *Ferrers dimension* of a digraph D is the minimum number of Ferrers digraphs whose intersection is D. By the last characterization above, Ferrers dimension of D equaling k is equivalent to being able to k-colour the 0's of A(D) so that the 0's in any 2X2 permutation submatrix get different colours. The edges missing from the ith Ferrers digraph then correspond to the positions with the ith colour. In general, let H(D) be the graph whose vertices correspond to the 0 positions in A(D), with vertices adjacent when they are the 0's of a 2X2 permutation submatrix. Note that H(D) is invariant under row or column permutations of A(D), and Ferrers dimension of D is $\varkappa(H(D))$.

The characterization of interval digraphs using the generalized linear ones property or partitionable zeros property implies that all interval digraphs have Ferrers dimension at most 2. However, there is no such bound for circular-arc digraphs, as we see next.

Theorem 4.6. *There are n-vertex circular-arc digraphs with Ferrers dimension n; i.e., Ferrers dimension of circular-arc digraphs is unbounded.*

Proof. Let D be an n-vertex digraph whose adjacency matrix A has exactly one 0 in each row and column. The Ferrers dimension of D is n, because H(D) is the complete graph on n vertices (no pair of these positions can be omitted by a single Ferrers digraph used in a collection intersecting to form D). Nevertheless, A has the circular ones property for rows, so Theorem 4.1 implies that D is a circular-arc digraph having a representation in which each terminal set is represented by a single point. Indeed, with n distinct points for the terminal sets, the source set for v can be the entire circle except the terminal point for its non-successor. ■

Given this result, it is natural to ask for the smallest Ferrers dimension of a digraph that is not a circular-arc digraph. The answer is 2, which follows readily from the existence of digraphs of Ferrers dimension 2 that are not interval digraphs and the next lemma.

Lemma 4.2. *Given a digraph D, let D′ be the digraph obtained from D by adding a vertex w and a loop at w (no other edge incident to w). Then D′ is a circular-arc digraph if and only if D is an interval digraph.*

Proof. In a circular-arc representation of D′, S_w and T_w must intersect. Since neither is required to intersect another arc, we may trim them to assume $S_w = T_w$. Now no other arc can intersect either of these without generating an unwanted edge. This means D must be an interval digraph.

Conversely, if D is an interval digraph, we can augment an interval representation by adding intersecting S_w and T_w in an unused portion of the line, so that D′ is also an interval digraph and hence a circular-arc digraph. ■

Example 4.4. *A digraph of Ferrers dimension 2 that is not a circular-arc digraph.*

In chapter - II, we proved that Ferrers dimension 2 is equivalent to the existence of independent row and column permutations of the adjacency matrix so that the resulting matrix has no 0 with a 1 both below it and to its right. Let D be the digraph with the adjacency matrix A(D) on the left below. It has Ferrers dimension 2, but we showed that [Sec.2.] it is not an interval digraph.

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}$$

Note that although D itself is not an interval digraph, it is a circular-arc digraph. On the right above, we have written the rows of the adjacency matrix in the order 3261457 and the column in the order 3216457. There are several stair partitions that exhibit the generalized circular ones property for this matrix; the simplest to describe puts the top four rows in U and the bottom three in L. Then the sets W_5 and W_6 bend around to absorb the 1's on the top that do not belong to the left-justified sets of 1's. By the Theorem 4.4, D is a circular-arc digraph. There does not seem to be a simultaneous row and column permutation allowing a suitable stair partition.

If we add a row at the top and a column at the left, consisting of all 0's except for a 1 in the upper left corner to the matrix A(D), then we obtain the adjacency matrix of the digraph D′ obtained by adding a single vertex with a loop. The matrix illustrates that D′ has Ferrers dimension 2, but by lemma 4.2 it is not a circular-arc digraph. Although the digraph D′ is not a circular-arc digraph, its complement is a circular-arc digraph. More generally, we have

Theorem 4.7. *The complement of any digraph with Ferrers dimension at most 2 is a circular-arc digraph.*

Proof. Begin with a Ferrers digraph D of dimension at most 2. Permute the rows and columns of its adjacency matrix so that no 0 has a 1 both below and to its right. Take the complement and reverse the order of the rows and the order of the columns. In the new matrix, no one has a 0 both above and to its left. let (L,U) be the stair partition of this matrix in which U is the entire matrix. If a 1 in position i,j has no 0 to its left (or above it), then this position belongs to V_i (or W_j), where V_i and W_j are defined for (L,U) as before. Hence (L,U) exhibits the generalized circular ones property for this matrix $A(\bar{D})$, and $\bar{D}$ is a circular-arc digraph. ∎

Although the complement of any Ferrers digraph is a Ferrers digraph, the digraphs of Ferrers dimension 2 are not closed under complementation. For example, the digraph consisting of a loop at each vertex or any union of disjoint cycles has a permutation matrix as its adjacency matrix and is intersection on two Ferrers digraphs, but its complement is not. Indeed, its complement digraph D has H(D) = K_n, so D has Ferrers dimension n and is not an interval digraph.

The fact that a trivial stair partition suffices to Prove Theorem 4.7 suggests that some stronger results may hold. Nevertheless, beyond Ferrers dimension 2, nothing can be guaranteed, as we show next.

Example 4.5. The two complementary digraphs on four vertices whose adjacency matrices appear below both have Ferrers dimension 3. but neither is a circular-arc digraph.

$$D_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

For each of these digraphs, $H(D_i)$ contains a triangle but is easy to 3-colour. We use ad hoc arguments to show they are not circular-arc digraphs. Let $\{x,y,z,w\}$ to denote the vertices in row and column order, and suppose there is a circular-arc representation.

For D_1, S_w and T_w must intersect, but since neither is required to intersect another arc, we may trim them to assume $S_w = T_w$. Now no other arc can intersect either of these without generating an unwanted edge. This means D_1 is a circular-arc digraph if and only if $D_1 - w$ is an interval digraph. As noted above D_1 is not an interval digraph.
For D_2, the intervals S_w and T_w must be disjoint. However, for $i \in \{x,y,z\}$ S_i and T_i intersect; select a point $a_i \in S_i \cap T_i$ from their intersection. We also know that S_i meets T_w and T_i meets S_w. Thus $S_i \cup T_i$ is an interval meeting both S_w and T_w, and it must cover a gap between them. There are only two such gaps. Assume by symmetry that $S_x \cup T_x$ and $S_y \cup T_y$ cover one gap between S_w and T_w. In moving from S_w to T_w along this gap, we may assume by symmetry that we reach a_x at least as early as a_y. Since T_y must meet S_w and S_x must meet T_w, this means that S_x and T_y both contain the arc from a_x to a_y, introducing the forbidden edge xy. ■

Although Example 4.4 and Theorem 4.7 show that the complement of a circular-arc digraph need not be a circular-arc digraph, we can make the following weaker remark :

If H_1 and H_2 are the subdigraphs of D generated by a stair-partition of A(D) yielding a circular-arc representation of D, i.e., $D = H_1 \cup H_2$ as in the proof of Theorem 4.4, then the union of the complements of H_1 and H_2 is a circular-arc digraph. The appropriate stair-partition can be obtained by reversing the order of the rows and columns in A(D) and rotating the stair-partition by π.