CHAPTER - III

# AN INTERVAL DIGRAPH IN RELATION TO ITS ASSOCIATED BIPARTITE GRAPHS

## 3.1. Introduction.

Recall that a digraph D(V,E) is a Ferrers digraph if $ab \in E$ and $cd \in E \Rightarrow ad \in E$ or $cb \in E$, for all $a,b,c,d \in V$. (Note that a or c may be equal to either b or d). It was Riguet [1951] who introduced Ferrers digraphs and characterized these digraphs as those in which the successor sets (or equivalently the predecessor sets) are linearly ordered by inclusion. Any digraph D is the intersection of a (finite) number of Ferrers digraphs and the minimum cardinality of such Ferrers digraphs is the Ferrers dimension (F.D) of D. The digraphs with F.D.2 were characterized independently by Cogis [1979] and also by Doignon, Ducamp and Falmagne [1984] in different contexts.

Ferrers digraphs were also characterized immediately from its definition by Riguet [1951] in terms of a forbidden submatrix of its adjacency matrix. A 2x2 permutation matrix i.e., a submatrix of the form $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is called an <u>obstruction</u> in the matrix. Alternatively, two edges xy and zt of D are said to belong to an obstruction, written $xy \overset{D}{*} zt$ or $xy*zt$, if none of xt and zy belong to D. i.e. $xy \in D$, $zt \in D$ but $xt \notin D$ and $zy \notin D$. Cogis calls them *F-incompatable*. A digraph D is a Ferrers digraph if and only if A(D) has no obstruction. In other words, the adjacency matrix has no 2x2 submatrix that is a permutation matrix. In order to characterize a digraph of F.D.2, Cogis [1979] defined an undirected graph H(D), the graph associated with D, whose vertices correspond to the 1's of the adjacency matrix of $\bar{D}$, the complement of D, with two such vertices (1's) joined by an edge if the corresponding 1's belong to an obstruction. Alternatively, the vertices of H(D) correspond to the 0's of A(D) with two such vertices (0's) joined by an edge if they belong to a submatrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ in A(D). Cogis [1979] proved that a finite digraph D has Ferrers dimension 2 at most iff H(D) is bipartite. Then he used this result to obtain a recognition algorithm for a digraph of F.D.2 in a polynomial time. The same characterization was obtained in a more general form by Doignon, Ducamp and Falmagne [1984] where the set is not restricted to be finite. The graph H(D) may have more than one (connected) component; besides it may have one or more isolated vertices (corresponding to the 1's of A($\bar{D}$) which do not belong to any obstruction). The graph obtained by deleting the isolated vertices from H(D) is denoted by $H_b$(D). It is called the bare graph associated with D (Doignon et.al.[1984]).

It was proved in the previous chapter that a digraph of F.D. 2 is equivalent to the existence of independent row and column permutations of the adjacency matrix so that the resulting matrix has no 0 with a 1 below it and another 1 to its right. In other words, corresponding to any zero in the rearranged matrix, either every entry below it is zero or every entry to its right is zero ('or' being inclusive). We shall refer to this property as <u>$F_2$-property of zeros</u> for the rearranged adjacency matrix of the digraph and the rearranged matrix as an <u>$F_2$-matrix</u> of the digraph. It is to be noted in this connection that in an $F_2$-matrix, a pair of zeros forming an obstruction must have the form $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, because the presence of the other form $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ violates the $F_2$-property of zeros. It was also shown that an interval digraph is necessarily a digraph of F.D.at most 2; but the converse is not true. As a matter of fact, it was proved that a digraph D is an interval digraph if and only if it is the intersection of two Ferrers digraphs whose union is complete or, equivalently, if and only if its complement $\bar{D}$ is the union of two disjoint Ferrers digraphs (since the complement of a Ferrers digraph is a Ferrers digraph).

With reference to a particular realization of $\bar{D}$ as the union of two Ferrers digraphs,(D being a digraph of F.D.2), we first introduce in this chapter the notion of interior edges of these two (Ferrers) digraphs. Then with the help of this concept we obtain some properties of a digraph of F.D.2 and then we show how the notion of interior edges are related to an interval digraph.

## 3.2. Interior edges

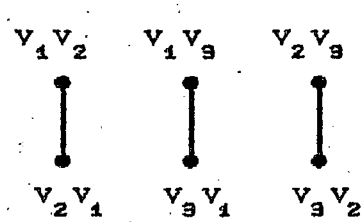
We begin with the following well-known theorem.

**Theorem 3.1.** (Cogis, Doignon et.al.) A digraph D is of Ferrers dimension at most 2 iff H(D) is bipartite.

Let D(V,E) be a digraph of F.D.2 so that H(D) is a bipartite graph. We shall denote the set of all isolated vertices of H(D) by **I**(H) or by **I**, and a bicolouration of $H_b(D)$ by (**R**, **C**). Recall that a colouration of a graph is an assignment of colours to its vertices so that no two adjacent points have the same colour. Naturally, a bicolurable graph uses two colours only. If $H_b(D)$ has more than one connected component $H_1,\ldots,H_p$, a bicolouration of $H_i$ will be denoted by $(R_i, C_i)$. It is evident that $\mathbf{R} = \bigcup_1^p \mathbf{R}_i$ and $\mathbf{C} = \bigcup_1^p \mathbf{C}_i$ for some labelling of the bicolouration $(\mathbf{R}_i, \mathbf{C}_i)$ of Hi. We shall also denote the elements of the sets **R**, **C**, $\mathbf{R}_i$, $\mathbf{C}_i$ or **I** by the corresponding capital letters R,C,$R_i$,$C_i$ or I respectively. We also note in this connection that the same symbols R,C,$R_i$,$C_i$ or I will be used to mean a 0 in A(D) and again a 1 in A($\bar{D}$) and the meaning will be evident from the context. The stable sets $R_i$ and $C_i$ will be called the fragments of H(D) (Cogis calls them p-colours). The two fragments $R_i$ and $C_i$ (for the same i) will be called the conjugate of each other. For a digraph D(V,E) of F.D.2, $\bar{D} = G_1(V,E_1) \cup G_2(V,E_2)$, where $G_1$ and $G_2$ are two Ferrers digraphs. Since $G_k$'s (k=1,2) are Ferrers digraphs, any two edges of $G_k$ cannot form an obstruction in $A(G_1)$ and so in $A(G)$. Since again $G_1$ and $G_2$ are subdigraphs of $\bar{D}$, two edges ab and cd forming an obstruction in $\bar{D}$ must not belong to the same $G_k$ $(k = 1,2)$. i.e. $ab \in E_1$ (or $E_2$) $\Rightarrow$ $cd \in E_2$ (or $E_1$). Thus if $H_b(D)$ has more than one component $H_i$ $(i = 1,\ldots,p)$, then given $G_1(V,E_1)$ and $G_2(V,E_2)$ whose union is $\bar{D}$ there exist some labelling $(R_i,C_i)$ of the bicolouration of $H_i$ such that $R = \bigcup_1^p R_i \subset E_1$ and $C = \bigcup_1^p C_i \subset E_2$. So if we want to cover $\bar{D}$ by two Ferrers digraphs, we should consider the fragments $(R_i,C_i)$ of $H(D)$ (which, in turn, yields a bicolouration of $H_b(D)$). On the other hand, however, any bicolouration of $H_b(D)$ does not necessarily lead to a covering of $\bar{D}$ by two Ferrers digraphs. It is easily verified by considering the simple digraph

| | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $v_1$ | 1 | 0 | 0 |
| $v_2$ | 0 | 1 | 0 |
| $v_3$ | 0 | 0 | 1 |

For the digraph D, H(D) is the graph consisting of three disconnected edges $\{v_1v_2, v_2v_1\}$, $\{v_1v_3, v_3v_1\}$ and $\{v_2v_3, v_3v_2\}$.




H(D)

*Fig. 3.1.*

If we consider $R_1 = \{v_1v_2\}$ $R_2 = \{v_3v_1\}$ $R_3 = \{v_2v_3\}$
$C_1 = \{v_2v_1\}$ $C_2 = \{v_1v_3\}$ $C_3 = \{v_3v_2\}$ ;
then the bicolouration $E_1 = R_1UR_2UR_3$ and $E_2 = C_1UC_2UC_3$ does not lead to a covering of $\bar{D}$ into two Ferrers digraphs whereas it does if we choose $E_1 = R_1UC_2UR_3$ and $E_2 = C_1UR_2UC_3$.

If any suitable bicolouration, however, yields a realization of $\bar{D}$ as the union of two Ferrers digraphs, then such a bicolouration will, in our thesis, be termed satisfactory bicolouration. Cogis [1979] adopted a constructive method to show the existence of a satisfactory bicolouration of $H_b(D)$. As a matter of fact, he obtained the particular bicolouration $(\mathcal{R},\mathcal{C})$ in such a way that adjoining all the edges of $I(H)$ to each of $\mathcal{R}$ and $\mathcal{C}$ yielded the required Ferrers digraphs realization $G_1$ and $G_2$ so that $\bar{D} = G_1UG_2$ where $G_1 = \mathcal{R} \cup I(H)$ and $G_2 = \mathcal{C} \cup I(H)$. This result was independently proved by Doignon et.al.[1984] in the more general case when the set of vertices is not necessarily finite. Indeed they also prove that in a certain restricted case any bicolouration of $H_b(D)$ very well serves the purpose.

In this chapter, by a configuration of an adjacency matrix A, we shall mean a submatrix of A obtained by any (independent) permutation of rows and of columns. But by a configuration of an $F_2$-matrix A, we shall, for convenience, mean a submatrix of A upto (independent) permutation of rows and columns so long as the rearranged permuted matrix retains its $F_2$-matrix structure (with the same labelling of $R_i$ and $C_i$). In order to prove Theorem 3.1, Cogis[1979] first obtained the following important property of a digraph of F.D.2. As we shall also often require this powerful property in our assertions, we state this property in the form of a proposition.

**Proposition 3.1.** *Let D(V,E) be a digraph of F.D.2 and A be a fragment of H(D); Also let both xx' and yy' are in A. Then*

(i) $\{xy', yx'\} \cap \{(I(H) \cup A\} \neq \phi$

(ii) if $xy', yx' \in \bar{E}$ then $\{xy', yx'\} \cap A \neq \phi$.

In the language of submatrix, this means that the presence of a configuration in A(D)

| | x' | y' |
|---|---|---|
| x | A | — |
| y | — | A |

implies that at least one '— 'must be an A or I and the presence of a configuration

| | x' | y' |
|---|---|---|
| x | A | 0 |
| y | 0 | A |

implies that at least one 0 must be an A.

While the recognition of a digraph of F.D.2 requires the realization of its complement as the union of two Ferrers digraphs $G_1$ and $G_2$, not necessarily disjoint, such that $\bar{D} = G_1 \cup G_2$, for an interval digraph recognition, however, the problem is to cover its complement by two Ferrers digraphs which should necessarily be disjoint, $\bar{D} = H_1 \cup H_2$, $H_1 \cap H_2 = \phi$. This is equivalent to adjoining every edge $I \in I(H)$ into only one of the two digraphs $G_1(V,R)$ and $G_2(V,C)$ for some bicolouration $(R,C)$ of $H_b(D)$ so that they become two disjoint Ferrers digraphs.

3.2.1. In the following, we prove some elementary properties of a digraph of F.D.2, which will be required in the sequel.

**Proposition 3.2.** *Let D be a digraph of F.D.2. Then a 0 of A(D) is an isolated vertex of H(D), iff there exists an $F_2$-matrix of D such that no 1 lies below or to the right of the corresponding zero in the matrix.*

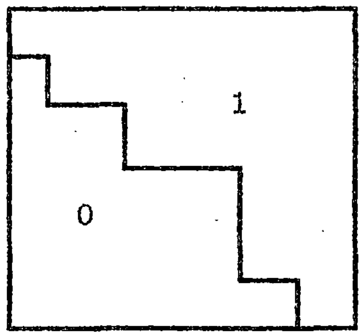
**Proof. Sufficency.** Let the rearranged matrix satisfying the $F_2$- property of zeros be denoted by $A = (a_{ij})$. Let a position $a_{ij}$ in A be 0 with all the elements to the right of $a_{ij}$ in the i-th row and all the elements below $a_{ij}$ in the j-th column being 0. Evidently, $a_{ij}$ cannot have any obstruction with any 0 to the right of j-th column or any 0 below i-th row. Now consider a zero above the ith row and to the left of the j-th column. Then since the matrix satisfies $F_2$-property, the two 0's cannot form an obstruction. Hence the zero corresponding to $a_{ij}$ is an isolated vertex in H(D).

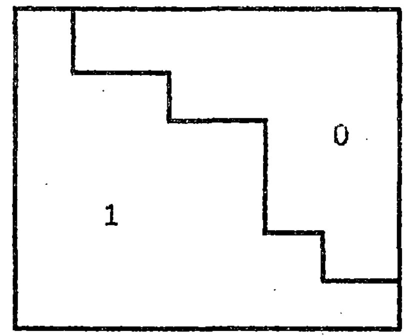
**Necessity.** Let D be a digraph of F.D.2 and let a zero of A(D) be an isolated vertex in H(D). Call it I. Rearrange the adjacency matrix to an $F_2$-matrix B. Then either every element to the right of I is 0 or every element below I is 0. First suppose that every element to the right of I is 0. We shall show that the row corresponding to I can be sufficiently shifted so as to satisfy the condition of the proposition. Let a position below I be 1. Since I does not belong to any obstruction all the 2X2 submatrix with these I and 1 must not be of the form $\begin{pmatrix}1 & I\\0 & 1\end{pmatrix}$. That is, it must be of one of the forms $\begin{pmatrix}1 & I\\1 & 1\end{pmatrix}$ $\begin{pmatrix}0 & I\\0 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & I\\1 & 1\end{pmatrix}$. Since the matrix is an $F_2$- matrix we can easily see that the row corresponding to I may be shifted to a position when all elements below I are 0 (and the $F_2$-property is retained).■

Recall from Theorem 2.7 that a digraph D is of F.D. at most 2 iff A(D) can be rearranged in the form of an $F_2$-matrix. Also recall from the proof of the same theorem that the zeros of the $F_2$-matrix are classified in such a way that R is the set of zeros having a 1 somewhere below them and C is the set of zeros having a 1 somewhere to its right. For any 2X2 submatrix forming an obstruction, the 0's must be an

R in the upper right and a C in the lower left and these are the only edges in the bipartite graph H(D), with two 0's having no 1 to the right or below generating isolated points. These R's together with the above 0's generating isolated points yields a Ferrers digraph $G_1$ and the C's together with those 0's yields another Ferrers digraph $G_2$ so that $\bar{D} = G_1 \cup G_2$. Again note that there may be an R which has no obstruction with a C and vice versa so that some R's and C's may again be isolated points in H(D). Hence if $H_b(D)$ has more than one component $H_i$, $H_b(D) = \cup Hi$ for some bicolouration $(R_i, C_i)$ of $H_i$ in the above $F_2$-matrix, any $R_i$ has only $R_j$'s and/or isolated vertices to its right and any $C_i$ has only $C_j$'s and/or isolated vertices below it. Also any two members $R_i$ and $C_i$ belonging to two conjugate fragments such that (R,C) is an edge in H(D) must appear as a submatrix $\begin{bmatrix} 1 & R_i \\ C_i & 1 \end{bmatrix}$ in the above $F_2$-matrix of A(D) ($R_i$ appear in the upper right and $C_i$ in the lower left).

Let (R,C) be a satisfactory bicolouration of $H_b(D)$ leading to a realization of $\bar{D} = G_1(V,E_1) \cup G_2(V,E_2)$ where $E_1 = R \cup I(H)$ and $E_2 = C \cup I(H)$. Let the rows and columns of $A(G_1)$ be so arranged that all the ones are clustered in the upper right. Similarly, the rows and columns of $A(G_2)$ be so arranged that all the ones are clustered in the lower left.

$A(G_1)=$ 


$A(G_2)=$ 


**Definition 3.1.** An edge corresponding to an $I \in G_1$ is said to be an interior edge of $G_1$, denoted by $I_r$, if there exists a configuration of the form $\begin{bmatrix} R & I \\ 0 & R \end{bmatrix}$ in $A(G_1)$; similarly, an $I \in G_2$ is said to be an interior edge of $G_2$, denoted by $I_c$, if there exists a configuration of the form $\begin{bmatrix} C & 0 \\ I & C \end{bmatrix}$ in $A(G_2)$. With reference to a particular realization of $\bar{D}$ as the union of $G_1$ and $G_2$, $\bar{D} = G_1 \cup G_2$, the set of all interior edges of $G_1$ will be called <u>interior of $G_1$</u> and will be denoted by $I_r\ (G_1)$ or $I_r$ and all interior edges of $G_2$ will be called <u>interior of $G_2$</u> and will be denoted by $I_c\ (G_2)$ or $I_c$. Note that the sets $I_r$ and $I_c$ are identified with reference to a particular realization of $\bar{D}$ and will change if the realization changes.

**Proposition 3.3.** *Let D be a digraph of F.D.2. Then for a satisfactory bicolouration of $H_b(D)$ the following two* *conditions are equivalent* :

(*i*) $I_r \cap I_c \neq \phi$ and

(*ii*) A(D) contains a configuration of the form

$$\begin{bmatrix} 1 & 1 & R \\ 1 & 1 & C \\ C & R & I \end{bmatrix}$$

(with respect to the same bicolouration).

**Proof.** *(ii)* $\Rightarrow$ *(i)* follows immediately from the definition of $I_r$ and $I_c$.

*(i)* $\Rightarrow$ *(ii)*. Let $I \in I_r \cap I_c$. Then this I belongs to the configurations $\begin{bmatrix} R & I \\ 0 & R \end{bmatrix}$ and $\begin{bmatrix} C & 0 \\ I & C \end{bmatrix}$ in $A(G_1)$ and $A(G_2)$ respectively. From the Proposition 3,1, it follows that the adjacency matrix $A(\bar{D})$ of $\bar{D}$ contains both the configurations $\begin{bmatrix} RI \\ 0R \end{bmatrix}$ and $\begin{bmatrix} C0 \\ IC \end{bmatrix}$ as well. Combining these two we can say that $A(\bar{D})$ has a configuration of the form

| | x′ | y′ | z′ |
|---|---|---|---|
| x | – | C | 0 |
| y | R | I | C |
| z | 0 | R | – |

where '–' means we are yet to conclude anything about whether it is 0 or 1 (exclusively or inclusively). Let, if possible, the '–' in the xx′ position be 1. Then this 1 must be either R or C or I. If it is R or I, then the configuration

| | x′ | y′ |
|---|---|---|
| x | – | C |
| z | 0 | R |

in $A(\bar{D})$ has a corresponding configuration in $A(G_1)$ given by

| | $x'$ | $y'$ |
|---|---|---|
| $x$ | 1 | 0 |
| $z$ | 0 | 1 |

This yields an obstruction in $A(G_1)$ which is not possible. Similarly, if it is C or I, then the configuration

| | $x'$ | $z'$ |
|---|---|---|
| $x$ | 1 | 0 |
| $y$ | 0 | 1 |

in $A(G_2)$ gives us an obstruction which is again not possible. So the '–' in the $xx'$ position must be 0. Similarly, the '–' in the $zz'$ position must be 0. Thus $A(D)$ has a configuration

| | $x'$ | $y'$ | $z'$ |
|---|---|---|---|
| $x$ | 0 | C | 0 |
| $y$ | R | I | C |
| $z$ | 0 | R | 0 |

or equivalently

| | $z'$ | $x'$ | $y'$ |
|---|---|---|---|
| $z$ | 0 | 0 | R |
| $x$ | 0 | 0 | C |
| $y$ | C | R | I |

and accordingly $A(D)$ has a configuration

| | | |
|---|---|---|
| 1 | 1 | R |
| 1 | 1 | C |
| C | R | I |

∎

**Proposition 3.4.** *Let $D$ be a digraph of F.D.2. If $Ir \cap Ic \neq \phi$ for a certain satisfactory bicolouration of $H_b(D)$ then the same is true for any other satisfactory bicolouration of $H_b(D)$.*

To prove this we require the following lemma:

Lemma 3.1. Let D be a digraph of F.D.2 and the rearranged $F_2$-matrix A have a configuration of any of the forms

$$\text{(i)}\quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & 1 & 1 & X \\ y & 1 & 1 & Y \\ z & R_1 & R_2 & I \end{array} \qquad \text{(ii)}\quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & 1 & 1 & X \\ y & 1 & 1 & Y \\ z & C_1 & C_2 & I \end{array}$$

$$\text{(iii)}\quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & 1 & 1 & R_1 \\ y & 1 & 1 & R_2 \\ z & X & Y & I \end{array} \qquad \text{(iv)}\quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & 1 & 1 & C_1 \\ y & 1 & 1 & C_2 \\ z & X & Y & I \end{array}$$

where X and Y are two distinct fragments of H(D) and I is such that no 1 lies below or to the right of it. Then in each case A has a configuration of the form

$$\begin{array}{|ccc} \hline 1 & 1 & R_i \\ 1 & 1 & C_j \\ C_k & R_m & I \end{array}$$

Proof. We shall prove that in cases (i) and (ii) A has a configuration of the form

$$\begin{array}{|ccc} \hline 1 & 1 & X \\ 1 & 1 & Y \\ C_k & R_m & I \end{array}$$

By taking the converse of the digraph D and by interchanging the labelling of $R_i$ and $C_i$ it will follow that in cases (iv) and (iii) A has a configuration

$$\begin{array}{|ccc} \hline 1 & 1 & R_k \\ 1 & 1 & C_m \\ X' & Y' & I \end{array}$$

where X' and Y' are the conjugate of X and Y respectively.

Then the existence of any of the four forms in A will imply the existence of the form

$$\left|\begin{array}{ccc} 1 & 1 & R_i \\ 1 & 1 & C_j \\ C_k & R_m & I \end{array}\right.$$

and the lemma will be proved.

Case(i). The matrix A has a submatrix

$$\begin{array}{c|cc} & w' & x' \\ \hline z & 1 & R_1 \\ w & C_1 & 1 \end{array}$$

This $C_1$ must be below and to the left of the given $R_1$, as noted in the introduction for a digraph of F.D.2. So A has a configuration

$$\begin{array}{c|cccc} & w' & x' & y' & z' \\ \hline x & - & 1 & 1 & X \\ y & - & 1 & 1 & Y \\ z & 1 & R_1 & R_2 & I \\ w & C_1 & 1 & - & - \end{array}$$

where '–' means we are yet to be definite whether this entry is 1 or 0. Since $H_1$ and $H_2$ are two distinct components of $H_b(D)$, so $xw'$ and $yw'$ positions are both 1 and $wy'$ position is 0. Again since X and Y are two distinct fragments of H(D), so $wz'$ is 0. So the above configuration takes the form

$$\begin{array}{c|cccc} & w' & x' & y' & z' \\ \hline x & 1 & 1 & 1 & X \\ y & 1 & 1 & 1 & Y \\ z & 1 & R_1 & R_2 & I \\ w & C_1 & 1 & 0 & 0 \end{array}$$

Again A has a submatrix

$$\begin{array}{c|ccc} & v' & x' & y' \\ \hline z & 1 & R_1 & R_2 \\ v & C_2 & - & 1 \end{array}$$

Since $H_1$ and $H_2$ are two distinct components, so vx′ is 0. Now two subcases arise regarding the positions of v-row and w-row :

(a) when w-row lies above v-row and (b) when v-row lies above w-row. Below we consider only the subcase (a); the other subcase (b) can be similarly proved and hence is omitted.

*Subcase(a) when w-row lies above v-row.* In this case A has a configuration

| | v′ | w′ | x′ | y′ | z′ |
|---|---|---|---|---|---|
| x | – | 1 | 1 | 1 | X |
| y | – | 1 | 1 | 1 | Y |
| z | 1 | 1 | $R_1$ | $R_2$ | I |
| w | – | $C_1$ | 1 | 0 | 0 |
| v | $C_2$ | – | 0 | 1 | 0 |

That vz′ is 0 follows from the fact that X and Y are distinct and vx′ is 0. Since wy′* vx′,they are not I's and since there is a 1 below wy′, it must be an $R_m$. Thus a configuration of A is

| | w′ | y′ | z′ |
|---|---|---|---|
| x | 1 | 1 | X |
| y | 1 | 1 | Y |
| w | $C_1$ | $R_m$ | 0 |

The case (i) will be proved if now we can show that wz′ is an I. If not, then there must be an entry pp′ such that wz′* pp′, so that two possibilities arising out of the obstruction are

| | p′ | z′ |
|---|---|---|
| w | 1 | 0 |
| p | 0 | 1 |

| | z′ | p′ |
|---|---|---|
| p | 1 | 0 |
| w | 0 | 1 |

We first consider the first possibility to arrive at a contradiction ; for the first possibility a configuration is

| | v′ | w′ | x′ | p′ | y′ | z′ |
|---|---|---|---|---|---|---|
| x | – | 1 | 1 | – | 1 | X |
| y | – | 1 | 1 | – | 1 | Y |
| z | 1 | 1 | $R_1$ | – | $R_2$ | I |
| w | – | $C_1$ | 1 | 1 | $R_m$ | 0 |
| p | – | – | – | 0 | – | 1 |
| v | $C_2$ | – | 0 | – | 1 | 0 |

The p′-column in the above configuration has been taken preceding to y′-column, because wy′ is $R_m$ and no 1 can lie to the right of a R. Since ww′ is $C_1$ and xw′, yw′, zw′ are all 1, w-row cannot occur preceding to any of x, y and z-rows. So none of the positions xp′ and yp′ is 0, because of the $F_2$-property of the configuration. Again, because X and Y are two distinct fragments of $H_b(D)$, none of these position can be 1. So wz′ is an I.

For the other possibility

| | z′ | p′ |
|---|---|---|
| p | 1 | 0 |
| w | 0 | 1 |

a possible configuration is

| | v′ | w′ | x′ | z′ | p′ | y′ |
|---|---|---|---|---|---|---|
| p | – | – | – | 1 | 0 | – |
| x | – | 1 | 1 | X | – | 1 |
| y | – | 1 | 1 | Y | – | 1 |
| z | 1 | 1 | R | I | – | R |
| w | – | $C_1$ | 1 | 0 | 1 | $R_m$ |
| v | $C_2$ | – | 0 | 0 | – | 1 |

The p′-column and so z′-column has been taken preceding to y'-column because wy' is $R_m$ and no 1 can lie to the right of it. Again the p-row has been taken above x and y rows because otherwise

| | z' | p' |
|---|---|---|
| x | X | 1 |
| y | Y | 1 |
| p | 1 | – |

violates $F_2$-property. There is no particular fixed ordering between x'-column and z'-column. Now, neither of xp' and yp' is 0 because of $F_2$-property and also since X and Y are distinct, none of them is 1. So wz' is an I and case (a) is proved.

Case(ii). The matrix A has a submatrix

| | y' | w' |
|---|---|---|
| w | 1 | $R_2$ |
| z | $C_2$ | 1 |

This $R_2$ must lie above and to the right of the given $C_2$, as observed in the introduction. So A has a configuration

| | x' | y' | w' | z' |
|---|---|---|---|---|
| x | 1 | 1 | – | X |
| y | 1 | 1 | – | Y |
| w | – | 1 | $R_2$ | – |
| z | $C_1$ | $C_2$ | 1 | I |

Since $H_1$ and $H_2$ are two distinct components of $H_b(D)$, wx' position is 0 and the positions xw' and yw' are 1. Again since X and Y are two distinct fragments of H(D), so wz' is 0. So A takes the form

| | x' | y' | w' | z' |
|---|---|---|---|---|
| x | 1 | 1 | 1 | X |
| y | 1 | 1 | 1 | Y |
| w | 0 | 1 | $R_2$ | 0 |
| z | $C_1$ | $C_2$ | 1 | I |

Again, A has a configuration

| | $x'$ | $y'$ | $v'$ |
|---|---|---|---|
| v | 1 | – | R |
| z | $C_1$ | $C_2$ | 1 |

This $R_1$ must lie above and to the right of the given $C_1$. Since $H_1$ and $H_2$ are distinct, $vy'$ is 0. Now two cases arise regarding the positions of v-row and w-row:

(a) when v-row lies above w-row. In this case, A has a configuration

| | $x'$ | $y'$ | $v'$ | $w'$ | $z'$ |
|---|---|---|---|---|---|
| x | 1 | 1 | – | 1 | X |
| y | 1 | 1 | – | 1 | Y |
| v | 1 | 0 | $R_1$ | – | – |
| w | 0 | 1 | – | $R_2$ | 0 |
| z | $C_1$ | $C_2$ | 1 | 1 | I |

The w-row has been taken below x and y-rows, because otherwise from

| | $x'$ | $y'$ |
|---|---|---|
| w | 0 | 1 |
| x or y | 1 | 1 |

it follows that the entry in $wx'$-position violates $F_2$-property. Now $vy' * wx'$. Since there is a 1 to the right of $wx'$-position, so this must be a $C_m$. Thus a configuration of A is

| | $x'$ | $w'$ | $z'$ |
|---|---|---|---|
| x | 1 | 1 | X |
| y | 1 | 1 | Y |
| w | $C_m$ | $R_2$ | 0 |

The case (ii) will be proved if now we can show that $wz'$ is an I. If not, then there must be an entry $pp'$ such that $wz' * pp'$, so that two possibilities arising out of this obstruction are

| | p' | z' |
|---|---|---|
| w | 1 | 0 |
| p | 0 | 1 |

| | z' | p' |
|---|---|---|
| p | 1 | 0 |
| w | 0 | 1 |

We first consider the first possibility to arrive at a contradiction ; For the first possibility a configuration is

| | x' | y' | v' | p' | w' | z' |
|---|---|---|---|---|---|---|
| x | 1 | 1 | – | – | 1 | X |
| y | 1 | 1 | – | – | 1 | Y |
| v | 1 | 0 | $R_1$ | – | – | – |
| w | $C_m$ | 1 | – | 1 | $R_2$ | 0 |
| p | – | – | – | 0 | – | 1 |
| z | $C_1$ | $C_2$ | 1 | – | 1 | I |

The p'-column in the above configuration has been taken preceding to w'-column, for otherwise

| | w' | p' |
|---|---|---|
| w | $R_2$ | 1 |
| z | 1 | – |

violates $F_2$-property. Since wx' is $C_m$ and xx', yx' are 1, w-row cannot occur preceding to any of x and y-rows. So none of the positions xp' and yp' is 0, because of the $F_2$-property of the configuration. Again, because X and Y are two distinct fragments of $H_b(D)$, none of the positions can be 1. So wz' is an I. For the other possibility

| | z' | p' |
|---|---|---|
| p | 1 | 0 |
| w | 0 | 1 |

a possible configuration is

| | x′ | y′ | v′ | z′ | p′ | w′ |
|---|---|---|---|---|---|---|
| p | – | – | – | 1 | 0 | – |
| x | 1 | 1 | – | X | – | 1 |
| y | 1 | 1 | – | Y | – | 1 |
| v | 1 | 0 | $R_1$ | – | – | – |
| w | $C_m$ | 1 | – | 0 | 1 | $R_2$ |
| z | $C_1$ | $C_2$ | 1 | I | – | 1 |

Since ww′ is $R_2$, p′-column and so z′-column has been taken preceding to w′-column. Again p-row has been taken above x and y-rows because otherwise

| | z′ | w′ |
|---|---|---|
| x | X | 1 |
| y | Y | 1 |
| p | 1 | – |

violates $F_2$-property. Now, neither of xp′ and yp′ is 0 because of the $F_2$-property and also since X and Y are distinct, none of them is 1. So wz′ is an I and case (a) is proved. ■

**Proof of Proposition 3.4.** Let, for a certain satisfactory bicolouration of $H_b(D)$, $I_r \cap I_c \neq \phi$ . Then it follows from the proposition 3.3 that A has a configuration

$$\begin{bmatrix} 1 & 1 & R_i \\ 1 & 1 & C_j \\ C_k & R_m & I \end{bmatrix}$$

for the given bicolouration. It may so happen that by a different satisfactory bicolouration, the above I fails to become an interior edge to both the realized Ferrers digraphs. This is possible only when the given configuration takes any of the four forms in the lemma 3.1 by the new bicolouration. But in those cases, the lemma shows that there exists a configuration of the form

$$\begin{bmatrix} 1 & 1 & R_i \\ 1 & 1 & C_j \\ C_k & R_m & I \end{bmatrix}$$

with reference to the new bicolouration and accordingly $I_r \cap I_c \neq \phi$ also for the new bicolouration. ■

**Theorem 3.2.** *Let D be a digraph of F.D.2 and let $H_1$ and $H_2$ are subdigraphs of D given by $H_1 = R \cup I_r$ and $H_2 = C \cup I_c$. Then both $H_1$ and $H_2$ are Ferrers digraphs.*

**Proof.** We will show that

$xx' \in H_i$, $yy' \in H_i \Rightarrow xy' \in H_i$ or $yx' \in H_i$ (i=1,2)

We shall prove it for the digraph $H_1$ and the other digraph will similarly follow. For the two edges $xx'$ and $yy'$ belonging to $H_1$ there are three possible alternatives : (i) both of them are R (ii) one is R and other is $I_r$ (iii) both of them are $I_r$. We consider the three cases below separately.

Case (i) Let

| | x' | y' |
|---|---|---|
| x | R | – |
| y | – | R |

be a configuration of A(D). Two subcases arise : either the positions $xy'$ and $yx'$ are both 0 or one only of them is 1, because presence of the configuration

| | x' | y' |
|---|---|---|
| x | R | 1 |
| y | 1 | R |

shows that $xx' * yy'$ which is not possible, since they have the same colour.

(a) Let $yx'$ be 1 then $xy'$ is not 1. So

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & 0 \\ y & 1 & R \end{array}$$

is a configuration of A(D). If $xy'$ is an I, then it follows that it is an $I_r$. If $xy'$ is not an I, then it follows that $xy'$ is R (Proposition 3.1).

(b) Now assume that both $xy'$ and $yx'$ are 0. i.e.

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & 0 \\ y & 0 & R \end{array}$$

Then either $xy'$ or $yx'$ is R (Proposition 3.1).

Case(ii) Let

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & - \\ y & - & I_r \end{array}$$

be a configuration of A(D). As earlier, two subcases arise:

(a) Let $yx'$ (or $xy'$) be 1 then $xy'$ (or $yx'$) is 0, and so the two possible configurations are

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & 0 \\ y & 1 & I_r \end{array} \quad \& \quad \begin{array}{c|cc} & y' & x' \\ \hline x & 1 & R \\ y & I_r & 0 \end{array}$$

We show below that when $yx'$ is 1 then $xy'$ is either R or $I_r$. Since $I_r$ is an interior edge, so A(D) must have a configuration

$$\begin{array}{c|cc} & z' & y' \\ \hline y & R & I_r \\ z & 1 & R \end{array}$$

So

| | $x'$ | $z'$ | $y'$ |
|---|---|---|---|
| $x$ | R | – | 0 |
| $y$ | 1 | R | $I_r$ |
| $z$ | – | 1 | R |

is a configuration of $A(D)$. The zero in the position $xy'$ is an I or R or C. If $xy'$ is a C then it can be seen that any possible permutation of the two rows and columns in the above configuration violates the condition of an $F_2$-matrix. Next when $xy'$ is an I, the configuration

| | $x'$ | $z'$ |
|---|---|---|
| $y$ | R | I |
| $z$ | – | R |

shows that $zx'$ is either 1 or R (Proposition 3.1). Now from the configuration

| | $x'$ | $z'$ |
|---|---|---|
| $y$ | 1 | R |
| $z$ | – | 1 |

,

it follows that $zx'$ is not R. So it must be 1. Hence

| | $x'$ | $y'$ |
|---|---|---|
| $x$ | R | I |
| $z$ | 1 | R |

shows that $xy'$ is an $I_r$ of $G_1$. The 0 in the other possible configuration

| | $y'$ | $x'$ |
|---|---|---|
| $x$ | 1 | R |
| $y$ | $I_r$ | 0 |

can be similarly seen to be either R or $I_r$. Thus in any case the 0 in the configuration is an R or $I_r$ and the theorem is proved for the case (a).

(b) Now assume that both $xy'$ and $yx'$ are 0, so that

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & 0 \\ y & 0 & I_r \end{array}$$

is a configuration of A(D). Both of them are not C (Proposition 3.1). If one is R then nothing is to be proved. Now, let one of them, say $xy'$, be I. The case when $yx'$ is I can similarly be taken care of. For the interior edge $I_r$ on $yy'$-position, A(D) must have a configuration

$$\begin{array}{c|cc} & z' & y' \\ \hline y & R & I_r \\ z & 1 & R \end{array}$$

So we have

$$\begin{array}{c|ccc} & x' & z' & y' \\ \hline x & R & - & I \\ y & 0 & R & I_r \\ z & - & 1 & R \end{array}$$

From the configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & I \\ z & - & R \end{array}$$

it follows that $zx'$ must be either 1 or R (Proposition 3.1). If $zx'$ is 1, then it immediately follows that $xy'$ is an interior edge $I_r$. If $zx'$ is R, then the configuration

$$\begin{array}{c|cc} & x' & z' \\ \hline y & 0 & R \\ z & R & 1 \end{array}$$

shows that $yx'$ is either R or $I_r$ [Case (i)].

Case (iii). Let

$$\begin{array}{c|cc} & x' & y' \\ \hline x & I_r & - \\ y & - & I_r \end{array}$$

be a configuration of A(D). As earlier, the positions $xy'$ and $yx'$ are both 0 or only one of them is 1.

(a) If $yx'$ is 1 then $xy'$ is 0 and

$$\begin{array}{c|cc} & x' & y' \\ \hline x & I_r & 0 \\ y & 1 & I_r \end{array}$$

is a configuration of A(D). For the interior edges $I_r$ on $xx'$ and $yy'$ positions, there must exist configurations of the form

$$\begin{array}{c|cc} & w' & x' \\ \hline w & 1 & R \\ x & R & I_r \end{array} \qquad \& \qquad \begin{array}{c|cc} & z' & y' \\ \hline z & 1 & R \\ y & R & I_r \end{array}$$

So A(D) has a configuration

$$\begin{array}{c|cccc} & z' & w' & x' & y' \\ \hline z & 1 & - & - & R \\ w & - & 1 & R & - \\ x & - & R & I_r & 0 \\ y & R & - & 1 & I_r \end{array}$$

The $z$-row and the $w$-row cannot coincide, because in that case the configuration

$$\begin{array}{c|cc} & z' & x' \\ \hline z{=}w & 1 & R \\ y & R & 1 \end{array}$$

shows that $wx' * yz'$, which is not possible since they have the same colour. Similarly, $z'$-column and $w'$-column are distinct. From the configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline w & R & - \\ y & 1 & I_r \end{array}$$

it follows [by case (ii)]that wy' is either R or $I_r$. So A(D) takes the form

$$\begin{array}{c|cccc} & z' & w' & x' & y' \\ \hline z & 1 & - & - & R \\ w & - & 1 & R & R\ /I_r \\ x & - & R & I_r & 0 \\ y & R & - & 1 & I_r \end{array}$$

Now the configuration

$$\begin{array}{c|cc} & w' & y' \\ \hline w & 1 & R\ /I_r \\ x & R & 0 \end{array}$$

shows that xy' is either R or $I_r$ [by cases (i) & (ii)].

The other possibility when xy' is 1 can be taken care of similarly.

(b) Next, let both xy' and yx' be 0.i.e. A(D) has a configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline x & I_r & 0 \\ y & 0 & I_r \end{array}$$

By Proposition 3.1, both xy' and yx' cannot be C. So one must be I or R. If it is R, then the theorem is proved. Let, one of xy' and yx', say, xy'; be I. For the interior edges $I_r$ on xx′ and yy′ positions, there must exist configurations of the form

| | w′ | x′ |
|---|---|---|
| w | 1 | R |
| x | R | $I_r$ |

| | z′ | y′ |
|---|---|---|
| z | 1 | R |
| y | R | $I_r$ |

in A(D). As earlier, z-row (z′-column) and w-row (w′-column) are distinct. So A(D) has a configuration

| | z′ | w′ | x′ | y′ |
|---|---|---|---|---|
| z | 1 | – | – | R |
| w | – | 1 | R | – |
| x | – | R | $I_r$ | I |
| y | R | – | 0 | $I_r$ |

For the configuration

| | w′ | y′ |
|---|---|---|
| x | R | I |
| y | – | $I_r$ |

if yw′ is 1 then it follows that xy′ is $I_r$ [by case(iia)] and the theorem is proved ; if yw′ is 0 then by case (iib), yw′ is either R or $I_r$. So A(D) takes the form

| | z′ | w′ | x′ | y′ |
|---|---|---|---|---|
| z | 1 | – | – | R |
| w | – | 1 | R | – |
| x | – | R | $I_r$ | I |
| y | R | R/$I_r$ | 0 | $I_r$ |

From the submatrix

| | w′ | x′ |
|---|---|---|
| w | 1 | R |
| y | R/$I_r$ | 0 |

it follows that yx′ is either R or $I_r$ [cases (i) & (ii)].

**Theorem 3.3.** *Let D be a digraph of F.D.2. If D is an interval digraph, then for any satisfactory bicolouration of $H_b(D)$,*

$$I_r \cap I_c = \phi.$$

Proof. Let, if possible, $I_r \cap I_c \neq \phi$ for some satisfactory bicolouration of $H_b(D)$. Then there exist an $I \in I(H)$ such that $I \in I_r \cap I_c$. It follows that I is an interior edge of both the Ferrers digraphs $G_1$ and $G_2$ where $G_1 = R \cup I(H)$ and $G_2 = C \cup I(H)$. First consider the case when $H_b(D)$ consists of one component only. Since the bicolouration of $H_b(D)$ is unique, R's and C's must belong exclusively to $G_1$ and $G_2$ respectively for any realization of $\bar{D}$ as the union of two Ferrers digraphs $G_1$ and $G_2$; it follows that the concerned I cannot be excluded from either of $G_1$ and $G_2$ for any such realization of $\bar{D}$. Hence $\bar{D}$ cannot be expressed as the union of two disjoint Ferrers digraphs and accordingly D cannot be an interval digraph.

Next, when $H_b(D)$ has more than one component, if $I_r \cap I_c \neq \phi$ for some satisfactory bicolouration, then the digraph $\bar{D}$ cannot be decomposed into two (disjoint) Ferrers digraphs with respect to the given bicolouration. Then the theorem follows from the Proposition 3.4 that $I_r \cap I_c \neq \phi$ for a bicolouration implies $I_r \cap I_c \neq \phi$ for any satisfactory bicolouration. ■

That the converse of the above theorem is not true follows from the following counter-example :

*Example 3.1. Consider the digraph D(V,E) whose adjacency matrix is given by*

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $v_2$ | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| $v_3$ | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 |
| $v_4$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_5$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_6$ | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 |
| $v_7$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $v_8$ | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |

This is a digraph of F.D.2 and $H_b(D)$ has only one component for this digraph. By labelling the 0's in terms of R's, C's and I's, A(D) takes the form

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | 1 | 1 | 1 | 1 | 1 | R | R | R |
| $v_2$ | 1 | 1 | 1 | 1 | C | 1 | I | R |
| $v_3$ | 1 | 1 | 1 | 1 | C | 1 | 1 | R |
| $v_4$ | 1 | R | R | R | I | R | I | R |
| $v_5$ | C | 1 | R | R | I | R | I | I |
| $v_6$ | C | 1 | 1 | 1 | C | 1 | C | 1 |
| $v_7$ | C | C | 1 | 1 | I | R | I | I |
| $v_8$ | C | C | C | 1 | C | 1 | I | I |

The set of interior edges of $G_1 = R \cup I(H)$ and $G_2 = C \cup I(H)$ are $I_r(G_1) = \{ v_4v_7, v_5v_7, v_5v_8, v_7v_8 \}$ and $I_c(G_2) = \{ v_5v_5, v_7v_5, v_7v_7, v_8v_7 \}$. It is seen that $I_r \cap I_c = \phi$ for this digraph. If $H_1 = R \cup I_r$ and $H_2 = C \cup I_c$, then $H_1 \cup H_2 \neq \bar{D}$ and the two edges $v_4v_5$ and $v_2v_7$ lie outside $H_1 \cup H_2$.

The following representation of $H_1$ and $H_2$ alongwith the two edges $v_4v_5$ and $v_2v_7$ in the form of a matrix makes it clear.

$H_1$ =

| | $v_1$ | $v_5$ | $v_2$ | $v_3$ | $v_4$ | $v_7$ | $v_6$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_4$ | | I | R | R | R | $I_r$ | R | R |
| $v_5$ | | | | R | R | $I_r$ | R | $I_r$ |
| $v_1$ | | | | | | R | R | R |
| $v_7$ | | | | | | | R | $I_r$ |
| $v_2$ | | | | | | | | R |
| $v_3$ | | | | | | I | | R |
| $v_6$ | | | | | | | | |
| $v_8$ | | | | | | | | |

$H_2$ =

| | $v_5$ | $v_1$ | $v_7$ | $v_2$ | $v_3$ | $v_4$ | $v_6$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | | | | | | | | |
| $v_4$ | I | | | | | | | |
| $v_2$ | C | | I | | | | | |
| $v_3$ | C | | | | | | | |
| $v_5$ | $I_c$ | C | | | | | | |
| $v_6$ | C | C | C | | | | | |
| $v_7$ | $I_c$ | C | $I_c$ | C | | | | |
| $v_8$ | C | C | $I_c$ | C | C | | | |

Since the bicolouration is unique, $H_1$ and $H_2$ must be contained exclusively in any two decomposed Ferrers digraphs of $\bar{D}$. Nevertheless, the edges $v_2v_7$ cannot be adjoined to any of $H_1$ or $H_2$ to make them Ferrers digraphs again. Hence it is not possible to cover $\bar{D}$ by two disjoint Ferrers digraphs and accordingly D is not an interval digraph.