CHAPTER - II

INTERVAL DIGRAPHS

2.1. Introduction.

The concepts of intersection graph and interval graph have been well studied for undirected graphs. Given a family of sets, each is assigned to a vertex, and the *intersection graph* of the family of sets has an edge between two of these vertices if and only if the corresponding sets intersect. A graph is an *interval graph* if it is the intersection graph of a family of intervals on the real line.

In this chapter, we introduce and study a natural analogue of these concepts for directed graphs D(V,E). We consider a family $\mathcal{F}$ of ordered pairs of sets, and to each ordered pair we assign a vertex v. The first set assigned to v is called its <u>source set</u> S_v, and the second is its <u>terminal set</u> (or *sink set*) T_v. The <u>intersection digraph</u> of a family of ordered pairs of sets is the digraph such that $uv \in E$ if and only if $S_u \cap T_v \neq \phi$. Note that loops are allowed, but there are no multiple edges. By analogy with interval graphs, we define an <u>interval digraph</u> to be an intersection digraph of a family of ordered pairs of intervals on the real line. Several characterizations are known for interval graphs; our

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The major part of this chapter has been published in <u>J. Gr. Th.</u> <u>13</u> (1989) 189-202 aim in this chapter is to give analogous characterizations for interval digraphs. In section 2.3, we obtain a characterization for interval digraphs similar to Fulkerson and Gross [1965] characterization for interval graphs, using a simultaneous consecutive ones property for the rows of two incidence matrices.

A class of intersection digraphs was introduced and studied by Maehara [1984]. He defined a <u>pointed set</u> to be a set S with a distinguished *'base point'* $b \in S$. Phrased in our terminology, he defined the <u>catch digraph</u> of a family of pointed sets $\{(S_v,b_v)\}$ to be the intersection digraph in which the source set for v is S_v and the sink set is b_v; i.e. $uv \in E$ iff $b_v \in S_u$. When the source sets are required to be intervals, this is a class of interval digraphs. Note that $b_v \in S_v$ forces a loop at each vertex. Dropping the requirement we get an intermediate family between the catch digraphs of intervals and the general interval digraphs which we call <u>interval-point digraph</u>.

Recall that the *adjacency matrix* A(D) of a digraph D with vertices numbered $v_1,\ldots,v_n$ is the 0,1-matrix with a 1 in position ij if and only if $v_i v_j$ is an edge. Maehara characterized the catch digraphs of families of pointed intervals as those whose adjacency matrix has the consecutive ones property for rows (without allowing column permutations). In section 2.3, we characterize the interval-point digraphs; dropping the requirement $b_v \in S_v$ corresponds to allowing column permutations in testing for the consecutive ones property of the adjacency matrix. In this connection we have to mention the paper by Prisner [1989] in which he also characterized interval catch digraphs. This characterization is quite analogous to Lekkerkerker- Boland [1962] characterization of interval graphs.

In section 2.4, we obtain more difficult characterizations of interval digraphs. We show that D is an interval digraph iff A(D) has (independent) row and column permutations so that every 0 can be replaced by one of {C,R} in such a way that R has all R's to its right and every C has all C's below it. At the same time, we characterize interval digraphs in terms of a special class of digraphs.

Recall that *Ferrers digraphs* are those whose successor sets are linearly ordered by inclusion. It is easy to see that the successor sets are linearly ordered by inclusion iff the predecessor sets are linearly ordered by inclusion, and that both are equivalent to the transformability of the adjacency matrix by independent row and column permutations to a

0,1-matrix in which the 1's are clustered in the lower left in the shape of a Ferrers diagram. We prove that D is an interval digraph if and only if it is the intersection of two Ferrers digraphs whose union is complete.

Intersections of Ferrers digraphs have been studied previously, but without the requirement that the union be complete. Recall that the *Ferrers dimension* of D is defined to be the minimum number of Ferrers digraphs whose intersection is D. By our characterization, the Ferrers dimension of an interval digraph is at most 2. The digraphs with Ferrers dimension 2 have been characterized, independently by Cogis [1979] and Doignon, Ducamp and Falmagne [1984] in different contexts. In section 2.5, we translate Cogis' condition to an adjacency matrix condition for Ferrers dimension 2 that is analogous to our adjacency matrix condition for interval digraphs. We then construct an example to show that not every digraph of Ferrers dimension 2 is an interval digraph.

Incidentally, in chapter-IV, we will obtain still another characterization of an interval digraph in terms of the partition of the adjacency matrix into two sectors.

2.2. Intersection digraphs

It is well-known that every finite undirected graph is an intersection graph of finite sets. The simplest construction is to use a set whose elements correspond to the edges of G and assign to each vertex the elements corresponding to its incident edges. Since vertices are adjacent if and only if they share an incident edge, G is the intersection graph of these finite sets. The analogous construction works for directed graphs. If S_v consists of the edges with v as source and T_v consists of the edges with v as terminus, then $S_u \cap T_v \neq \phi$ if and only if $uv \in E$.

For undirected graph, it is easy to obtain a more "efficient" representation by using cliques larger than single edges to cover the edges. Indeed, the <u>intersection number</u> i # (G) of an undirected graph G is defined to be the minimum size of a set U such that G is the intersection graph of subsets of U, and Erdös, Goodman, and Posa [1966] showed that the intersection number of G equals the minimum number of complete subgraphs needed to cover its edges. They also proved that $i\#(G) \leq \lfloor n^2/4 \rfloor$ for n-vertex graphs, achieved by $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

To develop analogous results for digraphs, we define a generalized complete bipartite subdigraph (abbreviated GBS) to be a subdigraph generated by vertex sets X,Y, whose edges are all *xy* such that $x \in X$, $y \in Y$. We say "generalized" because X,Y need not be disjoint, which means that loops may arise. Let the intersection number i#(D) of a digraph be the minimum size of U such that G is the intersection digraph of ordered pairs of subsets of U. The analogue of the Erdös, Goodman, Posa results is as follows :

Theorem 2.1. *The intersection of a digraph equals the minimum number of GBSs required to cover its edges, and the best possible upper bound on this is n for n-vertex graph.*

Proof. Suppose $\{(X_i,Y_i)\}$ with k members is a minimum collection of GBSs whose union is D. Let $S_v = \{i : v \in X_i\}$ and let $T_v = \{i : v \in Y_i\}$. Then $S_u \cap T_v \neq \phi$ if and only if $uv \in E$, and $i\#(D) \leq k$.

Conversely, if D is the intersection digraph of ordered pairs of subsets of U, where U has $i\#(D)$ elements, define $i\#(D)$ GBSs by $u \in X_i$, $v \in Y_i$ for some i, and $k \leq i\#(D)$.For the bound on i#(D), note that the set of edges with v as source is a GBS, so n disjoint GBSs can provide all the edges. This bound is achieved by the (directed) cycle of length n, in which there is no GBS with more than one edge.■

To see the importance of loops, note that the complete directed graph (with loops) is a GBS and has intersection number 1, but when the loops are forbidden the resulting digraph has intersection number $2\log_2|V|$. Also of interest are intersection graphs where the sets are required to be convex sets in Euclidean space. For undirected graphs, one-dimensional convex sets yield the interval graphs and three-dimensional convex sets allow all graphs to be represented. With two dimensional convex sets, all planar graphs and some others can be represented, but not all graphs. For example, Wegner [1967] showed that the graph obtained by subdividing each edge of K_5 is not an intersection graph of convex sets in the plane.

For digraphs, it is not clear whether the situation is analogous. Attempts to mimic Wegner's example fail. For example, consider the digraph obtained from K_5 by replacing each edge v_iv_j by a path $v_iu_{ij}v_j$, replacing each of the resulting 20 edges by a pair of oppositely oriented directed edges and adding a loop in each of the 15 vertices. This digraph is the intersection digraph of ordered pairs of convex sets in the plane; in fact, the source and sink sets can be segments for the u_{ij}'s and disks for the v_i's.

Therefore, we pose the question of whether every digraph is the intersection digraph of ordered pairs of convex sets in the plane. We devote the remainder of this chapter to representations using one-dimensional convex sets — i.e., intervals on the real line.

2.3. Characterizations by consecutive ones

We begin this section with the primary observation that an interval digraph property is hereditary in the sense that any induced subgraph of an interval digraph is also an interval digraph.

Theorem 2.2. *A sufficient condition for a digraph to be an interval digraph is that the adjacency matrix has the consecutive ones property for rows.*

Proof. Let D(V,E) be a digraph with n vertices whose adjacency matrix A has the consecutive ones property for rows. Then reordering the vertices of D, we may assume that the ones in each row of $A = (a_{ij})$ occur consecutively.

Let
$$x_i = \min \{ j \,/\, a_{ij} = 1\},$$
$$y_i = \max \{ j \,/\, a_{ij} = 1\}, \qquad (i = 1,\dots,n).$$

So considering $S_i = [x_i, y_i]$ and $T_i = [i]$, $(i = 1,\dots,n)$.

the digraph D can be seen to be represented by the family $\left\{ (S_i , T_i) \right\}$. ■

But the converse is not true. This follows from the following counter-example :

Example 2.1. Consider the digraph on four vertices with loops at v_2, v_3, v_4 and edges from these vertices to v_1 i.e. the digraph given in *Fig.* 2.1.

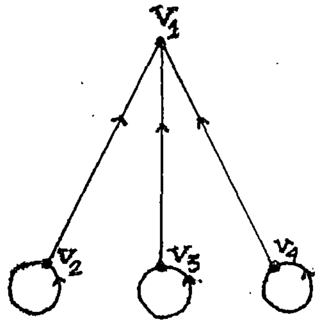

Fig. 2.1.

Its adjacency matrix is

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 1 \end{array}$$

This matrix does not have the consecutive ones property for rows ; but still it has the following interval representation :

i	1	2	3	4
S_i	ϕ	[1]	[2]	[3]
T_i	[1,3]	[1]	[2]	[3]

The above example leads us to characterize interval digraphs using singleton terminal sets; these are the *interval-point digraphs*.

Theorem 2.2. *D is an interval-point digraph if and only if its adjacency matrix has the consecutive ones property for rows.*

Proof. D has an interval-point representation if and only if there are sets $S(v)=[a(v),b(v)]$ and $T(v)=\{c(v)\}$ such that $uv \in E$ if and only if $c(v) \in S(u)$, where we may assume the $c(v)$'s are distinct. This is true if and only if numbering the vertices in increasing order of $c(v)$ exhibits the consecutive ones property for rows of $A(D)$. Furthermore, if there is a numbering $v_1,\dots,v_n$ that exhibits this, we may take $c(v_k) = k$, $a(v_k) = \min\{i: v_iv_k \in E\}$ to obtain an interval-point representation of D. ■

Note that the digraph given in Ex.2.1. is an interval digraph which is not an interval-point digraph.

Example 2.2. The smallest digraphs which are not interval digraphs are given in *Fig.* 2.2

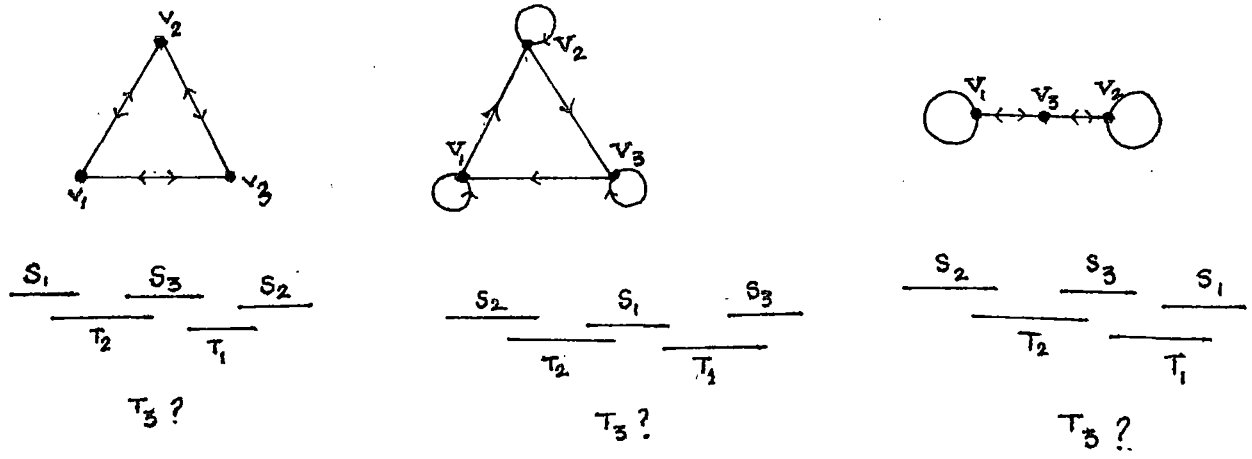

Fig. 2. 2

It is to be noted that adjacency matrices of all the three digraphs above are equivalent in the sense that one can be obtained from the other by a suitable permutation of rows (or columns).

We can also characterize interval digraphs using the consecutive ones property, but for this more general class we must consider incidence matrices involving vertices and GBSs. This is analogous to the Fulkerson, Gross characterization of interval graphs. Let $B = \{(X_k, Y_k)\}$ be a collection of GBSs whose union is D. We define the <u>vertex-source incidence matrix</u> for B(abbreviated V,X-matrix) to be the incidence matrix between the vertices and the source sets $\{X_k\}$. Similarly, the <u>vertex-terminus incidence matrix</u> for B (abbreviated V,Y-matrix) is the incidence matrix between the vertices and the terminal sets $\{Y_k\}$. Our first characterization of interval digraphs is the following :

Theorem 2.4. *D is an interval digraph if and only if there is a numbering of the GBSs in some covering* **B** *of* D *such that the ones in rows appear consecutively for both the* V, X*-matrix and* V, Y*-matrix of* D.

Proof. For sufficiency, consider such a **B** whose union is D, and let (X_k, Y_k) be a common numbering if the columns of the V,X and V,Y-matrices that exhibit the consecutive ones property for both. Assign $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$, where a_v, b_v, c_v, d_v are defined by $v \in X_k$ if and only if $a_v \leq k \leq b_v$ and $v \in Y_k$ if and only if $c_v \leq k \leq d_v$. Then $S_u \cap T_v \neq \phi$ if and only if $u \in X_k$ and $v \in Y_k$ for some k.

For necessity, consider a representation of D by a family $\{(S_v, T_v)\}$ of ordered pair of intervals. We may assume they are closed and have integer endpoints, with $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$. For each integer k belonging to any of these intervals, define a GBS $B_k = (X_k, Y_k)$ of the intersection digraph of the interval pairs by setting $X_k = \{v : k \in S_v\}$ and $Y_k = \{v : k \in T_v\}$. Then $S_u \cap T_v \neq \phi$ if and only if $u \in X_k$ and $v \in Y_k$ for some k, so the intersection digraph of the interval pairs is in fact the union of the specified GBSs. Furthermore, by construction the resulting V,X and V,Y-matrices have the simulteneous consecutive ones property for rows. ■

The Fulkerson-Gross characterization of interval graphs considers only a single matrix, whose columns correspond to all maximal cliques. Here we may not be able to include all maximal GBSs. For interval graphs, the Helly property of the real line guarantees that the intervals for vertices of any maximal clique must have a common intersection in any interval representation. For GBSs, this does not hold. In particular, the following example is an interval digraph for which the collection of all maximal GBSs does not exhibit the simulteneous consecutive ones property :

Example 2.3.

	v_1	v_2	v_3	v_4
v_1	1	1	1	1
v_2	1	0	0	1
v_3	0	1	0	1
v_4	0	0	1	1

The maximal GBSs of this digraph are (v_1v_2/v_1v_4), (v_1v_3/v_2v_4), (v_1v_4/v_3v_4), $(v_1/v_1v_2v_3v_4)$. The first three cover the edges of D, and the resulting V,X and V,Y-matrices are

	X_1	X_2	X_3
v_1	1	1	1
v_2	1	0	0
v_3	0	1	0
v_4	0	0	1

	Y_1	Y_2	Y_3
v_1	1	0	0
v_2	0	1	0
v_3	0	0	1
v_4	1	1	1

Note that columns cannot be added for the other maximal GBSs without destroying the consecutive ones property.

2.4 Characterization of interval digraph in terms of adjacency matrix

In this section, we prove two equivalent characterizations of interval digraphs; i.e. necessary and sufficient conditions for the existence of an intersection representation using ordered pairs of intervals. Both conditions are intimately connected with Ferrers digraphs.

As illustrated below, there are two natural partitions of the vertices associated with a Ferrers digraph, given by the differences of successive predecessor sets and the differences of the successive successor sets in their respective inclusion orderings. The <u>source partition</u> places v in the kth set (called A_k below) if v belongs to the kth largest predecessor set and no larger one, with the 0th set containing the vertices with no successors (i.e., in no predecessor set). Similarly, the <u>terminal partition</u> places v in the kth set (called D_k below) if v belongs to the kth smallest nonempty successor set and no smaller one, with the last set containing the vertices with no predecessors (i.e., in no successor set). Note that the 0th source set and/or last terminal set may be empty. In terms of the partitions, the condition for adjacency is $uv \in E$ if and only if $i \geq j$, where u is in the ith source set and v is in the jth terminal set.

$$\begin{array}{c|cccccccc}
 & & D_1 & & D_2 & & D_3 & & D_4 \\
\hline
A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
A_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
A_2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
A_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}$$

For undirected graphs, the analogous condition is that the adjacency sets form a chain by inclusion. This condition characterizes the graphs known as *threshold graphs*, introduced by Chvátal and Hammer. The relevance of Ferrers digraphs to interval digraphs is analogous to that between threshold graphs and interval graphs : G is a threshold graph if and only if both G and $\bar{G}$ are interval graphs. We also note that some of our reasonings in the proof below parallel works of Fishburn [1985] and Mirkin [1984] on interval orders.

We now prove the characterizations. We say that the union of two digraphs is complete if every ordered pair of vertices (including u=v) forms an edge in at least one of them. The difficult part of this result is to obtain an interval representation from the intersection of two Ferrers digraphs whose union is a complete digraph.

Theorem 2.5. The following conditions are equivalent :

(A) *D is an interval digraph.*

(B) *The rows and columns of the A(D) can be (independently) permuted so that each 0 can be replaced by one of {R,C} in such a way that every R has only R's to its right and every C has only C's below it.*

(C) *D is the intersection of two Ferrers digraphs whose union is complete.*

Proof.

<u>A implies B</u>. Consider an interval representation of D, with vertex v to which is assigned the source set [a(v),b(v)] and the sink set [c(v),d(v)]. Define two vertex numberings $u_1,\ldots,u_n$ and $w_1,\ldots,w_n$ by $u_k = v$ if a(v) is the kth smallest value among the a's, and $w_k = v$ if c(v) is the kth smallest value among the c's, with ties broken arbitrarily. Order the rows and columns of A(D) as $u_1,\ldots,u_n$ and $w_1,\ldots,w_n$.

If entry i,j of the resulting matrix is 0, then $a(u_i) > d(w_j)$ or $c(w_j) > b(u_i)$, but $a(v) \leq b(v)$ and $c(v) \leq d(v)$ implies these inequalities can not simultaneously hold. Relabel this position by C if $a(u_i) > d(w_j)$ and by R if $c(w_j) > b(u_i)$. Since the a's and c's increase with index, the first (second) inequality will continue to hold below (to the right of) this entry, so the condition of B holds.

B implies C. Consider such a permutation M of A(D); let Q_1, Q_2 denote the set of edges of $\bar{D}$ corresponding to positions of M labelled R,C respectively. Let F_1, F_2 be digraphs defined by $E(F_i) = E \cup Q_i$. Then F_1, F_2 are Ferrers digraphs (F_1 by a column permutation and F_2 by a row permutation) whose intersection is D and whose union is complete.

C implies A. Let F_1, F_2 be two Ferrers digraphs whose union is complete and whose intersection is D; we obtain an intersection representation of D by ordered pair of intervals. We will work with the complements of F_1, F_2, which we call H_1,H_2; their union is $\bar{D}$ and their intersection is empty. The complement of a Ferrers digraph is also a Ferrers digraph, as is apparent from the adjacency matrix. We could stick to F_1,F_2 by using a slightly different indexing of the source and terminal partitions, in which adjacency corresponds to strict inequality.

Let $A_0,\ldots,A_{p-1}$ be the source partition for H_1, and let $D_1,\ldots,D_p$ be its terminal partition, where A_0 and/or D_p may be empty. The adjacency matrix of the *converse* (also called *inverse*) of a digraph is the transpose of the adjacency matrix of the original digraph; thus the converse of a Ferrers digraph is a Ferrers digraph. Let $C_0,\ldots,C_{q-1}$ be the source partition and $B_1,\ldots,B_q$ the terminal partition for the converse of H_2, where B_q and/or C_0 may be empty. Thus adjacency in H_2 is characterized by $uv \in H_2$ if and only if $j \leq k$, where $u \in B_j$ and $v \in C_k$, and as usual $uv \in H_1$ if and only if $i \geq l$, where $u \in A_i$ and $v \in D_l$.

We will construct an interval representation of D by assigning numbers to the sets in these partitions : a_i, b_i, c_i, d_i respectively to A_i, B_i, C_i, D_i. We want to assign to vertex v the intervals $S_v = [a(v), b(v)]$, and $T_v = [c(v), d(v)]$, where $(a(v), b(v), c(v), d(v)) = (a_i, b_j, c_k, d_l)$ if $v \in A_i, B_j, C_k, D_l$; there are two requirements these numberings must satisfy.

First, S_v, T_v must be intervals, which requires $a_i \leq b_j$ and $c_k \leq d_l$ if $v \in A_i \cap B_j$ and $v \in C_k \cap D_l$. Next, given that $\{S_v\}$ and $\{T_v\}$ are intervals, we have $uv \in E$ in the intersection digraph if and only if $a(u) \leq d(v)$ and $b(u) \geq c(v)$. On the other hand, uv is a nonedge in H_1 and in H_2 if and only if $u \in A_i, B_j$, and $v \in C_k, D_l$, where $i < l$ and $j > k$. Thus, $\overline{H}_1 \cap \overline{H}_2$ will have the same edges as the intersection digraph of $\{(S_v, T_v)\}$ if a, d satisfy $a_i \leq d_l$ if and only if $i < l$ and b, c satisfy $b_j \geq c_k$ for $j > k$. To achieve this, it suffices that a, b, c, d be strictly increasing sequences with $a_i = d_i + 1$ and $c_i = b_i + 1$ for all $i \geq 1$. To summarize, it suffices to construct sequences a, b, c, d such that

(i) $a_i \leq b_j$ if $A_i \cap B_j \neq \phi$ and $c_k \leq d_l$ if $C_k \cap D_l \neq \phi$.

(ii) a, b, c, d are strictly increasing sequences.

(iii) $a_i = d_i + 1$ and $c_i = b_i + 1$ for $i \geq 1$

To construct these sequences, we first form a directed graph M on vertices corresponding to the sets of the partitions. Begin with directed paths $A_0, \ldots, A_{p-1}$, $B_1, \ldots, B_q$, $C_0, \ldots, C_{q-1}$, $D_1, \ldots, D_p$. Add edges A_iB_j when $A_i \cap B_j \neq \phi$, C_kD_l when $C_k \cap D_l \neq \phi$, B_iC_i for $1 \leq i \leq q-1$, and D_iA_i for $1 \leq i \leq p-1$. We claim M is acyclic, and will use this fact to construct the sequences. Suppose M has a cycle; it must pass through some A's, then some B's, then some C's and then some D's. It uses one edge A_iB_j and one edge C_kD_l; these correspond to vertices $u \in A_i \cap B_j$ and $v \in C_kD_l$ in D. Because the indices do not decrease on any other kind of edge, the existence of A_iB_i and C_kD_l in a cycle of M requires $j \leq k$ and $l \leq i$. This yields a contradiction, because it implies uv is an edge in both H_1 and H_2, which by hypothesis are disjoint.

Since M is acyclic, there exists an integer numbering $f: V(M) \longrightarrow \mathbb{N}$ (called a "topological ordering") such that $f(Y) > f(X)$ when $XY \in E(M)$. It remains only to show that we can choose f so that $f(A_i) = f(D_i) + 1$ and $f(C_i) = f(B_i) + 1$;

the desired sequences a,b,c,d then appear in the values of f. Note that this algorithm actually produces $a_i < b_j$ when $A_i \cap B_j \neq \phi$ and $c_k < d_l$ when $C_k \cap D_l \neq \phi$, rather than just $\leq$ and $\geq$. As a result, all intervals have length at least one; single- point intervals are not generated. It is usually possible to arrange the numbering to be less "spread out" by allowing equalities on these edges. We use this method of topological ordering here because the proof is short.

The natural algorithm for assigning the numbers is to assign to vertex X the number t if the longest path ending at X has t vertices (these numbers are easily computed by iteratively stripping off the vertices without predecessors). We claim that this numberings put $f(C_i) = f(B_i)+1$; the same argument works to show $f(A_i) = f(D_i) + 1$ also. For any C_i with $i \geq 1$, the predecessors are C_{i-1} and B_i. Since any path ending at B_i can be extended to C_i, we have $f(C_i) \geq f(B_i) + 1$. For the opposite inequality, consider a longest path ending at C_i. If it originates at C_0, then it visits only C's and the path from B_1 to B_i is shorter by one. Otherwise, it crosses from B_j to C_j for some $j \leq i$; now replacing the vertex sequence $C_j, \ldots, C_i$ at the end of the path by $B_{j+1}, \ldots, B_i$ yields a path ending at B_i that is shorter by one. In either case, we obtain $f(B_i) \geq f(C_i) - 1$.

It is to be noted that in order to obtain the sequences a,b,c,d , construction of the directed graph M is only an aid to finding them out and we can very well obtain them without constructing M. ■

Example 2.4. To illustrate these results, consider the adjacency matrix given below :

$$\begin{array}{c|ccccccccc}
 & v_2 & v_4 & v_8 & v_5 & v_3 & v_1 & v_6 & v_7 & v_9 \\
\hline
v_1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
v_3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
v_4 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
v_5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_6 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_7 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
v_8 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
v_9 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

By further row and column permutations, we obtain the partition of 0's into R's and C's that is condition (B). This immediately gives the *Ferrers* digraphs H_1 and H_2 whose union is D. This partition is not unique, and other partitions may lead to other interval representations.

$$H_1 = \begin{array}{cc|ccccccccc} & & \multicolumn{9}{c}{D} \\ & & v_8 & v_3 & v_4 & v_1 & v_2 & v_5 & v_6 & v_7 & v_9 \\ \hline & v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & v_3 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A & v_5 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & v_4 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & v_6 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & v_7 & C & C & C & 0 & 0 & 0 & 0 & 0 & 0 \\ & v_8 & C & C & C & C & 0 & 0 & 0 & 0 & 0 \\ & v_9 & C & C & C & C & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$H_2 = \begin{array}{cc|ccccccccc} & & \multicolumn{9}{c}{C} \\ & & v_2 & v_8 & v_4 & v_5 & v_3 & v_1 & v_6 & v_7 & v_9 \\ \hline & v_5 & 0 & 0 & R & R & R & R & R & R & R \\ & v_1 & 0 & 0 & 0 & 0 & 0 & R & R & R & R \\ & v_6 & 0 & 0 & 0 & 0 & 0 & R & R & R & R \\ B & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & R \\ & v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & R \\ & v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & R \\ & v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R \\ & v_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R \\ & v_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R \end{array}$$

Given H_1 and H_2, we move to the proof of (C) implies (A) to obtain the interval representation. The four vertex partitions are given by

i	0	1	2	3	4	5
A_i	v_1, v_2	v_3, v_5	v_4, v_6	v_7	v_8, v_9	
B_i		v_5	v_1, v_6	v_2, v_3, v_7	v_4, v_8, v_9	
C_i	v_2, v_8	v_4, v_5, v_3	v_1, v_6	v_7	v_9	
D_i		v_8	v_3	v_4	v_1	v_2, v_5, v_6, v_7, v_9

From the intersection $A_i \cap B_j$ and $C_k \cap D_l$, the crucial edges of the implication digraph are A_iB_i for i =1,2,3,4, C_iD_{i+1} for i = 0,1,4, and C_2D_4. The resulting topological ordering yields the following sequences :

i	0	1	2	3	4	5
a_i	1	3	7	8	11	
b_i		4	8	9	12	
c_i	1	5	9	10	13	
d_i		2	6	7	10	14

Picking out the a(v),b(v),c(v),d(v) for each vertex v yields the interval representation for D.

i	1	2	3	4	5	6	7	8	9
S_i	[1,8]	[1,9]	[3,9]	[7,12]	[3,4]	[7,8]	[8,9]	[11,12]	[11,12]
T_i	[9,10]	[1,14]	[5,6]	[5,7]	[5,14]	[9,14]	[10,14]	[1,2]	[13,14]

As mentioned earlier, there are less spread out representations than that generated by the algorithm in the proof. One such representation is

i	1	2	3	4	5	6	7	8	9
S_i	[1,4]	[1,6]	[3,6]	[4,8]	[1]	[4]	[6]	[8]	[8]
T_i	[5,7]	[1,9]	[2,3]	[2,5]	[2,9]	[5,9]	[7,9]	[2]	[9]

2.5. Interval digraphs and Ferrers dimension

As mentioned in the introduction, the Ferrers dimension of D is the minimum number of Ferrers digraphs whose intersection is D. Theorem 2.5 implies that Ferrers dimension at most 2 is a necessary condition for an interval digraph. In this section, we show that it is not a sufficient condition. Riguet [1951] originally defined Ferrers digraph by an algebraic condition. He showed it equivalent to the adjacency conditions we have described by obtaining a forbidden submatrix characterization of the adjacency matrix : D is a Ferrers digraph if and only if A(D) has no 2 by 2 submatrix that is a permutation matrix. This is equivalent to the inclusion condition on the successor or predecessor sets. Let us call such a forbidden submatrix an <u>obstruction</u>. Cogis [1979] defined a graph H(D) whose vertices correspond to the 0's of the adjacency matrix, with two such vertices joined by an edge if the corresponding 0's belong to an obstruction. He proved that D has Ferrers dimension at most 2 if and only if *H*(D) is bipartite. This equivalence yields a short proof of the permutation characterization of Ferrers dimension 2, because we can omit the more difficult step of showing that H(D) bipartite implies the other conditions.

Theorem 2.6. The following conditions are equivalent :

(A) D *has Ferrers dimension at most* 2.

(B) *The rows and columns of* A(D) *can be (independenly) permuted so that no* 0 *has a* 1 *both below it and to its right.*

(C) *The graph* H(D) *of couples in* D *is bipartite.*

Proof. The part (C) implies (A) is proved by Cogis [1979] and by Doignon, Ducamp and Falmagne [1984]. So we prove below only the other two parts.

(A) implies (B). Let F_1, F_2 be two Ferrers digraphs whose intersection is D, with adjacency matrices A_1, A_2. Let $u_1, \ldots, u_n$ be the row ordering of A_1 that, with some column ordering, put the 0's of A_1 in the lower left and its 1's in the upper right. Let $w_1, \ldots, w_n$ be the column ordering of A_2 that, with some row ordering, put the 0's of A_2 in the upper right and its 1's in the lower left. Put the rows of A(D) in the order $u_1, \ldots, u_n$ and its columns in the order $w_1, \ldots, w_n$. We denote the matrix position corresponding to vertex pair $u_i w_j$ as $M(u_i w_j)$, where M is any of $A_1, A_2, A(D)$. If $A(D)(u_i w_j) = 0$, then $D = F_1 \cap F_2$ implies $A_1(u_i w_j) = 0$ or $A_2(u_i w_j) = 0$. If $A_1(u_i w_j) = 0$, then $A_1(u_r w_j) = 0$ for all $r > i$, and hence $A(D)(u_r w_j) = 0$ for all $r > i$, even though this column may be in a different position in A_1 and A(D). Similarly, if $A_2(u_i w_j) = 0$, then the remainder of the row in A(D) is 0.

(B) implies (C). Permute the rows and columns of A(D) so that no 0 has a 1 both to its right and below. Let R be the set of 0's having a 1 somewhere below them, and let C be the set of 0's having a 1 somewhere to the right. For any 2 by 2 submatrix forming a couple, the 0's must be an R in the upper right and a C in the lower left; These are the only edges in H(D). Therefore H is bipartite, with the 0's having no 1 to the right or below generating isolated points.

Example 2.5. Consider the digraph with the adjacency matrix below. By this theorem, this digraph has Ferrers dimension at most 2. However, this matrix does *not* satisfy the condition of Theorem 2.5 for being an interval digraph. This is the subject of our final theorem.

$$\begin{array}{c|ccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
v_3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
v_4 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
v_5 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
v_6 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
v_7 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}$$

Theorem 2.7. *The interval digraphs are properly contained in the set of digraphs with Ferrers dimension at most* 2.

Proof. Any permutation of A(D) that satisfy condition (B) of Theoerem 2.5. also satisfy condition (B) of Theorem 2.6, so inclusion holds. For proper containment, we show that the digraph D of Example 2.5 is not an interval digraph.

We claim there is no way to permute the rows and columns of A(D) so as to satisfy condition (B) of Theorem 2.5. First, note that the 0's of any obstruction must receive different labels; i.e., they cannot be both R or both C. Therefore, when we consider the bipartite H(D), the partite sets of each component must be all R's or all C's. For this D, H(D) consists of one nontrivial component and one isolated vertex corresponding to (v_6, v_6). Leaving the assignment of this label unspecified, the two possibilities we must consider for the nontrivial component yield the assignments below.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & R & R \\ 1 & 1 & 1 & 1 & 1 & 1 & R \\ C & 1 & 1 & 1 & 1 & 1 & 1 \\ C & 1 & 1 & 1 & 1 & C & 1 \\ C & C & 1 & 1 & R & 0 & R \\ C & C & C & 1 & 1 & C & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & C & C \\ 1 & 1 & 1 & 1 & 1 & 1 & C \\ R & 1 & 1 & 1 & 1 & 1 & 1 \\ R & 1 & 1 & 1 & 1 & R & 1 \\ R & R & 1 & 1 & C & 0 & C \\ R & R & R & 1 & 1 & R & 1 \end{bmatrix}$$

Next, we obtain a forbidden configuration that appears in each of these assignments. Let a,b,c,d be rows and A,B,C,D be columns satisfying the following properties:

(1) R appears in positions (a,D),(b,D),(b,C), and the rest of rows a,b is 1.

(2) C appears in positions (d,A), (d,B), (c,B), and the rest of columns A,B is 1.

(3) Row c has at least two R's, and column C has at least two C's.

We claim that no ordering of the rows and columns of a labeled matrix containing rows a,b,c,d and columns A,B,C,D as specified can have only R's to the right of each R and only C's below each C. Suppose there is such an ordering. Row a forces column D to be right-most, and then row b forces column C to be next to it. Similarly, column A forces row d at the bottom, and then column B forces row c immediately above it. But now the next to last diagonal position must be both R and C, since row c has at least two R's and column C has at least two C's.

Consider the two potential assignments of R and C in A(D). For the assignment on the left, choose a,b,c,d to be rows 3,1,6,7, respectively, and A,B,C,D to be columns 3,1,6,7, respectively. For the assignment on the right, choose a,b,c,d to be rows 4,5,6,1, respectively, and A,B,C,D to be columns 4,5,6,1, respectively. In each case, these choices satisfy the requirements for the forbidden configuration.

We note that the above example contains a submatrix of the form

$$\begin{bmatrix} 1 & C & 1 \\ R & 0 & R \\ 1 & C & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & R & 1 \\ C & 0 & C \\ 1 & R & 1 \end{bmatrix}$$

and the presence of such configuration does not allow the 0 in the $v_\sigma v_\sigma$ position to become either an R or a C, and consequently it does not permit the given digraph to be an interval digraph. This poses the following questions :

Are the above conditions necessary for a digraph D to be a non interval digraph ? Are they sufficient ?
These are the moot points of our discussion in the next Chapter.