CHAPTER III

ON THE CALCULATIONS OF NONCOHERENT LINE CONTOURS.

INTRODUCTION:

In the calculation of contours of absorption lines formed by noncoherent scattering, the mean value $\bar{J}(t)$ of the average intensity $J_\nu(t)$, weighted by η_ν (the ratio of the line to the continuum absorption coefficient) at frequency ν is usually approximated by the value $J_{\nu_0}(t)$ at the line center. The center of the absorption line is formed at the uppermost layer of the atmosphere where the optical depth for the wing frequency is very small and the only possible redistribution takes place among the central frequencies (1) and hence it is little affected by the wing radiation. The approximation is thus justified. Miyamoto (2) derived the second approximation formula and estimated the accuracy of such approximation. Geovenelli (3) gave a more general approximation for $\bar{J}(t)$ viz.

$$\bar{J}(t) = J_c + a \left[J_0(t) - J_c \right] , \qquad (3.1)$$

where

$$a = \frac{\int (J_\nu - J_c) k_\nu \, d\nu}{(J_0(t) - J_c) \int k_\nu \, d\nu} , \qquad (3.2)$$

and J_ν is the total intensity i.e.

$$\int_{4\pi} I_\nu dw = J_\nu$$

so that $\bar{J}(t)$ is the weighted mean of the total intensity J_ν, J_c is the total intensity at the continuum, a is independent of depth but varies with the shape of the line within the limits $(1/2)^{1/2}$ and 1. Dasgupta (4) in a short note gave a sketchy outline of the exact solution of the problem of noncoherent scattering from altogether a new point of view. He used, instead of $\bar{J}(t)$, the noncoherent emission term J(t) which he called the pseudo-average intensity. On elaboration, his theoretically derived closed explicit form of J(t) comes out to be

$$J(t) = B_\nu(t) + \frac{a'}{2b'}\left(pE_2(t) - p'E_3(t) \right) \tag{3.3}$$

where

$$E_n(t) = \int_1^\infty \exp(-tx).x^{-n}\, dx \tag{3.4}$$

~~which~~ (3.3) is an approximate form but not a rational one. Since J(t) should be independent of a' and b', the probability of coherent and noncoherent scattering, we may treat a'/b' to be an arbitrary constant, say Y, so that equation (3.3) can may treat a'/b' to be an arbitrary constant, say γ, so that equation (3.3) can be put in the form

$$J(t) = B_\nu(t) + \gamma/2 \; (pE_2(t) - p'E_3(t)) \; , \qquad (3.5)$$

In this capter we study the usefulness of the approximations given by (3.1) and (3.5) for different values of γ. As most of the authors in this subject define J_ν to be the average intensity i.e.

$$\int_{4\pi} I_\nu dw = 4\pi J_\nu$$

we also maintain the tradition. However, this change of definition does not change the form of equations (3.1) and (3.2). We use, following Miyamoto[(2)], the notation $J(\xi,t)$ for $J_\nu(t)$ where ξ is the frequency difference from the line center measured in units of doppler width, i.e.

$$\xi = \frac{\nu - \nu_0}{\Delta}$$

so that

$$J_0(t) = J(0,t)$$

$$\eta_\nu = \eta(\xi) \quad , \quad \text{etc.}$$

More over, in this chapter we denote by t the optical depth in continuum.

Let $I(\xi,t,\mu)$ be the specific intensity of frequency ξ at optical depth t and in the direction $\theta = \cos^{-1}\mu$ with the normal to the photosphere. The equation of transfer in the Milne- Eddington model including the pressure noncoherent scattering is

$$\frac{\mu dI(\xi,t,\mu)}{dt} = \lambda^2(\xi)\, I(\xi,t,\mu) - \eta(\xi)J(t) - (p + p't) \quad ,(3.6)$$

where

$$\lambda^2(\xi) = 1 + \eta(\xi) \quad , \quad (3.7)$$

and J(t) is the pseudo average intensity.[4]
Multiplying (3.6) by dw, integrating and using the relations

$$\int I(\xi,t,\mu)\, dw = 4\pi\, J(\xi,t) \quad , \quad (3.8)$$

$$\int I(\xi,t,\mu)\, \mu\, dw = \pi F(\xi,t) \quad , \quad (3.9)$$

we get

$$\frac{dF(\xi,t)}{4\,dt} = \lambda^2(\xi)\,J(\xi,t) - \eta(\xi)\,J(t) - (p + p't)\ , \qquad (3.10)$$

Multiplying (3.6) by μdw , integrating and using (3.9) and relation

$$\int I(\xi,t,\mu)\,\mu^2\,dw = 4\pi\,K(\xi,t)\ , \qquad (3.11)$$

we get

$$\frac{dK(\xi,t)}{dt} = (1/4)\,\lambda^2(\xi)\,F(\xi,t)\ , \qquad (3.12)$$

From (3.10) , (3.12) and the Eddington's approximation

$$K(\xi,t) = (1/3)\,J(\xi,t)$$

we have

$$\frac{d^2 J(\xi,t)}{dt^2} = 3\lambda^2(\xi)\left[\lambda^2(\xi)J(\xi,t) - \eta(\xi)J(t) - (p + p't)\right]\ , \qquad (3.13)$$

Let us now approximate $J(t)$ by

$$J(t) = \bar{J}(t) = J_c + a(\ J(0,t) - J_c\ )\ , \qquad (3.14)$$

due to Geovenelli[3], so that we get from (3.13)

$$\frac{d^2\left[\frac{1}{a}\left\{J(t) - J_c(1-a)\right\}\right]}{dt^2} = 3\lambda^2(0)\left[\lambda^2(0).\frac{1}{a}\left\{J - J_c(1-a)\right\} - \eta(0)J(t) - (p + p't)\right] \quad ,\ldots \qquad (3.15)$$

Let us write

$$\alpha = 1 + \eta(0)\ (1 - a) \quad , \qquad (3.16)$$

$$D = 3\lambda^2(0).\ \alpha \quad , \qquad (3.17)$$

$$M = a\eta(0)\ J_c(1 - a)/\alpha \quad , \qquad (3.18)$$

$$N = a(p + p't)/\alpha \quad , \qquad (3.19)$$

$$B = J(t) - J_c(1 - a) \qquad (3.20)$$

so that (3.15) takes the form

$$\frac{d^2 B}{dt^2} = D(B - M - N) \quad , \qquad (3.21)$$

Since the second derivatives of both M and N with respect to t are zero, we can write

$$\frac{d^2 (B - M - N)}{dt^2} = D(B - M - N) \quad , \qquad (3.22)$$

Writing β for $B - M - N$ we have

$$\frac{d^2 \beta}{dt^2} = D\beta \quad , \qquad (3.23)$$

Solving the second order differential equation (3.23) we get the solution , since β does not increase exponentially with t.

$$\beta = - C_2 \exp(- \sqrt{D}t) \quad , \qquad (3.24)$$

where C_2 is the constant of integration.
The boundary condition

$$2\,\bar{J}(0) = \bar{F}(0)$$

at the surface gives the constant

$$C_2 = \frac{J_c(1 - a/\alpha) + a/\alpha\,(p - 2p'/3\lambda^2(0))}{1 + 2\sqrt{D}/3\lambda^2(0)}, \tag{3.25}$$

Hence

$$J(t) = J_c(1 - a/\alpha) + \frac{a}{\alpha}(p + p't) - C_2\exp(-Dt), \tag{3.26}$$

Substituting the value of $J(t)$ from (3.26) to (3.6) and solving,

$$I(\xi,0,\mu) = (\eta(\xi)/\lambda^2(\xi))\left\{J_c(1 - a/\alpha)\right\} + (a/\alpha + 1/\eta(\xi)).$$

$$(p + p'\mu/\lambda^2(\xi)) - \frac{C_2\eta(\xi)}{\mu\sqrt{D} + \lambda^2(\xi)}, \tag{3.27}$$

The residual intensity $r(\xi,0,\mu)$ is then

$$r(\xi,0,\mu) = (p + p'\mu)^{-1} . I(\xi,0,\mu) \tag{3.28}$$

When $a = 1$, the approximation (3.14) reduces to

$$J(t) = \bar{J}(t) = J(0,t) \quad , \tag{3.29}$$

$$\alpha = 1 \tag{3.30}$$

and equations (3.25) , (3.26) and (3.28) reduce respectively to

$$C_2 = \frac{p - 2p'/3\lambda^2(0)}{1 + 2/\sqrt{3}\lambda(0)} \tag{3.31}$$

$$J(t) = \bar{J}(t) = p + p't - C_2\exp(-\sqrt{3}\lambda(0)t) \quad , \tag{3.32}$$

$$r(\xi,0,\mu) = (p + p'\mu)^{-1} . I(\xi,0,\mu)$$

$$= (p + p'\mu)^{-1} . \left[p + p'\mu/\lambda^2(\xi) - \frac{\eta(\xi)\, C_2}{\mu\sqrt{3}\lambda(0) + \lambda^2(\xi)} \right] , \tag{3.33}$$

which are respectively equations (12) , (10) and (17) of Miyamoto(2).

Residual intensities of solar Calcium line are calculated with (3.28) for different values of ξ at μ = 1.0 and 0.25. Following Miyamoto(2)

$$\eta(0) = 8.75 \cdot 10^{6}$$

$$\eta(\xi) = 1.40 \cdot 10^{5} \cdot \xi^{-2}$$

$$p'/p = 4.2$$

From Table 1 of Geovenelli(3) it is seen that for $u \geq 0.9$ i.e. for medium and strong lines a is independent of u it is then a function of $\eta(\xi)$ only. The values of a is taken to be equal to 0.958 from that Table.The results are given in Table XVIII.

Table XVIII

ξ	r(1)	r(0.25)
100	0.1959	0.4901
200	0.2322	0.5131
300	0.3160	0.5662
400	0.4221	0.6335
500	0.5242	0.6983
600	0.6111	0.7533

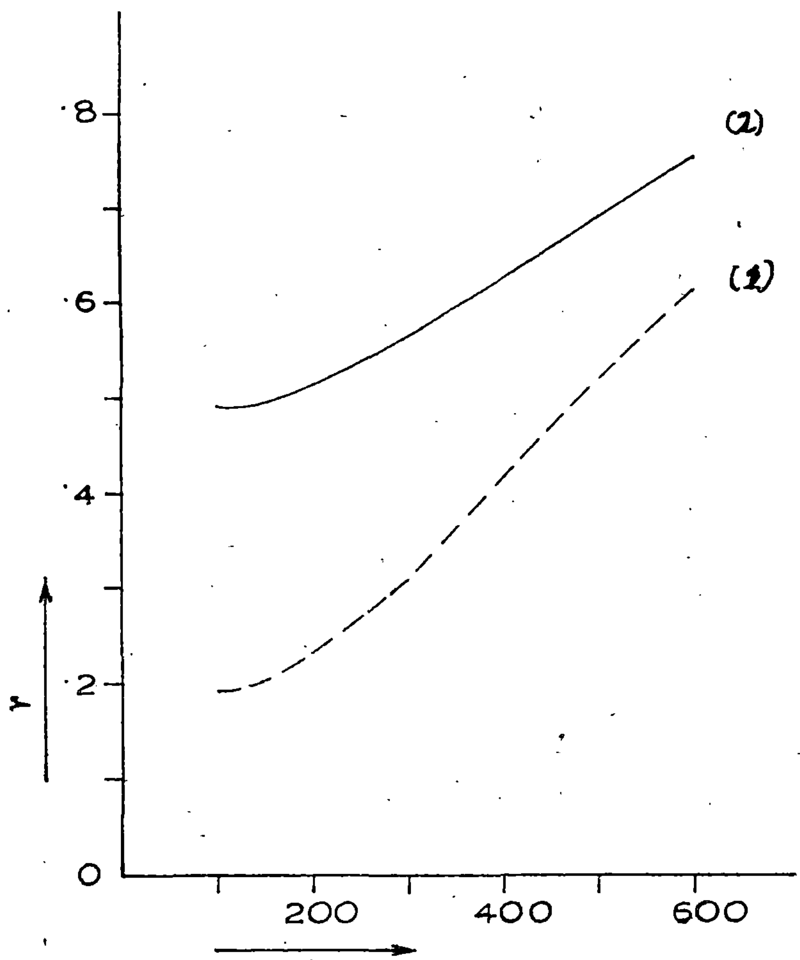

Fig. 2 : Contours solar of Ca K line calculated with [redacted] Geovenelli's form of $\overline{J}(t) = J_c + a\left[J_o(t) - J_c\right]$

(1) → Curve corresponding to $\mu = 1$

(2) → Curve corresponding to $\mu = 0.25$

The graph shows great divergence in the wing.

For the center of strong lines for which $\eta(0)$ has a large value, the residual intensity can approximately be written as

$$r(0,0,\mu) = 1/(1 + p'\mu/p) \quad , \qquad (3.34)$$

For Ca K line considered above, the central residual intensity at the center of the solar disc is 0.192 which agrees with the result given by Geovenelli (3) in Table 2.

Fron Table XVIII we find that the residual intensities calculated with the approximate form given by Geovenelli (3) are systemetically much less than the observed values in the wing region. The reason behind it might be that the contribution to scattering from the wing region has not been considered.(5) One interesting result obtained from(3.34) is that the residual intensity for centers of strong lines depend on $p'\mu/p$. The same result has also been obtained by Miyamoto.(7),(6)

We now replace J(t) by

$$J(t) = B_\nu(t) + \Upsilon/2\ (pE_2(t) - p'E_3(t)\ ) \quad , \qquad (3.35)$$

where $\Upsilon = a'/b'$ is the ratio of the probability of coherent to noncoherent scattering and $E_2(t)$ and $E_3(t)$ are the well known exponential integrals given by

$$E_n(t) = \int_1^{\infty} \exp(-tx)\, x^{-n}\, dx \quad , \qquad (3.36)$$

and

$$B_\nu(t) = p + p't \qquad (3.37)$$

is the Planck's function.

Substituting for $J(t)$ from (3.35) into (3.6) and using the Laplace transforms of $E_2(t)$ and $E_3(t)$ (8), we obtain the expression for emergent intensity as

$$I(\xi,0,\mu) = \frac{p}{4\lambda^2(\xi)} \Bigg[4\left\{\lambda^2(\xi) + p'\mu/p\right\} + 2\eta(\xi)\Upsilon$$

$$- (p'/p)\eta(\xi)\Upsilon\,(1 - 2\mu/\lambda^2(\xi)$$

$$- \frac{2\mu\eta(\xi)\Upsilon}{\lambda^2(\xi)} \left\{ 1 + \frac{p'\mu/p}{\lambda^2(\xi)} \right\} \log\,(1 + \lambda^2(\xi)/\mu \Bigg]$$

$$,\ldots \qquad (3.38)$$

The residual intensities for the solar K line of Calcium are then calculated for $\mu = 1.0$ and $\mu = 0.25$ for the cases $\Upsilon = 0.5$, $\Upsilon = 1.0$, and $\Upsilon = 1.5$ and are tabulated in Table XIX.

Table XIX

Residual intensities calculated with (3.38)

ξ	$\Upsilon = 0.5$		$\Upsilon = 1.0$		$\Upsilon = 1.5$	
	r(1)	r(0.25)	r(1)	r(0.25)	r(1)	r(0.25)
100	0.1356	0.2373	0.1866	0.3726	0.0863	0.1020
200	0.2894	0.4039	0.3251	0.4784	0.2537	0.2594
300	0.4510	0.5523	0.4749	0.5948	0.4272	0.4309
400	0.5713	0.6302	0.5984	0.6874	0.5657	0.5691
500	0.6787	0.7167	0.6904	0.7623	0.6671	0.6711
600	0.7492	0.7795	0.7579	0.8145	0.7406	0.7445
800	0.8412	0.8608	0.8463	0.8825	0.8360	0.8392

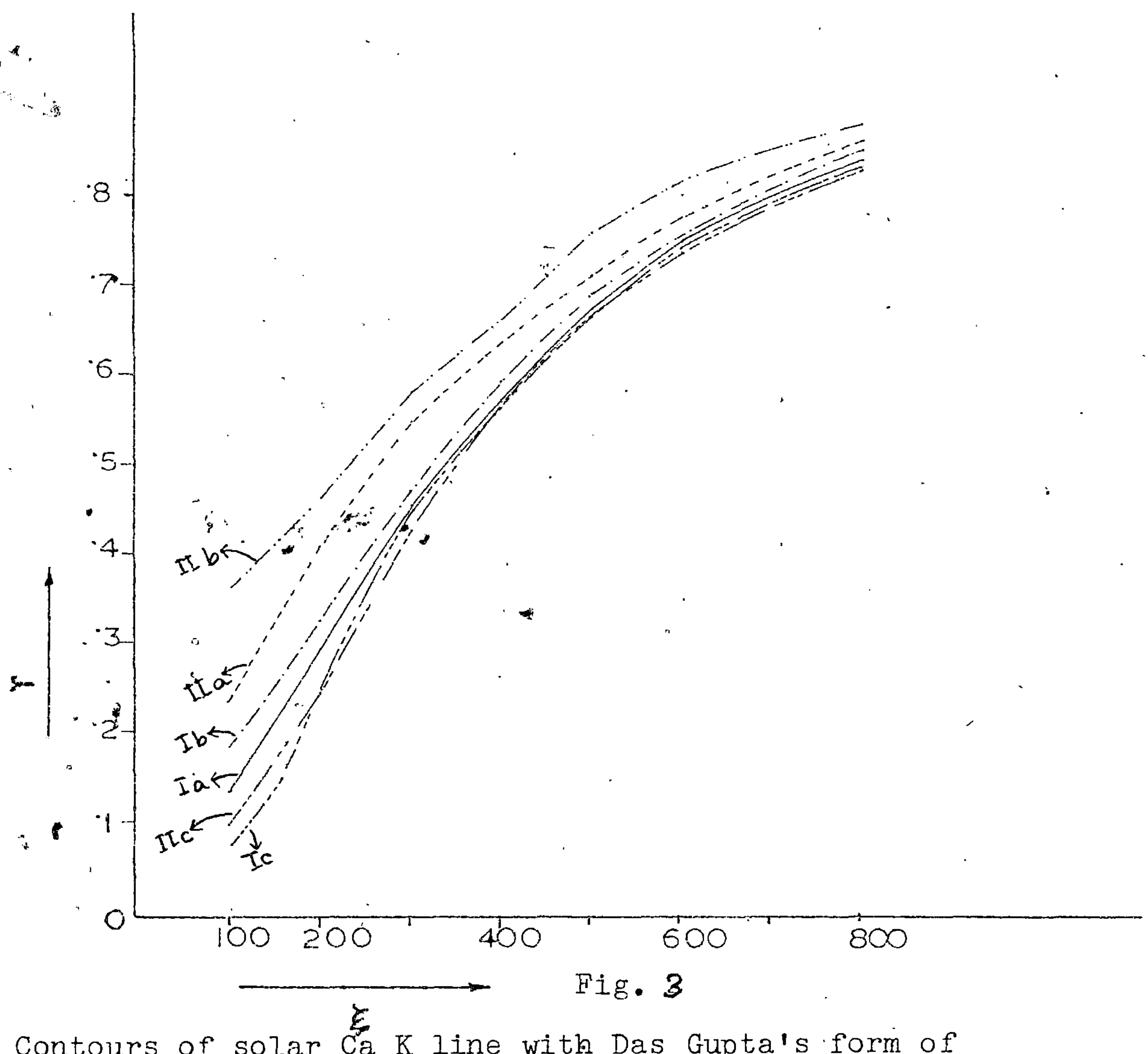

Fig. 3

Contours of solar Ca K line with Das Gupta's form of
$\overline{J}(t) = B_\nu(t) + \Upsilon/2(pE_2(t) - p'E_3(t))$.

$\mu = 1$
Ia corresponds to $\Upsilon = 0.5$
Ib corresponds to $\Upsilon = 1.0$
Ic corresponds to $\Upsilon = 1.5$

$\mu = 0.25$
IIa corresponds to $\Upsilon = 0.5$
IIb corresponds to $\Upsilon = 1.0$
IIc corresponds to $\Upsilon = 1.5$

For the center of strong fraunhorfer lines $\eta(0)$ is large so that we can approximately write

$$(1 + p'\mu/p)\, r(0,0,\mu) = 1 + \gamma/2\,(1 - p'/2p\,) \quad , \qquad (3.39)$$

For the three values of γ considered above the central residual intensities have been calculated and given in x Table XX.

Table XX

Central residual intensities of strong lines.

γ	r(1)	r(0.25)
0.5	0.1394	0.3636
1.0	0.0865	0.2195
1.5	0.0336	0.0854

It is seen from Table XIX that the wing of the strong lines are quite insensitive to the variation of γ but the inner core and particularly the center of the line can be made to approach the corresponding observed values by a appropiately adjusting the value of γ. From Table XX it is seen core and particularly the center of the line can be made to approach the corresponding observed values by appropiately adjusting the value of γ. From Table XX it is seen that the central residual intensity of Ca K line is nearer to the observed result (9) for $\gamma = 1.0$

References:

(1) Miyamoto, S. P.A.S.Japan, 5,142,1954. = Contr, Inst. Ap. Kyoto. No.39

(2) ———————— P.A.S.Japan, 6, 103, 1954. = Contr. Inst. Ap. Kyoto, No.46

(3) Geovenelli, R.G. M.N. 124, 221, 1962.

(4) Dasgupta, S.R. Ind. J. Phys. 39, 46, 1965.

(5) Geovenelli, R.G. Private communication.

(6) Miyamoto, S. P.A.S.Japan, 6, 140, 1954. = Contr. Inst. Ap. Kyoto, No. 49

(7) ———————— Zs. f. Ap. 35, 145 1954. = Contr. Inst. Ap. Kyoto, No. 53.

(8) Kourganoff, V.V. Basic Methods in Transfer Problems (Dover Publ.) 1963.

(9) Münch, G. Ap. J. 109, 275, 1949.