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Abstract

The labour in the calculation of residual intensities for doublets has been minimized by the use of our approximate form for the H-function. Finally, the residual intensities for a hypothetical triplet have been calculated.

1. *Introduction*

WOOLLEY and STIBBS (1) applied the theory of formation of absorption lines by coherent scattering to the case of interlocking without redistribution in M-E model and deduced the equation of transfer. They also obtained a solution using Eddington's approximation. BUSBRIDGE and STIBBS (2) applied the principle of invariance governing the law of diffuse reflection with a slight modification to solve the same problem. The expression for emergent intensity thus obtained involves Chandrasekhar's H-function within and outside the integral sign. They were afraid that the computational labour in the calculation of H-functions would be great and so avoided the calculation of residual intensities for triplets and higher multiplets.

Here in this paper we use our approximate form for the H-function (3) in the calculation of residual intensities from the expression obtained by BUSBRIDGE and STIBBS (2). We first calculate the residual intensities for doublets and compare the results with those of BUSBRIDGE and STIBBS (2). The satisfactory agreement encourages us to calculate the residual intensities for a hypothetical triplet.

2. *The Equation of Transfer*

The equation of transfer for the rth line of multiplets in the case of interlocking without redistribution is

$$\frac{\mu dI_r(\tau,\mu)}{d\tau}=(1+\eta_r)I_r(\tau,\mu)-(1+\epsilon\eta_r)(a+b\tau)$$
$$-(1-\epsilon)\alpha_r\sum_{p=1}^{k}\frac{1}{2}\eta_p\int_{-1}^{1}I_p(\tau,\mu')d\mu' \qquad r=1,2,\cdots k\,, \tag{1}$$

where $$\alpha_r=\eta_r/(\eta_1+\eta_2+\cdots+\eta_k)\,, \tag{2}$$

so that $$\alpha_1+\alpha_2+\cdots+\alpha_k=1\,, \tag{3}$$

and η_r, the ratio of line to the continuum absorption coefficient for the rth line, is independent of depth but is a function of frequency.

The coefficient of thermal emission, ϵ, is independent of both frequency and depth.

The expression for emergent intensity in the rth line obtained by BUSBRIDGE and STIBBS (2) by solving equation (1) is

$$I_r(0,\mu)=(a+bn_r\mu)H(n_r\mu)\left\{(\sum_{p=1}^{k}\alpha_p\lambda_p)^{1/2}+\frac{1}{2}\sum_{p=1}^{k}(\alpha_p n_p-\alpha_r n_r)(1-\lambda_p)\int_0^1\frac{\mu' H(n_p\mu')}{n_r\mu+n_p\mu'}d\mu'\right\}+\frac{1}{2}b\alpha_r n_r H(n_r\mu)\sum_{p=1}^{k}(1-\lambda_p)\int_0^1\mu' H(n_p\mu')d\mu' , \tag{4}$$

where

$$\lambda_r=\frac{1+\epsilon\eta_r}{1+\eta_r} \quad \text{and} \quad n_r=\frac{1}{1+\eta_r} .$$

The H-function for the rth line, $H(n_r\mu)$, is the solution of the integral equation,

$$\frac{1}{H(n_r\mu)}=(\sum_{p=1}^{k}\alpha_p\lambda_p)^{1/2}+\frac{1}{2}\sum_{p=1}^{k}\alpha_p(1-\lambda_p)\int_0^1\frac{n_p\mu' H(n_p\mu')}{n_r\mu+n_p\mu'}d\mu' . \tag{5}$$

Following CHANDRASEKHAR (4), BUSBRIDGE and STIBBS approximated $H(n_1\mu)$, the H-function for the first line in the multiplet, as

$$H(n_1\mu)=H(1-\alpha_1\lambda_1-\alpha_2\lambda_2-k_1,\mu) , \tag{6}$$

where

$$k_1=\frac{1}{2}\alpha_2(1-\lambda_2)(1-n_1/n_2) . \tag{7}$$

$H(n_2\mu)$ was calculated from $H(n_1\mu)$ using the relation

$$H(n_r\mu)=H\{n_1\cdot(n_r/n_1)\mu\} \qquad r=2,3\cdots . \tag{8}$$

We have approximated (3) the H-function as

$$H(\tilde{\omega}_0,\mu)=1+\frac{\alpha\mu}{1-\tilde{\omega}_0+2\mu\sqrt{1-\tilde{\omega}_0}} , \tag{9}$$

where α is a function of $\tilde{\omega}_0$ given by

$$\alpha=\tilde{\omega}_0(1-\tilde{\omega}_0)\exp\left[\tilde{\omega}_0\left(1+2\left|\tilde{\omega}_0-\frac{1}{2}\right|\right)\right] . \tag{10}$$

After a single iteration, the expression for $H(\tilde{\omega}_0,\mu)$ is obtained as

$$\frac{1}{H(\tilde{\omega}_0,\mu)}=1-\frac{\mu\tilde{\omega}_0}{4A(2\mu-A)}\left[\{(2A+\alpha)\mu-A^2\}2\ln\frac{\mu+1}{\mu}-A\alpha\ln\frac{A+2}{A}\right] , \tag{11}$$

where $A\equiv(1-\tilde{\omega}_0)^{1/2}$.

3. *The Calculation for a Doublet*

For a doublet $k=2$,

$$\eta_1:\eta_2=2:1 , \quad \alpha_1=2/3 , \quad \alpha_2=1/3 .$$

We write

$$\tilde{\omega}_0=1-\alpha_1\lambda_1-\alpha_2\lambda_2-k_1 , \tag{12}$$

so that $\tilde{\omega}_0$ is now a function of the parameters α_i and $\eta_i(i=1, 2)$. We then calculate $H(n_1\mu)$ with the help of equations (6) and (11) and tabulate it in Table 1.

TABLE 1. Values of the H-functions for a Doublet.

μ	Results Obtained with Our Approximation		Results of BUSBRIDGE and STIBBS	
	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_1\mu)$	$H(n_2\mu)$
0.0	1.0000	1.0000	1.000	1.0000
0.1	1.06016	1.07269	1.0598	1.0724
0.2	1.09353	1.11049	1.0935	1.1108
0.4	1.13703	1.15717	1.1382	1.1591
0.5	1.17139	1.18620	1.1679	1.1896
0.8	1.18620	1.20634	1.1896	1.2111
1.0	1.20189	1.22121	1.2064	1.2271

Values for $H(n_2\mu)$ calculated from relation (8) are also given in the same table. Results obtained by BUSBRIDGE and STIBBS (2) are also given in the same table for comparison.

In calculating the residual intensities from equation (4), the H-functions within the integral sign have been replaced by the right hand side of equation (9) and the values of H-functions outside the integral sign have been taken from Table 1. The case considered here is identical with one of the cases considered by BUSBRIDGE and STIBBS (2), i.e.,

$$\eta_1=1\ , \qquad \eta_2=1/2\ , \qquad \epsilon=0\ ,$$

$$\lambda_1=n_1=1/2\ , \quad \lambda_2=n_2=2/3\ .$$

Results are given in Table 2 where the corresponding results from Reference (2) are also given.

TABLE 2. Residual Intensities for a Doublet.

μ	Results Obtained with Our Approximation		Results of BUSBRIDGE and STIBBS	
	$R_1(\mu)$	$R_2(\mu)$	$R_1(\mu)$	$R_2(\mu)$
0.0	0.8352	0.8897	0.8360	0.8906
0.1	0.8245	0.8873	0.8258	0.8880
0.2	0.8018	0.8721	0.8024	0.8728
0.4	0.7583	0.8418	0.7597	0.8439
0.6	0.7273	0.8178	0.7260	0.8207
0.8	0.6970	0.7989	0.6996	0.8026
1.0	0.6753	0.7839	0.6786	0.7880

From Table 2 it appears that our approximation can be used to get the residual intensities for higher multiplets without much computational labour.

4. *Calculation for a Triplet*

For a triplet $k=3$,

$$\eta_1:\eta_2:\eta_3=5:3:1\,,\quad \alpha_1=5/9\,,\quad \alpha_2=1/3\,,\quad \text{and}\quad \alpha_3=1/9\,.$$

The case considered here is

$$\eta_1=5/9\,,\quad \eta_2=1/3\,,\quad \eta_3=1/9\,,\quad \epsilon=0\,,$$

$$\lambda_1=n_1=9/14\,,\quad \lambda_2=n_2=3/4\,,\quad \text{and}\quad \lambda_3=n_3=9/10\,.$$

Following BUSBRIDGE and STIBBS, $H(n_1\mu)$ in the case of a triplet can be approximated by

$$H(n_1\mu)\simeq H(1-\alpha_1\lambda_1-\alpha_2\lambda_2-\alpha_3\lambda_3-k_2,\,\mu)\,, \tag{13}$$

where

$$k_2=\frac{1}{2}[\alpha_2(1-\lambda_2)(1-n_1/n_2)+\alpha_3(1-\lambda_3)(1-n_1/n_3)]\,. \tag{14}$$

Writing $\tilde{\omega}_0$ for $1-\alpha_1\lambda_1-\alpha_2\lambda_2-\alpha_3\lambda_3-k_2$ in (13), we calculate the H-functions for the case considered and tabulated in Table 3. Residual intensities are then calculated and given in Table 4.

TABLE 3. Values of the H-functions for a Triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$
0.0	1.00000	1.00000	1.00000
0.1	1.03322	1.03666	1.04109
0.2	1.05516	1.05541	1.06119
0.4	1.07485	1.07558	1.08345
0.6	1.08599	1.09146	1.09832
0.8	1.09608	1.10128	1.10721
1.0	1.10356	1.10849	1.11403

TABLE 4. Residual Intensities for Triplet.

μ	$R_1(\mu)$	$R_2(\mu)$	$R_3(\mu)$
0.0	0.9099	0.9367	0.9741
0.1	0.8943	0.9091	0.9672
0.2	0.8733	0.9089	0.9626
0.4	0.8348	0.8663	0.9246
0.6	0.3034	0.8484	0.9184
0.8	0.7814	0.8336	0.9127
1.0	0.7642	0.8306	0.9124

From Table 4 we find that the residual intensities of the hypothetical line considered (which is a weaker one) has no drop near $\mu=0$. The weaker doublet considered by BUSBRIDGE and STIBBS (2) (case III) possesses the same property.

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It is to be noted (already pointed out in Reference (3)) that our apppoximate form for the H-function gives a good result in the range $0<\tilde{\omega}_0\leqslant 0.4$. The results in the range $0.4<\tilde{\omega}\leqslant 0.9$ are correct to 3 or 4 significant figures only, and in the range $0.9<\tilde{\omega}\leqslant 1$, the approximate form is not at all a good one. Therefore it can be used in the calculation of weak multiplets.

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