CHAPTER IV

APPROXIMATE SOLUTION OF EQUATION OF TRANSFER WITH EDDINGTON'S AMMENDED APPROXIMATION.

INTRODUCTION:

In solving the equation of transfer approximately, Eddington's approximation, also known as geometrical approximation, viz.

$$K = \frac{1}{3} J \tag{4.1}$$

has been largely utilised by different authors. Wilson (1) recently gave a three dimensional form of Eddington's approximation. More recently, Hueng (2) formulated the approximation in such a way that it helps one to discuss different orders of approximation. The use of this approximation introduces certain important errors. (3) Kourganoff (4) has shown that in the grey case and in the case of solar photosphere the relative error due to the application of Eddington's approximation is quite considerable, of the order of 30 percent for mean intensity. Eddington gave (vide Woolley and Stibbs (5)) another approximation called ammended second approximation viz.

$$K = (1/3)\ J.\ g(t) \tag{4.2}$$

where

$$g(t) = \frac{2 + 3t - 3E_4(t) + (9/2)E_5(t)}{2 + 3t - E_2(t) + (3/2)E_3(t)}, \tag{4.3}$$

$E_n(t)$'s being the wellknown exponential integral given by

$$E_n(t) = \int_1^{\infty} \frac{\exp(-tx)\,dx}{x^n} , \tag{4.4}$$

It is clear from (4.3) that

$$g(t) \to 17/14 \quad \text{as} \quad t \to 0 , \tag{4.5}$$

$$\to 1 \quad \text{as} \quad t \to \infty , \tag{4.6}$$

It can be expressed in the form

$$g(t) = \left(1 - \sum_j \alpha_j \exp(-\beta_j t) \right)^{-1} , \tag{4.7}$$

Bohm-Vitense[6] gave another form for g(t) viz.

$$g(t) = \left[1 - \frac{a.\exp(-bt)}{c + t} \right]^{-1} , \qquad (4.8)$$

where a, b and c are constants whose numerical values have been given by Bohm-Vitense[6] as

$$a = 0.1167 \quad , \; b = 1.972 \quad , \; c = 0.694 \qquad (4.8a)$$

In this chapter we solve the equation of transfer for noncoherent scattering using Eddington's Ammended approximation given by (4.2) with g(t) given by (4.8) and try to see to what advantage the ammended approximation can be used for the case under consideration.

Let $I(\xi,t,\mu)$ be the specific intensity of frequency difference

$$\xi = \frac{\nu - \nu_0}{\Delta}$$

at optical depth t and in the direction $\theta = \cos^{-1} \mu$ with the normal to the photosphere. The equation of transfer in the Milne-Eddington model for noncoherent scattering is[7]

$$\frac{\mu dI(\xi,t,\mu)}{dt} = \lambda^2(\xi)I(\xi,t,\mu) - \eta(\xi)\bar{J}(t) - (p + p't) \quad ,(4.9)$$

where

$$\bar{J}(t) = \frac{\int J(\xi,t)\ \eta(\xi)\ d\xi}{\int \eta(\xi)\ d\xi} \quad , \qquad (4.10)$$

$$\lambda^2(\xi) = 1 + \eta(\xi) \quad , \qquad (4.11)$$

$\eta(\xi)$ being the ratio of the line absorption coefficient to the continuum absorption coefficient at frequency difference ξ,

p and p', the coefficients in the linear expansion of Planck function.

Multiplying (4.9) by dw and integrating and using the relation

$$\int I(\xi,t,\mu)\ dw = 4\pi J(\xi,t) \quad , \qquad (4.12)$$

$$\int I(\xi,t,\mu)\ \mu\ dw = 4\pi H(\xi,t) \quad , \qquad (4.13)$$

we get

$$\frac{dH(\xi,t)}{dt} = \lambda^2(\xi)J(\xi,t) - \eta(\xi)\bar{J}(t) - (p + p't) \quad , \quad .... \qquad (4.14)$$

Multiplying (4.9) by $\mu d\omega$, integrating and using (4.13) and the relation

$$\int I(\xi,t,\mu)\, \mu^2\, d\omega = 4\pi K(\xi,t) \quad , \qquad (4.15)$$

we have

$$\frac{dK(\xi,t)}{dt} = \lambda^2(\xi) H(\xi,t) \quad , \qquad (4.16)$$

Differentiating (4.16) we get,

$$\frac{d^2 K(\xi,t)}{dt^2} = \lambda^2(\xi) \left[\lambda^2(\xi) J(\xi,t) - \eta(\xi)\bar{J}(t) - (p + p't) \right] \quad ,\ldots \qquad (4.17)$$

Now applying Eddington's ammended approximation given by (4.2) ,

$$\frac{d^2 \left\{ J(\xi,t) g(t) \right\}}{dt^2} = 3\lambda^2(\xi) \left[\lambda^2(\xi) J(\xi,t) - \eta(\xi)\bar{J}(t) - (p + p't) \right] \quad ,\ldots \qquad (4.18)$$

Multiplying (4.18) by $\eta(\xi)$, integrating over all frequencies and using the approximation

$$\bar{J}(t) = J(0,t) \quad , \tag{4.19}$$

we have

$$\frac{d^2\left\{\bar{J}(t).g(t)\right\}}{dt^2} = 3\lambda^2(0)\left[\bar{J}(t) - (p + p't)\right] \quad ; \tag{4.20}$$

or

$$\bar{J}''(t) + P\,\bar{J}'(t) + Q\,\bar{J}(t) = R \quad , \tag{4.21}$$

where

$$P = \frac{2g'(t)}{g(t)} \quad , \tag{4.22}$$

$$Q = \frac{g''(t) - 3\lambda^2(0)}{g(t)} \quad , \tag{4.23}$$

$$R = -3\lambda^2(0).(p + p't)/g(t) \quad , \tag{4.24}$$

Equation (4.21) is a linear nonhomogeneous differential equation of second degree with variable coefficients.

Solution of Equation (4.21):

Let y be the nontrivial solution of the homogeneous equation

$$\bar{J}''(t) + P\,\bar{J}'(t) + Q\,\bar{J}(t) = 0 \quad , \qquad (4.25)$$

then

$$y = \exp\left(-1/2 \int P dt \right)$$

$$= \exp\left(-\int \frac{g'(t)dt}{g(t)} \right)$$

$$= 1/g(t) \quad , \qquad (4.26)$$

Let $\bar{J}(\xi,t) = y \int u(t)\, dt$ be the general solution of the equation (4.21).

Differentiating $\bar{J}(\xi,t)$ with respect to t we have

$$\bar{J}'(t) = y\, u(t) + y' \int u(t)\, dt \quad , \qquad (4.27)$$

differentiating once more

$$\bar{J}''(t) = y\, u'(t) + 2y'u(t) + y'' \int u(t)\, dt \quad , \qquad (4.28)$$

Substituting for $\bar{J}(t)$, $\bar{J}'(t)$ and $\bar{J}''(t)$ into (4.23)

$$(y'' + Py' + Qy) \int u(t)\, dt + y\, u'(t) + (2y' + Py)\, u(t) = R \quad , \ldots \qquad (4.29)$$

since y is a solution of (4.25)

$$y'' + Py' + Qy = 0 \qquad (4.30)$$

Hence

$$u'(t) + (2y' + Py)u(t)/y = R/y \quad , \qquad (4.31)$$

which is a linear differential equation of degree one. The integrating factor for the equation (4.31) is

$$\text{I.F.} = \exp\left[\int \frac{2y' + Py}{y}\, dt \right]$$

$$= \exp\left[\log (y^2 \cdot g(t)^2) \right]$$

(4.32)

$$= 1 \quad , \tag{4.32}$$

hence

$$u(t) = \int \frac{R}{y}\, dt$$

$$= \int \frac{-3\lambda^2(0).(p + p't)\, dt}{y\ g(t)}$$

$$= -3\lambda^2(0) \left[pt + p't^2/2 \right] + C_1 \quad , \tag{4.33}$$

$\bar{J}(t)$ is then obtained as

$$\bar{J}(t) = 1/g(t) \left[C_2 + C_1 t - 3\lambda^2(0) \left\{ pt^2/2 + p't^3/6 \right\} \right] \quad ,\ldots \tag{4.34}$$

When $t \to \infty$, $g(t) \to 1$ and $\bar{J}(t) \to p + p't$
hence

$$p + p't = C_2 + C_1 t - 3\lambda^2(0) \left\{ pt^2/2 + p't^3/6 \right\} \quad , \tag{4.35}$$

equating the coefficients of equal powers of t

$$\left.\begin{array}{l} C_2 = p \\ C_1 = p' \end{array}\right| \qquad (4.36)$$

Hence

$$\bar{J}(t) = (p + p't) / g(t) \qquad (4.37)$$

Solution of Equation (4.9):

Substituting for $\bar{J}(t)$ from (4.37) into (4.9) we get,

$$\frac{\mu dI(\xi,t,\mu)}{dt} = \lambda^2(\xi)\, I(\xi,t,\mu) - (1 + \eta(\xi)/g(t))(p + p't) \quad ,\ldots \qquad (4.38)$$

so that the solution is obtained as

$$I(\xi,0,\mu) = \mu^{-1}\left[\int_0^\infty (1 + \eta(\xi)/g(t))(p + p't).\ \exp(-\lambda^2(\xi)t/\mu).\ dt\right]; \qquad (4.39)$$

Substituing for g(t) from (4.8) into (4.39) and integrating we have

$$I(\xi,0,\mu) = \mu\left\{p + \frac{p'\mu}{\lambda^2}\right\} + ae^{k} E_1(k)\,\eta(p'c - p)$$

$$- \frac{\eta p' a\mu}{\lambda^2 + \mu b} \qquad ,\ldots.. \qquad (4.40)$$

where

$$k = c(b + \lambda^2/\mu) \qquad ,\ldots.. \qquad (4.41)$$

CONTOURS OF SOLAR K LINE OF CALCIUM:

We now calculate the contour of Ca II K line with the data given by Miyamoto[7] which will help us to compare our results with those of Miyamoto.

Thus we take

$$p'/p = 4.2$$
$$\eta(\xi) = 1.40 \times 10^{5} \times \xi^{-2}$$

Values of a, b, and c are given by (4.8a) .

To evaluate the values ~~ogz~~ of $E_1(k)$ for large k we considered the asymptotic expansion of $E_n(k)$ given by (Ref.(8), equation 36.13)

$$E_n(k) = \frac{1}{ke^k} \quad 1 - \frac{n}{k} + \frac{n(n+1)}{k^2} - \ldots. \qquad (4.42)$$

The results are given in Table XXI where the corresponding results of Miyamoto (7) and Münch (9) (Münch's results are very close to Houtgast's (10) observation.) are also given.

Table XXI.

	$r(\xi,0,1)$			$r(\xi,0,0.2)$		
ξ	Results with (4.40)	Results of Miyamoto	Results of Münch	Results with (4.40)	Results of Miyamoto	Results of Münch
50	0.1743	0.2044	-----	0.4254	0.5233	-----
100	0.2159	0.2456	0.173	0.4923	0.5665	-----
200	0.3479	0.3716	0.336	0.5866	0.6431	-----
300	0.4815	0.5983	0.490	0.7139	0.7213	-----

Results in the Table XXI show that the use of Eddington's ammended approximatiin $K = (1/3)\ J.g(t)$ instead of $K = (1/3)\ J$ makes the contour more close to the observed one.

References:

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(5)	Woolley R.v.d.R. and Stibbs, D.W.N.	Outer Layers of a Star. Oxford Univ. Press. 1953.
(6)	Böhm-Vitense, E.	Zs. f. Ap. 57, 241, 1963.
(7)	Miyamoto, S.	P.A.S.Japan, 6, 103, 1954. = Contr. Inst. Ap. Kyoto, No. 46.
(8)	Kourganoff, V. V.	Basic Methods in Transfer Problems. Dover Publ. 1963.
(9)	Münch, G.	Ap. J. 109, 275, 1949.
(10)	Houtgast, J.	Dissertation (Utrecht) 1942.