CHAPTER

3

UNIQUENESS OF DIFFERENTIAL POLYNOMIALS OF A MEROMORPHIC FUNCTION CONCERNING WEIGHTED SHARING

3.1 Introduction, Definitions and Notations

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let $a \in S(f) \cap S(g)$. We say that f and g share a IM(CM) if $f - a$ and $g - a$ share the value 0 IM(CM).

The subject on sharing values between an entire function and its derivative was first studied by Rubel and Yang ([66]). In 1977, they proved the following result.

Theorem 3.1.1 *[66] Let f be a non-constant entire function. If f and $f^{(1)}$ share two distinct finite complex numbers a, b CM,*

The results of this chapter have been published in **The Mathematics Student (MS), see [64].**

then $f \equiv f^{(1)}$.

In 1979, Mues and Steinmetz ([51]) obtained the same result but in relax sharing condition as follows.

Theorem 3.1.2 *[51] Let* f *be a non-constant entire function. If* f *and* $f^{(1)}$ *share two distinct finite complex numbers* a, b *IM, then* $f \equiv f^{(1)}$.

Subsequently, similar considerations have been made with respect to higher derivatives and more general differential expressions as well. Above theorems motivate researchers to study the relation between an entire function and its derivative counterpart for one CM shared value. In this direction, in 1996, the following famous conjecture was proposed by Brück ([3]).

Conjecture 3.1.1 *[3] Let* f *be a non-constant entire function such that the hyper order* $\rho_2(f)$ *of* f *is not a positive integer or infinity, where*

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If f *and* $f^{(1)}$ *share a finite value* a *CM, then* $\frac{f^{(1)}-a}{f-a} = c$*, where* c *is a non-zero constant.*

Very recently many results have been published concerning the above conjecture ([8, 10, 16, 21, 40, 49, 50]). Next we recall the following definitions.

Definition 3.1.1 *[40, 89] Let* p *be a positive integer. Let* f *be a meromorphic function and* $a \in S(f)$.

(i) $\overline{N}_{p)}(r, \frac{1}{f-a})$ *denotes the counting function of those* a*-points of* f *whose multiplicities are not greater than* p*, where each* a*-point is counted only once.*

(ii) $\overline{N}_{(p}(r,\frac{1}{f-a})$ *denotes the counting function of those a-points of* f *whose multiplicities are not less than* p*, where each a-point is counted only once.*

(iii) $N_p(r,\frac{1}{f-a})$ *denotes the counting function of those a-points of* f*, where an a-point of* f *with multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$*.*

We denote $\delta_p(a,f)$ *by the quantity*

$$\delta_p(a,f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,\frac{1}{f-a})}{T(r,f)}.$$

Clearly $0 \leq \delta(a,f) \leq \delta_p(a,f) \leq \delta_{p-1}(a,f) \leq .... \leq \delta_2(a,f) \leq \delta_1(a,f) = \Theta(a,f)$*.*

Definition 3.1.2 *[1] Let* f *and* g *share* a *IM and let* z_0 *be a zero of* $f - a$ *of multiplicity* p *and a zero of* $g - a$ *of multiplicity* q*.*

(i) By $\overline{N}_L(r,\frac{1}{f-a})$ *we denote the reduced counting function of those a-points of* f *and* g *where* $p > q \geq 1$*;* $\overline{N}_L(r,\frac{1}{g-a})$ *is defined similarly.*

(ii) By $N_E^{1)}(r,\frac{1}{f-a})$ *the counting function of those a-points of* f *and* g *where* $p = q = 1$*, and*

(iii) by $\overline{N}_E^{(2}(r,\frac{1}{f-a})$ *the counting function of those a-points of* f *and* g *where* $p = q \geq 2$*, where each such a-points is counted only once.*

Relaxation of the sharing is done on the basis of the following notion of weighted sharing as introduced by Lahiri ([37, 38]).

Definition 3.1.3 *[37, 38] Let* l *be a non-negative integer or infinity and* $a \in S(f) \cap S(g)$*. We denote by* $E_l(a,f)$ *the set of all zeros of* $f - a$*, where a zero of multiplicity* m *is counted* m *times if* $m \leq l$ *and* $l + 1$ *times if* $m > l$*. If* $E_l(a,f) = E_l(a,g)$*, we say that* f*,* g *share the function* a *with weight* l*. We write* f *and* g

share (a,l) *to mean that* f *and* g *share the function* a *with weight* l. *Since* $E_l(a,f) = E_l(a,g)$ *implies that* $E_s(a,f) = E_s(a,g)$ *for any integer* $s\,(0 \leq s < l)$, *if* f, g *share* (a,l), *then* f, g *share* (a,s). *Moreover, we note that* f *and* g *share the function* a *IM or CM if and only if* f *and* g *share* $(a,0)$ *or* (a,∞) *respectively.*

In 2008, Zhang and L*ü* ([90]) proved the uniqueness of the n-th power of a meromorphic function and its k-th derivative sharing a small function.

Theorem 3.1.3 *[90] Let* k (≥ 1), n (≥ 1) *be integers and* f *be a non-constant meromorphic function. Also let* a $(\not\equiv 0,\infty)$ *be a small function with respect to* f. *Suppose* $f^n - a$ *and* $f^{(k)} - a$ *share* $(0,l)$. *If* $l = \infty$ *and*

$$(3+k)\Theta(\infty,f) + 2\Theta(0,f) + \delta_{2+k}(0,f) > 6 + k - n,$$

or, $l = 0$ *and*

$$(6+2k)\Theta(\infty,f) + 4\Theta(0,f) + 2\delta_{2+k}(0,f) > 12 + 2k - n,$$

then $f^n \equiv f^{(k)}$.

In the same paper, Zhang and L*ü* ([90]) posted the following question.

Question 3.1.1 *What will happen if* f^n *and* $\left(f^{(k)}\right)^s$ *share a small function ?*

In 2014, with the notion of weighted sharing of small function Banerjee and Majumder ([7]) proved two theorems. One of which improved Theorem 3.1.3 and both the theorems together answer the Question 3.1.1.

Theorem 3.1.4 *[7] Let* k (≥ 1), n (≥ 1) *be integers and* f *be a non-constant meromorphic function. Also let* a $(\not\equiv 0,\infty)$ *be a* *small function with respect to* f*. Suppose* $f^n - a$ *and* $f^{(k)} - a$ *share* $(0, l)$*. If* $l \geq 2$ *and*

$$(3+k)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > 6 + k - n,$$

or $l = 1$ *and*

$$(k + \frac{7}{2})\Theta(\infty, f) + \frac{5}{2}\Theta(0, f) + \delta_{2+k}(0, f) > 7 + k - n,$$

or $l = 0$ *and*

$$(6+2k)\Theta(\infty, f) + 4\Theta(0, f) + \delta_{2+k}(0, f) + \delta_{1+k}(0, f) > 12 + 2k - n,$$

then $f^n \equiv f^{(k)}$*.*

Theorem 3.1.5 *[7] Let* k (≥ 1)*,* n (≥ 1)*,* m (≥ 2) *be integers and* f *be a non-constant meromorphic function. Also let* a $(\not\equiv 0, \infty)$ *be a small function with respect to* f*. Suppose* $f^n - a$ *and* $(f^{(k)})^m - a$ *share* $(0, l)$*. If* $l \geq 2$ *and*

$$(3+2k)\Theta(\infty, f) + 2\Theta(0, f) + 2\delta_{1+k}(0, f) > 7 + 2k - n,$$

or $l = 1$ *and*

$$(2k + \frac{7}{2})\Theta(\infty, f) + \frac{5}{2}\Theta(0, f) + 2\delta_{1+k}(0, f) > 8 + 2k - n,$$

or $l = 0$ *and*

$$(6+3k)\Theta(\infty, f) + 4\Theta(0, f) + 3\delta_{1+k}(0, f) > 13 + 3k - n,$$

then $f^n \equiv (f^{(k)})^m$*.*

To state the next result we need the following definition.

Definition 3.1.4 *[27] Let* n_{0j}*,* n_{1j} *,*$n_{2j}, ..., n_{kj}$ *be non-negative integers. The expression*

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}}...(f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by f *of degree* $d(M_j) = \sum_{i=0}^{k} n_{ij}$ *and weight* $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$. *Let* $a_j \in S(f)$ *and* $a_j \not\equiv 0$ $(j = 1, 2, ..., t)$. *The sum* $P[f] = \sum_{j=1}^{t} a_j M_j[f]$ *is called a differential polynomial generated by* f *of degree* $\overline{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ *and weight* $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$. *The numbers* $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ *and* k *(the highest order of derivative of* f *in* $P[f]$*) are called respectively the lower degree and the order of* $P[f]$. *The differential polynomial* $P[f]$ *is said to be homogeneous if* $\overline{d}(P) = \underline{d}(P) = d$, *otherwise* $P[f]$ *is called a non-homogeneous differential polynomial. Also, we define* $Q := \max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\}$.

Regarding the uniqueness of f and $P[f]$, Bhoosnurmath and Kabbur ([12]) proved the following theorem.

Theorem 3.1.6 *[12] Let* f *be a non-constant meromorphic function and* a *be a small function of* f *such that* a $(\not\equiv 0, \infty)$. *Let* $P[f]$ *be a non-constant differential polynomial. If* f *and* $P[f]$ *share the value* a *IM and*

$$(2Q+6)\Theta(\infty, f) + (2 + 3\underline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,$$

then $f \equiv P[f]$.

Motivated by such uniqueness investigations, Charak and Lal ([20]) considered the possible extension of Theorems 3.1.4-3.1.6 in the direction of the question of Zhang and L$\ddot{u}$ ([90]) up to differential polynomial considering weighted sharing. In the following theorem we state the particular case $p(z) = z^n$ of the result of Charak and Lal.

Theorem 3.1.7 *[20] Let f be a non-constant meromorphic function and n be a positive integer and a $(\not\equiv 0,\infty)$ be a small function with respect to f. Let $P[f]$ be a non-constant differential polynomial. Suppose $f^n - a$ and $P[f] - a$ share $(0,l)$. If $l \geq 2$ and*

$$(3+Q)\Theta(\infty,f)+2\Theta(0,f)+\overline{d}(P)\delta(0,f) > Q+5+2\overline{d}(P)-\underline{d}(P)-n,$$

or, $l = 1$ and

$$(\frac{7}{2}+Q)\Theta(\infty,f)+\frac{5}{2}\Theta(0,f)+\overline{d}(P)\delta(0,f) > Q+6+2\overline{d}(P)-\underline{d}(P)-n,$$

or, $l = 0$ and

$$(6+2Q)\Theta(\infty,f)+4\Theta(0,f)+2\overline{d}(P)\delta(0,f) > 2Q+ 4\overline{d}(P)-2\underline{d}(P)+10-n,$$

then $f^n \equiv P[f]$.

Regarding the above results, a natural question is the following:

Question 3.1.2 *What can be said when $(f^n)^{(k)}$ share a small function with $P[f^q]$?*

The main theorem of this chapter answer the above question.

3.2 Lemmas

In this section we present some lemmas needed in the sequel. Let F, G be two non-constant meromorphic functions. We shall denote by H the following function

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right). \qquad (3.2.1)$$

Lemma 3.2.1 *Let q (≥ 1) be an integer. Let f be a non-constant meromorphic function and $P[f^q]$ be a differential polynomial of f^q. Then*

$$m\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m(r, \frac{1}{f^q}) + S(r, f).$$

Proof. *The proof can be obtain along the same line as the proof of Lemma* 2.4 *in [9].* ■

Lemma 3.2.2 *[4] Let f and g be two non-constant meromorphic functions.*

(i) If f and g share $(1, 0)$, then

$$\overline{N}_L\left(r, \frac{1}{f-1}\right) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + S(r, f).$$

(ii) If f and g share $(1, 1)$, then

$$2\overline{N}_L(r, \tfrac{1}{f-1}) + 2\overline{N}_L(r, \tfrac{1}{g-1}) + \overline{N}_E^{(2}(r, \tfrac{1}{f-1})$$
$$-\overline{N}_{f>2}(r, \tfrac{1}{g-1}) \leq N(r, \tfrac{1}{g-1}) - \overline{N}(r, \tfrac{1}{g-1}).$$

Lemma 3.2.3 *[91] Let f be a non-constant meromorphic function and p, k be two positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f),$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f),$$

where

$$\overline{N}(r, \frac{1}{f^{(k)}}) = N_1(r, \frac{1}{f^{(k)}}).$$

Lemma 3.2.4 *[6] Let F and G share $(1, l)$ and $H \not\equiv 0$, then*

$$N(r, H) \leq \overline{N}(r, F) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) + \overline{N}_L(r, \tfrac{1}{F-1})$$
$$+\overline{N}_L(r, \tfrac{1}{G-1}) + N_0(r, \tfrac{1}{F^{(1)}}) + N_0(r, \tfrac{1}{G^{(1)}}) + S(r, f).$$

Lemma 3.2.5 *[8] For any two non-constant meromorphic functions f and g,*

$$N_p(r, fg) \leq N_p(r, f) + N_p(r, g).$$

Lemma 3.2.6 *Let p, q be positive integers. Let f be a non-constant meromorphic function and $P[f^q]$ be a differential polynomial of f^q. Then*

$$N_p\left(r, \frac{1}{P[f^q]}\right) \leq q\overline{d}(P)N_{p+k}(r, \frac{1}{f}) + Q\overline{N}(r, f) + S(r, f).$$

Proof. For any non-constant meromorphic function f, $N_p(r, f) \leq N_s(r, f)$ if $p \leq s$. Now by Lemma 3.2.3 and Lemma 3.2.5 we have

$$N_p\left(r, \tfrac{1}{P[f^q]}\right) \leq \sum_{j=1}^{t} N_p\left(r, \tfrac{1}{M_j[f^q]}\right) + S(r, f)$$
$$= N_p\left(r, \tfrac{1}{M_1[f^q]}\right) + N_p\left(r, \tfrac{1}{M_2[f^q]}\right) + ... + N_p\left(r, \tfrac{1}{M_t[f^q]}\right) + S(r, f)$$
$$= N_p\left(r, \frac{1}{\prod_{i=0}^{k}\left((f^q)^{(i)}\right)^{n_{i1}}}\right) + N_p\left(r, \frac{1}{\prod_{i=0}^{k}\left((f^q)^{(i)}\right)^{n_{i2}}}\right) + ... +$$
$$+ N_p\left(r, \frac{1}{\prod_{i=0}^{k}\left((f^q)^{(i)}\right)^{n_{it}}}\right) + S(r, f)$$
$$\leq \sum_{i=0}^{k} n_{i1}N_p\left(r, \tfrac{1}{(f^q)^{(i)}}\right) + \sum_{i=0}^{k} n_{i2}N_p\left(r, \tfrac{1}{(f^q)^{(i)}}\right) + ... +$$
$$+ \sum_{i=0}^{k} n_{it}N_p\left(r, \tfrac{1}{(f^q)^{(i)}}\right) + S(r, f)$$
$$\leq \sum_{i=0}^{k}(n_{i1} + n_{i2} + ... + n_{it})\left\{N_{p+i}\left(r, \tfrac{1}{f^q}\right) + i\overline{N}(r, f^q)\right\}$$
$$+ S(r, f) \leq \max_{1\leq j\leq t}\left\{\sum_{i=0}^{k} n_{ij}qN_{p+k}\left(r, \tfrac{1}{f}\right)\right\}$$
$$+ \max_{1\leq j\leq t}\left(\sum_{i=0}^{k} i.n_{ij}\right)\overline{N}(r, f) + S(r, f)$$
$$= q\overline{d}(P)N_{p+k}(r, \tfrac{1}{f}) + Q\overline{N}(r, f) + S(r, f).$$

This completes the proof of the lemma. ■

Lemma 3.2.7 *Let n, k, q be positive integers such that $n \geq (k+2)$. Let f be a non-constant meromorphic function and $P[f^q]$ be a differential polynomial of f^q. Let us define $F = \frac{(f^n)^{(k)}}{a}$ and $G = \frac{P[f^q]}{a}$, where $a \in S(f)$. Then $FG \not\equiv 1$.*

Proof. *On contrary we assume* $FG \equiv 1$. *So* $(f^n)^{(k)}P[f^q] = a^2$. *Therefore* $N(r,f) = S(r,f)$ *and* $N(r,\frac{1}{f}) = S(r,f)$. *Now in view of Nevanlinna's first fundamental theorem and using Lemma 3.2.1 we have*

$$
\begin{gathered}
(n + q\overline{d}(P))T(r,f) \leq T\left(r, \frac{a^2}{f^{n+q\overline{d}(P)}}\right) + S(r,f) \\
\leq T\left(r, \frac{(f^n)^{(k)}}{f^n}.\frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + S(r,f) \leq T\left(r, \frac{(f^n)^{(k)}}{f^n}\right) \\
+T\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + S(r,f) \leq N\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + m\left(r, \frac{(f^n)^{(k)}}{f^n}\right) \\
+N\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + m\left(r, \frac{P[f^q]}{f^{q\overline{d}(P)}}\right) + S(r,f) \leq \\
k\left(\overline{N}(r,f) + \overline{N}(r,\tfrac{1}{f})\right) + N(r,P[f^q]) + q\overline{d}(P)N(r,\tfrac{1}{f}) \\
+q\left(\overline{d}(P) - \underline{d}(P)\right)T(r,f) + S(r,f).
\end{gathered}
$$

That is $(n + q\underline{d}(P))T(r,f) \leq S(r,f)$, *which is a contradiction.* ■

Lemma 3.2.8 *Let* q *be a positive integer. Let* f *be a meromorphic function and* $P[f^q]$ *be a differential polynomial generated by* f^q. *Then*

$$m(r, P[f^q]) \leq \overline{d}(P)m(r, f^q) + S(r,f).$$

Proof. *The proof can be obtain along the same line as the proof of Lemma* 2.6 *in [9].* ■

Lemma 3.2.9 *Let* q *be a positive integer. Let* f *be a non-constant meromorphic function and* $P[f^q]$ *be a differential polynomial of* f^q. *Then* $S(r, P[f^q])$ *can be replaced by* $S(r,f)$.

Proof. From Lemma 3.2.8 it is clear that $T(r, P[f^q]) = O(T(r,f))$ and so the lemma follows. ■

Lemma 3.2.10 *Let* q *be a positive integer. Let* f *be a meromorphic function with a pole of order* $p \geq 1$ *at* z_0. *If* $P[f^q]$ *be a* *differential polynomial of f^q whose coefficients are analytic at z_0, then $P[f^q]$ has a pole at z_0 of order at most $qp\overline{d}(P)+\Gamma_P-\overline{d}(P)$.*

Proof. Let $P[f]$ be defined as in Definition 3.1.4. Then by a simple calculation, $P[f^q]$ has a pole at z_0 and its order is at most

$$\begin{aligned}\max_{1\le j\le t}\left\{\sum_{s=0}^{k}(qp+s)n_{sj}\right\} &= \max_{1\le j\le t}\left\{(qp-1)\sum_{s=0}^{k}n_{sj}+\sum_{s=0}^{k}(s+1)n_{sj}\right\}\\ &\le (qp-1)\overline{d}(P)+\Gamma_P\\ &= qp\overline{d}(P)+\Gamma_P-\overline{d}(P).\end{aligned}$$

This completes the proof of the lemma. ■

3.3 Theorems

In this section we present the main result of the chapter.

Theorem 3.3.1 *Let $k\ (\ge 1)$, $n\ (\ge k+2)$, $q\ (\ge 1)$, $l\ (\ge 0)$ be integers and f be a non-constant meromorphic function. Also let $a\ (\not\equiv 0,\infty)\in S(f)$ and $P[f^q]$ be a non-constant differential polynomial such that $n\ne q\overline{d}(P)$. Suppose $(f^n)^{(k)}-a$ and $P[f^q]-a$ share $(0,l)$. If $l\ge 2$ and*

$$\begin{aligned}&(Q+3)\Theta(\infty,f)+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n)\\ &\qquad > Q+3+q\overline{d}(P),\end{aligned}\tag{3.3.1}$$

or, $l=1$ and

$$\begin{aligned}&(Q+\tfrac{k+7}{2})\Theta(\infty,f)+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n)\\ &\qquad +n\delta_{k+1}(0,f^n) > Q+\tfrac{k+7}{2}+q\overline{d}(P)+n,\end{aligned}\tag{3.3.2}$$

or $l=0$ and

$$(2Q+6+2k)\Theta(\infty,f)+q\overline{d}(P)\delta_{k+1}(0,f)+2n\delta_{k+1}(0,f^n)$$
$$+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n)>2Q+2k$$
$$+2q\overline{d}(P)+6+2n, \qquad (3.3.3)$$

then $(f^n)^{(k)}\equiv P[f^q]$.

Proof. Let $F=\frac{(f^n)^{(k)}}{a}$ and $G=\frac{P[f^q]}{a}$. Then $F-1=\frac{(f^n)^{(k)}-a}{a}$ and $G-1=\frac{P[f^q]-a}{a}$. Since $(f^n)^{(k)}$ and $P[f^q]$ share (a,l), it follows that F and G share $(1,l)$ except at zeros and poles of a. By Lemma 3.2.9 we have $S(r,P[f^q])=S(r,f)$. Therefore $\overline{N}(r,G)=\overline{N}(r,f)+S(r,f)=\overline{N}(r,F)$.

Suppose $H\not\equiv 0$. Then from (3.2.1) and Lemma 3.2.4 we have $m(r,H)=S(r,f)$ and

$$N_E{}^{1)}(r,\tfrac{1}{F-1})\le N(r,\tfrac{1}{H})+S(r,f)$$
$$\le T(r,H)+S(r,f)=N(r,H)+S(r,f)$$
$$\le \overline{N}(r,F)+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})+\overline{N}_L(r,\tfrac{1}{F-1})$$
$$+\overline{N}_L(r,\tfrac{1}{G-1})+N_0(r,\tfrac{1}{F^{(1)}})+N_0(r,\tfrac{1}{G^{(1)}})+S(r,f). \quad (3.3.4)$$

By Nevanlinna's second fundamental theorem we have

$$T(r,F)+T(r,G)\le 2\overline{N}(r,f)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G})$$
$$+\overline{N}(r,\tfrac{1}{G-1})-N_0(r,\tfrac{1}{F^{(1)}})-N_0(r,\tfrac{1}{G^{(1)}})+S(r,f), \quad (3.3.5)$$

where $N_0(r,\frac{1}{F^{(1)}})$ denotes the counting function of zeros of $F^{(1)}$ which are not the zeros of $F(F-1)$ and similarly $N_0(r,\frac{1}{G^{(1)}})$.

Case 1: $l \geq 1$. Then by (3.3.4), we obtain

$$\begin{aligned}
&\overline{N}\big(r, \tfrac{1}{F-1}\big) + \overline{N}\big(r, \tfrac{1}{G-1}\big) = N_E{}^{1)}\Big(r, \tfrac{1}{F-1}\Big) + \overline{N}_E{}^{(2}\Big(r, \tfrac{1}{F-1}\Big) \\
&+\overline{N}_L\big(r, \tfrac{1}{F-1}\big) + \overline{N}_L\big(r, \tfrac{1}{G-1}\big) + \overline{N}\big(r, \tfrac{1}{G-1}\big) + S(r, f) \\
&\leq \overline{N}(r, F) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) + 2\overline{N}_L\big(r, \tfrac{1}{F-1}\big) \\
&+2\overline{N}_L\big(r, \tfrac{1}{G-1}\big) + N_0(r, \tfrac{1}{F^{(1)}}) + N_0(r, \tfrac{1}{G^{(1)}}) + \overline{N}_E{}^{(2}\big(r, \tfrac{1}{F-1}\big) \\
&+\overline{N}\big(r, \tfrac{1}{G-1}\big) + S(r, f). \qquad (3.3.6)
\end{aligned}$$

Subcase 1.1: $l = 1$. In this case we have,

$$\begin{aligned}
\overline{N}_L\big(r, \tfrac{1}{F-1}\big) &\leq \tfrac{1}{2}N(r, \tfrac{1}{F^{(1)}}|F \neq 0) \\
&\leq \tfrac{1}{2}\overline{N}(r, F) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{F}) + S(r, f), \qquad (3.3.7)
\end{aligned}$$

where $N(r, \frac{1}{F^{(1)}}|F \neq 0)$ denotes the counting function of zeros of $F^{(1)}$ which are not the zeros of F.

Therefore by (3.3.7) and Lemmas 3.2.2, 3.2.3 we have

$$\begin{aligned}
&2\overline{N}_L(r, \tfrac{1}{F-1}) + 2\overline{N}_L(r, \tfrac{1}{G-1}) + \overline{N}_E{}^{(2}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) \\
&\leq N(r, \tfrac{1}{G-1}) + \overline{N}_L(r, \tfrac{1}{F-1}) + S(r, f) \leq N(r, \tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r, F) \\
&+\tfrac{1}{2}\overline{N}(r, \tfrac{1}{F}) + S(r, f) \leq N(r, \tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r, f) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{(f^n)^{(k)}}) \\
&+S(r, f) \leq N(r, \tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r, f) + \tfrac{1}{2}N_{k+1}(r, \tfrac{1}{f^n}) + \tfrac{k}{2}\overline{N}(r, f) \\
&+S(r, f) \leq N(r, \tfrac{1}{G-1}) + \tfrac{k+1}{2}\overline{N}(r, f) + \tfrac{1}{2}N_{k+1}(r, \tfrac{1}{f^n}) \\
&+S(r, f). \qquad (3.3.8)
\end{aligned}$$

So from (3.3.6) and (3.3.8), we get

$$\begin{aligned}
&\overline{N}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) \leq \overline{N}(r, f) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) \\
&+N(r, \tfrac{1}{G-1}) + \tfrac{k+1}{2}\overline{N}(r, f) + N_{k+1}(r, \tfrac{1}{f^n}) + N_0(r, \tfrac{1}{F^{(1)}}) + N_0(r, \tfrac{1}{G^{(1)}}) \\
&+S(r, f) \leq \tfrac{k+3}{2}\overline{N}(r, f) + \overline{N}_{(2}(r, \tfrac{1}{F}) + \overline{N}_{(2}(r, \tfrac{1}{G}) + T(r, G) \\
&+N_{k+1}(r, \tfrac{1}{f^n}) + N_0(r, \tfrac{1}{F^{(1)}}) + N_0(r, \tfrac{1}{G^{(1)}}) + S(r, f). \qquad (3.3.9)
\end{aligned}$$

Now from (3.3.5), (3.3.9) and Lemmas 3.2.3, 3.2.6 we get

$$T(r,F) \le \left(\tfrac{k+3}{2}+2\right)\overline{N}(r,f)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})$$
$$+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})+N_{k+1}(r,\tfrac{1}{f^n})+S(r,f)$$
$$\le \tfrac{k+7}{2}\overline{N}(r,f)+N_2(r,\tfrac{1}{F})+N_2(r,\tfrac{1}{G})+N_{k+1}(r,\tfrac{1}{f^n})+S(r,f)$$
$$\le \tfrac{k+7}{2}\overline{N}(r,f)+T(r,F)-nT(r,f)+N_{k+2}(r,\tfrac{1}{f^n})$$
$$+N_{k+1}(r,\tfrac{1}{f^n})+Q\overline{N}(r,f)+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})+S(r,f),$$

which implies

$$nT(r,f) \le \left(Q+\tfrac{k+7}{2}\right)\overline{N}(r,f)+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})$$
$$+N_{k+2}(r,\tfrac{1}{f^n})+N_{k+1}(r,\tfrac{1}{f^n})+S(r,f).$$

So

$$\left(Q+\tfrac{k+7}{2}\right)\Theta(\infty,f)+q\overline{d}(P)\delta_{k+2}(0,f)+n\delta_{k+2}(0,f^n)$$
$$+n\delta_{k+1}(0,f^n) \le Q+\tfrac{k+7}{2}+q\overline{d}(P)+n,$$

which contradicts (3.3.2).

Subcase 1.2: $l \ge 2$. In this case we have

$$2\overline{N}_L(r,\tfrac{1}{F-1})+2\overline{N}_L(r,\tfrac{1}{G-1})+\overline{N_E}^{(2}(r,\tfrac{1}{F-1})$$
$$+\overline{N}(r,\tfrac{1}{G-1}) \le N(r,\tfrac{1}{G-1})+S(r,f),$$

and from (3.3.6) we obtain

$$\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G-1}) \le \overline{N}(r,f)+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})$$
$$+N(r,\tfrac{1}{G-1})+N_0(r,\tfrac{1}{F^{(1)}})+N_0(r,\tfrac{1}{G^{(1)}})+S(r,f) \le \overline{N}(r,f)$$
$$+\overline{N}_{(2}(r,\tfrac{1}{F})+\overline{N}_{(2}(r,\tfrac{1}{G})+T(r,G)+N_0(r,\tfrac{1}{F^{(1)}})$$
$$+N_0(r,\tfrac{1}{G^{(1)}})+S(r,f). \qquad (3.3.10)$$

Now from (3.3.5), (3.3.10) and Lemmas 3.2.3, 3.2.6 we get

$$T(r,F) \leq 3\overline{N}(r,f) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G})$$
$$+\overline{N}_{(2}(r,\tfrac{1}{G}) + S(r,f) \leq 3\overline{N}(r,f) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G})$$
$$+S(r,f) \leq 3\overline{N}(r,f) + T(r,F) - nT(r,f) + N_{k+2}(r,\tfrac{1}{f^n})$$
$$+Q\overline{N}(r,f) + q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f}) + S(r,f),$$

which implies

$$nT(r,f) \leq (Q+3)\overline{N}(r,f) + N_{k+2}(r,\tfrac{1}{f^n}) + q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})$$
$$+S(r,f) \leq (Q+3)\{1-\Theta(\infty,f)\}T(r,f)$$
$$+n(1-\delta_{k+2}(0,f^n))T(r,f) + q\overline{d}(P)(1-\delta_{k+2}(0,f))T(r,f)$$
$$+S(r,f)$$

$$\Rightarrow (Q+3)\Theta(\infty,f)+q\overline{d}(P)\delta_{2+k}(0,f)+n\delta_{k+2}(0,f^n) \leq Q+3+q\overline{d}(P),$$

which violates assumption (3.3.1).

Case 2 : $l = 0$. Then we have

$$N_E^{1)}\big(r,\tfrac{1}{F-1}\big) = N_E^{1)}\big(r,\tfrac{1}{G-1}\big) + S(r,f),$$
$$\overline{N}_E^{(2}\big(r,\tfrac{1}{F-1}\big) = \overline{N}_E^{(2}\big(r,\tfrac{1}{G-1}\big) + S(r,f).$$

Now using (3.3.4), we have

$$\overline{N}(r,\tfrac{1}{F-1}) + \overline{N}(r,\tfrac{1}{G-1}) = N_E^{1)}(r,\tfrac{1}{F-1}) + \overline{N}_E^{(2}(r,\tfrac{1}{F-1})$$
$$+\overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1}) + \overline{N}(r,\tfrac{1}{G-1}) + S(r,f)$$
$$\leq N_E^{1)}(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{F-1}) + N(r,\tfrac{1}{G-1}) + S(r,f)$$
$$\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}_{(2}(r,\tfrac{1}{G}) + 2\overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1})$$
$$+N(r,\tfrac{1}{G-1}) + N_0(r,\tfrac{1}{F^{(1)}}) + N_0(r,\tfrac{1}{G^{(1)}}) + S(r,f). \quad (3.3.11)$$

Therefore from (3.3.5), (3.3.11) and Lemmas 3.2.2, 3.2.3 and 3.2.6 we get

$$T(r,F) \le 3\overline{N}(r,f) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G}) + \overline{N}_{(2}(r,\tfrac{1}{G})$$
$$+2\overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1}) + S(r,f) \le 3\overline{N}(r,f) + N_2(r,\tfrac{1}{F})$$
$$+N_2(r,\tfrac{1}{G}) + 2\overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1}) + S(r,f) \le 3\overline{N}(r,f)$$
$$+T(r,F) - nT(r,f) + N_{k+2}(r,\tfrac{1}{f^n}) + Q\overline{N}(r,f)$$
$$+q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f}) + 2\overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1}) + S(r,f),$$

which implies

$$nT(r,f) \le (Q+3)\overline{N}(r,f) + N_{k+2}(r,\tfrac{1}{f^n}) + q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f})$$
$$+2\overline{N}(r,\tfrac{1}{F}) + 2\overline{N}(r,F) + \overline{N}(r,\tfrac{1}{G}) + \overline{N}(r,G) + S(r,f)$$
$$\le (2Q+6+2k)\overline{N}(r,f) + q\overline{d}(P)N_{k+2}(r,\tfrac{1}{f}) + q\overline{d}(P)N_{k+1}(r,\tfrac{1}{f})$$
$$+2N_{k+1}(r,\tfrac{1}{f^n}) + N_{k+2}(r,\tfrac{1}{f^n}) + S(r,f)$$
$$\le (2Q+6+2k)\{1-\Theta(\infty,f)\}T(r,f)$$
$$+q\overline{d}(P)(1-\delta_{k+1}(0,f))T(r,f) + 2n(1-\delta_{k+1}(0,f^n))T(r,f)$$
$$+q\overline{d}(P)(1-\delta_{2+k}(0,f))T(r,f) + n(1-\delta_{2+k}(0,f^n))T(r,f)$$
$$+S(r,f).$$

Therefore

$$(2Q+6+2k)\Theta(\infty,f) + q\overline{d}(P)\delta_{k+1}(0,f) + 2n\delta_{k+1}(0,f^n)$$
$$+q\overline{d}(P)\delta_{k+2}(0,f) + n\delta_{k+2}(0,f^n) \le 2Q + 2k + 2q\overline{d}(P) + 6 + 2n,$$

which contradicts (3.3.3).

Next suppose $H \equiv 0$. Integrating twice we get

$$\frac{1}{F-1} = \frac{C}{G-1} + D, \tag{3.3.12}$$

where $C \neq 0$ and D are constants.

Here the following three cases can arise.

Case I: $D \neq 0, -1$. If z_0 be a pole of f with multiplicity p, by Lemma 3.2.10, F and G have a pole at z_0 with multiplicities $np + k$ and $qp\overline{d}(P) + \Gamma_P - \overline{d}(P)$ respectively. This contradicts (3.3.12). Thus it follows that $N(r, f) = S(r, f)$. Also it is clear that $\overline{N}(r, G) = \overline{N}(r, f) = S(r, f)$. From (3.3.12) we get

$$\frac{G-1}{C} = \frac{F-1}{D+1-DF}$$

Therefore $\overline{N}\left(r, \frac{1}{F-\frac{D+1}{D}}\right) = \overline{N}(r, G) = S(r, f)$.

Now by Nevanlinna second fundamental theorem and Lemma 3.2.3, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}(r, \tfrac{1}{F}) + \overline{N}\left(r, \frac{1}{F-\frac{D+1}{D}}\right) + S(r, f) \\ &\leq \overline{N}(r, F) + \overline{N}(r, \tfrac{1}{F}) + \overline{N}(r, G) + S(r, f) \\ &\leq 2\overline{N}(r, f) + T(r, F) - nT(r, f) + N_{k+1}(r, \tfrac{1}{f^n}) + S(r, f). \\ \Rightarrow nT(r, f) &\leq (k+1)\overline{N}(r, \tfrac{1}{f^n}) + S(r, f) \leq (k+1)T(r, f) + S(r, f), \end{aligned}$$

which contradicts $n \geq k + 2$.

Case II: $D = 0$. Then from (3.3.12), we get $G = CF - (C - 1)$. Therefore if $C \neq 1$ then $\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F-\frac{C-1}{C}})$.

Again by Nevannlina second fundamental theorem and Lemma 3.2.3, 3.2.6 we get

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}(r, \tfrac{1}{F}) + \overline{N}\left(r, \frac{1}{F-\frac{C-1}{C}}\right) \leq \overline{N}(r, f) \\ &+ T(r, F) - nT(r, f) + N_{k+1}(r, \tfrac{1}{f^n}) + \overline{N}(r, \tfrac{1}{G}) + S(r, f), \end{aligned}$$

which implies

$$\begin{aligned} &nT(r, f) \leq \overline{N}(r, f) + N_{k+1}(r, \tfrac{1}{f^n}) + Q\overline{N}(r, f) \\ &+ q\overline{d}(P)N_{k+1}(r, \tfrac{1}{f}) + S(r, f) \leq (Q+1)\{1 - \Theta(\infty, f)\}T(r, f) \\ &+ n\{1 - \delta_{k+1}(0, f^n)\}T(r, f) + q\overline{d}(P)\{1 - \delta_{k+1}(0, f)\}T(r, f) \\ &+ S(r, f) \end{aligned}$$

$\Rightarrow (Q+1)\Theta(\infty, f)+n\delta_{k+1}(0, f^n)+q\overline{d}(P)\delta_{k+1}(0, f) \leq Q+1+q\overline{d}(P)$, which is a contradiction to (3.3.3).

Therefore $C = 1$ and so $F \equiv G$ and hence $(f^n)^{(k)} \equiv P[f^q]$.

Case III: $D = -1$. Then from (3.3.12) we have

$$\frac{1}{F-1} = \frac{C}{G-1} - 1.$$

Therefore if $C \neq -1$, then $\overline{N}(r, \frac{1}{G}) = \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right)$. Proceeding as in Case II we get a contradiction. Therefore $C = -1$ and so $FG \equiv 1$, which is a contradiction by Lemma 3.2.7. ■

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