CHAPTER

2

UNIQUENESS OF MEROMORPHIC FUNCTIONS WHOSE DIFFERENTIAL POLYNOMIALS SHARE A NON-CONSTANT POLYNOMIAL

2.1 Introduction, Definitions and Notations

The following theorem in the value distribution theory is well known.

Theorem 2.1.1 *[2, 15]. Let f be a transcendental meromorphic function, $n \geq 1$ be an integer. Then $f^n f^{(1)} = 1$ has infinitely many solutions.*

Corresponding to Theorem 2.1.1, Fang and Hua [22], Yang and Hua [29] have obtained the following unicity theorem.

Theorem 2.1.2 *[22, 29] Let $f(z)$ and $g(z)$ be two non-constant*

The results of this chapter have been published in the **New Trends in Mathematical Sciences (NTMSCI), see [60].**

entire (meromorphic) functions, $n \geq 6$ ($n \geq 11$) be an integer. If $f^n f^{(1)}$ and $g^n g^{(1)}$ share 1 CM then either $f(z) = c_1 \exp(cz)$, $g(z) = c_2 \exp(-cz)$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

In 2002, Fang [23] proved the following theorems.

Theorem 2.1.3 *[23] Let $f(z)$ and $g(z)$ be two non-constant entire functions, let n, k be two positive integers with $n > 2k + 4$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 1 CM then either $f(z) = c_1 \exp(cz)$, $g(z) = c_2 \exp(-cz)$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem 2.1.4 *[23] Let f and g be two non-constant entire functions and let n, k be two positive integers with $n \geq 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM then $f \equiv g$.*

In same year Fang and Qiu [25] considered the fixed points sharing of entire functions and obtained the following uniqueness result.

Theorem 2.1.5 *[25] Let $f(z)$ and $g(z)$ be two non-constant entire functions and $n \geq 6$ be an integer. If $f^n(z)f^{(1)}(z)$ and $g^n(z)g^{(1)}(z)$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Later in 2004, Lin and Yi [42] proved the following theorem.

Theorem 2.1.6 *[42] Let f and g be two non-constant entire functions and $n \geq 7$ be an integer. If $f^n(f-1)f^{(1)}$ and $g^n(g-1)g^{(1)}$ share z CM, then $f \equiv g$.*

Zhang [92] extended Theorems (2.1.3–2.1.4) and proved the following results.

Theorem 2.1.7 *[92] Let $f(z)$ and $g(z)$ be two non-constant entire functions and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then either*
(1) $k = 1$, $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$, or
(2) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

Theorem 2.1.8 *[92] Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 2k + 6$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, then $f \equiv g$.*

Regarding Theorems 2.1.7–2.1.8, Xu et al [71] considered the case of meromorphic functions and proved the following theorems.

Theorem 2.1.9 *[71] Let f and g be two non-constant meromorphic functions and let n, k be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem 2.1.10 *[71] Let n, k be two positive integers with $n > 3k+12$, and f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$. If $(f^n(f-1))^{(k)}$, $(g^n(g-1))^{(k)}$ share z CM and f and g share ∞ IM, then $f \equiv g$.*

In view of Theorems 2.1.9–2.1.10 , Sahoo [68] obtained the following result for some more general non-linear differential polynomial.

Theorem 2.1.11 *[68] Let f and g be two non-constant meromorphic functions and let n, k and m be three positive integers with $n > 9k + 4m + 13$. Let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdot\cdot$ $+ a_1 w + a_0$, where $a_0 \neq 0$, $a_1, \cdot\cdot\cdot, a_m \neq 0$ are complex constants. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share z IM, f and g share ∞ IM, then $f = tg$ for a constant t such that $t^d = 1$, where $d = gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$; or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by*

$$R(f, g) = f^n(a_m f^m + ... + a_1 f + a_0) - g^n(a_m g^m + ... \\ ... + a_1 g + a_0). \tag{2.1.1}$$

Question 2.1.1 *It is natural to ask what happen if sharing fixed point in Theorem 2.1.11 is replaced by sharing a non-constant polynomial ?*

Corresponding to the above question Zhang and Xu [95] obtained the following result.

Theorem 2.1.12 *[95] Let f and g be two non-constant meromorphic functions, $p(z)$ be a non-constant polynomial of degree deg $p(z) = l \leq 5$, n, k and m be three positive integers with $n > 3k + m + 7$. Let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdot\cdot + a_1 w + a_0$ be a non-zero polynomial. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $p(z)$ CM, f and g share ∞ IM, then one of the following two cases holds:*

(1) $f = tg$ for a constant t such that $t^d = 1$, where $d = gcd(n + m, n + m - 1, ..., n + m - i, ..., n + 1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$;

(2) f and g satisfy the algebraic equation $R(f, g) = 0$, where

$R(f,g)$ is given by (2.1.1).
(3) $P(z)$ is reduced to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0,1,...,m\}$; if $p(z)$ is non-constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are three constants satisfying ${a_i}^2(c_1c_2)^{n+i}((n+i)c)^2 = -1$; if $p(z)$ is a constant b, then $f(z) = c_3 e^{cz}$, $g(z) = c_4 e^{-cz}$, where c_3, c_4 and c are three constants satisfying $(-1)^k {a_i}^2(c_3c_4)^{n+i}((n+i)c)^{2k} = b^2$.

In 2016, Sahoo [69] removed the restriction on the degree of the polynomial $p(z)$ in the above theorem and proved the following theorems.

Theorem 2.1.13 *[69] Let f and g be two transcendental meromorphic functions, $p(z)$ be a non-constant polynomial of degree l, and let n (≥ 1), k (≥ 1) and m (≥ 0) be three integers with $n > max\{3k+m+6, k+2l\}$. In addition, we suppose that either k, l are co-prime or $k > l$ when $l \geq 2$. Let $P(w)$ be defined as in the above theorem. If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share $p(z)$ CM; f and g share ∞ IM, then the following conclusions hold:*
(i) If $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ is not a monomial, then either $f = tg$ for a constant t that satisfies $t^d = 1$, where $d = \gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,...,m\}$; or f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(f,g)$ is given by (2.1.1). In particular, if $m = 1$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$, then $f = g$.
(ii) When $P(w) = a_mw^m$, or $P(w) = c_0$, then either $f = tg$ for some t such that $t^{n+m^*} = 1$, or $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant term such that $Q^{(1)}(z) = p(z)$; b_1, b_2 and b are three constants satisfying

$a_m{}^2(b_1b_2)^{n+m}((n+m)b)^2 = -1$ *or* $c_0{}^2(b_1b_2)^n(nb)^2 = -1$, *where* m^* *is same as in Lemma 2.2.12.*

Theorem 2.1.14 *[69] Let f and g be two transcendental meromorphic functions, $p(z)$ be a non-constant polynomial of degree l, and let n (≥ 1), k (≥ 1) and m (≥ 0) be three integers with $n > max\{9k + 4m + 11, k + 2l\}$. In addition, we suppose that either k, l are co-prime or $k > l$ when $l \geq 2$. Let $P(w)$ be defined as in the above theorem. If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share $p(z)$ IM; f and g share ∞ IM, then the conclusion of Theorem 2.1.13 holds.*

We removed the condition "f and g share ∞ IM" in Theorems 2.1.13–2.1.14 and obtained some results which are the main theorems of this chapter.

2.2 Lemmas

In this section we present some lemmas needed in the sequel.
Let F_1 and G_1 be non-constant meromorphic functions. We denote by H the following function:

$$H = \left(\frac{F_1^{(2)}}{F_1^{(1)}} - 2\frac{F_1^{(1)}}{F_1 - 1}\right) - \left(\frac{G_1^{(2)}}{G_1^{(1)}} - 2\frac{G_1^{(1)}}{G_1 - 1}\right). \qquad (2.2.1)$$

Lemma 2.2.1 *[77] Let f be a non-constant meromorphic function and let a_0, $a_1, \ldots\ldots$, $a_n (\not\equiv 0)$ be small functions with respect to f. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + \ldots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2.2 *[86] Let h be a non-constant entire function and let $f = e^h$. Let λ and μ be the order and lower order of h respectively.*

(i) If $\mu < \infty$, then μ is a positive integer, h is a polynomial of degree μ and $\lambda = \mu$.
(ii) If $\mu = \infty$, then h is transcendental and $\lambda = \mu$.

Lemma 2.2.3 *[43] Let f be a non-constant meromorphic function and s, k be two positive integers. Then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.2.4 *[86] Let f be a non-constant meromorphic function and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.2.5 *[39] Let $k \geq 1$ be an integer. If $N(r, \frac{1}{f^{(k)}}/f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ that are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*
$N(r, \frac{1}{f^{(k)}}/f \neq 0) \leq k\overline{N}(r, f) + N(r, \frac{1}{f}| < k) + k\overline{N}(r, \frac{1}{f}| \geq k) + S(r, f)$.

Lemma 2.2.6 *Let f and g be two non-constant transcendental meromorphic functions whose zeros and poles are of multiplicity at least s, where s is positive integer. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$, where $a_0(\neq 0), a_1, \cdots, a_m(\neq 0)$ are constants, and let n (≥ 1), k (≥ 1), m (≥ 0) be integers and $p(z)$ be a polynomial of degree l . If*

$$\lambda > \frac{1}{s} + \frac{m+n}{(m+n)s + 2k}, \tag{2.2.2}$$

where λ is the number of distinct roots of $P(w) = 0$, then

$$(f^n P(f))^{(k)} (g^n P(g))^{(k)} \neq p^2(z).$$

Proof. Suppose that

$$(f^nP(f))^{(k)}\,(g^nP(g))^{(k)} = p^2(z). \tag{2.2.3}$$

Let $P(z) = a_m(z-d_1)^{l_1}.(z-d_2)^{l_2}\cdots(z-d_\lambda)^{l_\lambda}$, where $\sum_{j=1}^{\lambda} l_j = m$, $1 \leq \lambda \leq m$, $d_i \neq d_j$ for $i \neq j$ and $1 \leq i, j \leq \lambda$, d_j's are non-zero constants and l_j's are positive integers for $j = 1, 2, .., \lambda$.

Let $z_0 \notin \{z : p(z) = 0\}$ be a zero of f with multiplicity p_0 (say). Then z_0 is a pole of g with multiplicity q_0 (say). From (2.2.3) we get, $np_0 - k = (m+n)q_0 + k \Rightarrow np_0 - k \geq (m+n)s + k$ and $p_0 \geq \frac{(m+n)s+2k}{n}$.

Let $z_1 \notin \{z : p(z) = 0\}$ be a zero of $P(f)$ of order p_1 (say) and a zero of $f - d_i$ of order q_i for some $i = 1, 2, ..., \lambda$. Then $p_1 = l_iq_i$ for some $i = 1, 2, ..., \lambda$ and z_1 is a pole of g with multiplicity $\overline{q}$ (say). So from (2.2.3) we get

$$\begin{aligned} q_il_i - k &= (n+m)\overline{q} + k \\ &\geq (n+m)s + k \\ i.e., q_i &\geq \frac{(n+m)s + 2k}{l_i}, \end{aligned}$$

for $i = 1, 2, .., \lambda$.

Let $z_2 \notin \{z : p(z) = 0\}$ be a zero of $(f^nP(f))^{(k)}$ of order p_2 but not a zero of $f^nP(f)$. Then from (2.2.3) z_2 is a pole of g of order ξ (say). Thus

$$p_2 = (n+m)\xi + k \geq (n+m)s + k.$$

Suppose that $z_3 \notin \{z : p(z) = 0\}$ be a pole of f then from (2.2.3) z_3 is a zero of $(g^nP(g))$ or a zero of $(g^nP(g))^{(k)}$. Therefore

$$\overline{N}(r, f) \leq \overline{N}(r, \tfrac{1}{g}) + \textstyle\sum_{j=1}^{\lambda} \overline{N}(r, \tfrac{1}{g-d_i})$$
$$+\overline{N}(r, \tfrac{1}{B^{(k)}}/B \neq 0) + S(r, g),$$

where $\overline{N}(r, \frac{1}{B^{(k)}}/B \neq 0)$ denotes the reduced counting function of those zeros of $B^{(k)}$ that are not the zeros of B, $B = g^n P(g)$. Now by Lemma 2.2.5 we have

$$\overline{N}(r, \tfrac{1}{B^{(k)}}/B \neq 0) \leq \tfrac{1}{(n+m)s+k} N(r, \tfrac{1}{B^{(k)}}/B \neq 0)$$
$$\leq \tfrac{1}{(n+m)s+k} \left\{ k\overline{N}(r, B) + N(r, \tfrac{1}{B}| < k) + k\overline{N}(r, \tfrac{1}{B}| \geq k) \right\}$$
$$+S(r, g) \leq \tfrac{k}{(n+m)s+k} \left\{ \overline{N}(r, B) + N_k(r, \tfrac{1}{B}) \right\} + S(r, g)$$
$$\leq \tfrac{k}{(n+m)s+k} \left\{ \overline{N}(r, \tfrac{1}{g}) + \textstyle\sum_{j=1}^{\lambda} \overline{N}(r, \tfrac{1}{g-d_i}) + \overline{N}(r, g) \right\} + S(r, g)$$

$$So, \quad \overline{N}(r, f) \leq \left(1 + \tfrac{k}{(n+m)s+k}\right) \left(\tfrac{n}{(m+n)s+2k} + \tfrac{m}{(m+n)s+2k} \right) T(r, g)$$
$$+ \tfrac{k}{((n+m)s+k)s} T(r, g) + S(r, g)$$
$$\leq \tfrac{1}{s} T(r, g) + S(r, g).$$

By Nevanlinna second fundamental theorem we have

$$\begin{aligned} \lambda T(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \sum_{j=1}^{\lambda} \overline{N}(r, \frac{1}{f - d_i}) + S(r, f) \\ &\leq \left(\frac{n}{(m+n)s + 2k} + \frac{m}{(m+n)s + 2k} \right) T(r, f) \\ &+ \frac{1}{s} T(r, g) + S(r, f) + S(r, g). \end{aligned} \tag{2.2.4}$$

Similarly,

$$\begin{aligned} \lambda T(r, g) &\leq \left(\frac{n}{(m+n)s + 2k} + \frac{m}{(m+n)s + 2k} \right) T(r, g) \\ &+ \frac{1}{s} T(r, f) + S(r, f) + S(r, g). \end{aligned} \tag{2.2.5}$$

Adding (2.2.4) and (2.2.5) we get

$$\left\{ \lambda - \frac{1}{s} - \frac{m+n}{(m+n)s + 2k} \right\} (T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),$$

which contradicts the hypothesis. This completes the proof of the Lemma. ∎

Lemma 2.2.7 *Let f and g be two non-constant meromorphic functions with $\sigma(f) < \infty$, $p(z)$ be non-constant polynomial of degree l and n, k, m be three positive integers with $n > max\{2k+2l, 2k(\sigma(f)-1)-(m+2l)\}$. In addition, we assume $k > l$ when $l \geq 2$. If*

$$(f^nP(f))^{(k)}\,(g^nP(g))^{(k)} = p^2, \qquad (2.2.6)$$

where $P(w) = a_mw^m$ or $P(w) = c_0$ then $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where b_1, b_2 and c are three constants satisfying $a_m{}^2(b_1b_2)^{n+m}((n+m)b)^2 = -1$ or $c_0{}^2(b_1b_2)^n(nb)^2 = -1$ and $Q(z)$ is a polynomial without constant term such that $Q^{(1)}(z) = p(z)$.

Proof. We first prove that

$$f \neq 0, g \neq 0. \qquad (2.2.7)$$

Let $P(w) = a_mw^m$. Then from (2.2.6) we get

$$\left(f^{n+m}\right)^{(k)}\left(g^{n+m}\right)^{(k)} = p^2(z) \qquad (2.2.8)$$

Suppose that z_0 is a zero of f of multiplicity r (say), but $p(z_0) \neq 0$. Then z_0 is a pole of g with multiplicity s_0 (say). From (2.2.8) we have
$(n+m)r - k = (n+m)s_0 + k \Rightarrow (n+m)(r-s_0) = 2k$,
which is a contradiction for $n > 2k + 2l$.
Now suppose that z_0 is a zero of f with multiplicity r_1 (say), if z_0 is not a pole of g then z_0 must be a zero of $p(z)$ of multiplicity l_0 (say). Then we have from (2.2.8) $(n+m)r_1 - k > 2l_0$, which is again a contradiction. If z_0 is a pole of g with multiplicity s_1 (say), then we have
$(n+m)r_1 - k = (n+m)s_1 + k + 2l_0$
$\Rightarrow (n+m)(r_1 - s_1) = 2k + 2l_0$, which is impossible. So, f has

no zeros.

Similarly it can be shown that g also has no zeros. Thus (2.2.7) is proved.

Next we prove that

$$N(r,f) = O(\log r), \qquad (2.2.9)$$
$$N(r,g) = O(\log r).$$

From (2.2.6), we have

$$(f^nP(f))^{(k)} = \frac{p^2(z)}{(g^nP(g))^{(k)}}. \qquad (2.2.10)$$

Now $N(r,(f^nP(f))^{(k)}) = N(r,f^nP(f)) + k\overline{N}(r,f^nP(f)) = (n+m)N(r,f) + k\overline{N}(r,f) + S(r,f)$. By Lemma 2.2.4

$$\begin{aligned} N(r,\frac{1}{(g^{n+m})^{(k)}}) &\leq N(r,\frac{1}{g^{n+m}}) + k\overline{N}(r,g^{n+m}) + O(\log r) \\ &= k\overline{N}(r,g) + O(\log r). \end{aligned}$$

Using the above inequality we get from (2.2.10)
$(n+m)N(r,f) + k\overline{N}(r,f) \leq k\overline{N}(r,g) + O(\log r)$.
Similarly we get
$(n+m)N(r,g) + k\overline{N}(r,g) \leq k\overline{N}(r,f) + O(\log r)$.
Combining we get $N(r,f) + N(r,g) = O(\log r)$.
Thus we obtain (2.2.9) which mean that f and g has at most finitely many poles.
Now we prove that $\sigma(f) = \sigma(g)$.
By Yamanoi [85] result of second fundamental theorem with $F = f^nP(f)$, $G = g^nP(g)$ we get

$$\begin{aligned} T(r,F^{(k)}) &\leq \overline{N}(r,F^{(k)}) + \overline{N}(r,\tfrac{1}{F^{(k)}}) + \overline{N}\left(r,\tfrac{1}{F^{(k)}-p(z)}\right) \\ &+(\epsilon+O(1))T(r,F) \leq \overline{N}(r,f) + \overline{N}(r,\tfrac{1}{F^{(k)}}) \\ &+\overline{N}\left(r,\tfrac{1}{F^{(k)}-p(z)}\right) + (\epsilon+O(1))T(r,F). \end{aligned}$$

Therefore $T(r,F^{(k)})-\overline{N}(r,\frac{1}{F^{(k)}}) \le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-p(z)}\right)+(\epsilon+O(1))T(r,F)$.

Using Lemma 2.2.3 we get

$T(r,F) \le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-p(z)}\right)+N_{k+1}(r,\frac{1}{F})+(\epsilon+O(1))T(r,F)$

$$\Rightarrow (n+m)T(r,f) \le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{G^{(k)}-p(z)}\right)$$

$$+(\epsilon+O(1))T(r,F) \le \overline{N}(r,f)+(k+1)(n+m)T(r,g)$$

$$+l\log r+(\epsilon+O(1))T(r,f)$$

$$\Rightarrow (n+m-l-1)T(r,f) \le (k+1)(n+m)T(r,g)$$

$$+(\epsilon+O(1))T(r,f).$$

Since $\epsilon<1$ then $T(r,f)=O(T(r,g))$.

Similarly, $T(r,g)=O(T(r,f))$ and hence

$$\sigma(f)=\sigma(g). \tag{2.2.11}$$

Then $f=\frac{e^{h(z)}}{r(z)}$, $g=\frac{e^{h_1(z)}}{q(z)}$ where $r(z)$ and $q(z)$ are polynomials with degree $deg(r(z))=r$, $deg(q(z))=q$, while $h(z)$ and $h_1(z)$ are non-constant entire functions.

By Lemma 2.2.2, $h(z)$ and $h_1(z)$ are polynomial with $deg(h(z))=deg(h_1(z))=h=\sigma(f)$.

Then we have $(f(z)^{n+m})^{(k)}=\frac{(m+n)e^{(n+m)h(z)}}{r^{n+m+k}(z)}R_k(z)$

and $(g(z)^{n+m})^{(k)}=\frac{(m+n)e^{(n+m)h_1(z)}}{q^{n+m+k}(z)}Q_k(z)$, where $R_k(z)$ and $Q_k(z)$ are polynomials.

From (2.2.6) we get $h(z)+h_1(z)=C$, where C is a constant. Furthermore, we have

$deg(R_k(z))+deg(Q_k(z))=deg(r^{n+m+k}(z))+deg(q^{n+m+k}(z))+2l$

$\Rightarrow k(r+h-1)+k(q+h-1)=q(n+m+k)+r(n+m+k)+2l$

$$\Rightarrow 2k(h-1)=(n+m)(q+r)+2l. \tag{2.2.12}$$

If $N(r,f)+N(r,g)\neq 0$ then $(q+r)\ge 1$.

From (2.2.12) we obtain $2k(h-1)\ge(n+m)+2l \Rightarrow n\le 2k(h-$

$1) - (m + 2l)$, which contradicts the given hypothesis.
Therefore $N(r, f) + N(r, g) = 0$, showing that both f and g are entire functions and so $r = q = 0$. From (2.2.12) we get $h = l+1$, $k = 1$ or $h = 2$, $k = l$.
Case 1: For $k = 1$, $h = l + 1$. We get $h^{(1)}(z) = bp(z)$, $h_1^{(1)}(z) = -bp(z)$, where $b \neq 0$ is a constant. This implies that $h(z) = bQ(z) + d_1$ and $h_1(z) = -bQ(z) + d_2$, where $Q(z)$ is a polynomial without constant term such that $Q^{(1)}(z) = p(z)$ and d_1, d_2 are constants. Therefore $f = b_1 e^{bQ(z)}$, $g = b_2 e^{-bQ(z)}$, where b_1, b_2 are constants satisfying the condition ${a_m}^2 (b_1 b_2)^{n+m} ((n+m)b)^2 = -1$.
Case 2: $h = 2$, $k = l$. This contradicts the hypothesis of the lemma.
The case $P(w) = c_0$ can be proved similarly. This completes the proof of the Lemma. ■

Lemma 2.2.8 {*p. 82 [86]*} *Let f_1 and f_2 be two non-constant meromorphic functions. If $c_1 f_1 + c_2 f_2 = c_3$, where c_1, c_2, c_3 are non-zero constants, then*

$$T(r, f_1) \leq \overline{N}(r, f_1) + \overline{N}(r, \frac{1}{f_1}) + \overline{N}(r, \frac{1}{f_2}) + S(r, f_1).$$

Lemma 2.2.9 *Let f and g be two non-constant meromorphic functions having zeros and poles of multiplicity at least s. Let k, m, n be three integers with $n > \frac{2k+1}{s} + m$ and let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ or $P(w) \equiv c_0$ $(\neq 0)$, where $a_0 \neq 0, a_1, \cdots, a_m \neq 0$ are complex constants. If $(f^n P(f))^{(k)} = (g^n P(g))^{(k)}$, then $f^n P(f) = g^n P(g)$.*

Proof. From the assumption, we get $f^n P(f) = g^n P(g) + r(z)$ where $r(z)$ is a polynomial of degree at most $k - 1$.

If $r(z) \not\equiv 0$ then by Lemma 2.2.8 we have

$$T(r, \frac{f^nP(f)}{r(z)}) \le \overline{N}(r, \frac{f^nP(f)}{r(z)}) + \overline{N}(r, \frac{r(z)}{f^nP(f)}) + \overline{N}(r, \frac{r(z)}{g^nP(g)}) + S(r,f) + S(r,g).$$

Therefore, we have

$$\begin{aligned} T(r, f^nP(f)) &\le T(r, \frac{f^nP(f)}{r(z)}) + (k-1)\log r + O(1) \\ &\le \overline{N}(r, \frac{f^nP(f)}{r(z)}) + \overline{N}(r, \frac{r(z)}{f^nP(f)}) + \overline{N}(r, \frac{r(z)}{g^nP(g)}) \\ &+ (k-1)\log r + S(r,f) + S(r,g) \le \overline{N}(r,f) \\ &+ \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) \\ &+ 2(k-1)\log r + S(r,f) + S(r,g). \end{aligned}$$

By Lemma 2.2.1 and $T(r,f) \ge s\log r +$ O(1), we have

$$(n+m)T(r,f) \le \left(m + \frac{2}{s} + \frac{2(k-1)}{s}\right) T(r,f) + (m + \frac{1}{s})T(r,g) + S(r,f) + S(r,g).$$

Similarly,

$$(n+m)T(r,g) \le \left(m + \frac{2}{s} + \frac{2(k-1)}{s}\right) T(r,g) + (m + \frac{1}{s})T(r,f) + S(r,f) + S(r,g).$$

By combining above two inequalities we get

$$\left\{(n+m) - (2m + \frac{2k+1}{s})\right\}(T(r,f) + T(r,g)) \le S(r,f) + S(r,g),$$

which is a contradiction. Hence $r(z) \equiv 0$ and so, $f^nP(f) = g^nP(g)$. ■

Lemma 2.2.10 *[29] Let f and g be two non-constant meromorphic functions, a be a finite non-zero constant. If f and g share a CM, then one of the following cases holds:*

(i) $T(r,f) \le N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r,f) + N_2(r,g) + S(r,f) +$

$S(r,g)$; same inequality holds for $T(r,g)$;
(ii) $fg = a^2$;
(iii) $f = g$.

Lemma 2.2.11 *[5] Let f and g be two non-constant meromorphic functions. If f and g share 1 IM and $H \not\equiv 0$, then*

$$\begin{aligned} T(r,f) &\leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + 2\overline{N}(r,\frac{1}{f}) \\ &+ 2\overline{N}(r,f) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned}$$

Lemma 2.2.12 *[68] Let f and g be two non-constant meromorphic functions, and n (≥ 1), k (≥ 1), m (≥ 0) be three integers. Suppose that $F_1 = \frac{(f^nP(f))^{(k)}}{p(z)}$ and $G_1 = \frac{(g^nP(g))^{(k)}}{p(z)}$. If there exists two non-zero constants c_1 and c_2 such that $\overline{N}(r, \frac{1}{F-c_1}) = \overline{N}(r, \frac{1}{G_1})$ and $\overline{N}(r, \frac{1}{G_1-c_2}) = \overline{N}(r, \frac{1}{F_1})$, then $n \leq 3k + m^* + 3$, where*

$$m^* = \begin{cases} m, & \text{if } P(f) \neq c_0. \\ 0, & \text{if } P(f) = c_0. \end{cases}$$

2.3 Theorems

In this section we present the main theorems of the chapter.

Theorem 2.3.1 *Let f and g be two transcendental meromorphic functions with $\sigma(f) < \infty$, whose zeros and poles are of multiplicity at least s, where s is positive integer. Let $p(z)$ be a polynomial of degree l and $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0(\neq 0), a_1, \cdots, a_m(\neq 0)$ are constants, n (≥ 1), k (≥ 1), m (≥ 0) be three integers satisfying $k > l$ when $l \geq 2$ and*

$$n > max\left\{2k + 2l, \frac{3k+8}{s} + m, 2k(\sigma(f) - 1) - (m + 2l)\right\}.$$

If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share $p(z)$ CM, then one of the following holds:

(i) If $P(w)$ is not a monomial and (2.2.2) hold, then either $f = tg$ for some t such that $t^d = 1$, where $d = \gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$; or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (2.1.1).

(ii) If $P(w) = a_m w^m$, or $P(w) = c_0$, then either $f = tg$ for a constant t such that $t^{n+m^*} = 1$, or $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant term such that $Q^{(1)}(z) = p(z)$ and b_1, b_2, b are three constants satisfying ${a_m}^2 (b_1 b_2)^{n+m}((n+m)b)^2 = -1$ or ${c_0}^2 (b_1 b_2)^n (nb)^2 = -1$, where m^* is same as in Lemma 2.2.12.

Proof. We discuss the following cases.
Case (i): Suppose $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$, where $a_0 (\neq 0), a_1, \cdots, a_m (\neq 0)$ are constants, is not a monomial. Let $F = (f^n P(f))^{(k)}$, $G = (g^n P(g))^{(k)}$ and $F^* = f^n P(f)$, $G^* = g^n P(g)$ and $F_1 = \frac{F}{p(z)}$, $G_1 = \frac{G}{p(z)}$. Since F_1, G_1 share 1 CM, by Lemma 2.2.10 one of the following subcases holds:
(a) $T(r, F_1) \leq N_2(r, \frac{1}{F_1}) + N_2(r, \frac{1}{G_1}) + N_2(r, F_1) + N_2(r, G_1) + S(r, F_1) + S(r, G_1)$; same inequality holds for $T(r, G_1)$;
(b) $F_1 G_1 = 1$;
(c) $F_1 = G_1$.
Subcase (a): We have

$$\begin{aligned} T(r, F) \leq & N_2(r, \tfrac{1}{F}) + N_2(r, \tfrac{1}{G}) + N_2(r, F) \\ & + N_2(r, G) + S(r, F) + S(r, G). \end{aligned} \quad (2.3.1)$$

By Lemma 2.2.3 with $s = 2$, we obtain

$$\begin{aligned} T(r, F^*) \leq & T(r, F) - N_2(r, \tfrac{1}{F}) \\ & + N_{k+2}(r, \tfrac{1}{F^*}) + S(r, F), \end{aligned} \quad (2.3.2)$$

and

$$N_2(r,\tfrac{1}{G}) \leq N_{k+2}(r,\tfrac{1}{G^*}) + k\overline{N}(r,G) + S(r,G). \qquad (2.3.3)$$

Using (2.3.1) and (2.3.3) in (2.3.2) we get

$$\begin{aligned} T(r,F^*) &\leq N_{k+2}(r,\tfrac{1}{F^*}) + N_{k+2}(r,\tfrac{1}{G^*}) + (k+2)\overline{N}(r,g) \\ &+2\overline{N}(r,f) + S(r,F) + S(r,G) \leq (k+2)\overline{N}(r,\tfrac{1}{f}) \\ &+N\left(r,\tfrac{1}{P(f)}\right) + (k+2)\overline{N}(r,\tfrac{1}{g}) + N\left(r,\tfrac{1}{P(g)}\right) \\ &+(k+2)\overline{N}(r,g) + 2\overline{N}(r,f) + S(r,f) + S(r,g) \\ &\leq \tfrac{k+2}{s}N(r,\tfrac{1}{f}) + N\left(r,\tfrac{1}{P(f)}\right) + \tfrac{k+2}{s}N(r,\tfrac{1}{g}) + N\left(r,\tfrac{1}{P(g)}\right) \\ &+\tfrac{k+2}{s}N(r,g) + \tfrac{2}{s}N(r,f) + S(r,f) + S(r,g). \end{aligned}$$

Now by Nevanlinna first fundamental theorem and Lemma 2.2.1 we get

$$\begin{aligned} (n+m)T(r,f) &\leq \left(\tfrac{2k+4}{s}+m\right)T(r,g) + \left(\tfrac{k+4}{s}+m\right)T(r,f) \\ &+S(r,f)+S(r,g). \end{aligned}$$

Similarly,

$$\begin{aligned} (n+m)T(r,g) &\leq \left(\tfrac{2k+4}{s}+m\right)T(r,f) + \left(\tfrac{k+4}{s}+m\right)T(r,g) \\ &+S(r,f)+S(r,g). \end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned} (n+m)\{T(r,f)+T(r,g)\} &\leq \left(\tfrac{3k+8}{s}+2m\right)\{T(r,f)+T(r,g)\} \\ &+S(r,f)+S(r,g), \end{aligned}$$

$$\Rightarrow \left\{n - \left(\frac{3k+8}{s}+m\right)\right\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction for $n > \left(\frac{3k+8}{s}+m\right)$.

Subcase (b): In this case we get a contradiction by Lemma 2.2.6.

Subcase (c): In this case we have

$$(f^nP(f))^{(k)} = (g^nP(g))^{(k)}.$$

Here $n > \left(\frac{3k+8}{s}+m\right) > (\frac{2k+1}{s}+m)$. So by Lemma 2.2.9 we get $f^nP(f) = g^nP(g)$,

$$i.e., f^n(a_mf^m+...+a_1f+a_0) = g^n(a_mg^m+...+a_1g+a_0). \quad (2.3.4)$$

Let $h = \frac{f}{g}$. If h is a constant, putting $f = gh$ in (2.3.4) we get

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+......+a_0g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = \gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$.
Thus $f = tg$ for some t such that $t^d = 1$, where $d = \gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$.
If h is not a constant then from (2.3.4) we see f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (2.1.1)
Case (ii): When $P(w) = a_mw^m$ or $P(w) = c_0$, where $a_m,\ c_0$ are non-zero complex constants. Proceeding as in Case (i) we obtain $F_1G_1 = 1$ or $F_1 = G_1$.
If $F_1G_1 = 1$ then Lemma 2.2.7 gives $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q^{(1)}(z) = p(z)$ and b_1, b_2, b are three constants satisfying $a_m{}^2(b_1b_2)^{n+m}((n+m)b)^2 = -1$ or $c_0{}^2(b_1b_2)^n(nb)^2 = -1$.
If $F_1 = G_1$ we obtain $f = tg$ for a constant t such that $t^{n+m^*} = 1$.
■

Following is the supportive example of Theorem 2.3.1 when $P(w)$ is not a monomial.

Example 2.3.1 *Let* $P(w) = a_4w^4 + a_0$*, where* $a_0, a_4 \in \mathbb{C}\setminus\{0\}$ *and* $p(z) = z^3 - 3z^2 + z - 1$*. Let* $f(z) = \frac{(e^z-a)^2}{(e^z-b)^3}$ *and* $g(z) = -i\frac{(e^z-a)^2}{(e^z-b)^3}$*, where* $a, b \in \mathbb{C}\setminus\{0\}$ *with* $a \neq b$*. Let* $k = 4, n = 16$*. Clearly* $s = 2$*. Now*

$$f^{16}P(f) = \frac{(e^z-a)^{32}}{(e^z-b)^{60}}\left\{a_4\,(e^z-a)^8 + a_0\,(e^z-b)^{12}\right\}$$

and

$$g^{16}P(g) = \frac{(e^z-a)^{32}}{(e^z-b)^{60}}\left\{a_4\,(e^z-a)^8 + a_0\,(e^z-b)^{12}\right\}.$$

Thus we see that f *and* g *are two non-constant meromorphic functions having zeros and poles of multiplicity at least* 2*, and* $[f^{16}P(f)]^{(4)}$ *and* $[g^{16}P(g)]^{(4)}$ *share the polynomial* $p(z)$ *CM with*

$$n > \max\left\{2k + 2l,\ \frac{3k+8}{s} + m^*,\ 2k(\sigma(f) - 1) - (m + 2l)\right\}.$$

We thus see that one of the conclusion $f \equiv tg$ *of Theorem 1.1.5 holds good where* $t^d = (-i)^{gcd(20,16)} = (-i)^4 = 1$*.*

The next is the supportive example of Theorem 2.3.1 when $P(w)$ is a monomial.

Example 2.3.2 *Let* $f(z) = \tan z$*, and* $g(z) = -\tan z$*,* $p(z) = a_2z^2 + a_0$*, where* $a_0, a_2 \in \mathbb{C}\setminus\{0\}$*,* $P(w) = w^2$*. Let* $n = 18$*. Clearly* $s = 1, l = 2, m^* = 2$ *and*

$$n > \max\left\{2k + 2l,\ \frac{3k+8}{s} + m^*,\ 2k(\sigma(f) - 1) - (m + 2l)\right\}.$$

We also see that $[f^{18}P(f)]^{(2)} = [f^{20}]^{(2)} = [\tan^{20} z]^{(2)}$ *and* $[g^{18}P(g)]^{(2)} = [g^{20}]^{(2)} = [\tan^{20} z]^{(2)}$ *share the polynomial* $p(z)$ *CM, and one of the conclusion* $f \equiv tg$ *of Theorem 1.1.5 holds good where* $t^d = (-1)^{gcd(20,18)} = (-1)^2 = 1$*.*

The following is the supportive example of Theorem 2.3.1 when $P(w) = c_0$.

Example 2.3.3 *Let*

$$f(z) = \frac{2+3i}{1-5i} e^{3z^3+2z^2-z+6} \text{ and } g(z) = \frac{1-5i}{2+3i} e^{-(3z^3+2z^2-z+6)}.$$

Let $n = 13$, $P(w) = c_0 = \dfrac{i}{13}$ *and* $k = 1$. *Here we see that* $b_1 = \dfrac{2+3i}{1-5i}$ *and* $b_2 = \dfrac{1-5i}{2+3i}$ *and* $b = 1$. *It is clear that*

$$c_0^2 (b_1 b_2)^n (nb)^2 = -1.$$

Let $Q(z) = 3z^3 + 2z^2 - z + 6$ *and* $p(z) = 9z^2 + 4z - 1$. *Clearly,* $Q'(z) = p(z)$. *Clearly*

$$n > \max\left\{2k + 2l,\ \frac{3k+8}{s} + m^*,\ 2k(\sigma(f) - 1) - (m + 2l)\right\}$$

is satisfied. We see that

$$\begin{aligned}
\left(f^{13}P(f)\right)^{(1)} &= \frac{i}{13}\left(\frac{2+3i}{1-5i}\right)^{13}\left(e^{13(3z^3+2z^2-z+6)}\right)^{(1)} \\
&= \frac{i}{13}\left(\frac{2+3i}{1-5i}\right)^{13} \cdot 13(9z^2+4z-1) \cdot e^{13(3z^3+2z^2-z+6)} \\
&= i\left(\frac{2+3i}{1-5i}\right)^{13} \cdot p(z) e^{13(3z^3+2z^2-z+6)}.
\end{aligned}$$

Similarly,

$$\left(g^{13}P(g)\right)^{(1)} = -i\left(\frac{1-5i}{2+3i}\right)^{13} \cdot p(z) e^{-13(3z^3+2z^2-z+6)}.$$

Clearly $\left(f^{13}P(f)\right)^{(1)}$ *and* $\left(g^{13}P(g)\right)^{(1)}$ *share the polynomial* $p(z)$ *CM. We see that* f *and* g *are of the forms* $f(z) = b_1 e^{bQ(z)}$ *and* $g(z) = b_2 e^{-bQ(z)}$ *with* $c_0^2 (b_1 b_2)^n (nb)^2 = -1$.

Theorem 2.3.2 *Let* f *and* g *be two transcendental meromorphic functions with* $\sigma(f) < \infty$, *whose zeros and poles are of multiplicity at least* s, *where* s *is a positive integer. Let* $p(z)$ *be a polynomial of degree* l *and* $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ *be* *a non-zero polynomial and* n (≥ 1), k (≥ 1), m (≥ 0) *be three integers satisfying* $k > l$ *when* $l \geq 2$ *and*

$$n > max\left\{2k + 2l, \frac{9k + 14}{s} + 4m, 2k(\sigma(f) - 1) - (m + 2l)\right\}.$$

If $(f^n P(f))^{(k)}$ *and* $(g^n P(g))^{(k)}$ *share* $p(z)$ *IM, then the conclusions of Theorem 2.3.1 hold.*

Proof. Case (i): Suppose $P(w)$ is not a monomial. Let $F = (f^n P(f))^{(k)}$, $G = (g^n P(g))^{(k)}$ and $F^* = f^n P(f)$, $G^* = g^n P(g)$ and $F_1 = \frac{F}{p(z)}$, $G_1 = \frac{G}{p(z)}$. Then F_1, G_1 share 1 IM.
We assume that $H \not\equiv 0$. So, by Lemma 2.2.11 we have

$$\begin{aligned} T(r, F_1) \leq\ & N_2(r, \tfrac{1}{F_1}) + N_2(r, \tfrac{1}{G_1}) + N_2(r, F_1) + N_2(r, G_1) \\ & +2\overline{N}(r, \tfrac{1}{F_1}) + 2\overline{N}(r, F_1) + \overline{N}(r, \tfrac{1}{G_1}) \\ & +\overline{N}(r, G_1) + S(r, f) + S(r, g). \end{aligned}$$

$$\begin{aligned} i;e., \quad T(r, F) \leq\ & N_2(r, \tfrac{1}{F}) + N_2(r, \tfrac{1}{G}) + N_2(r, F) + N_2(r, G) \\ & +2\overline{N}(r, \tfrac{1}{F}) + \overline{N}(r, \tfrac{1}{G}) + 2\overline{N}(r, F) \\ & +\overline{N}(r, G) + S(r, f) + S(r, g). \end{aligned} \tag{2.3.5}$$

Now by Lemma 2.2.3 with $s = 2$, we get

$$T(r, F^*) \leq T(r, F) - N_2(r, \frac{1}{F}) + N_{2+k}(r, \frac{1}{F^*}) + S(r, f). \tag{2.3.6}$$

and

$$N_2(r, \frac{1}{G}) \leq k\overline{N}(r, G) + N_{2+k}(r, \frac{1}{G^*}) + S(r, g). \tag{2.3.7}$$

Using (2.3.5) and (2.3.7) in (2.3.6) we have

$$\begin{aligned} T(r,F^*) &\le N_{2+k}(r,\frac{1}{F^*}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) \\ &+ 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,F) + \overline{N}(r,G) \\ &+ S(r,f) + S(r,g) \\ &\le N_{2+k}(r,\frac{1}{F^*}) + N_{2+k}(r,\frac{1}{G^*}) + k\overline{N}(r,G) + N_2(r,F) \\ &+ N_2(r,G) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,F) \\ &+ \overline{N}(r,G) + S(r,f) + S(r,g) \\ &\le (3k+4)\overline{N}(r,\frac{1}{f}) + 3N(r,\frac{1}{P(f)}) + (2k+3)\overline{N}(r,\frac{1}{g}) \\ &+ 2N(r,\frac{1}{P(g)}) + (2k+3)\overline{N}(r,g) + (2k+4)\overline{N}(r,f) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

By using Lemma 2.2.1 and Nevanlinna first fundamental theorem , we get

$$\begin{aligned} (n+m)T(r,f) \le \tfrac{3k+4}{s}T(r,f) + 3mT(r,f) + \tfrac{2k+3}{s}T(r,g) \\ +2mT(r,g) + \tfrac{2k+3}{s}T(r,g) + \tfrac{2k+4}{s}T(r,f) + S(r,f) \\ +S(r,g) \le \left(\tfrac{5k+8}{s} + 3m\right)T(r,f) + \left(\tfrac{4k+6}{s} + 2m\right)T(r,g) \\ +S(r,f) + S(r,g). \end{aligned} \quad (2.3.8)$$

Similarly,

$$\begin{aligned} (n+m)T(r,g) \le \left(\tfrac{5k+8}{s} + 3m\right)T(r,g) + \left(\tfrac{4k+6}{s} + 2m\right) \\ T(r,f) + S(r,f) + S(r,g). \end{aligned} \quad (2.3.9)$$

Combining (2.3.8) and (2.3.9) we get

$$\left\{n - (\frac{9k+14}{s} + 4m)\right\}(T(r,f) + T(r,g)) \le S(r,f) + S(r,g),$$

which is a contradiction as
$n > \max\left\{2k+2l, \frac{9k+14}{s} + 4m, 2k(\sigma(f)-1) - (m+2l)\right\}$.

Therefore $H \equiv 0$. Integrating twice we get

$$\frac{1}{F_1 - 1} = \frac{A}{G_1 - 1} + B, \tag{2.3.10}$$

where $A \neq 0$, B are constants. We now discuss the following three subcases:

Subcase (i): Let $B \neq 0$ and $A = B$. Then from (2.3.10) we get

$$\Rightarrow \frac{1}{F_1 - 1} = \frac{BG_1}{G_1 - 1}. \tag{2.3.11}$$

If $B = -1$ then from above equation we get

$$F_1 G_1 = 1$$

i;e.,

$$(f^n P(f))^{(k)} . (g^n P(g))^{(k)} = p^2(z),$$

a contradiction by Lemma 2.2.6.

If $B \neq -1$, then from (2.3.11) we have $\frac{1}{F_1} = \frac{BG_1}{(1+B)G_1 - 1}$ and so $\overline{N}(r, \frac{1}{G_1 - \frac{1}{1+B}}) = \overline{N}(r, \frac{1}{F_1})$.

Now by Nevanlinna second fundamental theorem we get

$$\begin{aligned} T(r, G_1) &\leq \overline{N}(r, \frac{1}{G_1}) + \overline{N}(r, \frac{1}{G_1 - \frac{1}{1+B}}) + \overline{N}(r, G_1) + S(r, G_1) \\ &\leq \overline{N}(r, \frac{1}{G_1}) + \overline{N}(r, \frac{1}{F_1}) + \overline{N}(r, G_1) + S(r, G_1). \end{aligned}$$

By Lemma 2.2.3 we obtain

$$\begin{gathered} T(r, G) \leq N_{k+1}(r, \tfrac{1}{f^n P(f)}) + k\overline{N}(r, f) + T(r, G) \\ +N_{k+1}(r, \tfrac{1}{g^n P(g)}) - (n + m)T(r, g) + \overline{N}(r, g) + S(r, g). \end{gathered}$$

Therefore,

$$\begin{gathered} (n + m)T(r, g) \leq (k + 1)\overline{N}(r, \tfrac{1}{f}) + N(r, \tfrac{1}{P(f)}) + k\overline{N}(r, f) \\ +(k + 1)\overline{N}(r, \tfrac{1}{g}) + N(r, \tfrac{1}{P(g)}) + \overline{N}(r, g) \\ +S(r, f) + S(r, g). \end{gathered}$$

$$\Rightarrow (n+m)T(r,g) \leq \left(\tfrac{2k+1}{s}+m\right)T(r,f)+\left(\tfrac{k+2}{s}+m\right)T(r,g)$$
$$+S(r,f)+S(r,g).$$

Similarly,

$$(n+m)T(r,f) \leq \left(\tfrac{2k+1}{s}+m\right)T(r,g)+\left(\tfrac{k+2}{s}+m\right)T(r,f)$$
$$+S(r,f)+S(r,g).$$

Combining above inequalities, we get

$$\left(n-(\frac{3k+3}{s}+m)\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction as
$n > \max\left\{2k+2l, \frac{9k+14}{s}+4m, 2k(\sigma(f)-1)-(m+2l)\right\}$.

Subcase (ii): Let $B \neq 0$ and $A \neq B$. Then from (2.3.10) we get $F_1 = \frac{(B+1)G_1-(B-A+1)}{BG_1+(A-B)}$ and so $\overline{N}(r, \frac{1}{G_1-\frac{B-A+1}{B+1}}) = \overline{N}(r, \frac{1}{F_1})$. Proceeding as in subcase (i) we get a contradiction.

Subcase (iii): Let $B = 0$ and $A \neq 0$. Then (2.3.10) gives $F_1 = \frac{G_1+A-1}{A}$ and $G_1 = AF_1-(A-1)$.
If $A \neq 1$, then we have $\overline{N}(r, \frac{1}{F_1-\frac{A-1}{A}}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G_1-(1-A)}) = \overline{N}(r, \frac{1}{F_1})$. Using the Lemma 2.2.12 we have $n \leq 3k+m+3$, a contradiction. Thus $A=1$ and hence $F_1 = G_1$

$$\Rightarrow \left(f^nP(f)\right)^{(k)} = \left(g^nP(g)\right)^{(k)}.$$

Here $n > \left(\frac{9k+14}{s}+4m\right) > \left(\frac{2k+1}{s}+m\right)$. So by Lemma 2.2.9 we get

$$f^nP(f) = g^nP(g),$$

$$i.e., f^n(a_mf^m+...+a_1f+a_0) = g^n(a_mg^m+...+a_1g+a_0). \quad (2.3.12)$$

Let $h = \frac{f}{g}$. If h is a constant, putting $f = gh$ in (2.3.12) we get

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+......+a_0g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = \gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$. Thus $f = tg$ for some t such that $t^d = 1$, where $d = \gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$.

If h is not a constant then from (2.3.12) we see that f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (2.1.1).

Case (ii): When $P(w) = a_m w^m$, or $P(w) = c_0$, where a_m, c_0 are non-zero complex constants. Proceeding as in Case (i) above we obtain $F_1 G_1 = 1$ or $F_1 = G_1$. If $F_1 G_1 = 1$ then Lemma 2.2.7 gives $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant term such that $Q^{(1)}(z) = p(z)$ and b_1, b_2, b are three constants satisfying ${a_m}^2 (b_1 b_2)^{n+m}((n+m)b)^2 = -1$ or ${c_0}^2 (b_1 b_2)^n (nb)^2 = -1$. If $F_1 = G_1$ we obtain $f = tg$ for a constant t such that $t^{n+m^*} = 1$ ■

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