CHAPTER

1 INTRODUCTION

The value distribution theory of entire and meromorphic functions which was developed by famous mathematician Rolf Nevanlinna is the back-bone of the thesis. The thesis mainly focuses on some uniqueness problems of entire and meromorphic functions and their derivatives.

In this chapter we give some definitions, notations and some basic as well as significant results of the value distribution theory.

Definition 1.1.1 *A meromorphic function in a domain $D \subseteq \mathbb{C}$ is a single valued function of one complex variable which is analytic in D except possibly for poles.*

Examples: $\frac{1}{z-1}$, $\frac{e^z}{z^2}$, $\cot z$ etc.

Definition 1.1.2 *A meromorphic function which has an essential singularity at the point at infinity is known as a transcendental meromorphic function.*

Examples: e^z, $\frac{e^z}{z}$ etc.

Definition 1.1.3 *A meromorphic function having no poles in the open complex plane is called an entire function.*

Examples: $\sin z$, e^z, $z^2 + 5$ etc.

Definition 1.1.4 *An entire function which has an essential singularity at the point at infinity is called a transcendental entire function.*

Examples: $\cos z$, e^z, $e^z \sin z$ etc.

The following theorem plays an important role in Nevanlinna value distribution Theory.

Theorem 1.1.1 ***(The Poisson-Jensen's Formula)***{ *p. 1 [27], p. 1 [79], p. 4 [86]*} *Suppose* $f(\zeta)$ *is meromorphic in* $|\zeta| \leq R(0 < R < \infty)$ *and that* a_μ $(\mu = 1, 2, ..., m)$ *and* b_ν $(\nu = 1, 2, ..., n)$ *are the zeros and poles of* $f(\zeta)$ *in* $|\zeta| \leq R$ *respectively. If* $z = re^{i\theta}$ *is a point in* $|\zeta| < R$*, distinct from* a_μ *and* b_ν*, then*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}d\varphi$$

$$+\sum_{\mu=1}^{m} log|\frac{R(z - a_\mu)}{R^2 - \overline{a}_\mu z}| - \sum_{\nu=1}^{n} log|\frac{R(z - b_\nu)}{R^2 - \overline{b}_\nu z}|.$$

Under the assumptions of the above theorem we obtain the following important special cases.

(i) If $f(\zeta)$ has neither zeros nor poles on $|\zeta| \leq R$, then for every point z with $|z| = r < R$, we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}d\varphi,$$

which is the so-called **Poisson's formula**.

(ii) If $f(0) \neq 0, \infty$, then we have **Jensen's formula** as follows

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|d\varphi - \sum_{\mu=1}^{m} \log\frac{R}{|a_\mu|} + \sum_{\nu=1}^{n} \log\frac{R}{|b_\nu|}.$$

Definition 1.1.5 {*p. 4 [27], p. 6 [79], p. 2 [86]*} *Let f be a non-constant meromorphic function in the open complex plane and $a \in \mathbb{C} \cup \{\infty\}$. Then*

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f)\log r,$$

is called the integrated counting function of the a-points of f, where $n(r, a; f)$ is the number of a-points of f in $|z| \leq r$ counted according to multiplicities. Sometimes $N(r, a; f)$ is denoted by $N(r, \frac{1}{f-a})$ and for $a = \infty$ by $N(r, f)$ or $N(r, \infty; f)$.

Definition 1.1.6 {*p. 3 [27], p. 5 [79], p. 1 [86]*} *For $x \geq 0$, we define*

$$\log^+ x = \max(\log x, 0) = \begin{cases} \log x, & \text{if } x \geq 1 \\ 0, & \text{if } 0 \leq x < 1. \end{cases}$$

Then

(i) $\log^+ x \geq 0$, for all $x \geq 0$,

(ii) $\log^+ x \geq \log^+ y$, for $x \geq y$,

(iii) $\log x = \log^+ x - \log^+ \frac{1}{x}$, for $x > 0$.

Definition 1.1.7 {*p. 4 [27], p. 6 [79], p. 2 [86]*} *Let f be a non-constant meromorphic function on $|z| \leq R$ $(0 < R < \infty)$. The function*

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\psi})|d\psi, \ 0 < r < R,$$

is called the proximity function of f. Every so often $m(r, f)$ is expressed as $m(r, \infty; f)$. Also for $a \neq \infty$

$$m(r, a; f) = m(r, \frac{1}{f-a}) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\psi}) - a|} d\psi.$$

Definition 1.1.8 {*p. 4 [27], p. 7 [79], p. 2 [86]*} *The Nevanlinna characteristic function of a meromorphic function f is*

$$T(r,f) = m(r,f) + N(r,f).$$

Clearly $T(r,f)$ is non-negative. The quantity $T(r,f)$ plays an important role in the Nevanlinna's value distribution theory of meromorphic functions.

Using Nevanlinna characteristic function Jensen's formula can be written as

$$T(r,f) = T(r,\frac{1}{f}) + \log|f(0)|,$$

where $f(0) \neq 0,\ \infty$. This shows that the characteristic functions of f and $\frac{1}{f}$ differ only by a constant.

In the following we list some properties of $m(r,f)$, $N(r,f)$ and $T(r,f)$.

Theorem 1.1.2 {*p. 5 [27], p. 7 [79], p. 3 [86]*} *If $f_\nu(z)$ $(\nu = 1, 2, ..., p)$ are meromorphic functions in $|z| \leq R$ $(0 < R < \infty)$, then*

1. $m\big(r, \prod_{\nu=1}^{p} f_\nu\big) \leq \sum_{\nu=1}^{p} m(r, f_\nu)$

2. $m\big(r, \sum_{\nu=1}^{p} f_\nu\big) \leq \sum_{\nu=1}^{p} m(r, f_\nu) + \log p$,

3. $N\big(r, \prod_{\nu=1}^{p} f_\nu\big) \leq \sum_{\nu=1}^{p} n(r, f_\nu)$,

4. $N\big(r, \sum_{\nu=1}^{p} f_\nu\big) \leq \sum_{\nu=1}^{p} n(r, f_\nu)$,

5. $T\big(r, \prod_{\nu=1}^{p} f_\nu\big) \leq \sum_{\nu=1}^{p} T(r, f_\nu)$,

6. $T\big(r, \sum_{\nu=1}^{p} f_\nu\big) \le \sum_{\nu=1}^{p} T(r, f_\nu) + \log p.$

Definition 1.1.9 *[14] Let* $[a, b] \subseteq \mathbb{R}$*. A function* $f : [a, b] \to \mathbb{R}$ *is said to be convex if for any two points* x_1 *and* x_2 *in* $[a, b]$

$$f(tx_2 + (1 - t)x_1) \le tf(x_2) + (1 - t)f(x_1),$$

whenever $0 \le t \le 1$.

In 1929 H. Cartan [13] gave the alternative expression of $T(r, f)$ as follows.

Theorem 1.1.3 $\{$*p.9 [86]*$\}$ *Let* f *be meromorphic in* $|z| < R$ $(0 < R \le \infty)$ *such that* $f(0) \ne \infty$*. Then for* $0 < r < R$,

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N\big(r, \frac{1}{f - e^{i\theta}}\big) d\theta + \log^+ |f(0)|.$$

This identity is known as **Cartan's Identity**.

From the above theorem it follows that

(i) $T(r, f)$ is a non-decreasing function of r and convex function of $\log r$.

We now state the first fundamental theorem of Nevanlinna in the following form.

Theorem 1.1.4 $\{$*p. 5 [27], p. 9 [79], p. 8 [86]*$\}$ **(Nevanlinna's First Fundamental Theorem)** *Let* f *be a non-constant meromorphic function defined in the open complex plane and let* a *be any complex number. Then*

$$m(r, a; f) + N(r, a; f) = T(r, f) + O(1),$$

where $O(1)$ *is a bounded quantity depending on* a*.*
This shows that the sum $m(r, a; f) + N(r, a; f)$ *for different values of* a *maintains a total, given by the quantity* $T(r, f)$ *which is invariant up to a bounded additive term involving* r*.*

Example 1. Suppose $P(z) = a_p z^p + a_{p-1} z^{p-1} + ... + a_0$ and $Q(z) = b_q z^q + b_{q-1} z^{q-1} + ... + b_0$, where $a_p(\neq 0), a_{p-1}, ..., a_0$ and $b_q(\neq 0), b_{q-1}, ..., b_0$ are complex numbers, and p, q are non-negative integers satisfying $p + q \geq 1$ such that $P(z)$ and $Q(z)$ have no common factors. Then $f(z) = \frac{P(z)}{Q(z)}$ is a non-constant rational function.

One can easily prove that

$$m(r, f) = \begin{cases} (p - q) \log r + O(1), & \text{if } p > q \\ O(1), & \text{if } p \leq q \end{cases}$$

and

$$N(r, f) = q \log r$$

holds for sufficiently large r. Thus

$$T(r, f) = \max(p, q) \log r + O(1).$$

Example 2. For the function $f(z) = e^z$, the Nevanlinna characteristic function is given by

$$T(r, f) = \frac{r}{\pi} + O(1).$$

Theorem 1.1.5 {*p. 11 [86]*} *If f is a transcendental meromorphic function in the complex plane, then*

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty.$$

Corollary 1.1.1 {*p. 13 [86]*} *Let f be a non-constant meromorphic function in the complex plane. Then f is a rational function if and only if*

$$\liminf_{r\to\infty} \frac{T(r, f)}{\log r} < \infty.$$

Definition 1.1.10 {*p. 16 [27], p. 12 [79], p. 10 [86]*} *Let $S(r)$ be a non-negative and non-decreasing real function defined* *in* (r_0, ∞)*,* $r_0 \geq 0$*. The order* λ *and lower order* μ *of the function* $S(r)$ *are defined as*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ S(r)}{\log r}, \ \mu = \liminf_{r\to\infty} \frac{\log^+ S(r)}{\log r},$$

respectively. It is clear from the definition that the order λ *and lower order* μ *always satisfy* $0 \leq \mu \leq \lambda \leq \infty$*. If* $\lambda = \infty$*, then* $S(r)$ *is said to be of infinite order.*

Definition 1.1.11 $\{$*p. 17 [27]*$\}$ *For* $0 < \lambda < \infty$*,*
let $L = \limsup_{r\to\infty} \frac{\log^+ S(r)}{r^\lambda}$*. Then we have the following possibilities.*
(i) If $L = 0$*, then* $S(r)$ *is minimal type.*
(ii) If $0 < L < \infty$*, then* $S(r)$ *is of mean type.*
(iii) If $L = \infty$*, then* $S(r)$ *is of maximal type.*
(iv) If $\int_{r_0}^{\infty} \frac{S(t)}{t^{\lambda+1}}dt$ *converges, then* $S(r)$ *is of convergence class.*

Now using the Nevanlinna's characteristic function $T(r, f)$ of f, the order and lower order of f are defined as follows.

Definition 1.1.12 $\{$*p. 12 [79], p. 10 [86]*$\}$ *Let* f *be a meromorphic function in the complex plane. The order* λ *and lower order* μ *of* f *are defined respectively by the order and lower order of* $T(r, f)$*, that is*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \mu = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

The function f is said to have maximal, mean or minimal type if the characteristic function $T(r, f)$ has this property.

Let f be non-constant entire function. Then the maximum modulus function of f on the circle $|z| = r$ is

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|.$$

The following theorem establishes the relationship between $M(r, f)$ and $T(r, f)$.

Theorem 1.1.6 {*p. 18 [27], p. 10 [79], p. 10 [86]*} *If f is a non-constant entire function, then*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f),$$

where $0 \leq r < R < \infty$.

The next theorem shows that the maximum modulus function $M(r, f)$ and the characteristic function $T(r, f)$, for an entire function f, have the same order.

Theorem 1.1.7 {*p. 18 [27]*} *If f is an entire function, then the functions $S_1(r) = \log^+ M(r, f)$ and $S_2(r) = T(r, f)$ have the same order. Also if λ $(0 < \lambda < \infty)$ is the common order, then $S_1(r)$ and $S_2(r)$ are together of minimal type, mean type, maximal type or of convergence class.*

Now we state the Nevalinna's second fundamental theorem, which is frequently used throughout the thesis.

Theorem 1.1.8 {*p. 15 [86]*} *Suppose that f is a non-constant meromorphic function in $|z| < R$ $(0 < R \leq \infty)$ and a_j $(j = 1, 2, ..., q)$ are q (≥ 2) distinct finite complex numbers. Then for* $0 < r < R$ $(0 < R \leq \infty)$, *we have*

$$m(r, f) + \sum_{j=1}^{q} m\big(r, \frac{1}{f - a_j}\big) \leq 2T(r, f) - N_1(r) + S(r, f),$$

where

$$N_1(r) = 2N(r, f) - N(r, f^{(1)}) + N(r, \frac{1}{f^{(1)}}),$$

and

$$S(r, f) = m\big(r, \frac{f^{(1)}}{f}\big) + m\big(r, \sum_{j=1}^{q} \frac{f^{(1)}}{f - a_j}\big) + O(1).$$

A meromorphic function g is said to be a logarithmic derivative if $g = \frac{f^{(1)}}{f}$ for some meromorphic function f. The following lemmas of logarithmic derivative estimate the quantity $S(r, f)$ in Theorem 1.1.8.

Lemma 1.1.1 {*p. 36 [27], p. 17 [79], p. 16 [86]*} *Suppose f is a non-constant meromorphic function in the whole complex plane such that* 0 *is neither a zero nor a pole of f. Then*

$$m\big(r, \tfrac{f^{(1)}}{f}\big) < 4\log^+ T(R, f) + 3\log^+ \tfrac{1}{R-r} + 4\log^+ R$$
$$+2\log^+ \tfrac{1}{r} + 4\log^+ \log^+ \tfrac{1}{|f(0)|} + 10$$

holds for $0 < r < R < \infty$.

Lemma 1.1.2 {*p. 21 [86]*} *Let f be a non-constant meromorphic function in the complex plane. Then*
(i) $m\big(r, \frac{f^{(1)}}{f}\big) = O(\log r)$ *as* $r \to \infty$*, if the order of f is finite.*
(ii) $m(r, \frac{f^{(1)}}{f}) = O(\log(rT(r, f)))$ *as* $r \to \infty$*, outside a set E of measure not greater than 2, if the order of f is infinity.*

From the above lemmas we get the following theorem which gives a sharp estimation of error term $S(r, f)$.

Theorem 1.1.9 {*p. 22 [86]*} *Suppose f be a non-constant meromorphic function in the complex plane and $S(r, f)$ be defined as in Theorem 1.1.8. Then we have*
(i) $\frac{S(r,f)}{T(r,f)} \to 0$ *as* $r \to \infty$*, for all values of r if the order of f is finite.*
(ii) If the order of f is infinite, then also $\frac{S(r,f)}{T(r,f)} \to 0$ *as* $r \to \infty$*, possibly outside a set E of finite linear measure.*

Definition 1.1.13 {*p. 42 [27], p. 31 [79], p. 23 [86]*} *Suppose f is a meromorphic function in* $|z| < R$ $(\leq \infty)$ *and a is any* *complex number. Then for* $0 < r < R$,

$$\overline{N}\big(r, \frac{1}{f-a}\big) = \int_0^r \frac{\overline{n}\big(t, \frac{1}{f-a}\big) - \overline{n}\big(0, \frac{1}{f-a}\big)}{t} dt + \overline{n}\big(0, \frac{1}{f-a}\big) \log r,$$

is called the reduced counting function of a-points of f*, where* $\overline{n}(r, 1/(f-a))$ *or equivalently* $\overline{n}(r, a; f)$ *is the number of distinct a-points of* f *in* $|z| \leq r$ *and*

$$\overline{n}(0, \frac{1}{f-a}) = \begin{cases} 0, & \textit{if } f(0) \neq a, \\ 1, & \textit{if } f(0) = a. \end{cases}$$

Sometimes the notation $\overline{N}(r, a; f)$ *is used in place of* $\overline{N}(r, \frac{1}{f-a})$.

Now we state the Nevanlinna's second fundamental theorem in more exact form in terms of reducing counting functions.

Theorem 1.1.10 {*p. 30 [79], p. 23 [86]*} *Let* f *be a non-constant meromorphic function in the complex plane and* $a_1,\ a_2, ...,\ a_q$ *be* q (≥ 3) *distinct values in the extended complex plane. Then*

$$(q-2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\big(r, \frac{1}{f-a_j}\big) + S(r, f),$$

where $S(r, f)$ *is the error term with the same properties as in Theorem 1.1.9.*

Let us now mention the following notations such as deficient value, deficiency and index of multiplicity which are used frequently throughout the thesis.

Definition 1.1.14 {*p. 42 [27], p. 24 [79], p. 32 [86]*} *Let* f *be a non-constant meromorphic function in the complex plane and* a *be any complex number. Then*

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)},$$

is called the deficiency or defect of a *of the function* f*. The complex number* a *is called a deficient value of* f *or an exceptional value of* f *in the sense of Nevanlinna if* $\delta(0, f) > 0$.

Also the quantities $\Theta(a, f)$ *and* $\theta(a, f)$ *are defined by*

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)},$$

and

$$\theta(a, f) = \liminf_{r\to\infty} \frac{N(r, \frac{1}{f-a}) - \overline{N}(r, \frac{1}{f-a})}{T(r, f)}.$$

From the above definition it is clear that $0 \leq \delta(a, f) \leq 1$, $0 \leq \Theta(a, f) \leq 1$ and $0 \leq \theta(a, f) \leq 1$.

Let us now present a theorem on deficient values for its greater significance which is also known as the Nevanlinna's second fundamental theorem on deficient values.

Theorem 1.1.11 {*p. 43 [27], p. 32 [79], p. 24 [86]*} *Suppose* f *be a non-constant meromorphic function in the complex plane. Then the set of values* a *for which* $\Theta(a, f) > 0$ *is countable, and on summing over all such values* a *gives*

$$\sum_a \{\delta(a, f) + \theta(a, f)\} \leq \sum_a \Theta(a, f) \leq 2.$$

Corollary 1.1.2 *Let* f *be a non-constant meromorphic function in the complex plane. Then there are at most countable many deficient values of* f *and the sum of deficiencies is at most* 2.

For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying the condition $S(r, f) = \circ(T(r, f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers.

We now state Milloux theorem which plays a significant role in the study of value distribution of derivative of meromorphic functions.

Theorem 1.1.12 ***(Milloux's theorem)***{*p. 55 [27], p. 36 [86]*} *Suppose that f is a non-constant meromorphic function in the complex plane and k is a positive integer, and let*

$$\Phi = \sum_{i=0}^{k} a_i f^{(i)},$$

where $a_1,\ a_2, ...,\ a_k$ are small functions of f. Then

$$m\big(r, \frac{\Phi}{f}\big) = S(r,f)$$

and

$$T(r,\Phi) \leq T(r,f) + k\overline{N}(r,f) + S(r,f) \leq (k+1)T(r,f) + S(r,f).$$

In the Nevanlinna's second fundamental theorem one can replace the counting functions for certain roots of $f(z) = a$ by roots of the equation $\Phi(z) = b$, where Φ is a polynomial defined as in the above theorem.

Theorem 1.1.13 {*p. 57 [27], p. 37 [86]*} *Let f be a non-constant meromorphic function in the complex plane. If Φ is not a constant, then*

$$T(r,f) < \overline{N}(r,f) + N\big(r, \frac{1}{f}\big) + \overline{N}\big(r, \frac{1}{\Phi - 1}\big) - N_0\big(r, \frac{1}{\Phi^{(1)}}\big) + S(r,f),$$

where $N_0(r, \frac{1}{\Phi^{(1)}})$ *is the counting function corresponding to zeros of* $\Phi^{(1)}$ *which are not zeros of* $\Phi - 1$.

From the above theorem, we get the following result.

Corollary 1.1.3 $\{$*p. 38 [86]*$\}$ *Suppose that* f *is a non-constant meromorphic function in the complex plane and* k *is a positive integer. Then*

$$T(r,f) < \overline{N}(r,f)+N(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}-1})-N(r,\frac{1}{f^{(k+1)}})+S(r,f).$$

Definition 1.1.15 *[86] Let* f *and* g *be two non-constant meromorphic functions and* $a \in \mathbb{C} \cup \{\infty\}$. *We denote by* $E(a,f)$ *the set* $E(a,f) = \{z \in \mathbb{C} : f(z) - a = 0\}$, *where each zero with multiplicity* m *is counted* m *times and by zeros of* $f - \infty$ *we mean poles of* f. *If we ignore multiplicities, then the set is denoted by* $\overline{E}(a,f)$. *We say that* f *and* g *share a IM (CM) provided that* $\overline{E}(a,f) = \overline{E}(a,g)$ $(E(a,f) = E(a,g))$. *If* $\frac{1}{f}$ *and* $\frac{1}{g}$ *share* 0 *IM (CM), then we say that* f *and* g *share* ∞ *IM (CM).*

Considering the problem of value sharing Nevanlinna proved the following theorem which is known as Nevanlinna's five-value theorem.

Theorem 1.1.14 $\{$*p. 156 [86]*$\}$ *Suppose* f *and* g *are two non-constant meromorphic functions and* a_j $(j = 1,\ 2,\ 3,\ 4,\ 5)$ *be five distinct values in the extended complex plane. If* $\overline{E}(a_j,f) = \overline{E}(a_j,g)$ *for* $j = 1,\ 2,\ 3,\ 4,\ 5$, *then* $f \equiv g$.

We end this section with Nevanlinna's four value theorem which can be stated as follows.

Theorem 1.1.15 $\{$*p. 122 [56]*$\}$ *Suppose* f *and* g *be distinct non-constant meromorphic functions and* a_j $(j = 1,\ 2,\ 3, 4)$ *be four* *distinct values. If f and g share a_j $(j = 1,\ 2,\ 3, 4)$ CM, then $f(z) = T(g(z))$, where T is a Möbius transformation such that two of the four values are fixed points and another two (are Picard exceptional values of f and g) exchange each other under T.*

The thesis contains eight chapters including the present chapter.

In chapter 2, we have studied the uniqueness of meromorphic functions whose differential polynomials share a non-constant polynomial.

In chapter 3, we discussed the uniqueness of $(f^n)^{(k)}$ and $P[f^q]$ when they share one small function with respect to f.

In chapter 4, we have investigated the uniqueness of differential polynomials of meromorphic functions sharing a small function. Here we have proved that either $P[f] \equiv P[g]$ or $P[f]P[g] \equiv a^2$ when $P[f]$ and $P[g]$ share (a, l).

In chapter 5, we have investigated the uniqueness of meromorphic functions when they share a set of roots of unity.

In chapter 6 and 7, using the notion of weakly weighted-sharing, we have studied the uniqueness of non-constant homogeneous differential polynomials $P[f]$ and $P[g]$ generated by meromorphic functions f and g respectively.

In chapter 8, we researched the uniqueness problem for higher order derivatives of meromorphic functions on annuli.

In the thesis we write Theorem a.b.c (or Definition a.b.c or Corollary a.b.c etc.) to indicate c-th theorem (or c-th Definition or c-th Corollary etc.) in the b-th section of the a-th chapter. The expressions (equalities and inequalities) are numbered in the format (a.b.c) indicating the c-th expression in the b-th section of the a-th chapter. Individual chapters have been presented in such a manner that they are more or less independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end.

***