CHAPTER

8

UNICITY THEOREM FOR HIGHER ORDER DERIVATIVES OF MEROMORPHIC FUNCTIONS ON ANNULI

8.1 Introduction, Definitions and Notations

In 1929, R. Nevanlinna [55] first investigated the uniqueness of meromorphic functions and obtained the famous five value theorem.

Theorem 8.1.1 *[55] If f and g are two non-constant meromorphic functions that share five distinct complex values a_1, a_2, a_3, a_4, a_5 IM, then $f \equiv g$.*

After Nevanlinna's five value sharing uniqueness theorem, many researchers [5, 18, 23, 38, 44, 46, 53, 73, 86, 87, 94] studied the uniqueness of meromorphic functions sharing one, two, three values or some sets on simply connected regions, namely, the whole

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complex plane, the unit disc and the angular domain. But there are many other sub-regions in the whole complex plane which are not simply connected, namely, the annuli, the m-punctured complex plane etc.

In the beginning of the last decade few researchers [31, 32, 33, 74] have obtained some results on the uniqueness theory of meromorphic functions on annuli. In the following we briefly discuss the basic definitions and notations of the Nevanlinna theory of meromorphic functions on the annulus proposed by Khrystiyanyn and Kondratyuk [32, 33].

Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$. For $1 < r < R_0 \leq +\infty$, let

$$N_1(r, f) = \int_{\frac{1}{r}}^{1} \frac{n_1(t, f)}{t} dt, \quad N_2(r, f) = \int_{1}^{r} \frac{n_2(t, f)}{t} dt,$$

$$m_0(r, f) = m(r, f) + m(\frac{1}{r}, f) - 2m(1, f),$$

$$N_0(r, f) = N_1(r, f) + N_2(r, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of f in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$ respectively. Similarly, for $a \in \mathbb{C} \cup \{\infty\}$ we have

$$\begin{aligned} \overline{N}_0(r, \frac{1}{f-a}) &= \overline{N}_1(r, \frac{1}{f-a}) + \overline{N}_2(r, \frac{1}{f-a}) \\ &= \int_{\frac{1}{r}}^{1} \frac{\overline{n_1}(t, \frac{1}{f-a})}{t} dt + \int_{1}^{r} \frac{\overline{n_2}(t, \frac{1}{f-a})}{t} dt \end{aligned}$$

in which each zero of the function $f - a$ is counted only once, where by zeros of $f - \infty$ we mean poles of f. In addition, we use $\overline{n}_1^{k)}(t, \frac{1}{f-a})$(or $\overline{n}_1^{(k}(t, \frac{1}{f-a})$) to denote the counting function of poles of the function $\frac{1}{f-a}$ with multiplicities $\leq k$(or $> k$) in

$\{z : t < |z| \leq 1\}$, each point counted only once. Similarly, we have the notations $\overline{N}_1^{k)}(t,f)$, $\overline{N}_1^{(k}(t,f)$, $\overline{N}_2^{k)}(t,f)$, $\overline{N}_2^{(k}(t,f)$, $\overline{N}_0^{k)}(t,f)$ and $\overline{N}_0^{(k}(t,f)$.

The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(r,f) = m_0(r,f) + N_0(r,f).$$

Definition 8.1.1 *We write $E(a,f) = \{z \in \mathbb{A} : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,f)$. We use $\overline{E}_{k)}(a,f)$ to denote the set of zeros of $f - a$ with multiplicities not greater than k, in which each zero is counted only once.*

Definition 8.1.2 *[17] Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Then f is said to be a transcendental or admissible on the annulus $\mathbb{A}$ if*

$$\limsup_{R\to\infty} \frac{T_0(R,f)}{\log R} = \infty,\ 1 < R < R_0 = +\infty,$$

or

$$\limsup_{R\to\infty} \frac{T_0(R,f)}{-\log(R_0 - R)} = \infty,\ 1 < R < R_0 = +\infty,$$

respectively.

Cao [17, 19] studied the uniqueness of meromorphic functions on annuli and obtained an analog version of Nevanlinna's five-value theorem.

Theorem 8.1.2 *[19] Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let a_j $(j = 1,2,...,q)$ be q distinct complex numbers in $\mathbb{C} \cup \{\infty\}$ and k_j $(j = 1,2,...,q)$ be positive integers or ∞ such that*

$$k_1 \geq k_2 \geq ... \geq k_q$$

and

$$\overline{E}_{k_j)}(a_j, f_1) = \overline{E}_{k_j)}(a_j, f_2),\ for\ j = 1, 2, ..., q.$$

Then

(i) if $q = 7$, *then* $f_1(z) \equiv f_2(z)$.

(ii) if $q = 6$ *and* $k_3 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.

(iii) if $q = 5$, $k_3 \geq 3$ *and* $k_5 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.

(iv) if $q = 5$ *and* $k_4 \geq 4$, *then* $f_1(z) \equiv f_2(z)$.

(v) if $q = 5$, $k_3 \geq 5$ *and* $k_4 \geq 3$, *then* $f_1(z) \equiv f_2(z)$.

(vi) if $q = 5$, $k_3 \geq 6$ *and* $k_4 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.

From Theorem 8.1.2 we get the following theorem.

Theorem 8.1.3 *[17] Let* f_1 *and* f_2 *be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* a_j $(j = 1, 2, 3, 4, 5)$ *be five distinct complex numbers in* $\mathbb{C} \cup \{\infty\}$. *If* $\overline{E}(a_j, f_1) = \overline{E}(a_j, f_2)$ *for* $j = 1, 2, 3, 4, 5$, *then* $f_1 \equiv f_2$.

Definition 8.1.3 *For* $B \subset \mathbb{A}$ *and* $a \in \mathbb{C} \cup \{\infty\}$, *we denote by* $\overline{N}_0^B(r, \frac{1}{f-a})$ *the reduced counting function of those zeros of* $f - a$ *on* $\mathbb{A}$ *which belong to the set* B.

In 2016 Xu and Wang [75] investigated the uniqueness of meromorphic functions on annuli and proved the following result.

Theorem 8.1.4 *[75] Let* f *and* g *be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* $a_1, ..., a_q$ $(q \geq 5)$ *be* q *distinct complex numbers or* ∞. *Suppose that* $k_1 \geq k_2 \geq ... \geq k_q$, m *are positive integers or infinity;* $1 \leq m \leq q$ *and* δ_j (≥ 0) $(j = 1, 2, ..., q)$ *are such that*

$$(1 + \frac{1}{k_m}) \sum_{j=m}^{q} \frac{1}{1 + k_j} + 3 + \sum_{j=1}^{q} \delta_j < (q - m - 1)(1 + \frac{1}{k_m}) + m.$$

Let $B_j = \overline{E}_{k_j)}(a_j, f) \setminus \overline{E}_{k_j)}(a_j, g)$ *for* $j = 1, 2, ..., q$. *If*

$$\overline{N}_0^{B_j}(r, a_j; f) \le \delta_j T_0(r, f)$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r,a_j;f)}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r,a_j;g)}$$
$$> \frac{k_m}{(1+k_m)\sum_{j=m}^{q} \frac{k_j}{1+k_j} - 2(1+k_m) + (m-2-\sum_{j=1}^{q} \delta_j)k_m},$$

then $f \equiv g$.

In this chapter we study the uniqueness of higher order derivaties of two meromorphic functions on annuli sharing five or more values.

8.2 Lemmas

In this section we state and prove some lemmas that will be needed to proof the main theorems.

Lemma 8.2.1 *[32] Let* f *be a non-constant meromomorphic function on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < r < R_0 \le +\infty$. *Then*

$$(i)\ T_0(r, f) = T_0\big(r, \tfrac{1}{f}\big),$$
$$(ii)\ \ \max\big\{T_0(r, f_1.f_2), T_0\big(r, \tfrac{f_1}{f_2}\big), T_0(r, f_1 + f_2)\big\}$$
$$\le T_0(r, f_1) + T_0(r, f_2) + O(1).$$

Lemma 8.2.2 *[32](***The first fundamental theorem)** *Let* f *be a non-constant meromomorphic function on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < r < R_0 \le +\infty$. *Then* $T_0\big(r, \frac{1}{f-a}\big) = T_0(r, f) + O(1)$ *for every fixed* $a \in \mathbb{C}$.

In 2005, the lemma on the logarithmic derivative on the annulus $\mathbb{A}$ was obtained by Khrystiyanyn and Kondratyuk [33].

Lemma 8.2.3 *[33] Let f be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$, and let $\lambda \geq 0$. Then*

$$m_0\big(r, \frac{f^{(1)}}{f}\big) = S_1(r, f),$$

where (i) in the case $R_0 = +\infty$,

$$S_1(r, *) = O(\log(rT_0(r, *)))$$

for $r \in (1, +\infty)$, except for the set Δ_r such that $\int_{\Delta_r} r^{\lambda-1}dr < +\infty$; (ii) if $R_0 < +\infty$, then

$$S_1(r, *) = O\big(\log\big(\frac{T_0(r, *)}{R_0 - r}\big)\big)$$

for $r \in (1, R_0)$, except for the set Δ'_r such that $\int_{\Delta'_r} \frac{dr}{(R_0-r)^{\lambda-1}} < +\infty$.

Lemma 8.2.4 *[17].***(The second fundamental theorem)** *Let f be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$. Let $a_1, a_2, ..., a_q$ be q distinct complex numbers in the extended complex plane. Then*

$$(q-2)T_0(r, f) < \sum_{j=1}^{q} \overline{N}_0\big(r, \frac{1}{f - a_j}\big) + S_1(r, f),$$

where $S_1(r, f)$ is stated as in Lemma 8.2.3.

Lemma 8.2.5 *[17]. Let f be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$. Let a be an arbitrary complex number and k be a positive integer. Then*

$$(i)\ \overline{N}_0\big(r, a; f\big) \leq \frac{k}{k+1}\overline{N}_0^{k)}\big(r, a; f\big) + \frac{1}{k+1}N_0\big(r, a; f\big),$$

$$(ii)\ \overline{N}_0\big(r, a; f\big) \leq \frac{k}{k+1}\overline{N}_0^{k)}\big(r, a; f\big) + \frac{1}{k+1}T_0(r, f) + O(1).$$

Lemma 8.2.6 *Let f be a transcendental meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq \infty$ and $a_1,\ a_2, ..., a_k$ be $k\ (\geq 3)$ distinct complex numbers. If for a non-negative integer n, $E(0, f) \subseteq E(0, f^{(n)})$, then $(k - 2 + o(1))T_0(r, f) \leq \sum_{j=1}^{k} \overline{N_0}(r, a_j; f^{(n)})$.*

Proof. *By the first fundamental theorem on annulus, we have*

$$T_0(r, f) = T_0(r, \tfrac{1}{f}) + O(1) \leq N_0(r, 0; f) + m_0(r, \tfrac{f^{(n)}}{f})$$
$$+m_0(r, \tfrac{1}{f^{(n)}}) + O(1) \leq N_0(r, 0; f) + T_0(r, f^{(n)})$$
$$-N_0(r, 0; f^{(n)}) + S_1(r, f). \qquad (8.2.1)$$

By the second fundamental theorem on annulus, we get

$$(k-1)T_0(r, f^{(n)}) \ \leq\ \overline{N_0}(r, f^{(n)}) + \sum_{j=1}^{k-1} \overline{N_0}(r, a_j; f^{(n)})$$
$$+\ \overline{N_0}(r, 0; f^{(n)}) + S_1(r, f).$$

Without loss of generality, we may assume that $a_k = 0$. Otherwise a suitable linear transformation will do the job. Then the above inequality reduces to

$$(k-1)T_0(r, f^{(n)}) \leq \overline{N_0}(r, f^{(n)}) + \textstyle\sum_{j=1}^{k} \overline{N_0}(r, a_j; f^{(n)})$$
$$+S_1(r, f), \qquad (8.2.2)$$

Using (8.2.2) in (8.2.1), we obtain

$$(k-1)T_0(r, f) \leq (k-1)N_0(r, 0; f) + \overline{N_0}(r, f^{(n)})$$
$$+\textstyle\sum_{j=1}^{k} \overline{N_0}(r, a_j; f^{(n)}) - (k-1)N_0(r, 0; f^{(n)}) + S_1(r, f),$$

$$\Rightarrow\ \ (k-1)T_0(r, f) \leq (k-1)N_0(r, 0; f) + \overline{N_0}(r, f)$$
$$+\textstyle\sum_{j=1}^{k} \overline{N_0}(r, a_j; f^{(n)}) - (k-1)N_0(r, 0; f^{(n)})$$
$$+S_1(r, f). \qquad (8.2.3)$$

Since $E(0,f) \subseteq E(0,f^{(n)})$, *we have from (8.2.3)*

$$(k-1)T_0(r,f) \le \overline{N_0}(r,f) + \sum_{j=1}^{k} \overline{N_0}(r,a_j;f^{(n)}) + S_1(r,f)$$

$\Rightarrow (k-2+o(1))T_0(r,f) \le \sum_{j=1}^{k} \overline{N_0}(r,a_j;f^{(n)})$.
This completes the proof of lemma. ■

To prove unicity theorem related to multiple values and derivatives of meromorphic functions on annuli, we need the following Xiong inequality for meromorphic functions on annuli.

Lemma 8.2.7 *[76]. Let* f *be a transcendental or admissible meromorphic function on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \le \infty$. *Let* a *be a finite complex number and* $b_1,\ b_2, ..., b_q$ *be* q *distinct finite non-zero complex numbers. Then for any positive integer* n *we have*

$$\begin{aligned} qT_0(r,f) &< \overline{N_0}(r,f) + qN_0(r,a;f) + \sum_{j=1}^{q} N_0(r,b_j;f^{(n)}) \\ &- (q-1)N_0(r,0;f^{(n)}) - N_0(r,0;f^{(n+1)}) + S_1(r,f), \end{aligned}$$

where $S_1(r,f)$ *is same as in Lemma 8.2.3.*

8.3 Theorems

In this section we present the main theorems of this chapter.

Theorem 8.3.1 *Let* $f_1,\ f_2$ *be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \le +\infty$. *Let* $a_j \in \mathbb{C} \cup \{\infty\}$ *be distinct for* $j = 1, 2, ..., k$ $(k \ge 5)$ *and* n *be a non-negative integer. Suppose* $E(a_j, f_1^{(n)}) \subset E(a_j, f_2^{(n)})$, *for* $j = 1, 2, ..., k$ *and* $E(0, f_i) \subseteq$

$E(0, f_i^{(n)})$ *for* $i = 1, 2$. *If*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} N_0(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} N_0(r, a_j; f_2^{(n)})} > \frac{n+1}{k-(n+3)},$$

then $f_1^{(n)} \equiv f_2^{(n)}$.

Proof.

By Xiong inequality for meromorphic functions on annuli, we have

$$\begin{aligned}(k-2)T_0(r, f_1) < \overline{N_0}(r, f_1) + (k-2)N_0(r, 0; f_1)\\ + \textstyle\sum_{j=1}^{k-2} N_0(r, a_j; f_1^{(n)}) - (k-3)N_0(r, 0; f_1^{(n)})\\ +S_1(r, f_1).\end{aligned} \quad (8.3.1)$$

Similarly,

$$\begin{aligned}(k-2)T_0(r, f_2) < \overline{N_0}(r, f_2) + (k-2)N_0(r, 0; f_2)\\ + \textstyle\sum_{j=1}^{k-2} N_0(r, a_j; f_2^{(n)}) - (k-3)N_0(r, 0; f_2^{(n)})\\ +S_1(r, f_2).\end{aligned} \quad (8.3.2)$$

Since, $E(0, f_i) \subseteq E(0, f_i^{(n)})$ for $i = 1, 2$ we get from (8.3.1) and (8.3.2)

$$\begin{aligned}(k-2)T_0(r, f_1) < \overline{N_0}(r, f_1) + \textstyle\sum_{j=1}^{k-2} N_0(r, a_j; f_1^{(n)}) + N_0(r, 0; f_1^{(n)})\\ +S_1(r, f_1),\end{aligned} \quad (8.3.3)$$

and

$$\begin{aligned}(k-2)T_0(r, f_2) < \overline{N_0}(r, f_2) + \textstyle\sum_{j=1}^{k-2} N_0(r, a_j; f_2^{(n)}) + N_0(r, 0; f_2^{(n)})\\ +S_1(r, f_2).\end{aligned} \quad (8.3.4)$$

Without loss of generality we assume that $a_k = \infty$, $a_{k-1} = 0$. First we assume that all a_j $(1 \le j \le k)$ are finite. Then from

(8.3.3) and (8.3.4) we get

$$(k-3)T_0(r,f_1) < \sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)}) + S_1(r,f_1), \tag{8.3.5}$$

and

$$(k-3)T_0(r,f_2) < \sum_{j=1}^{k-1} N_0(r,a_j;f_2^{(n)}) + S_1(r,f_2). \tag{8.3.6}$$

Assume that $f_1^{(n)} \not\equiv f_2^{(n)}$. Therefore from (8.3.5) and (8.3.6)

$$\begin{aligned}\textstyle\sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)}) &< N_0(r,0;f_1^{(n)} - f_2^{(n)}) \le T_0(r,f_1^{(n)}) \\ &+T_0(r,f_2^{(n)}) + O(1) \le (n+1)\{T_0(r,f_1) + T_0(r,f_2)\} + O(1) \\ &\le \left\{\tfrac{n+1}{k-3} + O(1)\right\}\left[\textstyle\sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)}) + \sum_{j=1}^{k-1} N_0(r,a_j;f_2^{(n)})\right]\end{aligned}$$

$\Rightarrow \liminf_{r\to\infty} \frac{\sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k-1} N_0(r,a_j;f_2^{(n)})} \le \frac{n+1}{k-(n+4)}$

i.e.,

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} N_0(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k} N_0(r,a_j;f_2^{(n)})} \le \frac{n+1}{k-(n+3)}, \tag{8.3.7}$$

which contradicts the hypothesis.

Similarly, when $a_k = \infty$, we get (8.3.7). Hence $f_1^{(n)} \equiv f_2^{(n)}$. This completes the proof. ■

Theorem 8.3.2 *Let f_1, f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$ and $a_j \in \mathbb{C} \cup \{\infty\}$ be distinct for $j = 1, 2, ..., k$ ($k \ge 5$). Suppose that $p_1 \ge p_2 \ge ... \ge p_k$ are positive integers or infinity and δ (≥ 0) is such that*

$$\frac{1}{p_1} + \left(1 + \frac{1}{p_1}\right)\sum_{j=2}^{k} \frac{1}{1+p_j} + 1 + \delta < \frac{k-2}{n+1}\left(1 + \frac{1}{p_1}\right)$$

for a non-negative integer n. *Let* $A_j = \overline{E}_{p_j)}(a_j, f_1^{(n)}) \backslash \overline{E}_{p_j)}(a_j, f_2^{(n)})$ *for* $j = 1, 2, ..., k$ *and* $E(0, f_i) \subseteq E(0, f_i^{(n)})$ *for* $i = 1, 2$. *If* $\sum_{j=1}^{k} \overline{N_0}^{A_j}(r, a_j; f_1^{(n)}) \leq \delta T_0(r, f_1^{(n)})$ *and*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})}$$

$$> \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j} - (n+1)\{(1+\delta)p_1+1\}},$$

then $f_1^{(n)} \equiv f_2^{(n)}$.

Proof. By Lemma 8.2.6, we have

$$(k-2+o(1))T_0(r, f_1) \leq \sum_{j=1}^{k} \overline{N_0}(r, a_j; f_1^{(n)}) \tag{8.3.8}$$

and

$$(k-2+o(1))T_0(r, f_2) \leq \sum_{j=1}^{k} \overline{N_0}(r, a_j; f_2^{(n)}). \tag{8.3.9}$$

From (8.3.8) and Lemma 8.2.5 we have

$$\begin{gathered}(k-2+o(1))T_0(r, f_1) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\ + \sum_{j=1}^{k} \frac{1}{1+p_j} N_0(r, a_j; f_1^{(n)}) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\ + \sum_{j=1}^{k} \frac{1}{1+p_j} T_0(r, f_1^{(n)}) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\ +(n+1)\left(\sum_{j=1}^{k} \frac{1}{1+p_j}\right) T_0(r, f_1)\end{gathered}$$

$$\begin{gathered}i.e., \quad \left\{k-2-(n+1)\sum_{j=1}^{k} \frac{1}{1+p_j} + o(1)\right\} T_0(r, f_1) \\ \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}).\end{gathered}$$

Similarly from (8.3.9) and Lemma 8.2.5 we get

$$\begin{gathered}\left\{k-2-(n+1)\sum_{j=1}^{k} \frac{1}{1+p_j} + o(1)\right\} T_0(r, f_2) \\ \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N_0}^{p_j)}(r, a_j; f_2^{(n)}).\end{gathered}$$

Let $B_j = \overline{E}_{p_j)}(a_j, f_1^{(n)}) \setminus A_j$ for $j = 1, 2, ..., k$. Now

$$\begin{aligned}
\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) &= \sum_{j=1}^{k} \overline{N_0}^{A_j}(r, a_j; f_1^{(n)}) + \sum_{j=1}^{k} \overline{N_0}^{B_j}(r, a_j; f_1^{(n)}) \\
&\leq \delta T_0(r, f_1^{(n)}) + N_0(r, 0; f_1^{(n)} - f_2^{(n)}) \\
&\leq (1+\delta)(n+1)T_0(r, f_1) + (n+1)T_0(r, f_2),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\Big\{k - 2 - (n+1)\sum_{j=1}^{k} \frac{1}{1+p_j} + o(1)\Big\} \sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\
\leq\ & (1+\delta)(n+1)\sum_{j=1}^{k} \frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\
+\ & (n+1)\sum_{j=1}^{k} \frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r, a_j; f_2^{(n)}).
\end{aligned}$$

Since $1 \geq \frac{p_1}{1+p_1} \geq \frac{p_2}{1+p_2} \geq ... \geq \frac{p_k}{1+p_k} \geq \frac{1}{2}$, we get from the above inequality

$$\begin{aligned}
&\Big\{k - 2 - (n+1)\sum_{j=1}^{k} \frac{1}{1+p_j} + o(1)\Big\} \sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\
\leq\ & (1+\delta)(n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \\
+\ & (n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_2^{(n)}),
\end{aligned}$$

$$i.e.,\ \Big\{k - 2 - (n+1)\textstyle\sum_{j=1}^{k} \frac{1}{1+p_j} - (1+\delta)(n+1)\frac{p_1}{1+p_1} + o(1)\Big\}$$
$$\textstyle\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)}) \leq (n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_2^{(n)}).$$

Therefore

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})}$$
$$\leq \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=1}^{k}\frac{1}{1+p_j}-(n+1)(1+\delta)p_1}$$

$$= \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=2}^{k}\frac{1}{1+p_j}-(n+1)\{(1+\delta)p_1+1\}},$$

which is a contradiction. Therefore $f_1^{(n)} \equiv f_2^{(n)}$. This completes the proof. ■

Theorem 8.3.3 . *Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, ..., a_k$ $(k \geq 5)$ be k distinct complex numbers or ∞. Suppose that $p_1 \geq p_2 \geq ... \geq p_k$, m $(1 \leq m \leq k)$ are positive integers or infinity and δ_j (≥ 0) $(j = 1, 2, ..., k)$ are such that*

$$\left(1+\frac{1}{p_m}\right)\sum_{j=m}^{k}\frac{1}{1+p_j} + \left(2+\sum_{j=1}^{k}\delta_j - m\right)$$
$$< \frac{\{k-2-(n+1)(m-1)\}}{n+1}\left(1+\frac{1}{p_m}\right),$$

where n is a non-negative integer. Let $A_j = \overline{E}_{p_j)}(a_j, f_1^{(n)}) \setminus \overline{E}_{p_j)}(a_j, f_2^{(n)})$ for $j = 1, 2, ..., k$. If

$$\overline{N}_0^{A_j}(r, a_j; f_1^{(n)}) \leq \delta_j T_0(r, f_1^{(n)})$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}_0^{p_j)}(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} \overline{N}_0^{p_j)}(r, a_j; f_2^{(n)})}$$
$$> \frac{(n+1)\frac{p_m}{1+p_m}}{k-2-(n+1)\left(m-1-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}+(1+\sum_{j=1}^{k}\delta_j)\frac{p_m}{1+p_m}\right)},$$

then $f_1^{(n)} \equiv f_2^{(n)}$.

Proof. By Lemma 8.2.6, we have

$$(k-2+o(1))T_0(r,f_1) \le \sum_{j=1}^{k} \overline{N_0}(r,a_j;f_1^{(n)}) \tag{8.3.10}$$

and

$$(k-2+o(1))T_0(r,f_2) \le \sum_{j=1}^{k} \overline{N_0}(r,a_j;f_2^{(n)}). \tag{8.3.11}$$

From (8.3.10), Lemma 8.2.5 and using $1 \ge \frac{p_1}{1+p_1} \ge \frac{p_2}{1+p_2} \ge ... \ge \frac{p_k}{1+p_k} \ge \frac{1}{2}$, we have

$$(k-2+o(1))T_0(r,f_1) \le \textstyle\sum_{j=1}^{k} \frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$+\textstyle\sum_{j=1}^{k}\frac{1}{1+p_j}N_0(r,a_j;f_1^{(n)}) \le \sum_{j=1}^{m-1}\left(\frac{p_j}{1+p_j}-\frac{p_m}{1+p_m}\right)\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$+\left(\textstyle\sum_{j=1}^{k}\frac{1}{1+p_j}\right)T_0(r,f_1^{(n)}) + \sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$+S_1(r,f_1) \le \textstyle\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)}) + (n+1)$$
$$\left((m-1)-\tfrac{(m-1)p_m}{1+p_m}+\textstyle\sum_{j=m}^{k}\frac{1}{1+p_j}\right)T_0(r,f_1)+S_1(r,f_1),$$

i.e.,

$$\left\{k-2-(n+1)\left((m-1)-\tfrac{(m-1)p_m}{1+p_m}+\textstyle\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}T_0(r,f_1)$$
$$\le \textstyle\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)}).$$

Similarly from (8.3.11) we get

$$\left\{k-2-(n+1)\left((m-1)-\tfrac{(m-1)p_m}{1+p_m}+\textstyle\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}T_0(r,f_2)$$
$$\le \textstyle\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)}).$$

Let $B_j = \overline{E}_{p_j)}(a_j,f_1^{(n)}) \setminus A_j$ for $j=1,2,...,k$. Now

$$\textstyle\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)}) = \sum_{j=1}^{k}\overline{N_0}^{A_j}(r,a_j;f_1^{(n)})$$
$$+\textstyle\sum_{j=1}^{k}\overline{N_0}^{B_j}(r,a_j;f_1^{(n)}) \le \sum_{j=1}^{k}\delta_j T_0(r,f_1^{(n)})$$
$$+N_0(r,0;f_1^{(n)}-f_2^{(n)}) \le (1+\textstyle\sum_{j=1}^{k}\delta_j)(n+1)T_0(r,f_1)$$
$$+(n+1)T_0(r,f_2),$$

i.e.,

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq (1+\sum_{j=1}^{k}\delta_j)(n+1)\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$+(n+1)\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)}).$$

Therefore we get from the above inequality

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq (1+\sum_{j=1}^{k}\delta_j)(n+1)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$+(n+1)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})$$

i.e.,

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}+(1+\sum_{j=1}^{k}\delta_j)\frac{p_m}{1+p_m}\right)\right\}$$
$$\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})\leq (n+1)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)}).$$

Hence

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})}$$
$$\leq\frac{(n+1)\frac{p_m}{1+p_m}}{\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}+(1+\sum_{j=1}^{k}\delta_j)\frac{p_m}{1+p_m}\right)\right\}},$$

which is a contradiction. Thus $f_1^{(n)}\equiv f_2^{(n)}$. This completes the proof of the theorem. ■

Corollary 8.3.1 *For $n=0$, Theorem 8.3.3 reduced to Theorem 8.1.4 .*

Corollary 8.3.2 *For $m=1$ and $\delta=\sum_{j=1}^{k}\delta_j$, Theorem 8.3.3 reduced to Theorem 8.3.2 .*

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