CHAPTER

7

SOME RESULTS ON UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS CONCERNING WEAKLY WEIGHTED SHARING

7.1 Introduction, Definitions and Notations

In 1997, Yi [84] studied the uniqueness of two non-constant meromorphic functions f and g when their n-th derivatives share the value 1 CM or IM and proved the following results.

Theorem 7.1.1 *[84] Let f and g be two non-constant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If*

$$2\delta(0, f) + (n + 4)\Theta(\infty, f) > n + 5$$

The results of this chapter have been published in **Journal of Classical Analysis (JCA), see [63].**

and

$$2\delta(0,g)+(n+4)\Theta(\infty,g)>n+5,$$

then either $f\equiv g$ *or* $f^{(n)}.g^{(n)}\equiv 1$.

Theorem 7.1.2 *[84] Let* f *and* g *be two non-constant meromorphic functions such that* $f^{(n)}$, $g^{(n)}$ *share the value* 1 *IM. If*

$$5\delta(0,f)+(4n+7)\Theta(\infty,f)>4n+11$$

and

$$5\delta(0,g)+(4n+7)\Theta(\infty,g)>4n+11,$$

then either $f\equiv g$ *or* $f^{(n)}.g^{(n)}\equiv 1$.

Let $n\geq 1$ be an integer. An expression of the form

$$L(f)=f^{(n)}+a_{n-1}f^{(n-1)}+....+a_0f, \qquad (7.1.1)$$

where $a_0,a_1,...,a_{n-1}$ are complex constants, is called a linear differential polynomial of f.

In 2015 Li and Li [48] considered the problem of replacing the n-th derivatives in the above theorems by linear differential polynomials and proved the following theorems.

Theorem 7.1.3 *[48] Let* f *and* g *be two non-constant entire functions. Suppose that* f, g *share the value* 0 *CM and* $L(f)$, $L(g)$ *share the value* 1 *CM and* $\delta(0,f)>\frac{1}{2}$. *If* $\rho(f)\neq 1$, *then either* $f\equiv g$ *or* $L(f).L(g)\equiv 1$.

Theorem 7.1.4 *[48] Let* f *and* g *be two non-constant entire functions. Suppose that* f, g *share the value* 0 *CM and* $L(f)$, $L(g)$ *share the value* 1 *IM and* $\delta(0,f)>\frac{4}{5}$. *If* $\rho(f)\neq 1$, *then either* $f\equiv g$ *or* $L(f).L(g)\equiv 1$.

When we consider $P[f]$ and $P[g]$ are non-constant homogeneous differential polynomials of f and g respectively, then we understand that the coefficients $a_k \in S(f) \cap S(g)$.

Recently Lahiri and Pal [41] extended the results of Li and Li [48] for homogeneous differential polynomials. They proved the following theorem.

Theorem 7.1.5 *[41] Let f and g be two non-constant meromorphic functions, $a\ (\not\equiv 0, \infty) \in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (6.1.1), are non-constant. If $P[f]$ and $P[g]$ share a IM and*

$$\min\left\{5\delta(0,f)+\tfrac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g)+\tfrac{4Q+7}{d}\Theta(\infty,g)\right\} > \tfrac{4Q+4d+7}{d},$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

Definition 7.1.1 *[38] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and l be a non-negative integer or ∞. We denote by $E_f(S,l)$ the set $E_f(S,l) = \bigcup_{a \in S} E_l(a,f)$. We say that f and g share the set S with weight l if $E_f(S,l) = E_g(S,l)$.*

In 2019 Pramanik [57] considered the weighted set sharing and proved the following result.

Theorem 7.1.6 *[57] Let f be a non-constant meromorphic function and $p(z)$ be a polynomial in z of degree $n\ (\geq 1)$ with $p(0) = 0$. Let $m\ (\geq 1)$, $l\ (\geq 0)$ be integers and $a\ (\not\equiv 0, \infty) \in S(f)$. Suppose $P[f]$ be a non-constant differential polynomial of f and $S_m = \{a,\ aw, ...,\ aw^{m-1}\}$. If $p(f)$ and $P[f]$ share (S_m, l) with one of the following conditions:*

(i) $l \geq 2$ *and*

$$(mQ+3)\Theta(\infty,f)+2n\Theta(0,p(f))+m\overline{d}(P)\delta(0,f)$$
$$> (mQ+3)+2m\overline{d}(P)-m\underline{d}(P)-(m-2)n,$$

(ii) $l = 1$ *and*

$$(mQ+\frac{7}{2})\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+\overline{d}(P)\delta(0,f)$$
$$> (mQ+\frac{7}{2})+(m+1)\overline{d}(P)-m\underline{d}(P)-(m-\frac{5}{2})n,$$

(iii) $l = 0$ *and*

$$(2mQ+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2m\overline{d}(P)\delta(0,f)$$
$$> (2mQ+6)+4m\overline{d}(P)-2m\underline{d}(P)-(m-4)n,$$

then $P[f] \equiv tp(f)$ *for some* t *such that* $t^m = 1$.

Suppose F and G share "$(1,l)$" and let z_0 be a zero of $F-1$ of multiplicity p and a zero of $G-1$ of multiplicity q. We define by $\overline{N}_{G>l+1}(r,1;F)$ the reduced counting function of those 1-points of F such that $q > l+1$; $\overline{N}_{F>l+1}(r,1;G)$ is defined similarly. Also we denote by $N_E^{1)}(r,1;F)$ the counting function of those 1-points of F and G where $p = q = 1$ and denote by $\overline{N}_E^{(2}(r,1;F)$ the counting function of those 1-points of F and G where $p = q \geq 2$, where each such zero is counted only once.

Using weakly weighted sharing, Lin and Lin [45] proved the following theorem.

Theorem 7.1.7 *[45] Let* n (≥ 1) *be an integer and let* k *be a non-negative integer or* ∞. *Let* f *be a non-constant meromorphic function and* $a \in S(f)$ *be such that* $a \not\equiv 0, \infty$. *If* f *and* $f^{(n)}$ *share*

"(a, k)" with one of the following conditions:
(i) $2 \leq k \leq \infty$ *and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

(ii) $k = 1$ *and*

$$\left(\frac{n+9}{2}\right)\Theta(\infty, f) + \frac{5}{2}\delta_{2+n}(0, f) > \frac{n}{2} + 6,$$

(iii) $k = 0$ *and*

$$(7 + 2n)\Theta(\infty, f) + 5\delta_{2+n}(0, f) > 2n + 11,$$

then $f \equiv f^{(n)}$.

Later in 2011, Xu and Hu [72] generalized Theorem 7.1.7 in the following manner.

Theorem 7.1.8 *[72] Let* n (≥ 1) *be an integer and let* k *be a non-negative integer or* ∞*. Let* f *be a non-constant meromorphic function and* $a \in S(f)$ *be such that* $a \not\equiv 0, \infty$*. Suppose* $L(f)$ *is defined as in (7.1.1). If* f *and* $L(f)$ *share "*(a, k)*" with one of the following conditions:*
(i) $2 \leq k \leq \infty$ *and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

(ii) $k = 1$ *and*

$$\left(\frac{7}{2} + n\right)\Theta(\infty, f) + \frac{3}{2}\delta_2(0, f) + \delta_{n+2}(0, f) > n + 5,$$

(iii) $k = 0$ *and*

$$(6 + 2n)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0, f) + 2\delta_{2+n}(0, f) > 2n + 10,$$

then $f \equiv L(f)$.

In 2019 Pramanik et al [58, 59] proved uniqueness of homogeneous differential polynomials $P[f]$ and $P[g]$ when they share "(a, l)".

Theorem 7.1.9 *[58] Let f and g be two non-constant meromorphic functions, a $(\not\equiv 0, \infty) \in S(f) \cap S(g)$. Suppose that $P[f]$ and $P[g]$, as defined by (6.1.1), are non-constant. If $P[f]$ and $P[g]$ share "(a, l)" with one of the following conditions:*
(i) $2 \leq l \leq \infty$ and

$$\min\left\{2\delta(0,f) + \tfrac{Q+4}{d}\Theta(\infty,f), 2\delta(0,g) + \tfrac{Q+4}{d}\Theta(\infty,g)\right\} > \tfrac{Q+d+4}{d},$$

(ii) $l = 1$ and

$$\min\left\{\tfrac{5}{2}\delta(0,f) + \tfrac{3Q+9}{2d}\Theta(\infty,f), \tfrac{5}{2}\delta(0,g) + \tfrac{3Q+9}{2d}\Theta(\infty,g)\right\} > \tfrac{3Q+3d+9}{2d},$$

(iii) $l = 0$ and

$$\min\left\{5\delta(0,f) + \tfrac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g) + \tfrac{4Q+7}{d}\Theta(\infty,g)\right\} > \tfrac{4Q+4d+7}{d},$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

Theorem 7.1.10 *[59] Let f and g be two transcendental meromorphic functions, a $(\not\equiv 0, \infty) \in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (6.1.1), are non-constant. If $P[f]$ and $P[g]$ share "(a, l)" with one of the following conditions:*
(i) $2 \leq l \leq \infty$ and

$$\min\{(Q+4)\Theta(\infty,f) + 2\delta_{2+p}(0,f), (Q+4)\Theta(\infty,g) + 2\delta_{2+p}(0,g)\} > 6 + Q - d,$$

(ii) $l = 1$ *and*

$$\min\{(3Q+9)\Theta(\infty,f)+5\delta_{2+p}(0,f),(3Q+9)\Theta(\infty,g)+5\delta_{2+p}(0,g)\}$$
$$> 3Q+14-2d,$$

(iii) $l = 0$ *and*

$$\min\{(4Q+7)\Theta(\infty,f)+5\delta_{2+p}(0,f),(4Q+7)\Theta(\infty,g)+5\delta_{2+p}(0,g)\}$$
$$> 4Q+12-d,$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

In this chapter we prove uniqueness of $P[f]$ and $P[g]$ depending upon a new condition involving the weight of sharing.

7.2 Lemmas

To prove the main theorem of the chapter we need the following lemmas.

Lemma 7.2.1 *[41] Let* f *be a non-constant meromorphic function and* $P[f]$ *be defined by (6.1.1), then*

$$(i)\ T(r,P) \leq dT(r,f) + Q\overline{N}(r,\infty;f) + S(r,f).$$
$$(ii)\ N(r,0;P) \leq T(r,P) - dT(r,f) + dN(r,0;f) + S(r,f),$$
$$\leq Q\overline{N}(r,\infty;f) + dN(r,0;f) + S(r,f).$$

Lemma 7.2.2 *[30] Let* f *be a transcendental meromorphic function and* $P[f]$ *be a homogeneous differential polynomial of degree* $d \geq 1$. *Then*

$$dT(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,1;F) + N(r,0;f^d)$$
$$-N_0(r,0;(P[f])^{(1)}) + S(r,f),$$

where $N_0(r,0;(P[f])^{(1)})$ *denotes the counting function corresponding to the zeros of* $(P[f])^{(1)}$ *which are not the zeros of* $P[f]$ *and* $P[f]-1$.

Lemma 7.2.3 *[77] Let* f *be a non-constant meromorphic function and let*

$$p(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0,$$

where $a_i \in S(f)$ *for* $i = 0,\ 1 \ldots n$, a_n $(\not\equiv 0)$, *be a polynomial of degree* n. *Then* $T(r,p(f)) = nT(r,f) + S(r,f)$.

Lemma 7.2.4 *Let* F *and* G *be non-constant meromorphic functions share "*$(1,l)$*", where* l *is a positive integer. If* $H \not\equiv 0$, *then*

$$T(r,F) \leq (1+\tfrac{1}{l})N(r,0;F) + (2+\tfrac{1}{l})\overline{N}(r,\infty;F) + N(r,0;G)$$
$$+2\overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

Proof. By a simple calculation we see that

$$\begin{aligned} N_E^{1)}(r,1;F) &\leq N(r,0;H) \leq T(r,H) + O(1) \\ &\leq N(r,\infty;H) + S(r,F) + S(r,G). \quad (7.2.1) \end{aligned}$$

Now $N(r,\infty;H) \leq \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_{G>l+1}(r,1;F)$
$$+\overline{N}_{F>l+1}(r,1;G) + \overline{N}_{(2}(r,0;F) + \overline{N}_{(2}(r,0;G) + \overline{N}_0(r,0;F^{(1)})$$
$$+\overline{N}_0(r,0;G^{(1)}) + S(r,F) + S(r,G). \quad (7.2.2)$$

By Nevanlinna second fundamental theorem, we have

$$T(r,F) + T(r,G) \leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F)$$
$$+\overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,1;G) - N_0(r,0;F^{(1)})$$
$$-N_0(r,0;G^{(1)}) + S(r,F) + S(r,G), \quad (7.2.3)$$

where $N_0(r,0;F^{(1)})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ which are not the zeros of F and $F-1$.

$N_0(r,0;G^{(1)})$ is defined similarly.

Note that F and G share "$(1,l)$". By (7.2.1), (7.2.2), we have

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)\leq N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)$$
$$+\overline{N}_L(r,1;G)+N_E^{(2}(r,1;F)+\overline{N}(r,1;G)=N_E^{1)}(r,1;F)$$
$$+\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)+N_E^{(2}(r,1;F)$$
$$+\overline{N}(r,1;G)\leq\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)$$
$$+\overline{N}_{(2}(r,0;G)+2\overline{N}_{G>l+1}(r,1;F)+2\overline{N}_{F>l+1}(r,1;G)$$
$$+N_E^{(2}(r,1;F)+\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})$$
$$+S(r,F)+S(r,G)\leq\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)$$
$$+\overline{N}_{(2}(r,0;G)+\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)$$
$$+N(r,1;G)+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)$$
$$+S(r,G)\leq\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)$$
$$+\overline{N}_{(2}(r,0;G)+\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)+T(r,G)$$
$$+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \quad (7.2.4)$$

Also We have,

$$\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)\leq\tfrac{1}{l}N(r,0;F^{(1)})+S(r,F)$$
$$\leq\tfrac{1}{l}N(r,0;F)+\tfrac{1}{l}\overline{N}(r,\infty;F)$$
$$+S(r,F)+S(r,G). \quad (7.2.5)$$

Now from (7.2.3), (7.2.4) and (7.2.5) we get

$$T(r,F)+T(r,G)\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}(r,0;F)$$
$$+\overline{N}_{(2}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_{(2}(r,0;G)+\tfrac{1}{l}N(r,0;F)$$
$$+\tfrac{1}{l}\overline{N}(r,\infty;F)+T(r,G)+S(r,F)+S(r,G).$$

Since $\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F)\leq N(r,0;F)$, we have

$$T(r,F)\leq(1+\tfrac{1}{l})N(r,0;F)+(2+\tfrac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)$$
$$+2\overline{N}(r,\infty;G)+S(r,F)+S(r,G).$$

This completes the proof of the lemma. ■

7.3 Theorems

In this chapter we prove the following theorem.

Theorem 7.3.1 *Let f and g be two non-constant meromorphic functions, $a\ (\not\equiv 0,\infty) \in S(f)\cap S(g)$. Suppose that $P[f]$ and $P[g]$, as defined by (6.1.1), are non-constant. If $P[f]$ and $P[g]$ share "(a,l)" with one of the following conditions:*
(i) $l=\infty$ and

$$\min\left\{2\delta(0,f)+\tfrac{Q+4}{d}\Theta(\infty,f),2\delta(0,g)+\tfrac{Q+4}{d}\Theta(\infty,g)\right\} > \tfrac{Q+d+4}{d}, \tag{7.3.1}$$

(ii) $0<l<\infty$ and

$$\begin{aligned}\min\ \{&\tfrac{(2l+1)d}{l}\delta(0,f)+(\tfrac{Q+1+2l}{l}+Q+2)\Theta(\infty,f),\\ &\tfrac{(2l+1)d}{l}\delta(0,g)+(\tfrac{Q+1+2l}{l}+Q+2)\Theta(\infty,g)\}\\ &> \tfrac{(l+1)d+Q+1}{l}+4+Q,\end{aligned} \tag{7.3.2}$$

(iii) $l=0$ and

$$\min\left\{5\delta(0,f)+\tfrac{4Q+7}{d}\Theta(\infty,f),5\delta(0,g)+\tfrac{4Q+7}{d}\Theta(\infty,g)\right\} > \tfrac{4Q+4d+7}{d}, \tag{7.3.3}$$

then either $P[f]\equiv P[g]$ or $P[f].P[g]\equiv a^2$.

Proof. Let

$$F=\frac{P[f]}{a},\ G=\frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share "(a,l)", it follows that F, G share "$(1,l)$" except at the zeros and poles of a.
Now we consider the following cases:
Case 1: $l=\infty$. By (i) of Theorem 7.1.9 we get the result.

Case 2: $0 < l < \infty$. Suppose $H \not\equiv 0$. From Lemma 7.2.4 we get

$$T(r,F) \leq (1+\tfrac{1}{l})N(r,0;F) + (2+\tfrac{1}{l})\overline{N}(r,\infty;F) + N(r,0;G)$$
$$+2\overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

Using Lemma 7.2.1 we obtain

$$dT(r,f) \leq \tfrac{Q+1+2l}{l}\overline{N}(r,\infty;f) + (2+Q)\overline{N}(r,\infty;g)$$
$$+\tfrac{(l+1)d}{l}N(r,0;f) + dN(r,0;g) + S(r,f) + S(r,g). \quad (7.3.4)$$

Similarly,

$$dT(r,g) \leq \tfrac{Q+1+2l}{l}\overline{N}(r,\infty;g) + (2+Q)\overline{N}(r,\infty;f)$$
$$+\tfrac{(l+1)d}{l}N(r,0;g) + dN(r,0;f) + S(r,f) + S(r,g). \quad (7.3.5)$$

Adding (7.3.4) and (7.3.5) we get

$$dT(r,f) + dT(r,g) \leq (\tfrac{Q+1+2l}{l} + Q + 2)\overline{N}(r,\infty;f)$$
$$+\tfrac{(2l+1)d}{l}N(r,0;f) + (\tfrac{Q+1+2l}{l} + Q + 2)\overline{N}(r,\infty;g)$$
$$+\tfrac{(2l+1)d}{l}N(r,0;g) + S(r,f) + S(r,g).$$

i.e.,

$$\left\{\tfrac{(2l+1)d}{l}\delta(0,f) + (\tfrac{Q+1+2l}{l} + Q + 2)\Theta(\infty,f) - \tfrac{(l+1)d+Q+1}{l} - 4 - Q\right\}T(r,f)$$
$$+\left\{\tfrac{(2l+1)d}{l}\delta(0,g) + (\tfrac{Q+1+2l}{l} + Q + 2)\Theta(\infty,g) - \tfrac{(l+1)d+Q+1}{l} - 4 - Q\right\}T(r,g)$$
$$\leq S(r,f) + S(r,g),$$

which contradicts assumption (7.3.2).

Thus $H \equiv 0$. Integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where A ($\neq 0$) and B are constants. Thus

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \quad (7.3.6)$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG-(B+1)}. \tag{7.3.7}$$

Next we consider the following three subcases:
Subcase 2.1: $B \neq 0, -1$. Then from (7.3.7) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 7.2.1 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{B+1}{B};G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + dN(r,0;g) \\ &+ \overline{N}(r,\infty;F) + S(r,G). \end{aligned}$$

$$\Rightarrow dT(r,g) \leq \overline{N}(r,\infty;f) + dN(r,0;g) + \overline{N}(r,\infty;g) \\ +S(r,f) + S(r,g). \tag{7.3.8}$$

If $A - B - 1 \neq 0$, then it follows from (7.3.6) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r, 0; G).$$

Again by Nevanlinna second fundamental theorem and Lemma 7.2.1 we have

$$T(r,F) \leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\frac{-A+B+1}{B+1};F)+S(r,F).$$

$$\begin{aligned} \Rightarrow dT(r,f) &\leq \overline{N}(r,\infty;f) + dN(r,0;f) + Q\overline{N}(r,\infty;g) \\ &+ dN(r,0;g) + S(r,f) + S(r,g). \end{aligned} \tag{7.3.9}$$

Combining (7.3.8) and (7.3.9), we get

$$\begin{aligned}T(r,f)+T(r,g) &\le N(r,0;f)+\tfrac{2}{d}\overline{N}(r,\infty;f)+2N(r,0;g)\\ &+\tfrac{Q+1}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\\ &\Rightarrow \left\{\delta(0,f)+\tfrac{2}{d}\Theta(\infty,f)-\tfrac{2}{d}\right\}T(r,f)\\ +\left\{2\delta(0,g)+\tfrac{Q+1}{d}\Theta(\infty,g)-\tfrac{Q+d+1}{d}\right\}T(r,g) &\le S(r,f)+S(r,g),\end{aligned}$$

which contradicts (7.3.2).

Therefore $A-B-1=0$. Then from (7.3.6) we have

$$\overline{N}(r,0;F+\frac{1}{B})=\overline{N}(r,\infty;G).$$

Again by Nevanlinna second fundamental theorem and (ii) of Lemma 7.2.1 we obtain.

$$\begin{aligned}T(r,F) &\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;F+\frac{1}{B})+S(r,F),\\ &\le \overline{N}(r,\infty;f)+T(r,F)-dT(r,f)+dN(r,0;f)\\ &+ \overline{N}(r,\infty;g)+S(r,f)+S(r,g),\end{aligned}$$

$$\begin{aligned}\Rightarrow dT(r,f) &\le \overline{N}(r,\infty;f)+dN(r,0;f)+\overline{N}(r,\infty;g)\\ &+S(r,f)+S(r,g).\end{aligned} \tag{7.3.10}$$

Combining (7.3.8) and (7.3.10) we get

$$\begin{aligned}T(r,f)+T(r,g) &\le N(r,0;f)+\tfrac{2}{d}\overline{N}(r,\infty;f)+N(r,0;g)\\ &+\tfrac{2}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\\ &\Rightarrow \left\{\delta(0,f)+\tfrac{2}{d}\Theta(\infty,f)-\tfrac{2}{d}\right\}T(r,f)\\ +\left\{\delta(0,g)+\tfrac{2}{d}\Theta(\infty,g)-\tfrac{2}{d}\right\}T(r,g) &\le S(r,f)+S(r,g),\end{aligned}$$

which violates assumption (7.3.2).

Subcase 2.2: $B=-1$. Then

$$G=\frac{A}{A+1-F}$$

and
$$F = \frac{(1+A)G - A}{G}.$$
If $A+1 \neq 0$, then
$$\overline{N}(r, A+1; F) = \overline{N}(r, \infty; G),$$
$$\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r, 0; F).$$
By similar argument as in Subcase 2.1 we get a contradiction.
Therefore $A+1=0$. Then $FG = 1 \Rightarrow P[f].P[g] \equiv a^2$.
Subcase 2.3: $B = 0$. Then (7.3.6) and (7.3.7) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$.
If $A-1 \neq 0$, then $\overline{N}(r, 0; A-1+F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, \frac{A-1}{A}; G) = \overline{N}(r, 0; F)$. Proceeding similarly as in Subcase 2.1 we get a contradiction. Therefore $A - 1 = 0$, i.e., $P[f] \equiv P[g]$.
Case 3: $l = 0$. By (iii) of Theorem 7.1.9 we get the result.
This completes the proof of the theorem. ■

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