CHAPTER

6

UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS CONCERNING WEAKLY WEIGHTED-SHARING

6.1 Introduction, Definitions and Notations

Let f and g be two non-constant meromorphic functions and $a \in S(f) \cap S(g)$.

Definition 6.1.1 *[45] Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities, and $N_0(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. By $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ we mean the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$ respectively. If*

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_E(r,a) = S(r,f) + S(r,g),$$

Some portion of the results of this chapter have been published in **Communications of the Korean Mathematical Society (CKMS)**, see [59] and remaining results have been published in **Matematychni Studii (MS)**, see [58].

then we say that f and g share a "CM". If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM".

Suppose F and G share 1 "IM". By $\overline{N}_L(r,1;F)$ we denote the counting function of 1-points of F whose multiplicities are greater than multiplicities of 1-points of G. $\overline{N}_L(r,1;G)$ is defined similarly.

Definition 6.1.2 *[45] Let k be a positive integer or ∞, and $a \in S(f)\cap S(g)$. Let f and g be two non-constant meromorphic functions sharing a "IM".*

(i) $\overline{N}^{E}_{k)}(r,a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}^{0}_{(k}(r,a)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k, where each a-point is counted only once.

Definition 6.1.3 *[45] For $a \in S(f)\cap S(g)$, if k is a positive integer or ∞, and*

$$\overline{N}_{k)}(r,a;f)+\overline{N}_{k)}(r,a;g)-2\overline{N}^{E}_{k)}(r,a)=S(r,f)+S(r,g),$$

$$\overline{N}_{(k+1}(r,a;f)+\overline{N}_{(k+1}(r,a;g)-2\overline{N}^{0}_{(k+1}(r,a)=S(r,f)+S(r,g),$$

or if $k=0$ and

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a)=S(r,f)+S(r,g),$$

then we say f and g weakly share a with weight k. Here, we write f, g share "(a,k)" to mean that f, g weakly share a with *weight k. Obviously, if f and g share "(a,k)", then f and g share "(a,p)" for any p $(0 \le p < k)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)" respectively.*

Definition 6.1.4 *[41] Let f be a non-constant meromorphic function. An expression of the form*

$$P[f] = \sum_{k=1}^{n} a_k \prod_{j=0}^{p} \left(f^{(j)}\right)^{l_{kj}}, \tag{6.1.1}$$

where $a_k \in S(f)$ for $k = 1, 2, \ldots\ldots, n$ and l_{kj} are non-negative integers for $k = 1, 2, \ldots\ldots, n; j = 0, 1, 2, \ldots., p$ and $d = \sum_{j=0}^{p} l_{kj}$, for $k = 1, 2, \ldots\ldots, n$ is called a homogeneous differential polynomial of degree d generated by f. Also we denote Q by the quantity $Q = max_{1\le k\le n} \sum_{j=0}^{p} j.l_{kj}$.

In 2006 Lin and Lin [45] first defined and used the concept of weakly-weighted sharing of functions to prove the uniqueness of a meromorphic function and its derivative. In this direction they proved the following theorems.

Theorem 6.1.1 *[45] Let $n \ge 1$ be an integer and $2 \le k \le \infty$. Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f and $f^{(n)}$ share "(a,k)" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

then $f \equiv f^{(n)}$.

Theorem 6.1.2 *[45] Let $n \ge 1$ be an integer and let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f and $f^{(n)}$ share "$(a,1)$" and*

$$\left(\frac{n+9}{2}\right)\Theta(\infty, f) + \frac{5}{2}\delta_{2+n}(0, f) > \frac{n}{2} + 6,$$

then $f \equiv f^{(n)}$.

Theorem 6.1.3 *[45] Let* $n \geq 1$ *be an integer and let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *If* f *and* $f^{(n)}$ *share "*$(a, 0)$*" and*

$$(7 + 2n)\Theta(\infty, f) + 5\delta_{2+n}(0, f) > 2n + 11,$$

then $f \equiv f^{(n)}$.

Xu and Hu [72] generalized Theorems 6.1.1–6.1.3 by considering linear differential polynomial in place of the n-th derivative.

Theorem 6.1.4 *[72] Let* $n \geq 1$ *be an integer and* $2 \leq k \leq \infty$. *Let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *Suppose* $L(f) = f^{(n)} + a_{n-1}f^{(n-1)} + ... + a_0 f$. *If* f *and* $L(f)$ *share "*(a, k)*" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

then $f \equiv L(f)$.

Theorem 6.1.5 *[72] Let* $n \geq 1$ *be an integer, let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *Suppose* $L(f)$ *be defined as in Theorem 6.1.4. If* f *and* $L(f)$ *share "*$(a, 1)$*" and*

$$\left(\frac{7}{2} + n\right)\Theta(\infty, f) + \frac{3}{2}\delta_2(0, f) + \delta_{n+2}(0, f) > n + 5,$$

then $f \equiv L(f)$.

Theorem 6.1.6 *[72] Let* $n \geq 1$ *be an integer, let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *Suppose* $L(f)$ *be defined as in Theorem 6.1.4. If* f *and* $L(f)$ *share "*$(a, 0)$*" and*

$$(6 + 2n)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0, f) + 2\delta_{2+n}(0, f) > 2n + 10,$$

then $f \equiv L(f)$.

Motivated by such uniqueness investigation, it is natural to consider the problem in a more general setting: Let f and g be any two non-constant meromorphic functions, $P[f]$ and $P[g]$ be non-constant homogenous differential polynomials of f and g respectively, and $a \in S(f) \cap S(g)$, $a \not\equiv 0, \infty$. If $P[f]$ and $P[g]$ share "(a, k)", then what will be the relation between $P[f]$ and $P[g]$? In this chapter we prove that under certain conditions either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

6.2 Lemmas

In this section, we present some necessary lemmas needed in the sequel.

Lemma 6.2.1 *[41] Let f be a non-constant meromorphic function and $P[f]$ be defined as in (6.1.1) then*

$$N(r, \infty, P) \leq dN(r, \infty, f) + Q\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 6.2.2 *[41] Let f be a non-constant meromorphic function and $P[f]$ be defined as in (6.1.1) then*

$$(i)\ T(r, P) \leq dT(r, f) + Q\overline{N}(r, \infty; f) + S(r, f).$$

$$\begin{aligned}(ii)\ N(r, 0; P) &\leq T(r, P) - dT(r, f) + dN(r, 0; f) + S(r, f)\\ &\leq Q\overline{N}(r, \infty; f) + dN(r, 0; f) + S(r, f).\end{aligned}$$

Lemma 6.2.3 *Let f be a transcendental meromorphic function and $P[f]$ be same as in (6.1.1). If $P[f] \not\equiv 0$ then we have*
(i) $N_2(r, 0; P) \leq N_{2+p}(r, 0; f) + Q\overline{N}(r, \infty; f) + S(r, f)$,
(ii) $N_2(r, 0; P) \leq N_{2+p}(r, 0; f) + T(r, P) - dT(r, f) + S(r, f)$.

Proof.

$$N_2(r,0;P) \leq N(r,0;P) - \sum_{k=3}^{\infty} \overline{N}(r,0;P| \geq k) = T(r,P)$$
$$-m(r,0;P) - \sum_{k=3}^{\infty} \overline{N}(r,0;P| \geq k) + O(1) \leq T(r,P)$$
$$-m(r,0;f^d) + m(r,\infty;\tfrac{P}{f^d}) - \sum_{k=3}^{\infty} \overline{N}(r,0;P| \geq k) + O(1)$$
$$\leq T(r,P) - dT(r,f) + N(r,0;f^d) - \sum_{k=3}^{\infty} \overline{N}(r,0;P| \geq k)$$
$$+S(r,f) \leq T(r,P) - dT(r,f) + N_{2+p}(r,0;f^d)$$
$$+\sum_{k=3+p}^{\infty} \overline{N}(r,0;f^d| \geq k) - \sum_{k=3}^{\infty} \overline{N}(r,0;P| \geq k)$$
$$+S(r,f) \leq T(r,P) - dT(r,f) + N_{2+p}(r,0;f) + S(r,f).$$

This proves (ii).

Now using Lemma 6.2.2 we have,

$$T(r,P) = N(r,\infty;P) + m(r,\infty;P) \leq m(r,\infty;f^d)$$
$$+m(r,\infty;\tfrac{P}{f^d}) + N(r,\infty;P) = dm(r,\infty;f) + N(r,\infty;P)$$
$$+S(r,f) \leq dm(r,\infty;f) + dN(r,\infty;f) + Q\overline{N}(r,\infty;f)$$
$$+S(r,f) = dT(r,f) + Q\overline{N}(r,\infty;f) + S(r,f).$$

Therefore , $N_2(r,0;P) \leq N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;f) + S(r,f)$. ■

Lemma 6.2.4 *[45] Let k be a non-negative integer or infinity, F and G be non-constant meromorphic functions sharing "$(1,k)$". Suppose $H \not\equiv 0$, be same as in (3.2.1).*
(i) If $2 \leq k \leq \infty$, then

$$T(r,F) \leq N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)$$
$$+S(r,F) + S(r,G).$$

(ii) If $k = 1$, then

$$T(r,F) \leq N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)$$
$$+\overline{N}_L(r,1;F) + S(r,F) + S(r,G).$$

(iii) If $k = 0$*, then*

$$T(r,F) \le N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)$$
$$+2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G).$$

The same inequality holds for $T(r,G)$*.*

Lemma 6.2.5 *[72] Let* F *and* G *be non-constant meromorphic functions sharing "*$(1,1)$*". Then*

$$\overline{N}_L(r,1;F) \le \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F).$$

Lemma 6.2.6 *[72] Let* F *and* G *be non-constant meromorphic functions sharing "*$(1,0)$*". Then*

$$\overline{N}_L(r,1;F) \le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + S(r,F).$$

Lemma 6.2.7 *[77] Let* f *be a non-constant meromorphic function and let*

$$p(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0,$$

where $a_i \in S(f)$ *for* $i = 0, 1, ..., n$*;* $a_n (\not\equiv 0)$*, be a polynomial of degree* n*. Then* $T(r, p(f)) = nT(r,f) + S(r,f)$*.*

6.3 Theorems

Now, we state the main theorems of this chapter.

Theorem 6.3.1 *Let* f *and* g *be two non-constant meromorphic functions,* $a\ (\not\equiv 0,\infty) \in S(f) \cap S(g)$*. Suppose that* $P[f]$ *and* $P[g]$ *as defined by (6.1.1) are non-constant. If* $P[f]$ *and* $P[g]$ *share "*(a,k)*" with one of the following conditions:*
(i) $2 \le k \le \infty$ *and*

$$\min\left\{2\delta(0,f) + \tfrac{Q+4}{d}\Theta(\infty,f), 2\delta(0,g) + \tfrac{Q+4}{d}\Theta(\infty,g)\right\} > \tfrac{Q+d+4}{d}, \tag{6.3.1}$$

(ii) $k = 1$ *and*

$$\min\left\{\tfrac{5}{2}\delta(0,f) + \tfrac{3Q+9}{2d}\Theta(\infty,f), \tfrac{5}{2}\delta(0,g) + \tfrac{3Q+9}{2d}\Theta(\infty,g)\right\} > \tfrac{3Q+3d+9}{2d}, \quad (6.3.2)$$

(iii) $k = 0$ *and*

$$\min\left\{5\delta(0,f) + \tfrac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g) + \tfrac{4Q+7}{d}\Theta(\infty,g)\right\} > \tfrac{4Q+4d+7}{d}, \quad (6.3.3)$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Proof. Let

$$F = \frac{P[f]}{a},\ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share "(a,k)", it follows that F, G share "$(1,k)$" except at the zeros and poles of a.

Let H be same as in Lemma 6.2.4. Suppose that $H \not\equiv 0$. Now we consider the following three cases:

Case 1: $2 \leq k \leq \infty$. From (i) of Lemma 6.2.4, we have

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N(r,0;F) + N(r,0;G) + S(r,F) + S(r,G).$$

Using (ii) of Lemma 6.2.2 in the above inequality, we obtain

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) + dN(r,0;f) + Q\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g)$$

and so

$$\begin{aligned} dT(r,f) &\leq 2\overline{N}(r,\infty;f) + dN(r,0;f) + (2+Q)\overline{N}(r,\infty;g) \\ &+ dN(r,0;g) + S(r,f) + S(r,g). \quad (6.3.4) \end{aligned}$$

Similarly,

$$\begin{aligned} dT(r,g) &\le 2\overline{N}(r,\infty;g) + dN(r,0;g) + (2+Q)\overline{N}(r,\infty;f) \\ &+ dN(r,0;f) + S(r,f) + S(r,g). \end{aligned} \tag{6.3.5}$$

Adding (6.3.4) and (6.3.5), we get

$$\begin{gathered} T(r,f) + T(r,g) \le 2N(r,0;f) + \tfrac{Q+4}{d}\overline{N}(r,\infty;f) + 2N(r,0;g) \\ +\tfrac{Q+4}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \\ \Rightarrow \left\{2\delta(0,f) + \tfrac{Q+4}{d}\Theta(\infty,f) - \tfrac{Q+d+4}{d}\right\} T(r,f) \\ +\left\{2\delta(0,g) + \tfrac{Q+4}{d}\Theta(\infty,g) - \tfrac{Q+d+4}{d}\right\} T(r,g) \le S(r,f) + S(r,g), \end{gathered}$$

which contradicts the hypothesis (6.3.1).

Therefore $H \equiv 0$ and integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where A ($\neq 0$) and B are constants. This gives

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \tag{6.3.6}$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{6.3.7}$$

Next we consider following three subcases:

Subcase 1: $B \neq 0, -1$. Then from (6.3.7) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r,\infty;F).$$

By using Nevanlinna second fundamental theorem and (ii) of Lemma 6.2.2 we get

$$\begin{gathered} T(r,G) \le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\tfrac{B+1}{B};G) + S(r,G) \\ \le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ \le \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + dN(r,0;g) \\ +\overline{N}(r,\infty;F) + S(r,f) + S(r,g), \end{gathered}$$

$$\Rightarrow dT(r,g) \leq \overline{N}(r,\infty;f) + dN(r,0;g) + \overline{N}(r,\infty;g)$$
$$+S(r,f) + S(r,g). \qquad (6.3.8)$$

If $A-B-1 \neq 0$, then it follows from (6.3.6) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r,0;G).$$

Applying Nevanlinna second fundamental theorem and Lemma 6.2.2, we obtain

$$T(r,F) \leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r, \frac{-A+B+1}{B+1}; F)+S(r,F),$$

$$\Rightarrow dT(r,f) \leq \overline{N}(r,\infty;f) + dN(r,0;f) + Q\overline{N}(r,\infty;g)$$
$$+dN(r,0;g) + S(r,f) + S(r,g). \qquad (6.3.9)$$

From (6.3.8) and (6.3.9), we get

$$T(r,f) + T(r,g) \leq N(r,0;f) + \tfrac{2}{d}\overline{N}(r,\infty;f) + 2N(r,0;g)$$
$$+\tfrac{Q+1}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g),$$
$$\Rightarrow \{\delta(0,f) + \tfrac{2}{d}\Theta(\infty,f) - \tfrac{2}{d}\}T(r,f) + \{2\delta(0,g)$$
$$+\tfrac{Q+1}{d}\Theta(\infty,g) - \tfrac{Q+d+1}{d}\}T(r,g) \leq S(r,f) + S(r,g),$$

which again contradicts (6.3.1).

Therefore $A-B-1=0$. Thus from (6.3.6) it follows that

$$\overline{N}(r,0;F+\frac{1}{B}) = \overline{N}(r,\infty;G).$$

By applying Nevanlinnas second fundamental theorem and (ii) of Lemma 6.2.2 we have

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,0;F+\tfrac{1}{B}) + S(r,F)$$
$$\leq \overline{N}(r,\infty;f) + T(r,F) - dT(r,f) + dN(r,0;f)$$
$$+\overline{N}(r,\infty;g) + S(r,f) + S(r,g),$$

$$\Rightarrow dT(r,f) \le \overline{N}(r,\infty;f) + dN(r,0;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \qquad (6.3.10)$$

Combining (6.3.8) and (6.3.10) we get

$$T(r,f)+T(r,g) \le N(r,0;f) + \tfrac{2}{d}\overline{N}(r,\infty;f) + N(r,0;g) + \tfrac{2}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g),$$

$$\Rightarrow \{\delta(0,f) + \tfrac{2}{d}\Theta(\infty,f) - \tfrac{2}{d}\}T(r,f) + \{\delta(0,g) + \tfrac{2}{d}\Theta(\infty,g) - \tfrac{2}{d}\}T(r,g) \le S(r,f) + S(r,g),$$

which violates assumption (6.3.1).

Subcase 2: $B=-1$. Then (6.3.6) and (6.3.7) we get

$$G = \frac{A}{A+1-F}$$

and

$$F = \frac{(1+A)G - A}{G}.$$

If $A+1 \ne 0$, then

$$\overline{N}(r, A+1; F) = \overline{N}(r,\infty;G),$$

$$\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r,0;F).$$

By similar argument as in Subcase 1, we get a contradiction. Therefore $A+1=0$, then

$$FG = 1 \Rightarrow P[f].P[g] = a^2.$$

Subcase 3: $B=0$. Then (6.3.6) and (6.3.7) gives $G = \frac{F+A-1}{A}$ and $F = AG+1-A$

If $A-1 \ne 0$, $\overline{N}(r,0;A-1+F) = \overline{N}(r,0;G)$ and $\overline{N}(r,\frac{A-1}{A};G) = \overline{N}(r,0;F)$. Proceeding similarly as in Subcase 1 we get a contradiction. Therefore $A-1=0$, then $F=G$ i.e.,

$$P[f] = P[g].$$

This complete the proof of Case 1.

Case 2: $k = 1$. From (ii) of Lemma 6.2.4 we have

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N(r,0;F) + N(r,0;G)$$
$$+\overline{N}_L(r,1;F) + S(r,F) + S(r,G).$$

Using Lemma 6.2.2 and Lemma 6.2.5, we get

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f)$$
$$+dN(r,0;f) + Q\overline{N}(r,\infty;g) + dN(r,0;g)$$
$$+\overline{N}_L(r,1;F) + S(r,f) + S(r,g),$$

$$i.e., \quad dT(r,f) \leq 2\overline{N}(r,\infty;f) + dN(r,0;f) + (2+Q)\overline{N}(r,\infty;g)$$
$$+dN(r,0;g) + \tfrac{1}{2}\overline{N}(r,\infty;f) + \tfrac{1}{2}Q\overline{N}(r,\infty;f) + \tfrac{1}{2}dN(r,0;f)$$
$$+S(r,f) + S(r,g) \leq \tfrac{5+Q}{2}\overline{N}(r,\infty;f) + \tfrac{3d}{2}N(r,0;f)$$
$$+(2+Q)\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g),$$

$$\Rightarrow dT(r,f) \leq \tfrac{5+Q}{2}\overline{N}(r,\infty;f) + \tfrac{3d}{2}N(r,0;f) + (2+Q)\overline{N}(r,\infty;g)$$
$$+dN(r,0;g) + S(r,f) + S(r,g). \qquad (6.3.11)$$

Similarly,

$$dT(r,g) \leq \tfrac{5+Q}{2}\overline{N}(r,\infty;g) + \tfrac{3d}{2}N(r,0;g) + (2+Q)\overline{N}(r,\infty;f)$$
$$+dN(r,0;f) + S(r,f) + S(r,g). \qquad (6.3.12)$$

Adding (6.3.11) and (6.3.12), we get

$$T(r,f) + T(r,g) \leq \tfrac{3Q+9}{2d}\overline{N}(r,\infty;f) + \tfrac{5}{2}N(r,0;f)$$
$$+\tfrac{3Q+9}{2d}\overline{N}(r,\infty;g) + \tfrac{5}{2}N(r,0;g) + S(r,f) + S(r,g),$$
$$\Rightarrow \{\tfrac{5}{2}\delta(0,f) + \tfrac{3Q+9}{2d}\Theta(\infty,f) - \tfrac{3Q+3d+9}{2d}\}T(r,f)$$
$$+\{\tfrac{5}{2}\delta(0,g) + \tfrac{3Q+9}{2d}\Theta(\infty,g) - \tfrac{3Q+3d+9}{2d}\}T(r,g) \leq S(r,f) + S(r,g),$$

which contradicts hypothesis (6.3.2).

Proceeding similarly as in Case 1 we get the result for this case.

Case 3: $k = 0$. From (iii) of Lemma 6.2.4 we have

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N(r,0;F) + N(r,0;G)$$
$$+2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G).$$

Using Lemma 6.2.2 and Lemma 6.2.6 in the above inequality we obtain

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f)$$
$$+dN(r,0;f) + Q\overline{N}(r,\infty;g) + dN(r,0;g) + 2\overline{N}(r,\infty;F)$$
$$+2\overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + S(r,f) + S(r,g),$$

i.e.,
$$dT(r,f) \leq 4\overline{N}(r,\infty;f) + dN(r,0;f) + (3+Q)\overline{N}(r,\infty;g)$$
$$+dN(r,0;g) + 2Q\overline{N}(r,\infty;f) + 2dN(r,0;f) + Q\overline{N}(r,\infty;g)$$
$$+dN(r,0;g) + S(r,f) + S(r,g),$$

$$\Rightarrow dT(r,f) \leq (4+2Q)\overline{N}(r,\infty;f) + 3dN(r,0;f)$$
$$+(3+2Q)\overline{N}(r,\infty;g) + 2dN(r,0;g)$$
$$+S(r,f) + S(r,g). \qquad (6.3.13)$$

Similarly,

$$dT(r,g) \leq (4+2Q)\overline{N}(r,\infty;g) + 3dN(r,0;g) +$$
$$(3+2Q)\overline{N}(r,\infty;f) + 2dN(r,0;f)$$
$$+S(r,f) + S(r,g). \qquad (6.3.14)$$

Combining (6.3.13) and (6.3.14), we obtain

$$T(r,f) + T(r,g) \leq \tfrac{4Q+7}{d}\overline{N}(r,\infty;f) + 5N(r,0;f)$$
$$+\tfrac{4Q+7}{d}\overline{N}(r,\infty;g) + 5N(r,0;g) + S(r,f) + S(r,g),$$
$$\Rightarrow \{5\delta(0,f) + \tfrac{4Q+7}{d}\Theta(\infty,f) - \tfrac{4Q+4d+7}{d}\}T(r,f)$$
$$+\{5\delta(0,g) + \tfrac{4Q+7}{d}\Theta(\infty,g) - \tfrac{4Q+4d+7}{d}\}T(r,g) \leq S(r,f) + S(r,g),$$

which contradicts hypothesis (6.3.3).

Approaching similarly as in Case 1 we get the result for this case. ■

Theorem 6.3.2 *Let f and g be two transcendental meromorphic functions, a $(\not\equiv 0, \infty) \in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$ are non-constant homogeneous differential polynomials defined by (6.1.1). If $P[f]$ and $P[g]$ share "(a,k)" with one of the following conditions:*
(i) $2 \leq k \leq \infty$ and

$$\min\{(Q+4)\Theta(\infty,f)+2\delta_{2+p}(0,f),(Q+4)\Theta(\infty,g)+2\delta_{2+p}(0,g)\} > 6+Q-d, \tag{6.3.15}$$

(ii) $k=1$ and

$$\min\{(3Q+9)\Theta(\infty,f)+5\delta_{2+p}(0,f),(3Q+9)\Theta(\infty,g)+5\delta_{2+p}(0,g)\} > 3Q+14-2d, \tag{6.3.16}$$

(iii) $k=0$ and

$$\min\{(4Q+7)\Theta(\infty,f)+5\delta_{2+p}(0,f),(4Q+7)\Theta(\infty,g)+5\delta_{2+p}(0,g)\} > 4Q+12-d, \tag{6.3.17}$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

Proof. Let

$$F = \frac{P[f]}{a},\ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share "(a,k)", it follows that F, G share "$(1,k)$" except at the zeros and poles of a.

Let H be same as in Lemma 6.2.4. Suppose that $H \not\equiv 0$. Now we consider the following three cases:

Case 1: $2 \leq k \leq \infty$. From (i) of Lemma 6.2.4, we get

$$T(r,F) \leq N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) \\ +N_2(r,0;G) + S(r,F) + S(r,G).$$

Using Lemma 6.2.3, we have

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) \\ +N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + S(r,F) + S(r,G),$$

and so

$$\begin{aligned} dT(r,f) &\leq 2\overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + (2+Q)\overline{N}(r,\infty;g) \\ &+ N_{2+p}(r,0;g) + S(r,f) + S(r,g). \qquad (6.3.18) \end{aligned}$$

Similarly,

$$\begin{aligned} dT(r,g) &\leq 2\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + (2+Q)\overline{N}(r,\infty;f) \\ &+ N_{2+p}(r,0;f) + S(r,f) + S(r,g). \qquad (6.3.19) \end{aligned}$$

Adding (6.3.18) and (6.3.19)

$$dT(r,f) + dT(r,g) \leq 2N_{2+p}(r,0;f) + (Q+4)\overline{N}(r,\infty;f) \\ +2N_{2+p}(r,0;g) + (Q+4)\overline{N}(r,\infty;g) + S(r,f) + S(r,g)$$

$$\Rightarrow \{2\delta_{2+p}(0,f) + (Q+4)\Theta(\infty,f) - (6+Q-d)\}\, T(r,f) \\ + \{2\delta_{2+p}(0,g) + (Q+4)\Theta(\infty,g) - (6+Q-d)\}\, T(r,g) \\ \leq S(r,f) + S(r,g),$$

which contradicts hypothesis (6.3.15).

Thus $H \equiv 0$. Integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where $A \neq 0$ and B are constants. Thus

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \qquad (6.3.20)$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG-(B+1)}. \qquad (6.3.21)$$

Next we consider the following three subcases:
Subcase 1. $B \neq 0, -1$. Then from (6.3.21) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 6.2.3 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\tfrac{B+1}{B};G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + N_{2+p}(r,0;g) \\ &\quad +\overline{N}(r,\infty;F) + S(r,G), \end{aligned}$$

$$\begin{aligned} i.e.,\ dT(r,g) &\leq \overline{N}(r,\infty;f) + N_{2+p}(r,0;g) + \overline{N}(r,\infty;g) \\ &\quad +S(r,f) + S(r,g). \end{aligned} \qquad (6.3.22)$$

If $A - B - 1 \neq 0$, then it follows from (6.3.20) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r, 0; G).$$

Again by Nevanlinna second fundamental theorem and Lemma 6.2.3 we have

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\frac{-A+B+1}{B+1};F) + S(r,F).$$

$$\begin{aligned} \Rightarrow\ dT(r,f) &\leq \overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;g) \\ &\quad +N_{2+p}(r,0;g) + S(r,f) + S(r,g). \end{aligned} \qquad (6.3.23)$$

Combining (6.3.22) and (6.3.23) we get

$$\begin{aligned}
dT(r,f)+dT(r,g) &\le N_{2+p}(r,0;f)+2\overline{N}(r,\infty;f)\\
&+2N_{2+p}(r,0;g)+(Q+1)\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\\
&\Rightarrow \{\delta_{2+p}(0,f)+2\Theta(\infty,f)-(3-d)\}\,T(r,f)\\
&+\{2\delta_{2+p}(0,g)+(Q+1)\Theta(\infty,g)-(Q+3-d)\}\,T(r,g)\\
&\le S(r,f)+S(r,g),
\end{aligned}$$

which again contradicts (6.3.15).

Hence $A-B-1=0$. Then from (6.3.20) we get

$$\overline{N}(r,0;F+\frac{1}{B})=\overline{N}(r,\infty;G).$$

Again by Nevanlinna second fundamental theorem and Lemma 6.2.3 we obtain

$$\begin{aligned}
T(r,F) &\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;F+\tfrac{1}{B})+S(r,F)\\
&\le \overline{N}(r,\infty;f)+T(r,F)-dT(r,f)+N_{2+p}(r,0;f)\\
&+\overline{N}(r,\infty;g)+S(r,f)+S(r,g).
\end{aligned}$$

$$\begin{aligned}
i.e.,\quad dT(r,f) &\le \overline{N}(r,\infty;f)+N_{2+p}(r,0;f)+\overline{N}(r,\infty;g)\\
&+S(r,f)+S(r,g). \qquad (6.3.24)
\end{aligned}$$

Combining (6.3.22) and (6.3.24) we have

$$\begin{aligned}
dT(r,f)+dT(r,g) &\le N_{2+p}(r,0;f)+2\overline{N}(r,\infty;f)\\
&+N_{2+p}(r,0;g)+2\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\\
&\Rightarrow \{\delta_{2+p}(0,f)+2\Theta(\infty,f)-(3-d)\}\,T(r,f)\\
&+\{\delta_{2+p}(0,g)+2\Theta(\infty,g)-(3-d)\}\,T(r,g)\\
&\le S(r,f)+S(r,g),
\end{aligned}$$

which violates our assumption (6.3.15).

Subcase 2. $B = -1$. Then

$$G = \frac{A}{A + 1 - F}$$

and

$$F = \frac{(1 + A)G - A}{G}.$$

If $A + 1 \neq 0$, then we have

$$\overline{N}(r, A + 1; F) = \overline{N}(r, \infty; G)$$

$$\overline{N}(r, \frac{A}{A + 1}; G) = \overline{N}(r, 0; F).$$

By similar argument as in Subcase 1 we have a contradiction. Therefore, $A + 1 = 0$. Thus $FG = 1 \Rightarrow P[f].P[g] \equiv a^2$.

Subcase 3. $B = 0$. Then (6.3.20) and (6.3.21) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$.

If $A - 1 \neq 0$, $\overline{N}(r, 0; A - 1 + F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, \frac{A-1}{A}; G) = \overline{N}(r, 0; F)$. Proceeding similarly as in Subcase 1, we get a contradiction. Therefore, $A - 1 = 0$, then $F \equiv G$ i.e., $P[f] \equiv P[g]$. This completes the proof of Case 1.

Case 2: $k = 1$. From (ii) of Lemma 6.2.4 we get

$$\begin{aligned} T(r, F) \leq\ & N_2(r, \infty; F) + N_2(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) \\ +\ & \overline{N}_L(r, 1; F) + S(r, F) + S(r, G). \end{aligned}$$

Using Lemma 6.2.3, we obtain

$$\begin{aligned} T(r, F) &\leq 2\overline{N}(r, \infty; F) + 2\overline{N}(r, \infty; G) + T(r, F) - dT(r, f) \\ &+N_{2+p}(r, 0; f) + Q\overline{N}(r, \infty; g) + N_{2+p}(r, 0; g) \\ &+\overline{N}_L(r, 1; F) + S(r, F) + S(r, G). \end{aligned}$$

Again by Lemma 6.2.3 and Lemma 6.2.5 we have

$$\begin{aligned}
dT(r,f) &\le 2\overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + (2+Q)\overline{N}(r,\infty;g)\\
&+N_{2+p}(r,0;g) + \tfrac{1}{2}\overline{N}(r,\infty;f) + \tfrac{1}{2}Q\overline{N}(r,\infty;f) + \tfrac{1}{2}N_{2+p}(r,0;f)\\
&+S(r,f) + S(r,g) \le \tfrac{5+Q}{2}\overline{N}(r,\infty;f) + \tfrac{3}{2}N_{2+p}(r,0;f)\\
&+(2+Q)\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + S(r,f) + S(r,g)\\
i.e.,\quad dT(r,f) &\le \tfrac{5+Q}{2}\overline{N}(r,\infty;f) + \tfrac{3}{2}N_{2+p}(r,0;f) + (2+Q)\overline{N}(r,\infty;g)\\
&+N_{2+p}(r,0;g) + S(r,f) + S(r,g).
\end{aligned}$$

Similarly

$$\begin{aligned}
dT(r,g) &\le \tfrac{5+Q}{2}\overline{N}(r,\infty;g) + \tfrac{3}{2}N_{2+p}(r,0;g) + (2+Q)\overline{N}(r,\infty;f)\\
&+N_{2+p}(r,0;f) + S(r,f) + S(r,g).
\end{aligned}$$

Adding the above two inequalities we get

$$\begin{aligned}
&dT(r,f) + dT(r,g) \le \tfrac{3Q+9}{2}\overline{N}(r,\infty;f) + \tfrac{5}{2}N_{2+p}(r,0;f)\\
&+\tfrac{3Q+9}{2}\overline{N}(r,\infty;g) + \tfrac{5}{2}N_{2+p}(r,0;g) + S(r,f) + S(r,g),\\
&\Rightarrow \left\{\tfrac{5}{2}\delta_{2+p}(0,f) + \tfrac{3Q+9}{2}\Theta(\infty,f) - \tfrac{3Q+14-2d}{2}\right\}T(r,f)\\
&+\left\{\tfrac{5}{2}\delta_{2+p}(0,g) + \tfrac{3Q+9}{2}\Theta(\infty,g) - \tfrac{3Q+14-2d}{2}\right\}T(r,g)\\
&\le S(r,f) + S(r,g),
\end{aligned}$$

which contradicts hypothesis (6.3.16).
Proceeding similarly as in Case 1, we get the conclusion for this case.

Case 3: $k = 0$. From (iii) of Lemma 6.2.4 we get

$$\begin{aligned}
T(r,F) &\le N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)\\
&+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G).
\end{aligned}$$

Using Lemma 6.2.3 and 6.2.6 we have

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f)$$
$$+N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + 2\overline{N}(r,\infty;F)$$
$$+2\overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + S(r,F) + S(r,G)$$

i.e.,
$$dT(r,f) \leq 4\overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + (3+Q)\overline{N}(r,\infty;g)$$
$$+N_{2+p}(r,0;g) + 2Q\overline{N}(r,\infty;f) + 2N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;g)$$
$$+N_{2+p}(r,0;g) + S(r,f) + S(r,g) \leq (4+2Q)\overline{N}(r,\infty;f)$$
$$+3N_{2+p}(r,0;f) + (3+2Q)\overline{N}(r,\infty;g) + 2N_{2+p}(r,0;g)$$
$$+S(r,f) + S(r,g).$$

Similarly,

$$dT(r,g) \leq (4+2Q)\overline{N}(r,\infty;g) + 3N_{2+p}(r,0;g)$$
$$+(3+2Q)\overline{N}(r,\infty;f) + 2N_{2+p}(r,0;f) + S(r,f) + S(r,g).$$

Combining the above two inequalities we get,

$$dT(r,f) + dT(r,g) \leq (4Q+7)\overline{N}(r,\infty;f) + 5N_{2+p}(r,0;f)$$
$$+(4Q+7)\overline{N}(r,\infty;g) + 5N_{2+p}(r,0;g) + S(r,f) + S(r,g),$$
$$\Rightarrow \{5\delta_{2+p}(0,f) + (4Q+7)\Theta(\infty,f) - (4Q+12-d)\}\, T(r,f)$$
$$+\{5\delta_{2+p}(0,g) + (4Q+7)\Theta(\infty,g) - (4Q+12-d)\}\, T(r,g)$$
$$\leq S(r,f) + S(r,g),$$

which contradicts hypothesis (6.3.17). Approaching similarly as in Case 1, we get the conclusion for this case. ■

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