CHAPTER

5

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING A SET OF ROOTS OF UNITY

5.1 Introduction, Definitions and Notations

In 1976, Gross [26] proposed the following question.

Does there exist a finite set $S \subseteq \mathbb{C}$ such that any two non-constant entire functions f and g sharing the set S must be $f \equiv g$?

Uniqueness theory of meromorphic functions sharing a set generalizes the uniqueness theory on sharing values and it is generally more difficult. We now explain set sharing of functions in the following definition.

Definition 5.1.1 *[86] For a meromorphic function f and a set $S \subseteq \mathbb{C}$, we define* $E_f(S) = \bigcup_{a\in S}\{z|f(z)-a=0, \textit{counting multiplicities}\}$, $\overline{E}_f(S) = \bigcup_{a\in S}\{z|f(z)-a=0, \textit{ignoring multiplicities}\}$.
If $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$, *then we say that f and g*

The results of this chapter have been published in **The Journal of The Indian Mathematical Society (JIMS), see [62].**

share the set S CM (IM). Evidently, if S contains only one element then it coincides with the usual definition of CM (IM) share value.

Using the notion of set sharing An and Khoai [34] proved the following theorem for uniqueness of meromorphic functions.

Theorem 5.1.1 *[34] Let f and g be two non-constant meromorphic functions. Let k, d, n be three positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share S CM, then one of the following holds.*
1. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.
2. $f = tg$ for some $t \in \mathbb{C}$ such that $t^{nd} = 1$.

In this chapter we relax the nature of sharing from CM to IM in Theorem 5.1.1 and also obtain a generalization of Theorem 5.1.1.

5.2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 5.2.1 *[29] Let f and g be two non-constant meromorphic functions. If $E_f(1) = E_g(1)$, then one of the following cases holds:*
(i) $T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$; same inequality holds for $T(r,g)$;
(ii) $fg = 1$;
(iii) $f = g$.

Lemma 5.2.2 *[86] Let f be a non-constant meromorphic function and let $a_0, a_1, ..., a_n (\not\equiv 0)$ be small functions with respect to*

f. Then

$$T\big(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_0\big) = nT(r,f) + S(r,f).$$

Lemma 5.2.3 *[43] Let f be a non-constant meromorphic function and s, k be two positive integers. Then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N_{s+k}\big(r, \frac{1}{f}\big) + S(r,f),$$

and

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N_{s+k}\big(r, \frac{1}{f}\big) + S(r,f).$$

Lemma 5.2.4 *[86] Let f be a non-constant meromorphic function and let k be a positive integer. Then*

$$T(r, f^{(k)}) \leq (k+1)T(r,f) + S(r,f).$$

Lemma 5.2.5 *[86] Let f be a non-constant meromorphic function and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N(r, \frac{1}{f}) + S(r,f).$$

Lemma 5.2.6 *[34] Let f be a non-constant meromorphic function and k, n be positive integers with $n \geq k+3$, $a \in \mathbb{C} \setminus \{0\}$. Then*

$$\frac{n-k-2}{n+k} T(r,f) \leq \overline{N}\left(r, \frac{1}{(f^n)^{(k)} - a}\right) + S(r,f).$$

Lemma 5.2.7 *Let f be a non-constant meromorphic function and $a\ (\not\equiv 0, \infty) \in S(f)$. Let k, m, n be three positive integers with $n \geq k+3$. Suppose $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ is a non-zero polynomial. Then*

$$(n-k-2)T(r,f) \leq \overline{N}\left(r, \frac{1}{(f^n P(f))^{(k)} - a}\right) + S(r,f).$$

Proof. Let $F = f^nP(f)$. By the Nevanlinna's second fundamental theorem, we have

$$T(r, F^{(k)}) \leq \overline{N}(r, f) + \overline{N}\big(r, \tfrac{1}{F^{(k)}}\big) + \overline{N}\left(r, \tfrac{1}{F^{(k)}-a}\right)$$
$$+(\epsilon + o(1))T(r, F), \qquad (5.2.1)$$

for all $\epsilon > 0$. By Lemma 5.2.2 and Lemma 5.2.3 with $s = 1$, we get from (5.2.1)

$$(n+m)T(r, f) \leq \overline{N}(r, f) + N_{k+1}\big(r, \tfrac{1}{F}\big) + \overline{N}\left(r, \tfrac{1}{F^{(k)}-a}\right)$$
$$+(\epsilon + o(1))T(r, f) \leq \overline{N}(r, f) + (k+1)\overline{N}\big(r, \tfrac{1}{f}\big)$$
$$+N_{k+1}\left(r, \tfrac{1}{P(f)}\right) + \overline{N}\left(r, \tfrac{1}{(f^nP(f))^{(k)}-a}\right)$$
$$+(\epsilon + o(1))T(r, f) \leq (k+2+m)T(r, f)$$
$$+\overline{N}\left(r, \tfrac{1}{(f^nP(f))^{(k)}-a}\right) + (\epsilon + o(1))T(r, f),$$
$$\Rightarrow \quad (n-k-2)T(r, f) \leq \overline{N}\left(r, \tfrac{1}{(f^nP(f))^{(k)}-a}\right) + (\epsilon + o(1))T(r, f).$$

Since $n \geq k+3$, it follows that $(f^nP(f))^{(k)}$ is not constant. Thus the proof is completed. ■

Lemma 5.2.8 *[34] Let f be a non-constant meromorphic function and k, m, n be three positive integers with $n > 2k$. Then*

$$(i)\ (n-2k)T(r, f) + kN(r, f) + N\left(r, \tfrac{f^{n-k}}{(f^n)^{(k)}}\right)$$
$$\leq T\big(r, (f^n)^{(k)}\big) + S(r, f).$$
$$(ii)\ N\left(r, \tfrac{f^{n-k}}{(f^n)^{(k)}}\right) \leq kT(r, f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 5.2.9 *Let f be a non-constant meromorphic function and k, m, n be three positive integers with $n > 3k + m$. Suppose $P(f)$ be defined as in Lemma 5.2.7. Then*

$$(i)\ (n-m-3k)T(r, f) + kN(r, f) + N\left(r, \tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)$$
$$\leq T(r, (f^nP(f))^{(k)}) + S(r, f).$$

(ii) $N\left(r, \dfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) \leq (m+k)T(r,f) + k\overline{N}(r,f) + S(r,f).$

Proof. (i) Since $P(f)$ is a polynomial of degree m, we have

$$N(r,(f^nP(f))^{(k)}) = (m+n)N(r,f) + k\overline{N}(r,f). \qquad (5.2.2)$$

Also $S(r,f) = S(r,f^n)$ and $m\big(r, \frac{f^{(k)}}{f}\big) = S(r,f)$.

$$\begin{aligned}(n+m-k)m(r,f) &= m\big(r, f^{n+m-k}\big) \leq m(r,(f^nP(f))^{(k)}) \\ &+m\left(r, \tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) + S(r,f) = m\big(r,(f^nP(f))^{(k)}\big) \\ &+T\left(r, \tfrac{(f^nP(f))^{(k)}}{f^{n+m-k}}\right) - N\left(r, \tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) + S(r,f). \qquad (5.2.3)\end{aligned}$$

Now

$$\begin{aligned}T\left(r, \tfrac{(f^nP(f))^{(k)}}{f^{n+m-k}}\right) &\leq T\left(r, \tfrac{(f^nP(f))^{(k)}}{f^{n+m}}\right) + kT(r,f) + S(r,f) \\ &= m\left(r, \tfrac{(f^nP(f))^{(k)}}{f^{n+m}}\right) + N\left(r, \tfrac{(f^nP(f))^{(k)}}{f^{n+m}}\right) + kT(r,f) \\ &+S(r,f) \leq mT(r,f) + k\overline{N}(r,f) + kT(r,f) \\ &+mN(r,\tfrac{1}{f}) + k\overline{N}(r,\tfrac{1}{f}) + S(r,f). \qquad (5.2.4)\end{aligned}$$

Using (5.2.4) in (5.2.3) we have

$$\begin{aligned}(n+m-k)m(r,f) &\leq m\big(r,(f^nP(f))^{(k)}\big) + (m+k)T(r,f) \\ &+k\overline{N}(r,f) + mN(r,\tfrac{1}{f}) + k\overline{N}(r,\tfrac{1}{f}) - N\left(r, \tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) \\ &+S(r,f). \qquad (5.2.5)\end{aligned}$$

By (5.2.2) and (5.2.5), we obtain

$$\begin{aligned}(n+m-k)T(r,f) + kN(r,f) &= (n+m-k)m(r,f) \\ &+(n+m)N(r,f) \leq N\big(r,(f^nP(f))^{(k)}\big) - k\overline{N}(r,f) \\ &+m\big(r,(f^nP(f))^{(k)}\big) + (m+k)T(r,f) + k\overline{N}(r,f) \\ &-N\left(r, \tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) + mN(r,\tfrac{1}{f}) + k\overline{N}(r,\tfrac{1}{f}) + S(r,f) \\ &= T(r,(f^nP(f))^{(k)}) - N\left(r, \tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) + (m+k)T(r,f) \\ &+mN(r,\tfrac{1}{f}) + k\overline{N}(r,\tfrac{1}{f}) + S(r,f),\end{aligned}$$

$$\Rightarrow\ (n-m-3k)T(r,f)+kN(r,f)+N\left(r,\frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)$$
$$\leq T\big(r,(f^nP(f))^{(k)}\big)+S(r,f).$$

(ii) Using (5.2.4) we get

$$N\left(r,\frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)\leq T\left(r,\frac{(f^nP(f))^{(k)}}{f^{n+m-k}}\right)\leq (m+k)T(r,f)$$
$$+k\overline{N}(r,f)+mN(r,\tfrac{1}{f})+k\overline{N}(r,\tfrac{1}{f})+S(r,f).$$

Thus the proof of the lemma is completed. ■

Lemma 5.2.10 *[96] Let f and g be two non-constant meromorphic functions. If $\overline{E}_f(1)=\overline{E}_g(1)$, then one of the following three cases holds:*
(i) $T(r,f)\leq N_2(r,\frac{1}{f})+N_2(r,\frac{1}{g})+N_2(r,f)+N_2(r,g)+2\overline{N}(r,f)+\overline{N}(r,g)+2\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+S(r,f)+S(r,g)$; same inequality holds for $T(r,g)$;
(ii) $fg=1$;
(iii) $f=g$.

Lemma 5.2.11 *[93] Let $f(z)$ and $g(z)$ be two non-constant entire functions and k, n be integers with $n>k$. If $(f^n)^{(k)}.(g^n)^{(k)}=h$, $h\in\mathbb{C}\setminus\{0\}$, then $f(z)=l_1e^{lz}$, $g(z)=l_2e^{-lz}$, where l_1, l_2 and l are three non-zero constants such that $(-1)^k(l_1l_2)^n(nl)^{2k}=h$.*

5.3 Theorems

In this section we prove the main theorems of the chapter.

Theorem 5.3.1 *Let f and g be two non-constant meromorphic functions. Let k, $d\geq 2$, n be three positive integers with $n>2k+\frac{8k+14}{d}$ and $S=\{a\in\mathbb{C}:a^d=1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share S IM, then one of the following holds.*

1. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three non-zero constants such that* $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.
2. $f = tg$ *for some* $t \in \mathbb{C}$ *such that* $t^{nd} = 1$.

Proof. Since $n \geq k+3$, from Lemma 5.2.6 with the value 1, it follows that $(f^n)^{(k)} = 1$ has infinitely many solutions. So $\overline{E}_{(f^n)^{(k)}}(S) \neq \phi$ and $\overline{E}_{(g^n)^{(k)}}(S) \neq \phi$. By $\overline{E}_{(f^n)^{(k)}}(S) = \overline{E}_{(g^n)^{(k)}}(S)$ we see that $((f^n)^{(k)})^d$ and $((g^n)^{(k)})^d$ share the value 1 IM. Applying Lemma 5.2.10 to $((f^n)^{(k)})^d$ and $((g^n)^{(k)})^d$ we get one of the following cases.
Case 1:

$$\begin{aligned} T(r,((f^n)^{(k)})^d) &\leq N_2\left(r,\tfrac{1}{((f^n)^{(k)})^d}\right) + N_2\left(r,\tfrac{1}{((g^n)^{(k)})^d}\right) \\ &+N_2\left(r,((f^n)^{(k)})^d\right) + N_2\left(r,((g^n)^{(k)})^d\right) + 2\overline{N}\left(r,((f^n)^{(k)})^d\right) \\ &+\overline{N}\left(r,((g^n)^{(k)})^d\right) + 2\overline{N}\left(r,\tfrac{1}{((f^n)^{(k)})^d}\right) + \overline{N}\left(r,\tfrac{1}{((g^n)^{(k)})^d}\right) \\ &+S\left(r,((f^n)^{(k)})^d\right) + S\left(r,((g^n)^{(k)})^d\right), \end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned} T(r,((g^n)^{(k)})^d) &\leq N_2\left(r,\tfrac{1}{((g^n)^{(k)})^d}\right) + N_2\left(r,\tfrac{1}{((f^n)^{(k)})^d}\right) \\ &+N_2\left(r,((g^n)^{(k)})^d\right) + N_2\left(r,((f^n)^{(k)})^d\right) + 2\overline{N}\left(r,((g^n)^{(k)})^d\right) \\ &+\overline{N}\left(r,((f^n)^{(k)})^d\right) + 2\overline{N}\left(r,\tfrac{1}{((g^n)^{(k)})^d}\right) + \overline{N}\left(r,\tfrac{1}{((f^n)^{(k)})^d}\right) \\ &+S\left(r,((f^n)^{(k)})^d\right) + S\left(r,((g^n)^{(k)})^d\right). \end{aligned} \tag{5.3.2}$$

By Lemmas 5.2.2, 5.2.4 and 5.2.8 we get

$$\begin{aligned} (n-2k)T(r,f) &\leq T(r,(f^n)^{(k)}) + S(r,f) \leq (k+1)T(r,f^n) \\ &+S(r,f) \leq (k+1)nT(r,f) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} (n-2k)T(r,g) &\leq T(r,(g^n)^{(k)}) + S(r,g) \leq (k+1)T(r,(g^n)^{(k)}) \\ &+S(r,g) \leq (k+1)nT(r,g) + S(r,g). \end{aligned}$$

Since

$$T\left(r,((f^n)^{(k)})^d\right) = dT\big(r,(f^n)^{(k)}\big) + S\big(r,(f^n)^{(k)}\big),$$

$$T\left(r,((g^n)^{(k)})^d\right) = dT\big(r,(g^n)^{(k)}\big) + S\big(r,(g^n)^{(k)}\big),$$

therefore we have

$$S\left(r,((f^n)^{(k)})^d\right) = S(r,(f^n)^{(k)}) = S(r,f)$$

and

$$S\left(r,((g^n)^{(k)})^d\right) = S(r,(g^n)^{(k)}) = S(r,g).$$

Now, if a is a pole of $((f^n)^{(k)})^d$ then $f(a)=\infty$ and multiplicity of pole is $\geq n+k \geq 2$. On the other hand, if a is a zero of $((f^n)^{(k)})^d$, then $((f^n)^{(k)})(a) = 0$ with multiplicity of zero ≥ 2. Therefore $N_2\left(r,((f^n)^{(k)})^d\right) = 2\overline{N}(r,f) \leq 2T(r,f)$.
By (ii) of Lemma 5.2.8 we get

$$\begin{aligned}\overline{N}\left(r,\tfrac{1}{(f^n)^{(k)}}\right) &\leq \overline{N}\left(r,\tfrac{1}{f^{n-k}}\right) + N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) \leq \overline{N}\left(r,\tfrac{1}{f}\right)\\ &+N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) \leq T(r,f) + N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) + S(r,f)\\ &\leq T(r,f) + kT(r,f) + k\overline{N}(r,f) + S(r,f)\\ &= (k+1)T(r,f) + k\overline{N}(r,f) + S(r,f).\end{aligned}$$

Similarly, $N_2(r,((g^nP(g))^{(k)})^d) = 2\overline{N}(r,g) \leq 2T(r,g)$.

$$\begin{aligned}\overline{N}\left(r,\frac{1}{(g^n)^{(k)}}\right) &\leq T(r,g) + N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right) + S(r,g)\\ &\leq (k+1)T(r,g) + k\overline{N}(r,g) + S(r,g).\end{aligned}$$

Using the above results, from (5.3.1) and (5.3.2) we get

$$\begin{aligned}T(r,((f^n)^{(k)})^d) &\leq 4(k+1)T(r,f) + 4k\overline{N}(r,f) + 2T(r,g)\\ &+2N\left(r,\tfrac{g^{n-k}}{(g^n)^{(k)}}\right) + k\overline{N}(r,g) + (k+1)T(r,g) + 4\overline{N}(r,f)\\ &+3\overline{N}(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{5.3.3}$$

Similarly

$$T(r,((g^n)^{(k)})^d) \le 4(k+1)T(r,g) + 4k\overline{N}(r,g) + 2T(r,f)$$
$$+2N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) + k\overline{N}(r,f) + (k+1)T(r,f) + 4\overline{N}(r,g)$$
$$+3\overline{N}(r,f) + S(r,f) + S(r,g). \qquad (5.3.4)$$

Adding (5.3.3) and (5.3.4) we obtain

$$T(r,((f^n)^{(k)})^d) + T(r,((g^n)^{(k)})^d) \le (5k+14)\{T(r,f)+T(r,g)\}$$
$$+5k\{\overline{N}(r,f)+\overline{N}(r,g)\} + 2N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) + 2N\left(r,\tfrac{g^{n-k}}{(g^n)^{(k)}}\right)$$
$$+S(r,f)+S(r,g). \qquad (5.3.5)$$

By Lemma 5.2.8 we get

$$d\left\{(n-2k)T(r,f) + k\overline{N}(r,f) + N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right)\right\} \le T(r,((f^n)^{(k)})^d)$$
$$+S(r,f) \qquad (5.3.6)$$

and

$$d\left\{(n-2k)T(r,g) + k\overline{N}(r,g) + N\left(r,\tfrac{g^{n-k}}{(g^n)^{(k)}}\right)\right\} \le T(r,((g^n)^{(k)})^d)$$
$$+S(r,g). \qquad (5.3.7)$$

From (5.3.5), (5.3.6) and (5.3.7) we get

$$d(n-2k)\{T(r,f)+T(r,g)\} + dk\{\overline{N}(r,f)+\overline{N}(r,g)\}$$
$$+dN\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) + dN\left(r,\tfrac{g^{n-k}}{(g^n)^{(k)}}\right) \le (8k+14)\{T(r,f)+T(r,g)\}$$
$$+2N\left(r,\tfrac{f^{n-k}}{(f^n)^{(k)}}\right) + 2k\{\overline{N}(r,f)+\overline{N}(r,g)\} + 2N\left(r,\tfrac{g^{n-k}}{(g^n)^{(k)}}\right)$$
$$+S(r,f)+S(r,g). \qquad (5.3.8)$$

Since $d \ge 2$ we have

$$dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right) \ge 2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right),$$

$$dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right) \ge 2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right),$$

and

$$dk\{\overline{N}(r,f)+\overline{N}(r,g)\} \geq 2k\{\overline{N}(r,f)+\overline{N}(r,g)\}.$$

Therefore from (5.3.8) we obtain

$$d(n-2k)\{T(r,f)+T(r,g)\} \leq (8k+14)\{T(r,f)+T(r,g)\}$$
$$+S(r,f)+S(r,g).$$

Therefore $d(n-2k) \leq 8k+14 \Rightarrow n \leq 2k+\frac{8k+14}{d}$, which is a contradiction.

Case 2: $((f^n)^{(k)})^d.((g^n)^{(k)})^d = 1$. Then we get $(f^n)^{(k)}.(g^n)^{(k)} = h$, where $h^d = 1$.

Let f has a zero of multiplicity α_1 (≥ 1) then g must have a pole. Let g has a pole of multiplicity β_1 (≥ 1). Thus $n\alpha_1 - k = n\beta_1 + k$, $\Rightarrow n(\alpha_1 - \beta_1) = 2k$, which is a contradiction. Therefore f has no zeros. Similarly, we can prove $g \neq 0$, $f \neq \infty$, $g \neq \infty$. Therefore f and g are non-constant entire functions. Thus using Lemma 5.2.11 to f and g we obtain $f = c_1e^{cz}$, $g(z) = c_2e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^k(c_1c_2)^n(nc)^{2k} = h$. Since $h^d = 1$ we obtain $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.

Case 3: $((f^n)^{(k)})^d = ((g^n)^{(k)})^d$. Thus we have $(f^n)^{(k)} = h(g^n)^{(k)}$ where $h^d = 1$. Set $e^n = h$. Then $(f^n)^{(k)} = ((eg)^n)^{(k)}$ and by similar arguments as in Case 3. of Theorem 1 [34], we have $f = seg$ with $s^n = 1$. Set $t = se$, then $t^{nd} = 1$. This completes the proof. ■

Theorem 5.3.2 *Let f and g be two non-constant meromorphic functions. Let k, m, n, d (≥ 2) be positive integers with $n > 3k+m+\frac{4k+4m+8}{d}$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. Suppose $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ is a non-zero polynomial. If*

$(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share S CM, then one of the following holds.
I) $P(w) \neq a_m w^m$ or $c_0(\neq 0)$. Then one of the following holds
I1) $(f^nP(f))^{(k)} . (g^nP(g))^{(k)} = h$, where $h^d = 1$,
I2) $(f^nP(f))^{(k)} = h\,(g^nP(g))^{(k)}$, where $h^d = 1$.
II) $P(w) = a_m w^m$ or $c_0(\neq 0)$. Then
II1) $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k {a_m}^2 (c_1c_2)^{n+m^*}((n+m^*)c)^{2k} = h$ or $(-1)^k c_0^2 (c_1c_2)^n (nc)^{2k} = h$, where $h^d = 1$.
II2) $f = tg$ for some t such that $t^{(n+m^*)d} = 1$, where

$$m^* = \begin{cases} m, & \text{if } P(f) \neq c_0. \\ 0, & \text{if } P(f) = c_0. \end{cases}$$

Proof. Since $n \geq k+3$, from Lemma 5.2.7 with the value 1, it follows that $(f^nP(f))^{(k)} = 1$ has infinitely many solutions. So $E_{(f^nP(f))^{(k)}}(S) \neq \phi$ and $E_{(g^nP(g))^{(k)}}(S) \neq \phi$. By $E_{(f^nP(f))^{(k)}}(S) = E_{(g^nP(g))^{(k)}}(S)$ we see that $\{(f^nP(f))^{(k)}\}^d$ and $\{(g^nP(g))^{(k)}\}^d$ share the value 1 CM. Applying Lemma 5.2.1 to F^d, G^d where $F = (f^nP(f))^{(k)}$ and $G = (g^nP(g))^{(k)}$ we get one of the following cases.
Case 1:

$$T(r,F^d) \leq N_2(r,\tfrac{1}{F^d}) + N_2(r,\tfrac{1}{G^d}) + N_2(r,F^d) + N_2(r,G^d) + S(r,F^d) + S(r,G^d), \tag{5.3.9}$$

and

$$T(r,G^d) \leq N_2(r,\tfrac{1}{F^d}) + N_2(r,\tfrac{1}{G^d}) + N_2(r,F^d) + N_2(r,G^d) + S(r,F^d) + S(r,G^d). \tag{5.3.10}$$

By Lemma 5.2.9 we obtain

$$(n-m-3k)T(r,f) \leq T(r,(f^nP(f))^{(k)}) \leq (k+1)T(r,f^nP(f)) \leq (k+1)(m+n)T(r,f) + S(r,f)$$

and

$$(n-m-3k)T(r,g) \le T(r,(g^nP(g))^{(k)}) \le (k+1)T(r,g^nP(g))$$
$$\le (k+1)(m+n)T(r,g)+S(r,g).$$

Since

$$T(r,\{(f^nP(f))^{(k)}\}^d) = dT(r,(f^nP(f))^{(k)}) + S(r,(f^nP(f))^{(k)})$$

and

$$T(r,\{(g^nP(g))^{(k)}\}^d) = dT(r,(g^nP(g))^{(k)}) + S(r,(g^nP(g))^{(k)}),$$

therefore we have

$$S(r,\{(f^nP(f))^{(k)}\}^d) = S(r,(f^nP(f))^{(k)}) = S(r,f)$$

and

$$S(r,\{(g^nP(g))^{(k)}\}^d) = S(r,(g^nP(g))^{(k)}) = S(r,g).$$

On the other hand, if a is a pole of $\{(f^nP(f))^{(k)}\}^d$ then $f(a)=\infty$ with multiplicity of pole of $\{(f^nP(f))^{(k)}\}^d$ is $\ge n+m+k>2$. Moreover, if a is a zero of $\{(f^nP(f))^{(k)}\}^d$, then $\{(f^nP(f))^{(k)}\}(a) = 0$ with multiplicity of zero of $\left\{(f^nP(f))^{(k)}\right\}^d \ge 2$ because $d \ge 2$. Therefore $N_2(r,\{(f^nP(f))^{(k)}\}^d) = 2\overline{N}(r,f) \le 2T(r,f)$.

$$N_2\left(r,\frac{1}{\{(f^nP(f))^{(k)}\}^d}\right) = 2\overline{N}\left(r,\frac{1}{(f^nP(f))^{(k)}}\right) \le 2\overline{N}\left(r,\frac{1}{f^{n+m-k}}\right)$$
$$+2\overline{N}\left(r,\frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right) \le 2\overline{N}(r,\tfrac{1}{f}) + 2N\left(r,\frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)$$
$$\le 2T(r,f)+2(m+k)T(r,f)+2k\overline{N}(r,f)+2mN(r,\tfrac{1}{f})$$
$$+2k\overline{N}(r,\tfrac{1}{f})+S(r,f) = (4m+4k+2)T(r,f)$$
$$+2k\overline{N}(r,f)+S(r,f).$$

Similarly,

$$N_2(r,\{(g^nP(g))^{(k)}\}^d) = 2\overline{N}(r,g) \le 2T(r,g),$$

and

$$N_2\left(r, \frac{1}{\{(g^nP(g))^{(k)}\}^d}\right) \le 2\overline{N}(r, \tfrac{1}{g}) + 2N\left(r, \frac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right)$$
$$\le (4m+4k+2)T(r,g) + 2k\overline{N}(r,g) + S(r,g).$$

Using the above results we get from (5.3.9) and (5.3.10)

$$T\left(r, \{(f^nP(f))^{(k)}\}^d\right) \le 2T(r,f) + (4m+4k+2)T(r,f)$$
$$+2k\overline{N}(r,f) + 4T(r,g) + 2N\left(r, \frac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right)$$
$$+S(r,f) + S(r,g), \qquad (5.3.11)$$

and

$$T\left(r, \{(g^nP(g))^{(k)}\}^d\right) \le 2T(r,g) + (4m+4k+2)T(r,g)$$
$$+2k\overline{N}(r,g) + 4T(r,f) + 2N\left(r, \frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)$$
$$+S(r,f) + S(r,g). \qquad (5.3.12)$$

Adding (5.3.11) and (5.3.12) we obtain

$$T\left(r, \{(f^nP(f))^{(k)}\}^d\right) + T\left(r, \{(g^nP(g))^{(k)}\}^d\right) \le (4m+4k+8)$$
$$\big\{T(r,f) + T(r,g)\big\} + 2k\{\overline{N}(r,f) + \overline{N}(r,g)\} + 2N\left(r, \frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)$$
$$+2N\left(r, \frac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right) + S(r,f) + S(r,g). \qquad (5.3.13)$$

By Lemma 5.2.9 we have

$$d\left\{(n-m-3k)T(r,f) + k\overline{N}(r,f) + N\left(r, \frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)\right\}$$
$$\le T\left(r, \{(f^nP(f))^{(k)}\}^d\right) + S(r,f), \qquad (5.3.14)$$

and

$$d\left\{(n-m-3k)T(r,g) + k\overline{N}(r,g) + N\left(r, \frac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right)\right\}$$
$$\le T\left(r, \{(g^nP(g))^{(k)}\}^d\right) + S(r,g). \qquad (5.3.15)$$

From (5.3.13), (5.3.14) and (5.3.15) we get

$$\begin{aligned} d(n-m-3k)\big\{T(r,f)+T(r,g)\big\}+dk\big\{\overline{N}(r,f)+\overline{N}(r,g)\big\}\\ +dN\left(r,\tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)+dN\left(r,\tfrac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right)\\ \leq (4m+4k+8)\{T(r,f)+T(r,g)\}+2k\{\overline{N}(r,f)\\ +\overline{N}(r,g)\}+2N\left(r,\tfrac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)+2N\left(r,\tfrac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right)\\ +S(r,f)+S(r,g). \end{aligned} \tag{5.3.16}$$

Since $d \geq 2$ we have

$$dN\left(r,\frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right)\geq 2N\left(r,\frac{f^{n+m-k}}{(f^nP(f))^{(k)}}\right),$$

$$dN\left(r,\frac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right)\geq 2N\left(r,\frac{g^{n+m-k}}{(g^nP(g))^{(k)}}\right),$$

and

$$dk\big\{\overline{N}(r,f)+\overline{N}(r,g)\big\}\geq 2k\big\{\overline{N}(r,f)+\overline{N}(r,g)\big\}.$$

Therefore from (5.3.16) we obtain

$$\begin{aligned} d(n-m-3k)\big\{T(r,f)+T(r,g)\big\}\leq (4m+4k+8)\{T(r,f)\\ +T(r,g)\}+S(r,f)+S(r,g). \end{aligned}$$

$\Rightarrow d(n-m-3k)\leq 4k+4m+8 \Rightarrow n\leq 3k+m+\frac{4k+4m+8}{d}$, which contradicts the hypothesis.

Case 2:

$$F^dG^d=1. \tag{5.3.17}$$

We consider the following two subcases:

Subcase 2.1: Suppose $P(z)\neq a_mz^m$ or c_0. Then

$$\{(f^nP(f))^{(k)}\}^d\{(g^nP(g))^{(k)}\}^d=1$$

$$\Rightarrow (f^nP(f))^{(k)}(g^nP(g))^{(k)}=h$$

where $h^d = 1$.
Subcase 2.2: Suppose $P(z) = a_m z^m$ or c_0.
Let $P(z) = a_m z^m$ then from (5.3.17) we get

$$a_m^2 (f^{n+m})^{(k)}.(g^{n+m})^{(k)} = h \tag{5.3.18}$$

for some h such that $h^d = 1$.
Let f has a zero of multiplicity α_1 (≥ 1) then from (5.3.18), g must have a pole. Let g has a pole of multiplicity β_1 (≥ 1). Thus $(n+m)\alpha_1 - k = (n+m)\beta_1 + k$ then $(n+m)(\alpha_1 - \beta_1) = 2k$, which is a contradiction. Therefore f has no zeros. Similarly, we can prove $g \neq 0$, $f \neq \infty$, $g \neq \infty$. Therefore f and g are non-constant entire functions. Thus by Lemma 5.2.11 we get $f = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^k a_m^2 (c_1 c_2)^{n+m} \{(n+m)c\}^{2k} = h$.

Let $P(z) = c_0 (\neq 0)$. Then proceeding similarly as above we obtain $f = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = h$, where $h^d = 1$.
Case 3: $F^d = G^d$. This implies

$$\begin{aligned} \{(f^n P(f))^{(k)}\}^d &= \{(g^n P(g))^{(k)}\}^d \\ \Rightarrow (f^n P(f))^{(k)} &= h(g^n P(g))^{(k)}, \end{aligned} \tag{5.3.19}$$

for some h such that $h^d = 1$.
Let $P(z) = a_m z^m$ or $c_0 (\neq 0)$ then from (5.3.19) we get $(f^{n+m^*})^{(k)} = h(g^{n+m^*})^{(k)}$. Set $e^{n+m^*} = h$. Then $(f^{n+m^*})^{(k)} = ((eg)^{n+m^*})^{(k)}$. Following similarly as in Case 3. of Theorem 1 [34], we get $f = seg$ with $s^{n+m^*} = 1$. Set $t = se$, then $t^{(n+m^*)d} = 1$. This completes the proof of the theorem. ■

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