CHAPTER

4

WEIGHTED SHARING AND UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

4.1 Introduction, Definitions and Notations

In 1976 Yang [78] asked to investigate the relationship between two transcendental entire functions f and g if f and g share the value 0 CM and $f^{(1)}$ and $g^{(1)}$ share the value 1 CM. Many authors including Shibazaki [67], Yi [80, 81], Yang and Yi [82], Hua [28], Mues and Reinders [52], Lahiri [35, 36] Wang, Lei and Chen [70], Majumder [54] studied the question and further generalizations of it.

In 1990 Yi [81] gave an answer to the question of Yang [78] as follows.

Theorem 4.1.1 *[81] Let f and g be two non-constant meromor-*

The results of this chapter have been published in **Southeast Asian Bulletin of Mathematics (SEABM), see [65].**

phic functions. If f, g share the value 0 CM and $f^{(n)}$, $g^{(n)}$ share the value 1 CM, and $2\delta(0;f)+(n+2)\Theta(\infty;f)>n+3$, then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

Yi [83] proved the following Theorem which improved Theorem 4.1.1.

Theorem 4.1.2 *[83] Let f and g be two non-constant meromorphic functions sharing the value ∞ CM. If $f^{(n)}$, $g^{(n)}$ share the value 1 CM and*

$$N(r,0;f)+N(r,0;g)+(n+2)\overline{N}(r,f)<(\lambda+o(1))T(r)$$

for $r\in I$, a set of infinite linear measure and λ is a positive constant <1, then $f^{(n)}.g^{(n)}\equiv 1$ unless $f\equiv g$.

In 1997 Yi [84] improved Theorems 4.1.1–4.1.2.

Theorem 4.1.3 *[84] Let f and g be two non-constant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 IM. If*

$$5\delta(0;f)+(4n+7)\Theta(\infty;f)>4n+11$$

and

$$5\delta(0;g)+(4n+7)\Theta(\infty;g)>4n+11,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

Let k be a positive integer. An expression of the form

$$P(f)=f^{(k)}+a_{k-1}f^{(k-1)}+....+a_0f,$$

where $a_0,a_1,...,a_{k-1}$ are complex constants, is called a linear differential polynomial of f.

Recently Li and Li [48] generalized the above theorems by replacing the derivatives with their corresponding linear differential polynomials.

Theorem 4.1.4 *[48] Let f and g be two non-constant entire functions. Suppose that f, g share the value 0 CM and $P(f)$, $P(g)$ share the value 1 IM and $\delta(0;f) > \frac{4}{5}$. If $\rho(f) \neq 1$, then either $f \equiv g$ or $P(f).P(g) \equiv 1$.*

In this chapter we proved the uniqueness of non-constant homogeneous differential polynomials $P[f]$ and $P[g]$ under some conditions using the idea of weighted sharing.

4.2 Lemmas

In this section we present some lemmas which are required in the sequel.

Lemma 4.2.1 *[50] Let f be non-constant meromorphic function and $P[f]$ be a homogeneous differential polynomial of degree d and weight Γ, and let p be a positive integer. If $P[f] \not\equiv 0$ and $\Gamma \geq (k+2)d-(p+1)$, we have*

(i) $N_p(r,0;P) \leq T(r,P) - dT(r,f) + N_{p+\Gamma-d}(r,0;f^d) + S(r,f)$.

(ii) $N_p(r,0;P) \leq (\Gamma-d)\overline{N}(r,\infty;f) + N_{p+\Gamma-d}(r,0;f^d) + S(r,f)$.

Lemma 4.2.2 *[20] Let F and G be non-constant meromorphic functions sharing $(1,l)$, where l is a non-negative integer. Then* $\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \leq N(r,1;G)-\overline{N}(r,1;G)+S(r,F)+S(r,G)$, *when* $l=1$ *and*
$2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \leq N(r,1;G)-\overline{N}(r,1;G)+S(r,F)+S(r,G)$, *when* $l \geq 2$.

Lemma 4.2.3 *[8] Let F and G be non-constant meromorphic functions sharing $(1,l)$, where l is a non-negative integer. Then*
$\overline{N}_L(r,1;F) \leq \dfrac{1}{2}\overline{N}(r,\infty;F) + \dfrac{1}{2}\overline{N}(r,0;F) + S(r,F)$, *when* $l \geq 1$

$$\overline{N}_L(r,1;F) \le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + S(r,F), \;\; when \; l=0.$$

Lemma 4.2.4 *[20] Let F and G be non-constant meromorphic functions sharing $(1,0)$. Then*

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\le \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;G)\\ &+\overline{N}_{(2}(r,0;F)+2\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+N(r,1;G)\\ &+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G).\end{aligned}$$

Lemma 4.2.5 *[8] Let F and G be non-constant meromorphic functions share $(1,l)$, where l is a non-negative integer and $H \not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty,H) &\le \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;G)\\ &+\overline{N}_{(2}(r,0;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F^{(1)})\\ &+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G).\end{aligned}$$

Lemma 4.2.6 *[8] If F and G be non-constant meromorphic functions sharing $(1,l)$, where l is a non-negative integer and $H \not\equiv 0$. Then*

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\le \overline{N}(r,\infty;H)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)\\ &+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 4.2.7 *[30] Let f be a transcendental meromorphic function, $P[f]$ be a homogeneous differential polynomial of degree $d \ge 1$. Then*

$$\begin{aligned}dT(r,f) &\le \overline{N}(r,\infty;f)+\overline{N}(r,1;F)+N(r,0;f^d)\\ &-N_0(r,0;(P[f])^{(1)})+S(r,f),\end{aligned}$$

where $N_0(r,0;(P[f])^{(1)})$ denotes the counting function corresponding to the zeros of $(P[f])^{(1)}$ which are not the zeros of $P[f]$ and $P[f]-1$.

4.3 Theorems

In this section we discuss the main results of the chapter.

Theorem 4.3.1 *Let f and g be two non-constant meromorphic functions, a $(\not\equiv 0,\infty) \in S(f)\cap S(g)$. Suppose $P[f]$ and $P[g]$ are non-constant homogeneous differential polynomials of degree d, weight Γ and order k satisfying $\Gamma \geq (k+2)d-2$. If $P[f]$ and $P[g]$ share (a,l) with one of the following conditions:*
(i) $l \geq 2$ and

$$\min\ \big\{(4+\Gamma-d)\Theta(\infty,f)+2d\delta_{2+\Gamma-d}(0,f^d),\\ (4+\Gamma-d)\Theta(\infty,g)+2d\delta_{2+\Gamma-d}(0,g^d)\big\} > \Gamma+4, \qquad (4.3.1)$$

(ii) $l=1$ and

$$\min\ \big\{\tfrac{9+3\Gamma-3d}{2}\Theta(\infty,f)+2d\delta_{2+\Gamma-d}(0,f^d)+\tfrac{d}{2}\delta_{1+\Gamma-d}(0,f^d),\\ \tfrac{9+3\Gamma-3d}{2}\Theta(\infty,g)+2d\delta_{2+\Gamma-d}(0,g^d)\\ +\tfrac{d}{2}\delta_{1+\Gamma-d}(0,g^d)\big\} > \tfrac{3\Gamma+9}{2}, \qquad (4.3.2)$$

(iii) $l=0$ and

$$\min\ \big\{(7+4\Gamma-4d)\Theta(\infty,f)+2d\delta_{2+\Gamma-d}(0,f^d)+3d\delta_{1+\Gamma-d}(0,f^d),\\ (7+4\Gamma-4d)\Theta(\infty,g)+2d\delta_{2+\Gamma-d}(0,g^d)\\ +3d\delta_{1+\Gamma-d}(0,g^d)\big\} > 7+4\Gamma, \qquad (4.3.3)$$

then either $P[f]\equiv P[g]$ or $P[f].P[g]\equiv a^2$.

Proof. Let
$$F=\frac{P[f]}{a},\ G=\frac{P[g]}{a}.$$
Since $P[f]$ and $P[g]$ share (a,l), it follows that F, G share $(1,l)$ except at the zeros and poles of a.

Now we consider the following two cases:

Case 1: $H \not\equiv 0$. Assume that $l \geq 1$. By Nevanlinna second fundamental theorem, Lemma 4.2.5 and Lemma 4.2.6 we have

$$\begin{aligned}T(r,F)+T(r,G) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)\\ &+\overline{N}(r,0;G)+\overline{N}(r,\infty;H)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)\\ &+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)-\overline{N}_0(r,0;F^{(1)})-\overline{N}_0(r,0;G^{(1)})\\ &+S(r,F)+S(r,G) \leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_2(r,0;F)\\ &+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\ &+\overline{N}(r,1;G)+S(r,F)+S(r,G). \qquad (4.3.4)\end{aligned}$$

Subcase 1.1: Let $l \geq 2$. Then using Lemma 4.2.1 and Lemma 4.2.2, we get from (4.3.4)

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_2(r,0;F)\\ &+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\ &+\overline{N}(r,1;G)+S(r,F)+S(r,G) \leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\ &+N_2(r,0;F)+N_2(r,0;G)+N(r,1;G)+S(r,F)+S(r,G),\\ \Rightarrow\ T(r,F) &\leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)\\ &+N_2(r,0;G)-m(r,1;G)+S(r,F)+S(r,G)\\ \text{i.e.,}\ dT(r,f) &\leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{2+\Gamma-d}(r,0;f^d)\\ &+N_{2+\Gamma-d}(r,0;g^d)+(\Gamma-d)\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\ &\leq 2\overline{N}(r,\infty;f)+N_{2+\Gamma-d}(r,0;f^d)+N_{2+\Gamma-d}(r,0;g^d)\\ &+(2+\Gamma-d)\overline{N}(r,\infty;g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly we get,

$$\begin{aligned}dT(r,g) &\leq 2\overline{N}(r,\infty;g)+N_{2+\Gamma-d}(r,0;g^d)+N_{2+\Gamma-d}(r,0;f^d)\\ &+ (2+\Gamma-d)\overline{N}(r,\infty;f)+S(r,f)+S(r,g).\end{aligned}$$

Adding above two inequalities we get

$$dT(r,f)+dT(r,g)\le(4+\Gamma-d)\overline{N}(r,\infty;f)+2N_{2+\Gamma-d}(r,0;f^d)$$
$$+2N_{2+\Gamma-d}(r,0;g^d)+(4+\Gamma-d)\overline{N}(r,\infty;g)+S(r,f)+S(r,g)$$
$$\Rightarrow\{(4+\Gamma-d)\Theta(\infty,f)+2d\delta_{2+\Gamma-d}(0,f^d)-(4+\Gamma)\}T(r,f)+$$
$$\{(4+\Gamma-d)\Theta(\infty,g)+2d\delta_{2+\Gamma-d}(0,g^d)-(4+\Gamma)\}T(r,g)$$
$$\le S(r,f)+S(r,g),$$

which contradicts (4.3.1).

Subcase 1.2: Let $l=1$. From (4.3.4) and Lemmas 4.2.1–4.2.3 we get

$$T(r,F)+T(r,G)\le2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_2(r,0;F)$$
$$+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)$$
$$+\overline{N}(r,1;G)+S(r,F)+S(r,G)\le2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)$$
$$+N_2(r,0;F)+N_2(r,0;G)+N(r,1;G)+\overline{N}_L(r,1;F)$$
$$+S(r,F)+S(r,G),$$

$$\Rightarrow\quad T(r,F)\le2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_2(r,0;F)$$
$$+N_2(r,0;G)-m(r,1;G)+\tfrac{1}{2}\overline{N}(r,\infty;F)+\tfrac{1}{2}\overline{N}(r,0;F)$$
$$+S(r,F)+S(r,G).$$

Therefore,

$$dT(r,f)\le\tfrac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{2+\Gamma-d}(r,0;f^d)$$
$$+N_{2+\Gamma-d}(r,0;g^d)+(\Gamma-d)\overline{N}(r,\infty;g)+\tfrac{1}{2}N_{1+\Gamma-d}(r,0;f^d)$$
$$+\tfrac{\Gamma-d}{2}\overline{N}(r,\infty;f)+S(r,f)+S(r,g)$$

$$i.e.,\quad dT(r,f)\le\tfrac{5+\Gamma-d}{2}\overline{N}(r,\infty;f)+N_{2+\Gamma-d}(r,0;f^d)$$
$$+N_{2+\Gamma-d}(r,0;g^d)+(2+\Gamma-d)\overline{N}(r,\infty;g)$$
$$+\tfrac{1}{2}N_{1+\Gamma-d}(r,0;f^d)+S(r,f)+S(r,g).\tag{4.3.5}$$

Similarly,

$$dT(r,g) \le \tfrac{5+\Gamma-d}{2}\overline{N}(r,\infty;g) + N_{2+\Gamma-d}(r,0;f^d)$$
$$+N_{2+\Gamma-d}(r,0;g^d) + (2+\Gamma-d)\overline{N}(r,\infty;f)$$
$$+\tfrac{1}{2}N_{1+\Gamma-d}(r,0;g^d) + S(r,f) + S(r,g). \qquad (4.3.6)$$

Adding (4.3.5) and (4.3.6) we get

$$dT(r,f) + dT(r,g) \le \tfrac{9+3\Gamma-3d}{2}\overline{N}(r,\infty;f) + 2N_{2+\Gamma-d}(r,0;f^d)$$
$$+2N_{2+\Gamma-d}(r,0;g^d) + \tfrac{9+3\Gamma-3d}{2}\overline{N}(r,\infty;g) + \tfrac{1}{2}N_{1+\Gamma-d}(r,0;f^d)$$
$$+\tfrac{1}{2}N_{1+\Gamma-d}(r,0;g^d) + S(r,f) + S(r,g)$$
$$\Rightarrow \{\tfrac{9+3\Gamma-3d}{2}\Theta(\infty,f) + 2d\delta_{2+\Gamma-d}(0,f^d) + \tfrac{d}{2}\delta_{1+\Gamma-d}(0,f^d)$$
$$-\tfrac{9+3\Gamma}{2}\}T(r,f) + \{\tfrac{9+3\Gamma-3d}{2}\Theta(\infty,g) + 2d\delta_{2+\Gamma-d}(0,g^d)$$
$$+\tfrac{d}{2}\delta_{1+\Gamma-d}(0,g^d) - \tfrac{9+3\Gamma}{2}\}T(r,g) \le S(r,f) + S(r,g),$$

which contradicts our assumption (4.3.2).

Subcase 1.3: Let $l = 0$. By Nevanlinna second fundamental theorem, Lemma 4.2.1 and Lemma 4.2.4 we get

$$T(r,F) + T(r,G) \le \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F)$$
$$+\overline{N}(r,0;G) + \overline{N}(r,1;F) + \overline{N}(r,1;G) - \overline{N}_0(r,0;F^{(1)})$$
$$-\overline{N}_0(r,0;G^{(1)}) + S(r,F) + S(r,G) \le 2\overline{N}(r,\infty;F)$$
$$+2\overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + N(r,1;G)$$
$$+2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G),$$
$$\Rightarrow T(r,F) \le 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N_2(r,0;F)$$
$$+N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,0;F) + \overline{N}(r,\infty;G)$$
$$+\overline{N}(r,0;G) + S(r,F) + S(r,G),$$

that is,

$$\begin{aligned} dT(r,f) &\leq 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{2+\Gamma-d}(r,0;f^{d}) \\ &+ N_{2+\Gamma-d}(r,0;g^{d})+(\Gamma-d)\overline{N}(r,\infty;g)+2N_{1+\Gamma-d}(r,0;f^{d}) \\ &+ 2(\Gamma-d)\overline{N}(r,\infty;f)+(\Gamma-d)\overline{N}(r,\infty;g) \\ &+ N_{1+\Gamma-d}(r,0;g^{d})+S(r,f)+S(r,g) \\ &\leq (4+2\Gamma-2d)\overline{N}(r,\infty;f)+(3+2\Gamma-2d)\overline{N}(r,\infty;g) \\ &+ N_{2+\Gamma-d}(r,0;f^{d})+N_{2+\Gamma-d}(r,0;g^{d})+2N_{1+\Gamma-d}(r,0;f^{d}) \\ &+ N_{1+\Gamma-d}(r,0;g^{d})+S(r,f)+S(r,g). \qquad (4.3.7) \end{aligned}$$

Similarly,

$$\begin{aligned} dT(r,g) &\leq (4+2\Gamma-2d)\overline{N}(r,\infty;g)+(3+2\Gamma-2d)\overline{N}(r,\infty;f) \\ &+ N_{2+\Gamma-d}(r,0;g^{d})+N_{2+\Gamma-d}(r,0;f^{d})+2N_{1+\Gamma-d}(r,0;g^{d}) \\ &+ N_{1+\Gamma-d}(r,0;f^{d})+S(r,f)+S(r,g). \qquad (4.3.8) \end{aligned}$$

Adding (4.3.7) and (4.3.8) we get

$$\begin{gathered} dT(r,f)+dT(r,g)\leq(7+4\Gamma-4d)\overline{N}(r,\infty;f)+(7+4\Gamma-4d) \\ \overline{N}(r,\infty;f)+2N_{2+\Gamma-d}(r,0;f^{d})+2N_{2+\Gamma-d}(r,0;g^{d}) \\ +3N_{1+\Gamma-d}(r,0;f^{d})+3N_{1+\Gamma-d}(r,0;g^{d})+S(r,f)+S(r,g), \\ \Rightarrow\{(7+4\Gamma-4d)\Theta(\infty,f)+2d\delta_{2+\Gamma-d}(0,f^{d})+3d\delta_{1+\Gamma-d}(0,f^{d}) \\ -(7+4\Gamma)\}T(r,f)+\{(7+4\Gamma-4d)\Theta(\infty,g)+2d\delta_{2+\Gamma-d}(0,g^{d}) \\ +3d\delta_{1+\Gamma-d}(0,g^{d})-(7+4\Gamma)\}T(r,g)\leq S(r,f)+S(r,g), \end{gathered}$$

which contradicts (4.3.3).

Case 2: $H\equiv 0$ and integrating twice we obtain

$$\frac{1}{G-1}=\frac{A}{F-1}+B,$$

where $A\neq 0$ and B are constants. Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)} \qquad (4.3.9)$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG-(B+1)}. \tag{4.3.10}$$

Next we consider the following three subcases:
Subcase 2.1: $B \neq 0, -1$. Then from (4.3.10) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 4.2.1 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\tfrac{B+1}{B};G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + N_{1+\Gamma-d}(r,0;g^d) \\ &\quad + \overline{N}(r,\infty;F) + S(r,G), \end{aligned}$$

$$\begin{aligned} i.e., \; dT(r,g) \leq \overline{N}(r,\infty;f) + N_{1+\Gamma-d}(r,0;g^d) + \overline{N}(r,\infty;g) \\ + S(r,f) + S(r,g). \end{aligned} \tag{4.3.11}$$

If $A-B-1 \neq 0$, then it follows from (4.3.9) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r, 0; G).$$

Again by Nevanlinna second fundamental theorem and Lemma 4.2.1 we obtain

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r, \frac{-A+B+1}{B+1}; F) + S(r,F)$$

$$\begin{aligned} \Rightarrow dT(r,f) \;&\leq\; \overline{N}(r,\infty;f) + N_{1+\Gamma-d}(r,0;f^d) + (\Gamma-d)\overline{N}(r,\infty;g) \\ &+\; N_{1+\Gamma-d}(r,0;g^d) + S(r,f) + S(r,g). \end{aligned} \tag{4.3.12}$$

Combining (4.3.11) and (4.3.12) we get

$$dT(r,f)+dT(r,g)\le N_{1+\Gamma-d}(r,0;f^d)+2\overline{N}(r,\infty;f)$$
$$+2N_{1+\Gamma-d}(r,0;g^d)+(1+\Gamma-d)\overline{N}(r,\infty;g)+S(r,f)+S(r,g)$$
$$\Rightarrow\{2\Theta(\infty,f)+d\delta_{1+\Gamma-d}(0,f^d)-2\}T(r,f)+$$
$$\{(1+\Gamma-d)\Theta(\infty,g)+2d\delta_{1+\Gamma-d}(0,g^d)-(1+\Gamma)\}T(r,g)$$
$$\le S(r,f)+S(r,g),$$

which contradicts (4.3.1)-(4.3.3).

Hence $A-B-1=0$. Then by (4.3.9)

$$\overline{N}(r,0;F+\frac{1}{B})=\overline{N}(r,\infty;G).$$

Again by Nevanlinna second fundamental theorem

$$\begin{aligned}T(r,F)&\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;F+\frac{1}{B})+S(r,F)\\&\le \overline{N}(r,\infty;f)+T(r,F)-dT(r,f)+N_{1+\Gamma-d}(r,0;f^d)\\&+\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\end{aligned}$$

$$i.e.,\ dT(r,f)\le\overline{N}(r,\infty;f)+N_{1+\Gamma-d}(r,0;f^d)+\overline{N}(r,\infty;g)$$
$$+S(r,f)+S(r,g). \tag{4.3.13}$$

Combining (4.3.11) and (4.3.13), we have

$$dT(r,f)+dT(r,g)\le N_{1+\Gamma-d}(r,0;f^d)+2\overline{N}(r,\infty;f)$$
$$+N_{1+\Gamma-d}(r,0;g^d)+2\overline{N}(r,\infty;g)+S(r,f)+S(r,g)$$
$$\Rightarrow\{2\Theta(\infty,f)+d\delta_{1+\Gamma-d}(0,f^d)-2\}T(r,f)+$$
$$\{2\Theta(\infty,g)+d\delta_{2+\Gamma-d}(0,g^d)-2\}T(r,g)$$
$$\le S(r,f)+S(r,g),$$

which violates (4.3.1)-(4.3.3).

Subcase 2.2: $B = -1$. Then

$$G = \frac{A}{A+1-F}$$

and

$$F = \frac{(1+A)G - A}{G}.$$

If $A + 1 \neq 0$, then we obtain

$$\overline{N}(r, A+1; F) = \overline{N}(r, \infty; G),$$

$$\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r, 0; F).$$

By similar argument as in Subcase 2.1, we have a contradiction. Therefore, $A + 1 = 0$. Then $FG = 1 \Rightarrow P[f].P[g] \equiv a^2$.

Subcase 2.3: $B = 0$. Then (4.3.9) and (4.3.10) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$.

If $A - 1 \neq 0$, $N(r, 0; A - 1 + F) = N(r, 0; G)$ and $N(r, \frac{A-1}{A}; G) = N(r, 0; F)$. Proceeding similarly as in Subcase 2.1, we get a contradiction. Therefore, $A - 1 = 0$. Then $F \equiv G$ i.e., $P[f] \equiv P[g]$. This completes the proof of Theorem 4.3.1. ■

Theorem 4.3.2 *Let f and g be two non-constant meromorphic functions, a $(\not\equiv 0, \infty) \in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$ are non-constant homogeneous differential polynomials of degree d, weight Γ and order k satisfying $\Gamma \geq (k+2)d - 2$. Let f and g share the values 0 CM and ∞ IM. If $P[f]$ and $P[g]$ share (a, l) with one of the following conditions:*

(i) $l \geq 2$ and

$$(4 + \Gamma - d)\Theta(\infty, f) + 2d\delta_{2+\Gamma-d}(0, f^d) > \Gamma + 4,$$

(ii) $l = 1$ and

$$\frac{9 + 3\Gamma - 3d}{2}\Theta(\infty, f) + 2d\delta_{2+\Gamma-d}(0, f^d) + \frac{d}{2}\delta_{1+\Gamma-d}(0, f^d) > \frac{3\Gamma + 9}{2},$$

(iii) $l=0$ *and*

$$(7+4\Gamma-4d)\Theta(\infty,f)+2d\delta_{2+\Gamma-d}(0,f^{d})+3d\delta_{1+\Gamma-d}(0,f^{d})>7+4\Gamma,$$

then either $P[f]\equiv P[g]$ *or* $P[f].P[g]\equiv a^{2}$.

Proof. Let

$$F=\frac{P[f]}{a},\ G=\frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a,l), it follows that F, G share $(1,l)$ except at the zeros and poles of a. By Lemma 4.2.1 and Lemma 4.2.7, we get

$$\begin{aligned} dT(r,f) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,1;F)+N(r,0;f^{d})+S(r,f)\\ &= \overline{N}(r,\infty;g)+\overline{N}(r,1;G)+N(r,0;g^{d})+S(r,f)\\ &\leq (1+\Gamma+d)T(r,g)+S(r,f)+S(r,g). \qquad (4.3.14)\end{aligned}$$

Similarly,

$$dT(r,g)\leq(1+\Gamma+d)T(r,f)+S(r,f)+S(r,g). \qquad (4.3.15)$$

From (4.3.14) and (4.3.15) we get $S(r,f)=S(r,g)$. The rest of the proof is similar to that of Theorem 4.3.1. ■

The following Theorem follows from Theorem 4.3.2 and so we omit its proof.

Theorem 4.3.3 *Let* f *and* g *be two non-constant entire functions,* $a\ (\not\equiv 0,\infty)\in S(f)\cap S(g)$. *Suppose* $P[f]$ *and* $P[g]$ *are non-constant homogeneous differential polynomials of degree* d, *weight* Γ *and order* k *satisfying* $\Gamma\geq(k+2)d-2$. *If* f *and* g *share the value* 0 *CM,* $P[f]$ *and* $P[g]$ *share* (a,l) *with one of the following conditions:*

(i) $l\geq 2$ *and*

$$\delta_{2+\Gamma-d}(0,f^{d})>\frac{1}{2},$$

(ii) $l = 1$ *and*

$$2\delta_{2+\Gamma-d}(0, f^d) + \frac{1}{2}\delta_{1+\Gamma-d}(0, f^d) > \frac{3}{2},$$

(iii) $l = 0$ *and*

$$2\delta_{2+\Gamma-d}(0, f^d) + 3\delta_{1+\Gamma-d}(0, f^d) > 4,$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Corollary 4.3.1 *Let* f *and* g *be two non-constant entire functions such that* f *and* g *share the value* 0 *CM. Let* $P(f)$ *and* $P(g)$ *be non-constant linear differential polynomials of* f *and* g *respectively. If* $P(f)$ *and* $P(g)$ *share* $(1, l)$ *with one of the following conditions:*
(i) $l \geq 2$ *and*

$$\delta_{k+2}(0, f) > \frac{1}{2},$$

(ii) $l = 1$ *and*

$$2\delta_{k+2}(0, f) + \frac{1}{2}\delta_{k+1}(0, f) > \frac{3}{2},$$

(iii) $l = 0$ *and*

$$2\delta_{k+2}(0, f) + 3\delta_{k+1}(0, f) > 4,$$

then either $f \equiv g$ *or* $P(f)P(g) \equiv 1$ *under any one of the following conditions:*
(i) $\rho(f) \neq 1$,
(ii) $\rho(f) = 1$ *and*
(a) f *has at most finite number of zeros, or*
(b) f *has infinitely many zeros and* f *is of minimal type.*

Proof. By Theorem 4.3.3 we get either $P(f) \equiv P(g)$ or $P(f).P(g) \equiv 1$. Let $P(f) \equiv P(g)$. Then $P(f - g) \equiv 0$. Proceeding similarly as in the proof of Corollary 1.2 of Lahiri and Pal [41], we can prove that $f \equiv g$. ■

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