Chapter 1

Introduction and Review

1.1 Introduction

General Relativity (GR) is arguably considered as the most elegant theory in physics. GR pronounces gravitation as the manifestation of space-time geometry. GR has been extremely successful at describing observations and passed all experimental tests conducted so far. GR based cosmology is the cornerstone of the current hot Big Bang description of our Universe. However, some unexpected components turn out to make up most of the Universe's mass-energy budget in the GR description of the observations at large scales.

Recent cosmological observations suggest that the Universe is undergoing a phase of accelerated expansion. The explanation of such accelerated expansion in the purview of general relativity requires the presence of a large amount of some exotic form of energy density with negative pressure, the so-called dark energy. On the other hand, the amount of luminous matter in galaxies is found insufficient to explain the observed galactic rotation curves and thereby the existence of non-luminous or dark matter, that neither interacts with radiation nor with the conventional matter except through the gravitational field or through some feeble interaction, has to be assumed. This dark matter component is also required to be non-relativistic (i.e. cold) in view of structure formation. It appears from a wide variety of astrophysical observations that ordinary baryonic matter constitutes just 4.9% of the energy density in the Universe while dark matter composes about 26.8% and the dark energy contributes most – about 68.3% of the energy density in the Universe [1].

Dark energy models in the framework of general relativity suffer from fine-tuned, unnatural properties as will be elaborated in the subsequent sections. On the other hand, despite extensive efforts, dark matter is still undetected. The nature of dark matter is also not clear. Dark energy and dark matter are two of the major outstanding issues in physics and cosmology today.

1.2 General Relativity:

The journey of exploring the laws of nature had crossed a milestone when Nicolus Copernicus predicted the actual planetary motion and his student Galileo Galilei proved his teacher's prediction by his revolutionary discovery of Telescope. The laws of planetary motion by Johannes Kepler, created a perception about motion inside the solar system which was given a proper and generalized dimension by Isac Newton with his revolutionary Theory of Gravitation [2]. The invention of the telescope and the theory of Gravitation explored the gateway of gathering knowledge about the phenomena not only beyond the Earth but beyond the solar system also. Newton's law of gravitation ($F = \frac{Gm_1m_2}{r^2}$, where F stands for gravitational force, G is the gravitational constant ($= 6.674 \times 10^{-11} Nm^2Kg^{-2}$), m_1 and m_2 are the masses of the particles and r is the distance between these two particles) was highly successful in explaining planetary dynamics. Transforming the Newton's equation into the form of gravitational field using Poisson's equation, one finds:

$$\nabla^2\phi(r,t) = 4\pi G\rho(r,t)$$

where, ∇ is the spatial Laplace operator, $\phi(r,t)$ is the gravitational scalar potential, and $\rho(r,t)$ is the density of the gravitating object. The above expression shows that the gravitational potential varies only with spatial derivatives, not with time derivative, i.e. if the matter distribution varies, the gravitational potential changes instantaneously with the infinite speed which was considered as a prime drawback of Newton's theory of gravitation.

Newton's law of dynamics is based on Galilean transformation, but the constancy of speed ($c = 1/\sqrt{\mu_0\epsilon_0}$) of an electromagnetic wave in Maxwell's theory cannot be explained by the Galilean transformation. If we consider that Galilean transformation and Maxwell's equation both are correct, an absolute frame of reference (called 'ether') had to be introduced where the electromagnetic waves can be propagated at speed $c = 1/\sqrt{\mu_0 \epsilon_0}$. But Michelson-Morley experiment using optical interferometer, invented by Michelson himself, didn't find any evidence of the existence of an absolute reference frame, rather the experiment showed the constancy of speed of light irrespective of the motion of the observer.

In the beginning of the twentieth century, Albert Einstein formulated the Spacial Theory of Relativity (STR) [3] based on two simple postulates: (a) the laws of physics are same in all inertial frames and (b) the speed of light in free space has same value c in all inertial frame. Probably the most revolutionary effect of these two postulates is that space and time are intertwined leading to a single continuum known as space-time. A point to be noted that the special theory of relativity rests on Euclidean geometry and is valid only for inertial observers.

Based on the Principle of Equivalence, Principle of General Covariance and generalizing the Euclidean space-time continuum of special relativity to curved (Riemannian) space-time geometry, Einstein formulated General Theory of Relativity (GR) [4] during the period 1907-15. The curved geometry is essentially described through metric tensor ($g_{\mu\nu}$) which is related to incremental line element as $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. GR describes gravity not as a force but as a geometric property of space-time. Gravity is a warping of space-time as per GR.

The field equations of GR are given by:

$$G_{\mu\nu} = -\frac{8\pi G T_{\mu\nu}}{c^4}, \tag{1.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is the energy-momentum tensor for matter.

Ricci tensor $R_{\mu\nu}$ can be expressed in terms of metric tensor $g_{\mu\nu}$ via Riemannian connection $\Gamma^\lambda_{\mu\nu}$ as,

$$R_{\mu\nu} = \frac{\delta \Gamma^\lambda_{\lambda\mu}}{\delta x^\nu} - \frac{\delta \Gamma^\lambda_{\mu\nu}}{\delta x^\lambda} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} \tag{1.2}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\delta g_{\sigma\mu}}{\delta x^\nu} - \frac{\delta g_{\mu\nu}}{\delta x^\sigma} - \frac{\delta g_{\nu\sigma}}{\delta x^\mu} \right) \tag{1.3}$$

For a given energy-momentum tensor, one would look for solution of the Einstein field equations in terms of metrics that determine the space time geometry for the given source.

A simple but important case when an observer is at a location outside the source. All the components of energy momentum tensor are zero ($T_{\mu\nu} = 0$) outside the source. The Einstein field equations in such a situation turn into

$$R_{\mu\nu} = 0 \tag{1.4}$$

The general static spherically symmetric metric is given by [5];

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.5}$$

where $B(r)$ and $A(r)$ are two unknown metric coefficients which are the function of r only.

The well known static spherically symmetric vacuum ($T_{\mu\nu} = 0$) solution of the above Einstein's field equation (1.1), is the Schwarzschild solution

$$ds^2 = [1 - \frac{2m}{r}]dt^2 - [1 - \frac{2m}{r}]^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.6}$$

where $m = \frac{MG}{c^2}$ and M stands for the mass of the gravitating object, G is the gravitational constant and c is the speed of light.

The predictions of GR have been tested by a variety of experiments with increasingly high precision and the theory has passed all such tests conducted till now.

1.3 Dark Energy

Einstein studied the nature of the Universe by using his field equations. On the apparent observational basis, it was thought that the astronomical objects like stars, galaxies are static, i.e. these are not moving at all. He found from his theory that the nature of the Universe is dynamic. The same conclusion was also reached by Friedmann [6] and Lemaitre [7] by studying the nature of the Universe using Einstein's field equation and employing Robertson-Walker metric. To tally with the contemporary thought of static Universe, Einstein introduced a constant in his field equations, called Einstein's Cosmological constant.

The Friedmann-Lemaitre-Robertson-Walker (FLRW) model of the Universe is outlined below. If one considers that the Universe is homogeneous and isotropic, it can be described by the following generic metric :

$$d\tau^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2] \tag{1.7}$$

where $a(t)$ is the cosmological scale factor, and k signifies the curvature of the Universe. The above metric is known after Robertson and Walker.

Solving the equation (1.7) using Einstein's field equations the following expressions are found,

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4}{3}\pi G(\rho + 3p) \tag{1.8}$$

and

$$\frac{\dot{a}^2(t) + k}{a^2(t)} = \frac{8\pi G\rho}{3} \tag{1.9}$$

Where ρ is the effective mass density and p is the pressure.

In 1929, Hubble collected the red-shift versus luminosity distance data of different Galaxies. During the experimental observation of the measurement of the redshift of nearer Galaxies by Hubble and his team,, distance vs redshift relation was found linear when the value of redshift is less than 0.1. The mathematical base of the Hubble experiment is as follows [8]:

The expression of apparent luminosity (l) is given by,

$$l = \frac{L}{4\pi {d_L}^2} \tag{1.10}$$

where L is the absolute luminosity of a source at a distance of d_L.

At large distance, especially in cosmological distances, the expression needs to be modified for the following reasons:

1. If light from a distant luminous object reaches the Earth at time t_0, then the effective area of the sphere drawn around the luminous object and passing through the Earth will be equal to $4\pi r^2 a^2(t_0)$, where r is the coordinate distance between the earth and the luminous light source.

2. The rate of arrival of the photons is lower than the rate at which they are emitted by the redshift factor $a(t_1)/a(t_0) = 1/(1+z)$.

3. the energy $h\nu_0$ of a received photon in the Earth is less than the energy $h\nu_1$ of the emitted photon from the light source by the same redshift factor $1/(z+1)$.

Therefore the effective apparent luminosity on the Earth can be expressed by,

$$l = \frac{L}{4\pi r^2 a^2(t_0)(1+z)^2} \tag{1.11}$$

Comparing with the equation (1.10) and (1.11), one can express,

$$d_L = a(t_0)r(1+z) \tag{1.12}$$

When $z << 1$, the relation between luminosity distance and redshift can be expressed as power series in the form of the redshift $1+z = a(t_0)/a(t_1)$ and the look-back time $t_0 - t_1$, is given by,

$$z = H_0(t_0 - t_1) + \frac{1}{2}(q_0 + 2){H_0}^2(t_0 - t_1)^2 + ... \tag{1.13}$$

where H_0 is the Hubble constant ($H_0 = \dot{a_0}/a_0$), and q_0 is the deceleration parameter, expressed by,

$$q_0 = \frac{-1}{{H_0}^2 a(t_0)} \frac{d^2a(t)}{dt^2} \tag{1.14}$$

The expression (1.13) can be inverted to express the Hubble constant (H_0) in the form of redshift,

$$H_0(t_0 - t_1) = z - \frac{1}{2}(q_0 + 2)z^2 + ... \tag{1.15}$$

The coordinate distance r can be expressed from the relation $[\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dx}{\sqrt{1-kx^2}}]$,

$$\frac{t_0 - t_1}{a(t_0)} + \frac{H_0(t_0 - t_1)^2}{2a(t_0)} + ... = r + ... \tag{1.16}$$

dots in the right hand side denotes the third and higher order terms of r. Using the equation (1.15), the solution is found,

$$ra(t_0)H_0 = z - \frac{1}{2}(1 + q_0)z^2 + ... \tag{1.17}$$

Which gives the expression of luminosity distance,

$$d_L = H_0^{-1}[z + \frac{1}{2}(1 - q_0)z^2 + ...] \tag{1.18}$$

For small z, the higher-order terms in z can be neglected and the above equation turns to Hubble's relation. For higher red-shifts, the higher-order terms as well as the deceleration parameter $q_0 = -\ddot{a}(t_0)\frac{at_0}{\dot{a}^2(t_0)}$ will come in consideration. Determination of the value H_0 and q_0 is a big challenge in astronomy because it will help us to know the dynamic nature of the Universe. The expression of luminosity distance is not useful for the redshifts of the order of unity as power series expansion will not be a smart approach in such a scenario. In that case, we have to adopt the measurement technique through the dynamic theory of expansion. To achieve this, the approach of FLRW model has been adopted using Einstein's field equation and Robertson-Walker metric as mentioned earlier through equations (1.7), (1.8) and (1.9):

Critical density plays an important role to define the state of Universe and the critical density (ρ_0) is defined by the density of the Universe which makes the curvature of the Universe flat, i,e. $k = 0$. The Universe is considered as closed if $\rho > \rho_0$ and open if $\rho < \rho_0$.

From equation (1.9), one can get the expression of critical density,

$$\rho_0 = \frac{3H_0}{8\pi G} \tag{1.19}$$

The expressions of proper energy density for different states of Universe is given by the relation $\rho \propto a^{-3-3w}$, where ω is the constant of the equation of state$(= p/\rho)$:

Considering non-relativistic matter: $p = 0$

$$\rho = \rho_0 (a(t)/a_0)^{-3} \tag{1.20}$$

For relativistic matter: $p = \rho/3$

$$\rho = \rho_0 (a(t)/a_0)^{-4} \tag{1.21}$$

Considering vacuum energy: $p = -\rho$

$$\rho = \rho_0 \tag{1.22}$$

Recent measurements indicate that the Universe is flat, i.e., $k \simeq 0$. Therefore the expression of effective energy density considering the mixture of non-relativistic matter, relativistic matter and vacuum energy, given by,

$$\rho = \frac{3{H_0}^2}{8\pi G}[\Omega_\Lambda + \Omega_M (\frac{a_0}{a})^3 + \Omega_R (\frac{a_0}{a})^4] \tag{1.23}$$

where present epoch energy densities of vacuum (ρ_{Λ_0}), non-relativistic matter (ρ_{M_0}) and relativistic matter (ρ_{R_0}) are given by,

$$\rho_{\Lambda_0} = \frac{3{H_0}^2 \Omega_\Lambda}{8\pi G} \tag{1.24}$$

$$\rho_{M_0} = \frac{3{H_0}^2 \Omega_M}{8\pi G} \tag{1.25}$$

$$\rho_{R_0} = \frac{3{H_0}^2 \Omega_R}{8\pi G} \tag{1.26}$$

The equation (1.9) suggests,

$$\Omega_\Lambda + \Omega_M + \Omega_R = 1 \tag{1.27}$$

Now using equation (1.9) and (1.23), one may write:

$$\begin{aligned} dt &= \frac{dx}{H_0 x\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4}}} \\ &= \frac{-dz}{H_0(1+z)\sqrt{\Omega_\Lambda + \Omega_M (1+z)^3 + \Omega_R (1+z)^4}} \end{aligned} \tag{1.28}$$

where $x = a/a_0 = 1/(1+z)$ and z signifies the redshift. From the above equation the expression of the co-ordinate distance $(r(z))$ of the source can be deduced and employing the relation of co-ordinate distance and luminosity distance ($d_L(z) = a_0 r(z)(1+z)$), the expression of luminosity distance can be found,

$$d_L(z) = \frac{1+z}{H_0}\int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{1.29}$$

From the above relation, it is quite clear that if the variation of luminosity distance with redshift can be determined experimentally, the value of Ω_{Λ_0}, Ω_{M_0} and Ω_{R_0} can be deduced analytically. In the late 1990s, this job was done by two independent Supernovae search teams led by Riess(1998) and Perlmutter(1999) [9]. They explored the evidence of accelerating Universe by the survey of type Ia Supernovae as shown in figure (1.1). As the peak brightness of Supernovae is quite uniform, the object was selected as a standard candle. Considering higher red-shift Supernovas, each of the observatory teams found that distance vs redshift relation is not linear as demonstrated in the equation, whereas the relation found about to linear for the observed Supernovas of redshift less than 0.1. From the high red-shift Ia-SNa, it was found that the earlier expansion rate was slower than that is today and with the measurement of luminosity distance of low redshift Supernovae observation and statistical data analysis of density parameters showed that the present era is dark energy dominated era with flat curvature. Observations on Cosmic Microwave Background radiation also support the geometrical nature of the Universe. The search for the biggest mystery of physics begins from there, the source of the energy behind this accelerating Universe is still unrevealed and the energy is known as Dark Energy.

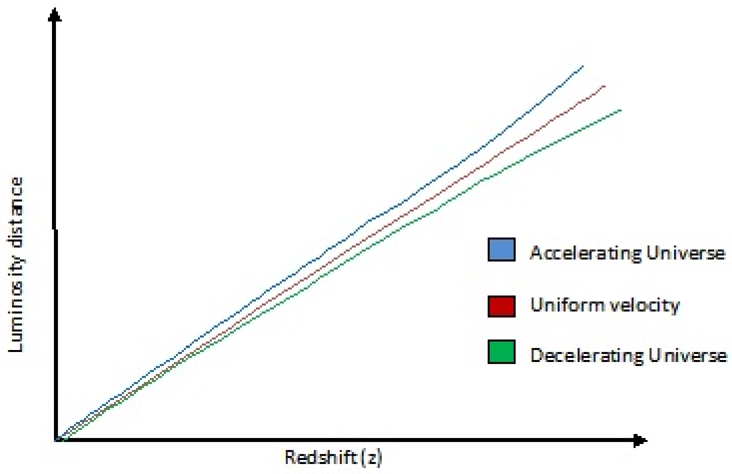

FIGURE 1.1: Luminosity distance vs. cosmological redshift variation curve for type Ia Supernovae

1.4 Dark matter

In 1919, during the observation of Solar Eclipse near the Hyades star cluster, the gravitational deflection angle of the light from the stars indicated the existence of extra mass. Dutch astronomer Jacobus Kapteyn predicted the existence of extra mass by using stellar velocities [10]. In 1930s the concept became stronger when F. Zwicky was calculating the stellar velocities of the Coma cluster by using Virial theorem and he found the evidence of extra unseen mass, addressed as dark matter [11].

After few decades, the strong evidence of Dark Matter was provided by Vera Rubin and Kent Ford [12] with an observation by spectrograph measuring the radius vs velocity curve of the edge of Andromeda Galaxy (a spiral galaxy) and they found that the rotation curve is almost flat.

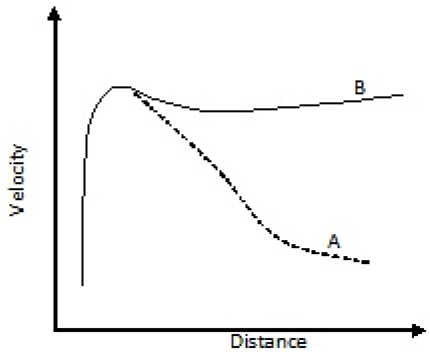

FIGURE 1.2: Flat rotation curve of a spiral galaxy

The dotted line in figure (1.2) was the desired rotation curve without any existence of Dark Matter but Rubin and Ford found the continuous lined flat curve which indicates the presence of extra masses(Dark Matter) in spiral galaxies.

Gravitational lensing of light by massive objects coming from a distance source (like as quasar) is considered as the strong evidence of the presence of Dark Matter. Measuring the distortion geometry due to gravitational lensing, the total mass of the lensing object can be deduced [13]. The Dark Matter distribution has been deduced using the gravitational lensing phenomenon.

Several probable explanations have been provided to theorize the entities, Dark Energy and Dark Matter.

1.5 Candidates to explain Dark Energy:

The FLRW model (equations (1.8) and (1.9)) suggests for expansion of the Universe with time. Any deceleration of the Universe can easily be explained in terms of the deceleration parameter. The effective density of Universe (ρ) and pressure (p) are positive quantities considering that it is composed of normal matters and radiation and the expansion should not be accelerated with time as dictate by equation (1.8). But after the discovery of the accelerating Universe, the perception was changed and it has led inclusion of a new component of the energy-momentum tensor of the Universe having negative pressure, addressed as a dark energy component.

A significant candidate of dark energy component is Cosmological Constant (Λ) which was first-time introduced by Einstein himself to balance the dynamic nature of the Universe. The Einstein's field equations can be expressed as,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\lambda}_{\lambda} = -8\pi G T_{\mu\nu} \tag{1.30}$$

With the introduction of Λ the energy-momentum tensor $T_{\mu\nu}$ can be replaced by effective energy momentum tensor $T_{\mu\nu} + \Lambda g_{\mu\nu}$ and consequently equation (1.8) and (1.9) will be re-written as,

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4}{3}\pi G(\rho + 3p) + \frac{\Lambda}{3} \tag{1.31}$$

and

$$\frac{\dot{a}^2(t) + K}{a^2(t)} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{1.32}$$

Equation (1.31) shows the contribution of Λ is negative to the pressure term and hence it exhibits repulsive nature. The energy which causes the repulsion is greater than the gravitational energy, resulting the cosmological expansion with acceleration.

The energy associated with Λ can be explained by the vacuum energy in particle physics. But the problem is that the value of Cosmological Constant ($10^{-120} m^{-2}$)

in quantum physics is many order smaller than the cosmologically observed value ($10^{-52}m^{-2}$). Considering the perfect fluid equation of state, the value of the constant of the equation of state($\omega = p/\rho$, where p and ρ stand for pressure and energy density respectively) is found -1 for cosmological constant whereas the energy and matter densities vary in different rates throughout the history of Universe. The variable constant of the equation of state can not be explained by the cosmological constant model of dark energy. The problem is known as the coincidence problem.

The coincidence problem of cosmological constant model has been attended by using Scalar-field models of dark energy. Instead of a fix constant of equation of state which arises considering cosmological constant model of dark energy, the periphery can be widen by considering the situation that equation of state can vary with time as mentioned in inflationary cosmology. There are several approaches of scalar-field dark energy models which includes Quintessence [14]; [15]; [16], Phantom [17]; [18], K-essence [19]; [20]; [21]; [22], Chaplygin gas [23]; [24], modified f(r) gravity models [25]; [26] etc.

In Quintessence model of dark energy, the constant of equation of state(ω) is represented as,

$$\omega = \frac{p}{\rho} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} \tag{1.33}$$

where ϕ represents the scalar-field and $V(\phi)$ stands for the potential energy.

In this model the equation (1.33) shows that the value of ω evolves from 1/3 to -1. For matter dominating era $\omega = 0$, for radiation dominating era $\omega = 1/3$ and for $-1 < \omega \leq -1/3$, the accelerating universe expansion reflects.

On the other hand, for negative kinetic energy, the ω evolves $\omega < -1$ region which is known as the Phantom model of dark energy. In this scenario, the expansion rate will be increased with time and once the expansion rate will exceed the limit of the speed of light, the observable objects of the Universe will unable to interact with each other. This hypothetical condition of the Universe is known as Big Rip.

In the quintessence model of dark energy, the potential energy of the scalar field is used to explain the acceleration of the Universe, whereas it is also possible to arise the condition of accelerating Universe by altering the kinetic energy of the scalar field. This kinetic energy dependent scalar-field explanation of the accelerating expansion is known K-essence model of dark energy. But all these scalar field theories have their own periphery and limitations. Therefore, some alternative concepts of dark energy models have also been proposed.

DGP (Dvali-Gabadadze-Porrati) theory, based on brane-world model, has given an alternative proposal of acceleration of Universe [27]. In the brane-world model, an extra 5th dimension has been introduced where (3+1) Minkowskian dimension is embedded till a certain distance ($r_* = (r_0^2 r_g)^{1/3}$). The general relativistic effects can be successfully explained within the threshold distance r_* but beyond that distance, the 5th dimension is introduced where large distance phenomena like the cosmological expansion with acceleration can be explained without taking the non-zero vacuum energy in consideration. But the stability of this concept has been questioned by the critics [28].

1.6 Candidates to explain Dark Matter:

Basically two kinds of explanations are there to be represented as the candidates of dark matter. One is matter contributions that are not detected yet and another one is the alternative theories to explain the dark matter phenomena like flat rotation curve of spiral galaxies and gravitational lensing etc without the need of any dark matter.

1.6.1 Matter representation of dark matter:

The cosmic baryonic density can be derived by CMBR (Cosmic Microwave Background Radiation) temperature anisotropies, which suggests $\Omega_{bar} = 0.045$ whereas $\Omega_m = 0.3$. This signifies that most of the matters are non-baryonic dark matter. In fact, the density of luminous matter (Ω_{lum}) less than the Ω_{bar}, i.e., $\Omega_{lum} < \Omega_{bar}$, that means, some baryonic dark matters also exist which is yet to be revealed. This implies that both baryonic and non-baryonic matters contribute to dark matter. Further, the study of structure formation in the Universe demands that dark matter particles should be non-relativistic (cold dark matter).

In baryonic components, like faint stars, cold gas clouds, Rydberg matter etc, have been predicted as dark matter components. The constitutes of non-baryonic dark matter candidates include neutrinos, axions, mirror matters, black hole, etc.

Dark matter in the form of Massive Compact Halo Objects (MACHOs) is proposed in the literature. Low mass stars like brown and red dwarfs may constitute the baryonic dark matter if they located at large distances or at the dark halo of galaxies. Having low mass, Brown dwarfs cannot initiate the thermonuclear reaction, and red dwarfs are massive enough to burn hydrogen in their cores. These can contribute as dark matter but the quantity is too less than the total estimated dark matter. Molecular hydrogen gas, which is treated as cold gas, is also difficult to detect and considered as dark matter candidate. Rydberg matter, a dark matter candidate, is low density condensed phase of matter which is highly transparent of light due to highly excited state and extremely long lifetime. Because of their invisibility, black holes are also proposed as viable MACHOs. Microlensing surveys, however, suggest that the mass density of MACHOs is not sufficient to explain the required amount of dark matter.

Weak interacting particles are considered for dark matter particles as they cannot be detectable by telescopes. Neutrinos are only known dark matters which contribute significantly to cosmic energy density and are detected in nature. However, the mass density of neutrinos is not large enough to explain the dark matter fraction of the cosmic average density. Axions, which are introduced to solve the problem of CP violation in particle physics and are interact weakly, are also proposed as a candidate for dark matter.

Among the weakly interacting particles WIMP or Weakly Interacting Massive Particles are the most favored candidate for dark matter. Beyond the Standard Models, several theories predict the existence of WIMP. For instance, the lightest supersymmetric particle in supersymmetric theories may act as WIMP. Other possible WIMPs include the lightest particle in Little Higgs models, lightest Kaluza-Klein particle, etc.

1.6.2 Alternative models to explain dark matter effects

Among the alternative approaches to explain the dark matter consequences include MOND (Modified Newtonian Dynamics) [29], Conformal theory based on Weyl gravity [30], modified f(r) gravity [31] etc. These models explain dark matter effects without invoking any dark matter.

MOND is a modification of Newtonian dynamics to explains the flat rotation curve which is considered as a dark matter effect as earlier discussed. This concept was proposed by M. Milgrom in 1983. The base of this modification is to segregate into two sections based on of high and low acceleration. As per this proposal, The dynamics of an object under a gravitating object follow the Newtonian behavior at high acceleration, whereas it shows deep-MOND behavior at low acceleration. The MOND equation of force is given by,

$$F = m\mu(x)(\frac{a}{a_0})a \tag{1.34}$$

where F is the Newtonian force, m is the mass of the object, a is the acceleration, $\mu(x)$ is known as interpolating function, a_0 is the constant which denotes the transition between Newtonian and MOND domain. To synch with Newtonian mechanics, the condition will be as followed,

$$\mu(x) \rightarrow 1 \qquad \text{for} \qquad x \gg 1$$

whereas, the following condition will be obeyed to be consistent with the dark matter observation,

$$\mu(x) \rightarrow x \qquad \text{for} \qquad x \ll 1$$

If an object of mass m moving around a gravitating object of mass M in a circular orbit with linear velocity v, then we get,

$$\frac{GMm}{r^2} = \frac{m(\frac{v^2}{r})^2}{a_0}$$

$$\Rightarrow v^4 = GMa_0$$

The above expression from MOND describes the flat rotation curve of dark matter effect but it can not construct a satisfactory cosmological model and other observed property of galaxy clusters.

Mannheim and Kazanas [30] proposed an alternative model of dark matter effect based on conformal invariant Weyl gravity. They have found the following metric which explain the flat rotation curve of spiral galaxies:

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.35}$$

where,

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \kappa r^2 \tag{1.36}$$

β, γ, and k stand for the integration constants. Putting the value of $k = \gamma = 0$, the metric provides the Schwarzschild metric, and when $\gamma = 0$, it will give SDS metric. γ is the parameter in the above metric which represents the dark matter effects, and the value of γ was found $10^{-26}m^{-1}$ based on the observational data of several galaxies.

Grumiller also proposed a model for gravity at large distances based on modified general relativity and proposed a metric to define large distance phenomena like dark energy and dark matter [31].

Another proposal to explain dark matter is based on f(r) gravity, which was proposed by H. A. Buchdahl in 1970 [32]. This is basically modified general relativity, which is a family of theories based on several circumstances. An arbitrary function has been introduced which gives the freedom to explain the dark sector effects.

1.7 Objectives of the present work

Does dark sector really exist or the observations pertaining to dark sector simply hint a problem with general relativity? What are the nature of dark sector ? The dark sector is still dark despite a long effort. If exists, dark energy/matter is likely to affect the gravitational phenomena in all distance scales including the local scales. As the evidences of dark sector so far are found only in large distance scale observations, the study of effects of dark energy/matter on local gravitational phenomena are important not only to confirm their presence but it may also help to understand the nature of the dark sector. Already several analysis have been performed so far in this direction, as will be reviewed in the next section, but certain aspects have not been addressed adequately.

In this thesis work we have examined the influences of dark energy and dark matter on different local gravitational phenomena critically considering different models of dark energy and dark matter. Emphasis will be given to discriminate the models of dark energy and dark matter by comparing theoretical predictions with the observations. We shall particularly investigate the influence of dark sector on several gravitational phenomena like gravitational time delay, gravitational time advancement, gravitational lensing etc. We shall construct static spherically symmetric metric for galactic halos based on flat rotation curve and cold dark matter approximation and shall examine whether such model is consistent with gravitational lensing observations. We shall also check whether some alternative dark matter models are consistent with Tully-Fisher relation or not.

1.8 Current status of studies on local gravitational influences of dark sector

The observation of gravitational influences on a few observables, namely perihelion shift of planets, bending of light by gravitating object, the time delay due to gravitating object and red-shift of photons, in the Solar System neighborhood provide the classical evidences in favor of the theory of GR. The influences of dark sector have been studied so far on all such classical gravitational observables which impose some constraints on dark sector parameters. However, solar system experiments put only upper bound on the dark sector parameters compared to the value obtained in cosmological observations.

The parametrized post-Newtonian (PPN) formalism is usually employed to describe the gravitational theories in a weak gravitational field equation (1.6) [5]. The PPN description provides the advantage of comparing predictions of GR with those from several alternative metric theory of gravity. However, the PPN formalism cannot, in general, accommodate the effect of the dark sector.

The geodesic equations for general static spherically symmetric metric as given in equation (1.5) give

$$A(r)(\frac{dr}{dp})^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E \tag{1.37}$$

where p is a parameter describing the trajectory and proportional to the proper time(τ),

$$J = r^2 \frac{d\phi}{dp}$$

and E is a constant which is equal to zero for photons and greater than zero for the material particles. Replacing the dp by the expression $d\phi$, we get,

$$\frac{A(r)}{r^4}(\frac{dr}{d\phi})^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2} \tag{1.38}$$

If the external space time geometry due to the gravitating object is described by Schwarzschild metric equation (1.6), the above equation can be expressed as,

$$\frac{1}{r^4}(\frac{dr}{d\phi})^2 + \frac{B(r)}{r^2} - \frac{1}{b^2} = 0 \tag{1.39}$$

where, $E = 0$ for light trajectory and $b = r^2 \frac{d\phi}{dp}$, addressed as impact parameter.

Expressing $u = 1/r$ and differentiating equation (1.39) with respect to ϕ, we get the second order differential equation,

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \tag{1.40}$$

The general solution of the above equation is given by,

$$u = \frac{sin\phi}{R} + \frac{3m}{2R^2}(1 + \frac{1}{3}cos2\phi) \tag{1.41}$$

where, R is related with the closest distance(r_0) of light trajectory from the centre of the gravitating object by the expression,

$$\frac{1}{r_0} = \frac{1}{R} + \frac{m}{R^2} \tag{1.42}$$

The above equations are crucial to examine the influences of Schwarzschild geometry on different gravitational phenomena.

In the presence of cosmological constant (Λ), the exterior space-time due to a static spherically symmetric mass distribution is Schwarzschild-de Sitter (SDS) metric which is described by equation (1.5) with

$$B_\Lambda(r) = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2 \tag{1.43}$$

and

$$A_\Lambda(r) = [1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2]^{-1} \tag{1.44}$$

B(r) and A(r) are replaced by $B_\Lambda(r)$ and $A_\Lambda(r)$ respectively in equation (1.5).

For SDS geometry, ignoring higher order terms in Λ, the orbit equation reads,

$$\frac{d^2u}{d\phi^2} = \frac{m}{L^2} - u + 3mu^2 - \frac{\Lambda}{3L^2u^2} \tag{1.45}$$

The above equation is employed to determine the effect of Λ on various gravitational phenomena.

1.8.1 Influences of Dark sectors on perihelion shift of planets

The ability to explain perihelion shift of planets is a prominent success of general theory of relativity. At perihelia(r_-) and aphelia (r_+) of the orbit, r reaches at minimum and maximum with respect to the angular displacement and thus the $dr/d\phi$ vanishes at these two points. Applying this condition in equation (1.38) [5], we get,

$$\frac{1}{r_\pm} - \frac{1}{J^2B(r_\pm)} = -\frac{E}{J^2} \tag{1.46}$$

For the Schwarzschild metric in equation (1.6), the expression of precession of perihelia shift of planets becomes [33],

$$\Delta\phi_{Sch} = \pi\frac{6m}{L} \tag{1.47}$$

where $L(= l(1-e^2))$ is the semi-latus rectum of the elliptical orbit, e and l stand for the eccentricity and length of the semi-major axis of the orbit respectively.

The PPN metric is essentially an expansion about the Minkowski metric (g_{ij}) in terms of some dimensionless small gravitational (Newtonian) potential (U, ψ, φ) so that in isotropic coordinates

$$g_{00} = -1 + 2U - 2\beta U^2 + ... \tag{1.48}$$

$$g_{ij} = \delta_{ij}(1 + 2\gamma U + ...) \tag{1.49}$$

Where,

$$U(x,t) = \int \frac{\rho(x^{'},t)}{|x-x'|}d^3x' \tag{1.50}$$

where γ and β are first PPN parameters.

For the PPN metric, the expression of perihelion shift is given by [34],

$$\delta\phi = \frac{6m\pi}{L}(\frac{1}{3}(2+2\gamma-\beta) + \frac{1}{6}(2\alpha_1 - \alpha_2 + \alpha_3 + 2\kappa)\eta + \frac{JR^2}{2mL}) \tag{1.51}$$

where m is the mass of two-body system, i.e., m_1 and m_2 are the masses of two objects then $m \equiv m_1 + m_2$ and $\eta = m_1 m_2/m^2$, R and J are the mean radius of the oblate body and dimensionless measure of its quadrupole moment respectively, γ and β are the PPN parameters and α_1, α_2, α_3 and κ parameters are dependent on the ratio of masses of two-body system. The parameters α_1, α_2, α_3 and κ will be negligible for the mass of Mercury.

The contribution of Λ leads an additional shift over the Schwarzschild expression [35], [36], [37], [38], [39], [40].

$$\Delta\phi_{\Lambda} = \Delta\phi_{Sch} + \frac{\pi c^2\Lambda l^3}{m}(1-e^2)^3 \tag{1.52}$$

Where $\Delta\phi_{Sch}$ is the perihelion shift due to Schwarzschild geometry as given in equation (1.47).

Several elaborated works on the effect of the cosmological constant on perihelion shift are discussed in the literature. Kerr et. al. found the general expression for effect of Λ on pericentre precession considering the arbitrary orbital eccentricity [41]. Iorio investigated the effect of the cosmological constant on perihelion precession for several solar planets in the frame-work of SDS space-time. Miraghaei and Nouri-Zonoz studied the perihelion shift of Mercury on the Newtonian limit of SDS metric and found the effect of Λ on perihelion shift [40].

Arakida studied the effect of the cosmological constant on the perihelion shift of planets and found a general expression for all orbital eccentricity [42].

The perihelion shift of planets due to alternative dark matter and dark energy models have been addressed by several authors.

One of the significant alternatives of Einstein's theory of general relativity is provided by Weyl gravity where conformal invariance of space-time has been used. The static spherically symmetric metric solution of Weyl gravity was obtained by Mannhein and Kazanas [30] which is found consistent with the experimental tests of gravitation in a weak gravitational field. As mentioned in equation (1.35) and (1.36), the static spherically symmetric vacuum solution of conformal gravity is given by:

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.53}$$

where,

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \kappa r^2 \tag{1.54}$$

β, γ, and k stand for the integration constants. Putting the value of $k = \gamma = 0$, the metric provides the Schwarzschild metric and when $\gamma = 0$, it will give SDS metric. An explanation of the flat rotation curve of spiral galaxies has been provided by this metric solution of Weyl Conformal gravity which can be presented as an alternate solution of Dark matter problem.

The Precession of perihelion shift of planets was investigated using the above metric [43] and the expression of perihelion shift found for Weyl gravity is:

$$\delta\phi \approx \frac{6\pi\beta}{l(1-e^2)} + \frac{3\pi}{\beta}\kappa l^3(1-e^2)^3 - \frac{\pi}{\beta}\gamma l^2(1-e^2)^2 \tag{1.55}$$

where, β stands for $MG/c^2(\equiv m)$ and κ is equivalent to the cosmological constant $(\Lambda/3)$. If the equation (1.55) can be investigated minutely, it can be observed that the first term of right-hand side denotes the Schwarzschild term of perihelion precession of planets whereas the second term is the contribution for the cosmological constant and third term has been appeared due to the effect of γr term in the equation (1.54).

The perihelion shift has also been studied for quintessence model [44], MOND [45]; [46], f(r) gravity models [47]; [48].

Using the solar quadruple moment $J = (2.2 \pm 0.1) \times 10^{-7}$ [49] and substituting the orbital elements and constants for Mercury in solar orbit, the expression of perihelion shift is found,

$$\delta\phi = 42.''98\left(\frac{1}{3}(2 + 2\gamma - \beta) + 3 \times 10^{-4}\frac{J}{10^{-7}}\right) \tag{1.56}$$

Messenger spacecraft provided a significantly improved knowledge about orbital motion. Adopting the Cassini boundary limit of γ, the bound of β is given by $\beta - 1 = (-4.1 \pm 7.8) \times 10^{-5}$.

To detect the influence of Cosmological constant comparing with the Schwarzschild term, the perihelion shift of Mercury of 43" per century, is in full agreement of Einstein's theory of General Relativity with the accuracy of $430\mu as$ and the cosmological constant was constrained up to 10^{-41} order approximately [50]; [37] and achieved up to 10^{-42} order with -0.0036 ± 0.005 arc-secs accuracy level [51]. Including Sun's angular momentum and uncertainty of solar quadrupole moment, the Λ was constrained up to 10^{-43} order [52], i.e., 10^{-9} more precession level need to achieve to get the effect of dark energy. And to detect the dark matter effect, the ratio between Schwarzschild term and dark matter contribution (i.e. γ contribution) is very important and it is approximately in the order of 10^{-11} [43].

1.8.2 Influences of dark sectors on gravitational deflection of light:

Gravitational deflection of electromagnetic wave provides a prime evidence in favor of general relativity. The expression for deflection angle of electromagnetic wave due to a gravitating object (lens), coming from a source to an observer situated at r distance from the centre of the gravitating object, can be deduced from the geodesic equations (from the general equation of motion as mentioned in equation (1.38), considering $E = 0$ for electro-magnetic wave) which is given by,

$$\phi(r) - \phi_\infty = \int_r^\infty A^{1/2}(r)[(\frac{r}{r_0})^2(\frac{B(r_0)}{B(r)}) - 1]^{-1/2}\frac{dr}{r} \tag{1.57}$$

Implementing several metric solutions on the above expression, the gravitational deflection angle for the different gravitational models can be obtained.

For Schwarzschild metric the above expression of gravitational deflection becomes:

$$\Delta\phi_{sch} = \frac{4m}{r_0} \tag{1.58}$$

where the closest approach of the e.m. wave trajectory is denoted as r_0.

1.8.2.1 Approaches to deduce gravitational deflection angle on several dark sector models:

The early studies concluded that there should not be any effect of Λ on gravitational bending of light [35], [53], [37]. The motion of electromagnetic wave in SDS space-time can be described through the Lagrangian $\mathcal{L}$ of the space-time:

$$2\mathcal{L} = B_\Lambda(r)\dot{t}^2 - B_\Lambda^{-1}(r)\dot{r}^2 - r^2\dot{\phi}^2 \tag{1.59}$$

where $B_\Lambda(r) = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2$ and dot stands for the differentiation with respect to the affine parameter (λ). The motion is restricted to the $\theta = \pi/2$ plane. The conserved quantities, E (energy) and angular momentum (l'), can thus be expressed as

$$E \equiv B_\Lambda(r)\frac{dt}{d\lambda} \tag{1.60}$$

$$l' \equiv r^2\frac{d\phi}{d\lambda} \tag{1.61}$$

The null geodesic equation for the space time is given by,

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}[\frac{1}{b^2} - \frac{B_\Lambda(r)}{r^2}]^{-1/2} \tag{1.62}$$

where, $b \equiv l'/E$ which denotes impact factor in Schwarzschild space-time expression (1.39) as $\frac{dt}{dp} = -\frac{1}{B(r)}$ can be deduced from the Lagrangian. The second order differentiation leads,

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \tag{1.63}$$

That is exactly the same as the path equation in Schwarzschild geometry as equation (1.40). Note that the path equation does not involve Λ. Consequently the orbit equation will be same to the orbit equation for Schwarzschild metric in equation (1.41),

$$u = \frac{sin\phi}{R} + \frac{3m}{2R^2}(1 + \frac{1}{3}cos2\phi) \tag{1.64}$$

Absence of Λ in the above expression apparently suggests that there should not be any effect of the cosmological constant on the deflection of light. The bending angle in Schwarzschild space-time is estimated considering the limit $r \rightarrow \infty$ in the light orbital equation, and the angle between the two asymptotic directions gives the total deflection angle. For SDS space-time, however, $r \rightarrow \infty$ makes no sense. The de-Sitter horizon is $r_\Lambda = \sqrt{3/\Lambda}$ as may be obtained from the SDS metric. Rindler and Ishak thus proposed an alternative solution in which the angle is evaluated through the tangent on the light trajectory with the co-ordinate direction at a given arbitrary point. Subsequently, they obtained the expression for deflection angle in de-Sitter geometry as follows:

$$\Delta\phi_{\Lambda Rindler} = 2[\frac{2m}{R} - \frac{4m^3}{R^3} - \frac{\Lambda R^3}{12m}] \tag{1.65}$$

In SDS geometry expressed in equation (1.43) and (1.44), the tangent of the angle (ψ) of the light trajectory made with the coordinate point at a given point, is given by [54], [55],

$$tan\psi = rB_\Lambda(r)^{1/2}|\frac{d\phi}{dr}| \tag{1.66}$$

The above equation can be written for the null geodesics [56],

$$tan\psi = [\frac{B_\Lambda(r_0)}{B_\Lambda(r)}\frac{r^2}{r_0^2} - 1]^{-1/2} \tag{1.67}$$

Avoiding the higher order of m and Λ,

$$tan\psi = \frac{r_0}{r} + \frac{m}{r} - \frac{mr_0}{r^2} - \frac{\Lambda r_0 r}{6} + \frac{\Lambda r_0^3}{6r} \tag{1.68}$$

When $r >> r_0$ and the angles ψ and ϕ are very small and avoiding the higher order terms of m, Λ and r_0/r, the expression of deflection angle will be,

$$\Delta\phi_\Lambda = 2[\frac{2m}{r_0} - \frac{mr_0}{r^2} - \frac{\Lambda r_0 r}{6} + \frac{\Lambda r_0^3}{6r}] \tag{1.69}$$

Generalizing the results of Rindler and Ishak [54], Bhadra et. al. [55] calculated the angle between the lensed light trajectory at the source and the observer location as follows,

$$\Delta\phi_\Lambda = \frac{4m}{r_0} - mr_0(\frac{1}{d_{LS}^2} + \frac{1}{d_{LO}^2}) - \frac{\Lambda r_0}{6}(d_{LO} + d_{LS}) + \frac{\Lambda r_0^3}{6}(\frac{1}{d_{LO}} + \frac{1}{d_{LS}}) \tag{1.70}$$

where, d_{LO} and d_{LS} is the coordinate distances of gravitating object from observer and source respectively. For a small angle, R can be replaced as r_0.

Additionally, Bhadra et. al. [55] have given importance to a reference object to study the bending of a light trajectory by a gravitating object. Considering the reference object, it has been found that the contribution of cosmological constant (Λ) is dependent on the distance between the source and the reference object.

Sereno [57], [58] has also supported the local coupling between the mass of the lens and the Λ in the expression of deflection angle in SDS metric. As per Sereno the gravitational deflection angle is expressed by as follows:

$$\begin{aligned}\Delta\phi_{Sereno} \approx \pi - \tfrac{4m}{b} + b(\tfrac{1}{r_{LS}} + \tfrac{1}{r_{LO}}) - \tfrac{15m^2\pi}{4b^2} - \tfrac{128m^3}{3b^3} + \tfrac{b^3}{6}(\tfrac{1}{{r_{LS}}^3} + \tfrac{1}{{r_{LO}}^3}) \\ - \tfrac{3465m^4\pi}{64b^4} - \tfrac{3584m^5}{5b^5} - \tfrac{2mb}{{r_\Lambda}^2} - \tfrac{mb^3}{4}(\tfrac{1}{{r_{LS}}^4} + \tfrac{1}{{r_{LO}}^4}) \\ + \tfrac{3b^5}{40}(\tfrac{1}{{r_{LS}}^5} + \tfrac{1}{{r_{LO}}^5}) - \tfrac{b^3}{2r_\Lambda}(\tfrac{1}{r_{LS}} + \tfrac{1}{r_{LO}})\end{aligned} \tag{1.71}$$

b stands for impact parameter and can be replaced by r_0 for a small deflection angle.

Schucker supported the approach of Rindler and Ishak and found the effect of cosmological constant due to isolated spherical mass without using lens equation [59]. Lake also supported the work and showed the effect of the cosmological constant using two opposite sources [60]. Bhattacharya et. al. used the Rindler-Ishak procedure to analyze the gravitational deflection of light using the Einstein-Strauss vacuole model with cosmological constant [61].

In the contrary, some authors questioned the contribution of Λ on the gravitational deflection of light. Khriplovich and Pomeransky demonstrated that it doesn't affect practically on gravitational lensing using Friedmann-Robertson-Walker coordinates [62]. Park also concluded that no correction was needed involving cosmological constant by solving null geodesic equations [63].

Ishak derived the contribution of the cosmological constant on gravitational deflection from the gravitational potential and Fermat's principle [64]. He further found the Λ contribution on geometrical time delay term for the bending of light.

Miraghaei and Nouri-Zonoz studied the gravitational deflection on the Newtonian limit of SDS metric and found the effect of Λ on general relativistic approach [40].

Arakida and Kasai re-examined the effect of the cosmological constant on gravitational deflection of light and showed that the Λ appears in the orbital equation of light [65]. Aghili et. al. studied the effect of cosmological constant for time varying cosmological expansion, i.e. when Hubble constant varies with time[66].

Biressa et. al. studied the effect of cosmological constant on gravitational lensing to calculate projected mass of lens including cosmological constant [67].

Butcher argued accepting the cosmological constant correction on gravitational lensing of light that the effect is negligible in the practical way as it is smaller than the uncertainty from unlensed distances [68].

Guenouche and Zouzou investigated the gravitational lensing in the framework of the Einstein-Straus solution with positive cosmological constant considering closed Universe [69].

The local influence of gravitational deflection has been studied in scaler field model of dark energy [70], where a spherically symmetric static metric was developed in quintessence model of dark energy and studied the effect on gravitational deflection. The metric they have developed is as follows:

$$ds^2 = (1 - \frac{2m}{r} - \frac{\alpha}{r^{3w+1}})dt^2 - (1 - \frac{2m}{r} - \frac{\alpha}{r^{3w+1}})^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta\phi^2) \quad (1.72)$$

where α is a constant of integration and w is the constant of the equation of state which varies $-1 \leq w < 0$ where$-1 \leq w < -1/3$ shows the nature of dark energy dominating accelerating universe and $w = 0$ signifies matter-domination and $w = 1/3$ radiation domination.

Using the quintessence based metric, the second order equation of motion is given by:

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 + \frac{3\alpha(w+1)u^{2w+2}}{2} \quad (1.73)$$

Solving the equation analytically, they have found the solution of the above second order equation for different values of the constant of equation of state w, for example, for $w = -1/3$,

$$\Delta\phi_{quintessence} = \frac{4m}{r_0} + \frac{4m}{r_0(1-\alpha)^{3/2}} \quad (1.74)$$

and for $w = -1$, which is equivalent to the cosmological constant model, no influence of cosmological constant is noticed, i.e. at $w = -1$, the gravitational deflection angle term turns into a pre-Rindler-Ishak expression of gravitational deflection in SDS space-time.

$$\Delta\phi_{quintessence} = \Delta\phi_{\Lambda} = \frac{4m}{r_0} \tag{1.75}$$

Rectifying the evaluation process by applying Rindler-Ishak [54] approach, the influence of dark energy in the quintessence metric has been deduced [71].

On the other hand, the gravitational deflection is the prime evidence of dark matter. This phenomenon has been used as a tool to verify different approaches of dark matter effect associated models. As mentioned earlier Modified Newtonian Dynamics(MOND) is a significant model to represent dark matter effects. Gravitational bending in MOND, has been studied by several physicists [72], [73], [74].

The gravitational deflection angle in MOND, is given by:

For $r_0 > r_c$,

$$\Delta\phi_{MOND} = \pi\frac{\sqrt{Ga_0M}}{c^2} \tag{1.76}$$

For $r_0 \leq r_c$,

$$\Delta\phi_{MOND} = \frac{2GM}{c^2r_0}\sqrt{\frac{r_c - r_0}{r_c + r_0}} + \frac{2GM}{c^2r_c}\sqrt{\frac{r_c - r_0}{r_c + r_0}} + \frac{2\sqrt{Ga_0M}}{c^2}sin^{-1}\frac{r_0}{r_c} \tag{1.77}$$

For $r_c \to \infty$, the above expression turns into Newtonian expression of gravitational deflection angle,

$$\Delta\phi_{Newtonian} = \frac{2GM}{c^2r_0} \tag{1.78}$$

where r_0 is the impact parameter of light trajectory and r_c is the critical radius of Newtonian mechanic and MOND in flat rotation curve, expressed by $r_c = \sqrt{GM(r_c)/a_0}$, a_0 is a constant called critical accelerating parameter, (M is the effective mass of gravitating object) [75].

It should be mentioned that the general relativistic correction of factor '2' has also been adopted in MOND expression for the expression of the deflection angle.

Now considering the conformal Weyl gravity metric as mentioned in equations (1.35) and (1.36) [30], the expression of gravitational deflection angle was studied [76] and then further reexamined by Sultana and Kazanas [77], based on the approach by Rindler and Ishak [54].

$$\Delta\phi_{Weyl} = \frac{4m}{b} - \frac{2m^2\gamma}{b} - \frac{\kappa b^3}{2m} \tag{1.79}$$

where b stands for the impact parameter and can be replaced by closest approach(r_0) of light trajectory from the centre of the gravitating object and γ represents the dark matter effect and κ is equivalent to cosmological constant $(\Lambda/3)$. But objection raised by Cattani et. al. [78] due to negative contribution of dark matter effect which is represented by γ, where the contribution should be enhancement effect on Schwarzschild term of lensing angle. They analyzed the issue and explained that the actual conformal metric as mentioned in equation (1.36) is given as follows:

$$B(r) = \alpha - \frac{2m}{r} + \gamma r - \kappa r^2 \tag{1.80}$$

where $\alpha = (1 - 6m\gamma)^{1/2}$ and $\alpha = 1$ approximated for the distances neither too large nor too small. But no such approximation is made in this work. As per this work, negative contribution of γ was appeared due to avoiding the first order terms associated with $\alpha \neq 1$ and the expression of gravitational deflection for Weyl gravity considering all the first order terms of γ, is given by,

$$\Delta\phi_{weyl} = \frac{4m}{r_0} - \frac{\kappa r_0^2}{2m} + \frac{15m^2\gamma}{r_0} \tag{1.81}$$

The above expression of the deflection angle shows the positive contribution of γ term which holds the practical dark matter influence.

Sultana reexamined the gravitational deflection of light on conformal Weyl gravity to get the 2nd order contribution of γr and found that the contribution is insignificant [79].

Lim and Wang derived an exact solution for gravitational lensing using static spherically symmetric metric for SDS and Mannhein-Kazanas metric of Weyl gravity both [80].

The gravitational lensing effect was also studied in f(r) gravity model [81], [47], [48].

Starting from Eddington and his co-workers, several attempts have been made to measure gravitational deflection angle. For the PPN metric, the of gravitational deflection is given by [34],

$$\delta\phi = \frac{1}{2}(1+\gamma)\frac{4m}{r_0}(\frac{1+cos\psi}{2}) \tag{1.82}$$

where m is the mass of the gravitating object, ψ is the angle between observer to lens line and incoming direction of photon to the observer and γ is the first PPN parameter that varies from theory to theory. For example, for Schwarzschild metric $\gamma = 1$.

Eddington and his co-workers first time attempted the experimental observation [82] and they found the deflection angle with 30 percent accuracy and the result was scattered between one half and twice the Einstein value of lensing angle. However, the scenario has been changed after the development of radio interferometry measurements. The very long baseline radio interferometer (VLBI) provided improved precession level of the deflection angle. The modern techniques have the capability to produce the accuracy more than 100 micro-second.

The solar system gravitational bending observations do not put stringent constraint on Λ; to detect the influence of cosmological constant the precession level of measuring bending angle needs to be approximately 10^{-18} times higher than the precision level of detecting contribution of Schwarzschild term (from equation (1.70)) if the source is situated at kpc distance [55]. However, the contribution of Λ to the deflection angle can be larger than the second-order term in the deflection angle lensed by pure Schwarzschild geometry for several cluster lens systems [64]. The effect of Weyl model of dark matter (equation (1.81)), is negligibly small [77].

1.8.3 The influences of dark sector on gravitational time delay:

The gravitational time delay is a phenomenon where the object under gravity suffers time delay when it moves under the influence of gravitating object, if we compare the total traveling time of the object, required in absence of the gravitating object. Shapiro first proposed the phenomenon of gravitational time delay and carried out a measurement with Lincoln Laboratory collaboration using a radar signal that traveled to a planet and reflected back to earth [83]. To derive the theoretical expression of gravitational time delay, again the general equation of motion, equation (1.37), is used. Replacing dp by dt using the relation $\frac{dt}{dp} = \frac{1}{B(r)}$, one obtains

$$\frac{A(r)}{B^2(r)}(\frac{dr}{dt})^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E \tag{1.83}$$

For light trajectory, $E = 0$ and $\frac{dr}{dt}$ must be vanished at the closest approach of light trajectory (at $r = r_0$), so equation (1.83) gives,

$$J^2 = \frac{r_0^2}{B(r_0)} \tag{1.84}$$

Therefore, the equation of motion for light trajectory, is given by,

$$\frac{A(r)}{B^2(r)}(\frac{dr}{dt})^2 + (\frac{r_0}{r})^2\frac{1}{B(r_0} - \frac{1}{B(r)} = 0 \tag{1.85}$$

From the above equation, the time required to travel for a light beam from r_0 to r or vice-versa is given by,

$$t(r, r_0) = \int_{r_0}^{r} (\frac{A(r)/B(r)}{1 - \frac{B(r)}{B(r_0)}(\frac{r_0}{r})^2})^{1/2} dr \tag{1.86}$$

For Schwarzschild metric, we get

$$t_{Sch}(r, r_0) \simeq \sqrt{r^2 - r_0^2} + 2mln(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}) + m(\frac{r - r_0}{r + r_0})^{1/2} \tag{1.87}$$

The first term in the above expression denotes the time required for light to travel in a straight line with unit velocity of light. Second and third terms reflect the gravitational contribution of traveled time, and positive term expresses the time delay effect.

1.8.3.1 Approaches to deduce gravitational time delay on several dark sector models:

The influences of dark energy and dark matter have been studied in several works. Kagramanova et al [37] studied the influence of dark energy in SDS metric as mentioned in equation (1.43) and (1.44). As per the study, the expression of gravitational time delay is

$$t_\Lambda(r, r_0) \simeq \sqrt{r^2 - r_0^2} + 2mln(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}) + m(\frac{r - r_0}{r + r_0})^{1/2}$$
$$+\frac{\Lambda}{18}[(\sqrt{r^2 - r_0^2})(2r^2 + r_0^2) + 3m(4r\sqrt{r^2 - r_0^2} + r_0^2(2 + \frac{\sqrt{r^2 - r_0^2}}{r + r_0}))] \tag{1.88}$$

where Cosmological constant(Λ) associated term reflects the dark energy contribution on gravitational time delay in SDS space-time.

On the other hand, Asada examined the gravitational time delay of light in several modified gravity models [84]. He introduced a general static spherically symmetric metric, represented as,

$$A(r) \approx 1 - \frac{2m}{r} + A_m r^m \tag{1.89}$$

and

$$B(r) \approx 1 + \frac{2m}{r} + B_n r^n \tag{1.90}$$

where $m = GM/c^2$, M is the mass of gravitating object and A_m, B_n, m and n are varies with dark energy model to model. For example, when $n = 2$, $A_n = -B_m = -\Lambda/3$, the metric represents Schwarzschild-De-Sitter(SDS) metric and for $n = 1/2, A_n = -2B_n = \pm 2\sqrt{m/r_c^2}$, it shows the DGP model of dark energy.

Asada deduced the expression of gravitational time by considering the radio signal transmitted from earth (situated at r_E co-ordinate distance), reflected back from a reflector, situated at r_R co-ordinate distance and r_o is the closest approach of the signal's trajectory from the centre of the gravitating object and the expression is given by:

$$\delta t = 2(\sqrt{r_E^2 - r_o^2} + \sqrt{r_R^2 - r_0^2}) + 2m(2ln\frac{r_E + \sqrt{r_E^2 - r_0^2}}{r_0}$$
$$+2ln\frac{r_R + \sqrt{r_R^2 - r_0^2}}{r_0} + \sqrt{\frac{r_E - r_0}{r_E + r_0}} + \sqrt{\frac{r_R - r_0}{r_R + r_0}}) + \delta t_{DE} \qquad (1.91)$$

where δt_{DE} denotes dark energy effect contribution in time delay expression, expressed as (for $n = m > 0$),

$$\delta t_{DE} = r_0^{n+1}(\int_1^{R_E} + \int_1^{R_R})dR \times (-A_n\frac{R^{n+3} - 2R^{n+1} + R}{(R^2-1)^{3/2}} + B_n\frac{R^{n+1}}{\sqrt{R^2-1}}) \qquad (1.92)$$

where $R \equiv r/r_0$, $R \equiv r_E/r_0$ and $R_R \equiv r_R/r_0$. And taking $r_R >> r_0$ and $n \neq 0$, the following expression was obtained by Asada,

$$\delta t_{DE} = \frac{B_n - A_n}{n+1}(r_E^{n+1} + r_R^{n+1}) + \frac{B_n + A_n}{2(n-1)}(r_E^{n-1} + r_R^{n-1} - 2r_0^{n-1})r_0^2 + O(r_0^4) \qquad (1.93)$$

The above equation is a generalized expression of dark energy effect in gravitational time delay for different models with different values of A_n, m and n as mentioned earlier.

Schucker and Zaimen studied the effect of cosmological constant on gravitational time delay for an isolated spherical mass [85].

Ishak derived the contribution of the cosmological constant on gravitational time delay from the gravitational potential and Fermat's principle [64]. He also found the Λ contribution on geometrical time delay term.

Guenouche and Zouzou investigated the gravitational time delay in the framework of the Einstein-Straus solution with positive cosmological constant considering closed Universe [69].

Effect of dark energy on Gravitational time delay was also studied in quintessence model of dark energy [86]. Considering the quintessence metric as mentioned in equation (1.72), the quintessence model based expression of gravitational time delay is given by,

$$t_{Quintessence}(r, r_0) = t_{Sch} + \int_{r_0}^{r} (1 - \frac{r_0^2}{r^2})^{-1/2} [\frac{r^{3\omega+1} - r_0^{3\omega+1}}{r_0^{3\omega-1} r^{3\omega+1}(r^2 - r_0^2)} + \frac{2}{r^{3\omega+1}}] \frac{\alpha}{2} \quad (1.94)$$

The quintessence term in the above equation was solved for different values of ω. For example, if $\omega = -1/3$, the quintessence associated term will be $\alpha r\sqrt{1 - \frac{r_0^2}{r^2}}$ and for $w = -1$, which is actually signifies the cosmological constant model, that will be $\frac{\alpha r}{6}(2r^2 + r_0^2)\sqrt{1 - \frac{r_0^2}{r^2}}$ which supports the cosmological constant associated expression of gravitational time delay when $\alpha \equiv \Lambda/3$ and avoiding higher-order and multiplication terms of Λ and m.

The gravitational time delay was also studied under the influence of dark matter environment which is provided in Weyl gravity by Mannhein and Kazanas [30] and the metric represented by the equation (1.35) and (1.36). The effect of dark matter on gravitational time delay was studied using the conformal metric by Ederly and Paranjape [76] and found the expression of time delay, to travel for a radar signal from r_0 to r distance considering the centre of the gravitating object(of mass M) as co-ordinate centre, as follows,

$$\begin{aligned} t_{weyl}(r, r_0) \simeq & \sqrt{r^2 - r_0^2} + 2\beta ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0} + \beta\sqrt{\frac{r - r_0}{r + r_0}} \\ & -\frac{\gamma}{2}(\frac{r^3 - r_0^3}{\sqrt{r^2 - r_0^2}}) + \frac{\kappa}{6}(2r^2 - r_0^2)\sqrt{r^2 - r_0^2} \end{aligned} \quad (1.95)$$

As mentioned earlier, β in above equation (1.95) stands for $GM/c^2(\equiv m)$, γ represents the dark matter effect and κ is equivalent to cosmological constant($\Lambda/3$). The above equation (1.95) reflects the dark matter effect associated term as well as the Schwarzschild metric related term as mentioned in equation (1.87) and dark energy associated Λ term as shown earlier in equation (1.88).

Farrugia et. al. studied the gravitational time delay in f(r) gravity model [48].

If a radar signal is sent to a planet or satellite from Earth and passes through the vicinity of the Sun, the expression of gravitational time delay under PPN metric is given by [34],

$$t(r, r_0) = \frac{1}{2}(1+\gamma)[240 - 20ln(\frac{r_0^2}{r})]\mu s, \tag{1.96}$$

Several high precession measurements were made using radar signal passing near the conjunction of a gravitating object after the discovery of the significant consequence of general relativity by Irwin Shapiro in 1964. A round trip travel time, through the vicinity of the gravitating object, is to be measured to get the gravitational time delay value and fit the value of γ can be found based on least square fit method, which depends on which space-time metric has been adopted, by using equation (1.96). To measure the gravitational time delay by Sun as a gravitating object for a radar signal, few artificial satellites like Voyager-2, Mariners 6 and 7, Viking Mars landers and orbiters, Cassini spacecraft, were used as re-transmitters of the radar signal and Mercury, Venus or Saturn was used as reflectors.

The gravitational time delay measurements restrict up to ($\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$) γ by Cassini spacecraft taking Saturn as a re-transmitter and reflector [87].

The gravitational time delay relates with frequency shift. The relative change of frequency,

$$y = \frac{\nu(t) - \nu_0}{\nu_0}$$

where ν_0 is the emitted frequency of the wave and $\nu(t)$ is the received frequency at the Earth. The Schwarzschild contribution to the change of frequency (y) is in the order of 10^{-10} order and the Cassini spacecraft measured in the order of 10^{-13}.

The time delay measurements though provide most stringent constraint on the PPN parameter γ but restrict Λ loosely; the Cassini observations suggest $\Lambda \leq 10^{-24}\ m^{-2}$ [37]. The dark matter parameter in equation (1.89) is constrained upto

$10^{-23}cm^{-1}$ order using the Shapiro time delay [76] where the value of γ is in the order of $10^{-28}cm^{-1}$.

1.8.4 The influences of dark sector on gravitational frequency-shift:

The concept of gravitational frequency-shift arises theoretically from the concept of proper time($d\tau$) which is defined by the time interval measured by the clock of an observer in rest, i.e. spatial co-ordinate interval $dx^i = 0$. The expression of proper time is given by,

$$d\tau = \sqrt{g_{00}}dx^0 \tag{1.97}$$

$\sqrt{g_{00}} = B(r)$ in equation (1.5).

If we compare the proper time interval at two distinct point of space but both correspond to the same interval of co-ordinate time, then the ratio of proper time interval is given by,

$$\frac{d\tau_1}{d\tau_2} = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_2)}} \tag{1.98}$$

where the $d\tau_1$ and $d\tau_2$ are the proper time interval at x_1 and x_2 position respectively. And considering ν_1 and ν_2 are the frequencies of a photon at x_1 and x_2 points respectively, then the above equation can be expressed as,

$$\nu_2\sqrt{g_{00}(x_2)} = \nu_1\sqrt{g_{00}(x_1)} \tag{1.99}$$

The above equation expresses the frequency shift under the gravitational influence. Using Schwarzschild metric ($g_{00} = B(r) = 1 - \frac{2m}{r}$) as mentioned equation-(1.6), the expression of gravitational frequency shift will be as follows:

$$\frac{\nu_2}{\nu_1} \simeq 1 - \frac{m}{r_2} + \frac{m}{r_1} \tag{1.100}$$

The higher order terms of m/r_1 and m/r_2 are avoided, where $m = GM/c^2$ and ν_1 and ν_2 are the frequencies of same photon traveling from r_1 and r_2 co-ordinate distance respectively.

1.8.4.1 Approaches to deduce gravitational frequency-shift on several dark sector models:

The gravitational frequency shift has been also studied in several models of dark sectors. Kagramanova.et.al. [37] and Sereno. et.al. [38] have studied the gravitational frequency shift in Cosmological constant model of dark energy by using SDS metric as shown in equation (1.43) and (1.44).

The expression of gravitational shift in SDS metric is given by,

$$\frac{\nu_2}{\nu_1} \simeq 1 - \frac{m}{r_2} + \frac{m}{r_1} - \frac{\Lambda}{6}(r_2^2 - r_1^2) \tag{1.101}$$

The higher order terms of m/r_1 and m/r_2 are avoided,, where $m = GM/c^2$ and ν_1 and ν_2 are the frequencies of same photon traveling from r_1 and r_2 co-ordinate distance respectively and Λ stands for the cosmological constant.

The gravitational frequency shift was also studied in Quintessence model dark energy using the metric as mentioned earlier in equation (1.72) and expression of gravitational frequency shift was found by [70], [88],

$$\frac{\nu_2}{\nu_1} \simeq 1 - \frac{m}{r_2} + \frac{m}{r_1} + \Delta\nu_{quintessence} \tag{1.102}$$

where,

$$\Delta\nu_{quintessence} = \frac{\alpha}{2}\left(\frac{1}{r_2^{3\omega+1}} - \frac{1}{r_1^{3\omega+1}}\right) \tag{1.103}$$

The significance of ω, α are mentioned earlier in equation (70).

Farrugia et. al. investigated the gravitational frequency shift in f(r) gravity model [48].

The gravitational frequency shift under the PPN metric is expressed by,

$$\Delta\nu = (1+\beta)\frac{\Delta U}{c^2} \tag{1.104}$$

where β is the PPN parameter. The first time the gravitational frequency shift was successfully measured in Pound-Rebka-Snider experiment of 1960-1965 using gamma-ray photon at Harvard University.

In recent times, an advanced hydrogen maser clock, placed on International Space Station and an atomic clock based on Cesium called PHARAO (Project D'Horloge Atomique par Refroidissement d'Atomes en Orbit) are used to measure the gravitational frequency shift under the Atomic Clock Ensemble in Space(ACES) project.

The precession level has been achieved so far up to 10^{-15} order using clock comparison and 10^{-15} H-maser in GP-A redshift measurement [89]. But to sense the effect of dark energy, the accuracy must be reached at least 10^{-38} order [37].

1.8.5 Gravitational wave and a wider aspect to detect the influences of dark sector:

The theory of general relativity suggests that the ripple of space-time perturbation will travel in the form of a wave in the transverse direction of propagation, which can be expressed as follows:

$$(-\frac{\delta^2}{\delta t^2} + c^2\Delta^2)h_{\mu\nu} = 0 \tag{1.105}$$

where, $h_{\mu\nu}$ is a very weak perturbation of space-time metric, nearly Minkowsky metric in Spacial Relativity, c stands for speed of light, $\Delta = (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})$, the spatial second order differential operator. No component of the metric perturbation ($h_{\mu\nu}$) is found in direction of wave propagation.

Gravitational effects have been tested so far in different distance scales and gravity strength areas (like weak and strong gravity regions). Gravitational wave astronomy has been explored the possibilities to test the gravitation in large scale and strong field regime as it can travel a large distance without any interruption, unlike electromagnetic wave. The recent detection of gravitational wave by LIGO, has opened up the window to explore the reality of dark sector and many other unresolved astronomical problems. Total five binary black-hole [90], [91], [92], [93], [94], [95] and a binary neutron star [96] sourcing GW have been detected so far.

A recent observation by advanced LIGO and Virgo detectors, a strong signal of gravitational wave event GW170817 has been detected from a merger of binary neutron stars [97] and a gamma-ray(GRB170817A) was also detected from the same region of gravitational wave source by the same LIGO-Virgo detectors. The detection of GW170817 was the first multi-messenger astronomical observation from where both gravitational wave and electromagnetic wave have been detected. These observations enable to be used as the sources of standard siren which able to measure the astronomical distances of the sources using gravitational waves. Measuring distances of the source by siren and red-shift of the electromagnetic wave, the Hubble constant can be measured and using this way, the dynamic nature of the Universe can be analyzed with high precision and existence and effect of dark energy will be re-verified. On the other hand, dark matter, in the form of axions or ultra-light bosons, form clouds around a black-hole, which is observable with gravitational waves [97].