Chapter - 4

Spontaneous Faraday Rotation in a Laser Produced Plasma

4.1 Introduction

Plasma is a highly nonlinear optical medium. With high-power pulsed lasers, nonlinear optical effects in plasmas are easily observable. They are hardly avoidable in laser heating of plasmas and in laser induced fusion work. As a charged fluid, a plasma is readily perturbed by external fields. So, extremely strong and complex response of intense laser fields can yield many nonlinear optical phenomena in plasmas [Bloembergen (1972), Svelto (1974), Shen (1976), Stix (1992)]. Such nonlinear effects arise from the nonlinear optical response of the plasma constituent to the applied fields.

It is well known that when a ray of light is made incident on a crystal of Iceland Spar, it is refracted into two rays. One is called the Ordinary ray (O-ray) while the other is known as the Extraordinary ray (E-ray). Both rays are plane polarized whose vibrations are perpendicular to each other. This phenomenon in which a single incident ray is refracted into two rays is called double refraction or birefringence. Such phenomenon is inevitable when a ray of light enters into anisotropic media because, it instantaneously splits up into two separate components which travel with different polarizations. These two rays are consequently reflected to different extent, and follow different paths that lead to rotation of plane of polarization. In isotropic media, this effect is not observed because the plane of polarization remains unchanged. Nevertheless, when a ray of light enters into the same medium in presence of a static magnetic field, the medium no longer remains isotropic and eventually it shows the birefringence properties (Faraday effect and Cotton-Mount effect).

This Chapter is based on the paper "Spontaneous Faraday Rotation due to Strong Laser Radiation in a " by B. Bhattacharyya, P. Mulser and Urmi Sanyal, Physics Letters A, **249,** pp 324-329(1998).

Moreover, strong electric field can induce double reflection or birefringence in normally isotropic medium (Kerr effectand Pockels effects). For example, an elliptically polarized wave will be treated as the superposition of two plane polarized waves having different amplitudes and phase difference $\pi/2$. If it propagates in either an electro-optically (Kerr effect) or a magneto-optically (Faraday effect) active isotropic (or anisotropic) medium, an intensity induced rotation of their plane of polarization will be observed. Chiao and Godine [1969] showed that an intense elliptically polarized light wave undergoes rotation in Kerr-active molecular medium, This rotation has also been observed first in liquid in the pretransitional behaviour of the field induced molecular alignment in isotropic nematic substance [Wong and Shen (1974)]. We now study the nonlinear corrections of an electromagnetic wave, in particular laser light in a plasma phenomena logical employing the birefringence properties leading to magneto optical Faraday effect and induced magnetization.

Plasma acts as an optically active medium. When a constant magnetic field acts parallel to the direction of propagation of a monochromatic transverse wave in cold nondissipative plasma, the wave is also split into two waves of different wave numbers in the plasma. This is the effect of birefringence in a magnetized plasma. Of these two waves, one (E_R) rotates
anticlockwise and the other (E_L) does clockwise. E_R is right circularly polarized wave (or Extraordinary) wave, E_L is the left circularly polarized wave (or Ordinary wave). If the observer is looking anti-parallel to the applied field, electrons appear to gyrate anticlockwise whereas ions gyrate clockwise. If the resultant electric field of E_R and E_L is received by a polarized antenna then the plane of polarization is found to be rotated through an angle Φ. This angle Φ is known as a Faraday rotation angle.

A plane polarized wave can be resolved in a plasma into two circularly polarized waves with different plane of polarizations that, consequently, follow the different dispersion rates and, subsequently, turn out to be induced magnetic fields in plasma. On the other hand, if the plane of polarization of those waves remain same, i.e., their dispersion rates are equal, then the induced magnetization will cancel each other [Ginzburg (1963), Krall and Trivelpiece (1973), Shen (1991), Stix (1992)]. This induced magnetic field is called Inverse Faraday Effect (IFE) [Pomeau and Quemada (1967), Stiger and Woods (1972), Talin etal., (1975), Abdullaev and Frolov (1981), Chakraborty etal., (1990), Bera. etal., (1992), Bhattacharyya (1994), Horovitz etal., (1995), Sheng and Meyr-ter-Vehn (1996)]. In plasmas, the rotating electric field of a circularly polarized wave drives the electrons and the ions into circular orbits, and the velocity of which is proportional to the electric field of the wave. So, the magnetic moment is proportional to the radiation intensity. Therefore, the magnitude of the induced fields can be estimated from magnetic moment. In this chapter, we will develop a theory useful for the study of the evolution of nonlinear Faraday rotation by the method of birefringence and the evolution of the induced magnetization in the presence as well as in the absence of ambient magnetic fields in a plasma.

In Chapter-3, we have discussed the mechanism of spontaneous and simultaneous generation of the axial (poloidal) and lateral (toroidal) magnetic field for a cold, nondissipative, two-component plasma consisting of both electrons and ions. However, we have not considered the relativistic effect. In fact, electrons and ions become relativistic in laser induced plasma [Mulser (1980), Hora (1991), Kruer (1988), Mima etal., (1994)]. So, the generation of magnetic fields, in particular lateral fields, due to relativistic effects in plasmas have been studied here [Bhattacharyya and Chakraborty (1979), Pukhov and Meyer-ter-Vehn (1996)].

4.2 Formulation of the problem

(I) Basic assumptions and relevant equations:- We have assumed that the waves ar sinusoidal, i.e., the perturbed field variables are harmonic in nature. The plasma is assumed t be cold, i.e., thermal velocities of electrons and ions are much less than the velocity of light such that $v_{the,i} << c$, where v_{the} and v_{thi} represent the thermal velocities of electrons and ions respectively, and c is the velocity of light. Plasma is also to be assumed homogeneous with mobile electrons and ions. The incident laser is an electromagnetic wave of such high intensity that the motion of electrons and ions becomes weakly relativistic, but the power of the radiation does not exceed the threshold power limit for the appearance of self-focussing and self-trapping mechanisms [Chiao etal., (1994), Shen (1976), Max (1976, 1982), Kruer (1988)]; also, the difference between the velocities of electrons and ions is much less than the velocity of the light , so that, the instabilities of the system can be minimized. We also assume that there is no first harmonic density fluctuation due to interaction of waves with the plasma. Nonlinearly excited second harmonic density fluctuation exists and its effect on stimulated Raman scattering (SRS) and stimulated Brillouin scattering (SBS) will be visible only in an order higher than three [Newell and Moloney (1992), Kruer (1988)]. So, these effects are neglected. Self-action effects arising from ponderomotive force and thermal instabilities are neglected because pressure variation and thermal velocities are ignored. Two circularly polarized waves propagate parallel to the direction of static and uniform magnetic field. The problem of nonlinear propagation of an intense wave is solved in closed form [Maker and Terhune (1965), Chiao and Godine (1969), Decoster (1978)], in so far as the wave processes remain quasi-monochromatic [Arons and Max (1974), Chakraborty etal., (1984)]. Under the above set of assumptions and using the fluid model, we get the following basic equations:-

The equations of momentum for electron and ion as,

$$[\frac{\partial}{\partial t} + (\mathbf{v_e}.\nabla)]\mathbf{p_e} = -\frac{e}{m_e}\mathbf{E} - \frac{e}{m_e c}(\mathbf{v_e}\times\mathbf{H}) - \nu_e(\mathbf{v_e}-\mathbf{v_i}) \quad (4.2.1),$$

$$[\frac{\partial}{\partial t} + (\mathbf{v_i}.\nabla)]\mathbf{p_i} = \frac{e}{m_i}\mathbf{E} + \frac{e}{m_i c}(\mathbf{v}\times\mathbf{H}) - \nu_i(\mathbf{v_i}-\mathbf{v_e}) \quad (4.2.2),$$

the continuity equations for electron and ion as,

$$\frac{\partial n_e}{\partial t} + \nabla.(n_e\mathbf{v}_e) = 0 \quad (4.2.3)$$

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i \mathbf{v}_i) = 0 \tag{4.2.4}$$

and the four Maxwell's equations as,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \tag{4.2.5}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi e}{c}(n_i \mathbf{v}_i - n_e \mathbf{v}_e) \tag{4.2.6}$$

$$\nabla.\mathbf{E} = 4\pi e(n_i - n_e) \tag{4.2.7}$$

$$\nabla.\mathbf{H} = 0 \tag{4.2.8}$$

where the subscripts e and i represent the species of electrons and ions respectively; $\mathbf{p}_e$, $\mathbf{v}_e$, n_e, m_e stand for relativistic momentum, velocity, density and mass of electrons with negatively charge e respectively; $\mathbf{p}_i$, $\mathbf{v}_i$, n_i, m_i are those of ions having positive charge e. **E**, **H,** c are the electric field, the magnetic field and the velocity of light respectively; ν_e and ν_i are the collision frequencies of electrons with ions and ions with electrons respectively.

For weak relativistic effect, i.e., when $v_{e,i}^2 << c^2$ holds, then the relativistic momentum for electrons and ions can be expressed as

$\mathbf{p}_{e,i} = m_{e,i}\, \mathbf{v}_{e,i}\, (1 - v_{e,i}^2/c^2)^{-½} \cong m_{e,i}\, \mathbf{v}_{e,i}\, (1 - v_{e,i}^2/c^2)$.

Hence, the equations (4.2.1) and (4.2.2) can be written as

$$\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e.\nabla)\mathbf{v}_e + \frac{\partial}{\partial t}\frac{v_e^2 \mathbf{v}_e}{2c^2} = -\frac{e}{m_e}\mathbf{E} - \frac{e}{m_e c}(\mathbf{v}_e \times \mathbf{H}) - \nu_e(\mathbf{v}_e - \mathbf{v}_i) \tag{4.2.9}$$

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i.\nabla)\mathbf{v}_i + \frac{\partial}{\partial t}\frac{v_i^2 \mathbf{v}_i}{2c^2} = \frac{e}{m_i}\mathbf{E} + \frac{e}{m_i c}(\mathbf{v} \times \mathbf{H}) - \nu_i(\mathbf{v}_i - \mathbf{v}_e) \tag{4.2.10}$$

(II) <u>A method using rotating complex coordinates</u>:- From general terminology, any vector **A** (say) can be written as

$$A = A_y + iA_z \quad \text{and} \quad \overline{A} = A_y - iA_z \tag{4.2.11}$$

where A_y and A_z are the transverse components of the vector **A** and so, the propagation vector along x-axis. In fact, apparently, the first relation of (4.2.11) is the complex conjugate of its second relation, and vice-versa. Nevertheless, phenomenological, A and $\overline{A}$ act respectively as the left and right circularly polarized components of the vector **A**.

Using the relations (4.2.11), in the equations (4.2.9) and (4.2.10), we have the electron momentum equation as

$$\frac{\partial v_e}{\partial t} + \frac{e}{m_e}E + \nu_e(v_e - v_i) - i\Omega_e v_e = -v_{ex}\frac{\partial}{\partial x}v_e - \frac{ie}{m_e c}v_{ex}H - \frac{\partial}{\partial t}\left(\frac{v_e^2 \overline{v}_e}{2c^2}\right) \tag{4.2.12}$$

$$\frac{\partial \overline{v}_e}{\partial t} + \frac{e}{m_e}\overline{E} + \nu_e(\overline{v}_e - \overline{v}_i) + i\Omega_e \overline{v}_e = -v_{ex}\frac{\partial}{\partial x}\overline{v}_e + \frac{ie}{m_e c}v_{ex}\overline{H} - \frac{\partial}{\partial t}\left(\frac{\overline{v}_e^2 v_e}{2c^2}\right) \tag{4.2.13}$$

and also the ion momentum equation as

$$\frac{\partial v_i}{\partial t} - \frac{e}{m_i}E + \nu_i(v_i - v_e) + i\Omega_i v_i = -v_{ix}\frac{\partial}{\partial x}v_i + \frac{ie}{m_i c}v_{ix}H - \frac{\partial}{\partial t}\left(\frac{v_i^2 \overline{v}_i}{2c^2}\right) \tag{4.2.14}$$

$$\frac{\partial \overline{v}_i}{\partial t} - \frac{e}{m_i}\overline{E} + \nu_i(\overline{v}_i - \overline{v}_e) - i\Omega_i \overline{v}_i = -v_{ix}\frac{\partial}{\partial x}\overline{v}_i - \frac{ie}{m_i c}v_{ix}\overline{H} - \frac{\partial}{\partial t}\left(\frac{\overline{v}_i^2 v_i}{2c^2}\right) \tag{4.2.15}$$

Further, using the relation (4.2.11), the transverse components of (4.2.5) can be written as

$$\frac{\partial E}{\partial x} + \frac{1}{c}\frac{\partial H}{\partial t} = 0 \tag{4.2.16}$$

$$\frac{\partial \overline{E}}{\partial x} - \frac{1}{c}\frac{\partial \overline{H}}{\partial t} = 0 \tag{4.2.17}$$

Similarly, the transverse components of (4.2.6) give

$$i\frac{\partial H}{\partial x} - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi e}{c}(n_i v_i - n_e v_e) \tag{4.2.18}$$

$$i\frac{\partial \bar{H}}{\partial x} + \frac{1}{c}\frac{\partial \bar{E}}{\partial t} = -\frac{4\pi e}{c}(n_i \bar{v}_i - n_e \bar{v}_e) \tag{4.2.19}$$

Moreover, the longitudinal part of the momentum equation (4.2.9) for electron is

$$\frac{\partial v_{ex}}{\partial t} + \frac{e}{m_e}E_x = \frac{ie}{2m_e c}(\bar{v}_e H - v_e \bar{H}) - \nu_e(v_{ex} - v_{ix}) \tag{4.2.20}$$

and the same for ions be written, from (4.2.10), as

$$\frac{\partial v_{ix}}{\partial t} - \frac{e}{m_i}E_x = -\frac{ie}{2m_i c}(\bar{v}_i H - v_i \bar{H}) - \nu_i(v_{ix} - v_{ex}) \tag{4.2.21}$$

Also the longitudinal component of (4.2.6) gives

$$\frac{\partial E_x}{\partial t} = -4\pi e\,(n_e v_{ex} - n_i v_{ix}) \tag{4.2.22}$$

Simplifying the equations (4.2.7) and (4.2.8) for longitudinal components, we have

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e) \tag{4.2.23}$$

and

$$\frac{\partial H}{\partial x} = 0 \tag{4.2.24}$$

4.3 Linearized equations and solutions

The above set of equations from (4.2.12) to (4.2.24) (i.e., the equations from (4.2.3) to (4.2.10) along with (4.2.11) are to be solved by the successive approximation scheme

[Bellman (1964), Ames (1965)) which has already been elucidated in Chapter-2. We then have the first order equations of continuity for electrons and ions as

$$\frac{\partial n_{e1}}{\partial t} + n_0 \frac{\partial . v_{ex1}}{\partial x} = 0 \tag{4.3.1}$$

$$\frac{\partial n_{i1}}{\partial t} + n_0 \frac{\partial v_{ix1}}{\partial x} = 0 \tag{4.3.2}$$

and the same order momentum equations for electrons and ions are

$$\frac{\partial v_{e1}}{\partial t} + \frac{e}{m_e} E_1 - i\Omega_e v_{e1} + \nu_e (v_{e1} - v_{i1}) = 0 \tag{4.3.3}$$

$$\frac{\partial \bar{v}_{e1}}{\partial t} + \frac{e}{m_e} \bar{E}_1 + i\Omega_e \bar{v}_{e1} + \nu_e (\bar{v}_{e1} - \bar{v}_{i1}) = 0 \tag{4.3.4}$$

$$\frac{\partial v_{i1}}{\partial t} - \frac{e}{m_i} E_1 - i\Omega_i \bar{v}_{i1} + \nu_i (v_{i1} - v_{e1}) = 0 \tag{4.3.5}$$

$$\frac{\partial \bar{v}_{i1}}{\partial t} - \frac{e}{m_i} \bar{E}_1 - i\Omega_i \bar{v}_{i1} + \nu_i (\bar{v}_{i1} - \bar{v}_{e1}) = 0 \tag{4.3.6}$$

We also have the first order Maxwell's equations as

$$\frac{\partial E_1}{\partial x} - \frac{i}{c} \frac{\partial H_1}{\partial t} = 0 \tag{4.3.7}$$

$$\frac{\partial \bar{E}_1}{\partial x} + \frac{i}{c} \frac{\partial \bar{H}_1}{\partial t} = 0 \tag{4.3.8}$$

$$\frac{\partial H_1}{\partial x} + \frac{i}{c} \frac{\partial E_1}{\partial t} = -i \frac{4\pi e n_0}{c} (v_{i1} - v_{e1}) \tag{4.3.9}$$

$$\frac{\partial \bar{H}_1}{\partial x} - \frac{i}{c} \frac{\partial \bar{E}_1}{\partial t} = i \frac{4\pi e n_0}{c} (\bar{v}_{i1} - \bar{v}_{e1}) \tag{4.3.10}$$

Now, we start with the linearized wave solution of the electric field [Krall and

Trivelpiece (1973)]$

$$\mathbf{E}_1 = \frac{mc\omega}{2e}\{(\hat{y}+i\hat{z})(\alpha e^{i\theta_r}+\bar{\beta}e^{-i\bar{\theta}_l})+(\hat{y}-i\hat{z})(\bar{\alpha}e^{-i\bar{\theta}_r}+\beta e^{i\theta_l})\} \quad (4.3.11)$$

where $\alpha = (ea/mc\omega)$, $\beta = (eb/mc\omega)$ are the dimensionless amplitudes of the two circularly polarized waves of the electric field, $m = \frac{m_e m_i}{m_e + m_i}$ is the reduced mass. Phase of these waves are taken as $\theta_r = k_r x - \omega t$ and $\theta_l = k_l x - \omega t$, where k_r and k_l are the wave numbers of right and left circularly polarized waves respectively and ω is the wave frequency. The form of these two waves would be chosen such that they reduce to an elliptically polarized wave in an unmagnetized plasma with same phase (i.e.$\theta_l = \theta_r$). But, for a magnetized plasma, they will be treated as left and right circularly polarized waves with different phases θ_l and θ_r respectively.

Using the relation (4.2.11), the equation (4.3.11) can be rewritten as

$$E_1 = \frac{mc\omega}{2e}(\bar{\alpha}e^{-i\bar{\theta}_r}+\beta e^{i\theta_l}) \quad (4.3.12)$$

$$\bar{E}_1 = \frac{mc\omega}{2e}(\alpha e^{i\theta_r}+\bar{\beta}e^{-i\bar{\theta}_l}) \quad (4.3.13)$$

Solving the equations (4.3.7) and (4.3.8) together with the relations (4.3.12) and (4.3.13), respectively, we have the linearized magnetic fields as

$$H_1 = i\frac{mc\omega}{2e}(\bar{n}_r\bar{\alpha}e^{-i\bar{\theta}_r}+n_l\beta e^{i\theta_l}) \quad (4.3.14)$$

$$\bar{H}_1 = -i\frac{mc\omega}{e}(n_r\bar{\alpha}e^{i\theta_r}+\bar{n}_l\bar{\beta}e^{-i\bar{\theta}_l}) \quad (4.3.15)$$

where $n_{r,l} = (k_{r,l}c/\omega)$, $\bar{n}_{r,l} = (\bar{k}_{r,l}c/\omega)$.

Using the relation (4.3.12), in the equations (4.3.3) and (4.3.5), we have

$$\frac{\partial v_{e1}}{\partial t} - i\Omega_e v_{e1} + \nu_e v_{e1} - \nu_e v_{i1} = -\frac{e}{m_e}E_1 = c\omega M_e[\bar{\alpha}e^{-i\bar{\theta}_r}+\beta e^{i\theta_l}] \quad (4.3.16)$$

and $$-\nu_e v_{e1} + \frac{\partial v_i}{\partial t} + i\Omega_i v_{i1} + \nu_i v_{i1} = \frac{e}{m_e} E_1 = c\omega M_i [\bar{\alpha} e^{-i\bar{\theta}_r} + \beta e^{i\theta_l}] \quad (4.3.17)$$

Solving equations (4.3.16) and (4.3.17), we have the linearized velocity of electrons as

$$v_{e1} = -ic[-\frac{\{M_e(1+Y_i-iZ_i)+iM_iZ_e\}\bar{\alpha}e^{-i\bar{\theta}_r}}{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i} + \frac{\{M_e(1-Y_i+iZ_i)-iM_iZ_e\}\bar{\beta}e^{i\theta_l}}{(1+Y_e+iZ_e)(1-Y_i+iZ_i)+Z_eZ_i}] \quad (4.3.18)$$

and that of ions as

$$v_{i1} = ic[-\frac{\{M_i(1-Y_e-iZ_e)+iM_eZ_i\}\bar{\alpha}e^{-i\theta_r}}{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i} + \frac{\{M_i(1+Y_e+iZ_e)-iM_eZ_i\}\beta e^{i\theta_l}}{(1+Y_e+iZ_e)(1-Y_i+iZ_i)+Z_eZ_i}] \quad (4.3.19)$$

where, $Y_e = (\Omega_e/\omega)$, $Y_i = (\Omega_i/\omega)$, $Z_e = (\nu_e/\omega)$, $Z_i = (\nu_i/\omega)$, $M_e = m/m_e$, $M_i = m/m_i$

In short, the equations (4.3.18) and (4.3.19) can be rewritten as

$$v_{e1} = -ic(\bar{C}_1\bar{\alpha}e^{-i\bar{\theta}_r} + C_2\beta e^{i\theta_l}) \quad (4.3.20)$$

and

$$v_{i1} = ic(\bar{C}_3\bar{\alpha}e^{-i\bar{\theta}_r} + C_4\beta e^{i\theta_l}) \quad (4.3.21)$$

where $$\bar{C}_1 = \frac{\{M_e(1+Y_i-iZ_i)+iM_iZ_e\}}{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i},$$

$$C_2 = \frac{\{M_e(1-Y_i+iZ_i)-iM_iZ_e\}}{(1+Y_e+iZ_e)(1-Y_i+iZ_i)+Z_eZ_i},$$

$$\bar{C}_3 = \frac{\{M_i(1-Y_e-iZ_e)+iM_eZ_i\}}{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i},$$

and $$C_4 = \frac{\{M_i(1+Y_e+iZ_e)-iM_eZ_i\}}{(1+Y_e+iZ_e)(1-Y_i+iZ_i)+Z_eZ_i}.$$

Similarly, substituting (4.3.13) in the equations (4.3.4) and (4.3.6) and solving them, we get the following expressions of the complex conjugates of v_{e1}, and v_{i1} as

$$\overline{v}_{e1} = ic(C_1\alpha e^{i\theta_r} + \overline{C}_2\overline{\beta}e^{-i\overline{\theta}_r}) \quad (4.3.22)$$

and

$$\overline{v}_{i1} = -ic(C_3\alpha e^{i\theta_r} + \overline{C}_4\overline{\beta}e^{-i\overline{\theta}_r}) \quad (4.3.23)$$

where, $\overline{C}_4 = \dfrac{\{M_i(1+Y_e-iZ_e)+iM_eZ_i\}}{(1+Y_e-iZ_e)(1-Y_i-iZ_i)+Z_eZ_i}$,

$$C_1 = \frac{\{M_e(1+Y_i+iZ_i)-iM_iZ_e\}}{(1-Y_e+iZ_e)(1+Y_i+iZ_i)+Z_eZ_i},$$

$$\overline{C}_2 = \frac{\{M_e(1-Y_e-iZ_e)+iM_iZ_e\}}{(1+Y_e-iZ_e)(1-Y_i-iZ_i)+Z_eZ_i},$$

and $C_3 = -\dfrac{\{M_i(1-Y_e+iZ_e)-iM_iZ_e\}}{(1-Y_e+iZ_e)(1+Y_i+iZ_i)+Z_eZ_i}$

It may be noted that the wave is transverse and so the solenoidal conditions $\nabla.\mathbf{E}_1 = 0$ implies $\nabla.(\mathbf{v}_{e1}, \mathbf{v}_{i1}, \mathbf{H}_1) = 0$. Consequently, from the first order equations (4.2.20) to (4.2.24) together with (4.3.1) and (4.3.2), we have the first order electron and ion densities as $n_{e1} = 0$ and $n_{i1} = 0$ respectively. Hence, there is no change in densities of electrons and ions in the linearized approximation that agrees well with our basic assumption.

4.4 Linear dispersion relation

Combining the equations (4.3.7) and (4.3.9), we have

$$\frac{\partial^2 E_1}{\partial t^2} - c^2\frac{\partial^2 E_1}{\partial x^2} + 4\pi e n_0\frac{\partial}{\partial t}(v_{i1} - v_{e1}) = 0 \quad (4.4.1)$$

and also combining the equations (4.3.3) and (4.3.5) we get

$$4\pi e^2 n_0\frac{\partial}{\partial t}(v_{i1} - v_{e1}) = D^2(\omega_{pe}^2+\omega_{pi}^2)\,E_1/[(D-i\Omega_e+\nu_e)(D+i\Omega_i+\nu_i) - \nu_e\nu_i\,] \quad (4.4.2)$$

Eliminating $(v_{i1} - v_{e1})$ from (4.4.1) and (4.4.2), we have

$$\frac{\partial^2 E_1}{\partial t^2} - c^2\frac{\partial^2 E_1}{\partial x^2} + (\omega_{pe}{}^2+\omega_{pi}{}^2)\frac{\partial^2 E_1}{\partial t^2} /[(\frac{\partial}{\partial t} - i\Omega_e+\nu_e)(\frac{\partial}{\partial t}+i\Omega_i+\nu_i) - \nu_e\nu_i] = 0 \qquad (4.4.3)$$

Putting $E_1 = \bar{a}e^{-i\bar{\theta}_r} + be^{i\theta_l}$ in equation (4.4.3), and then collecting the coefficients of like powers of $e^{i\theta_l}$ and also of $e^{-i\bar{\theta}_r}$ from both sides of (4.4.3), we get

$$n_l^2 = 1 - \frac{(X_e+X_i)}{\{(1+Y_e+iZ_e)(1-Y_i+iZ_i)+Z_eZ_i\}} \qquad (4.4.4)$$

and $$\bar{n}_r^2 = 1 - \frac{(X_e+X_i)}{\{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i\}} \qquad (4.4.5)$$

Similarly, after simplifying the equations (4.3.4), (4.3.6), (4.3.8) and (4.3.10); we get

$$\frac{\partial^2 \overline{E_1}}{\partial t^2} - c^2\frac{\partial^2 \overline{E_1}}{\partial x^2} + (\omega_{pe}{}^2+\omega_{pi}{}^2)\frac{\partial^2 \overline{E_1}}{\partial t^2} /[(\frac{\partial}{\partial t} - i\Omega_e+\nu_e)(\frac{\partial}{\partial t}+i\Omega_i+\nu_i) - \nu_e\nu_i] = 0 \qquad (4.4.6)$$

and also using $E_1 = ae^{i\theta_r} + \bar{b}e^{-i\bar{\theta}_l}$ in (4.4.6) and then collecting the coefficients of like powers of $e^{-i\theta_r}$ and $e^{-i\bar{\theta}_l}$ from both the sides of (4.4.6), we get

$$n_r^2 = 1 - \frac{(X_e+X_i)}{\{(1-Y_e+iZ_e)(1+Y_i+iZ_i)+Z_eZ_i\}} \qquad (4.4.7)$$

and $$\bar{n}_l^2 = 1 - \frac{(X_e+X_i)}{\{(1+Y_e-iZ_e)(1-Y_i-iZ_i)+Z_eZ_i\}} \qquad (4.4.8)$$

The relations (4.4.4), (4.4.5), (4.4.7) and (4.4.8) are known as dispersion relations in linear approximation of the propagation of two circularly polarized electromagnetic waves in a magnetized dissipative plasma. These are amplitude independent dispersion relations.

Ignoring the collisional effects (or the dissipation terms) i.e. $Z_e = Z_i \cong 0$, then the dispersion relations reduce to,

$$n_r^2 = 1 - \frac{(X_e + X_i)}{(1 - Y_e)(1 + Y_i)} \tag{4.4.9}$$

$$n_l^2 = 1 - \frac{(X_e + X_i)}{(1 + Y_e)(1 - Y_i)} \tag{4.4.10}$$

The above relations (4.4.9) and (4.4.10) are also known as the linearized dispersion relations of right and left circularly polarized electromagnetic waves respectively in a magnetized and collisionless plasma.

4.5 Second order field equations and solutions

The second order equations for continuity of electrons and ions are derived from the equations (4.2.3) and (4.2.4) respectively by an approximation scheme [Bellman (1964)] used in the earlier Chapter-2:

$$\frac{\partial n_{e2}}{\partial t} + n_0 \frac{\partial v_{e2x}}{\partial x} = 0 \tag{4.5.1}$$

$$\frac{\partial n_{i2}}{\partial t} + n_0 \frac{\partial v_{i2x}}{\partial x} = 0 \tag{4.5.2}$$

Similarly, from the equations (4.2.9) and (4.2.10) the second order momentum equations of electrons and ions can be written as

$$\frac{\partial v_{e2x}}{\partial t} + \frac{e}{m_e} E_{2x} + \nu_e (v_{e2x} - v_{i2x}) = i \frac{e}{2 m_e c} (\bar{v}_{e1} H_1 - v_{e1} \bar{H}_1) \tag{4.5.3}$$

$$\frac{\partial v_{i2x}}{\partial t} - \frac{e}{m_i} E_{2x} + \nu_i (v_{i2x} - v_{e2x}) = -i \frac{e}{2 m_i c} (\bar{v}_{i1} H_1 - v_{i1} \bar{H}_1) \tag{4.5.4}$$

and also the second order Maxwell's equations reduce to

$$\frac{\partial E_{2x}}{\partial t} = 4 \pi e n_0 (v_{e2x} - v_{i2x}) \tag{4.5.5}$$

$$\frac{\partial E_{2x}}{\partial x} = -4\pi e(n_{e2} - n_{i2}) \tag{4.5.6}$$

$$H_{2x} = 0 \tag{4.5.7}$$

Simplifying the equations (4.5.3) and (4.5.5), we have

$$(D^2+\omega_{pe}^2+\nu_e D)v_{e2x}-(\omega_{pe}^2+\nu_e D)v_{i2x} = SEV_2 \tag{4.5.8}$$

where $SEV_2 = i(\frac{e}{2m_e c})D(\bar{v}_{e1}H_1 - v_{e1}\bar{H}_1)$

and also from equations (4.5.4) and (4.5.5), we have

$$(D^2+\omega_{pi}^2+\nu_i D)v_{i2x}-(\omega_{pi}^2+\nu_i D)\, v_{e2x} = SIV_2 \tag{4.5.9}$$

where $SIV_2 = -i(\frac{e}{2m_i c})D(\bar{v}_{i1}H_1 - v_{i1}\bar{H}_1)$

Simplifying the equations (4.5.8) and (4.5.9), we have the equation for second order velocity of electrons as

$$[(D^2+\omega_{pe}^2+\nu_e D)(D^2+\omega_{pi}^2+\nu_i D)-(\omega_{pe}^2+\nu_e D)(\omega_{pi}^2+\nu_i D)]v_{e2x}$$
$$= (D^2+\omega_{pi}^2+\nu_i D)SEV_2+(\omega_{pe}^2+\nu_e D)SIV_2 \tag{4.5.10}$$

and that of ions as

$$[(D^2+\omega_{pe}^2+\nu_e D)(D^2+\omega_{pi}^2+\nu_i D)-(\omega_{pe}^2+\nu_e D)(\omega_{pi}^2+\nu_i D)]v_{i2x}$$
$$= (D^2+\omega_{pe}^2+\nu_e D)SIV_2+(\omega_{pi}^2+\nu_i D)SEV_2 \tag{4.5.11}$$

The terms of SEV_2 and SIV_2 can be simplified as:

$$SEV_2 = -(c\omega^2 M_e)[(C_1 n_l - C_2 n_r)\alpha\beta e^{i(\theta_r+\theta_l)} - (\bar{C}_1\bar{n}_l - \bar{C}_2\bar{n}_r)\bar{\alpha}\bar{\beta} e^{-i(\bar{\theta}_r+\bar{\theta}_l)}]$$

and $SIV_2 = -(c\omega^2 M_i)[(C_3 n_l - C_4 n_r)\alpha\beta e^{i(\theta_r+\theta_l)} + (\bar{C}_3\bar{n}_l - \bar{C}_4\bar{n}_r)\bar{\alpha}\bar{\beta} e^{-i(\bar{\theta}_r+\bar{\theta}_l)}]$

Solving the equation (4.5.10), we get the second order velocity of electrons as

$$v_{e2x} = -\frac{c}{4}[C_{11}\alpha\beta e^{i(\theta_r+\theta_l)} + \bar{C}_{11}\bar{\alpha}\bar{\beta} e^{-i(\bar{\theta}_r+\bar{\theta}_l)}] \tag{4.5.12}$$

where, $C_{11} = [(\frac{1}{M_e})(C_1 n_L - C_2 n_R)(X_i - 2iZ_i - 4) + (\frac{1}{M_i})(C_3 n_L - C_4 n_R)(X_e - 2iZ_e)](\frac{1}{\varsigma})$,

$\varsigma = 4 - (X_e + X_i) + 2i(Z_e + Z_i)$, and $\overline{C_{11}}$ is the complex conjugate of C_{11}.

Similarly, the second order velocity of ions can be derived by solving (4.5.11) as

$$v_{i2x} = -\frac{c}{4}[C_{22}\alpha\beta e^{i(\theta_r+\theta_l)} + \bar{C}_{22}\bar{\alpha}\bar{\beta} e^{-i(\bar{\theta}_r+\bar{\theta}_l)}] \tag{4.5.13}$$

where $C_{22} = [(\frac{1}{M_e})(C_1 n_L - C_2 n_R)(X_i - 2iZ_i) + (\frac{1}{M_i})(C_3 n_L - C_4 n_R)(X_e - 2iZ_e - 4)](\frac{1}{\varsigma})$

and $\overline{C_{22}}$ is the complex conjugate of C_{22}.

Solving the equations (4.5.1) and (4.5.2) together with (4.5.12) and (4.5.13), we get the second order electron and ion densities as

$$n_{e2} = -\frac{n_0}{8}[C_{11}(n_r + n_l)\alpha\beta e^{i(\theta_r+\theta_l)} + \bar{C}_{11}(\bar{n}_r + \bar{n}_l)\bar{\alpha}\bar{\beta} e^{-i(\bar{\theta}_r+\bar{\theta}_l)}] \tag{4.5.14}$$

and

$$n_{i2} = -\frac{n_0}{8}[C_{22}(n_r + n_l)\alpha\beta e^{i(\theta_r+\theta_l)} + \bar{C}_{22}(\bar{n}_r + \bar{n}_l)\bar{\alpha}\bar{\beta} e^{-i(\bar{\theta}_r+\bar{\theta}_l)}] \tag{4.5.15}$$

It is evident that the second order calculations for E_2 and $\overline{E_2}$ are as simple as those of the first order fields. Moreover, the dispersion relations in the second order do not show any new aspect because of the fact that the linearized solutions contain first harmonic terms in one hand and on the other hand the second order solutions contribute only second harmonic terms. So, we consider the third order equations corrected upto first order in the next section for studying nonlinear dispersion relations.

4.6 Derivation of nonlinear dispersion relation

Using the relation (4.2.11), the third order momentum equations for electrons and ions can be written as

$$\left(\frac{\partial}{\partial t}-i\Omega_e+\nu_e\right)v_{e3}-\nu_e v_{i3}+\frac{e}{m_e}E_3=-v_{e2x}\frac{\partial v_{e1}}{\partial x}-i\frac{e}{m_e c}v_{e2x}H_1-\frac{\partial}{\partial t}\frac{v_{e1}^2\bar{v}_{e1}}{2c^2} \tag{4.6.1}$$

$$\left(\frac{\partial}{\partial t}+i\Omega_e+\nu_e\right)\bar{v}_{e3}-\nu_e \bar{v}_{i3}+\frac{e}{m_e}\bar{E}_3=-v_{e2x}\frac{\partial \bar{v}_{e1}}{\partial x}-i\frac{e}{m_e c}v_{e2x}\bar{H}_1-\frac{\partial}{\partial t}\frac{\bar{v}_{e1}^2 v_{e1}}{2c^2} \tag{4.6.2}$$

$$\left(\frac{\partial}{\partial t}+i\Omega_i+\nu_i\right)v_{i3}-\nu_i v_{e3}-\frac{e}{m_i}E_3=-v_{i2x}\frac{\partial v_{i1}}{\partial x}+i\frac{e}{m_i c}v_{i2x}H_1-\frac{\partial}{\partial t}\frac{v_{i1}^2\bar{v}_{i1}}{2c^2} \tag{4.6.3}$$

$$\left(\frac{\partial}{\partial t}-i\Omega_i+\nu_i\right)\bar{v}_{i3}-\nu_i \bar{v}_{e3}-\frac{e}{m_i}\bar{E}_3=-v_{i2x}\frac{\partial \bar{v}_{i1}}{\partial x}-i\frac{e}{m_i c}v_{i2x}\bar{H}_1-\frac{\partial}{\partial t}\frac{\bar{v}_{i1}^2 v_{i1}}{2c^2} \tag{4.6.4}$$

The first, second and third terms of the right hand side of (4.6.1) and (4.6.2) represent the substantial derivatives of electron momentum, Lorentz force and relativistic effect of electrons, respectively, whereas all terms on the right hand side of (4.6.3) and (4.6.4) refer to ions.

Also, using the same relation (4.2.11), the third order Maxwell's equations can be written as

$$\frac{1}{c}\frac{\partial H_3}{\partial t}+\frac{\partial E_3}{\partial x}=0 \tag{4.6.5}$$

$$\frac{1}{c}\frac{\partial \bar{H}_3}{\partial t}-\frac{\partial \bar{E}_3}{\partial x}=0 \tag{4.6.6}$$

$$\frac{1}{c}\frac{\partial E_3}{\partial t}-i\frac{\partial H_3}{\partial x}+\frac{4\pi e n_0}{c}(v_{i3}-v_{e3})=-\frac{4\pi e}{c}(n_{i2}v_{i1}-n_{e2}v_{e1}) \tag{4.6.7}$$

$$\frac{1}{c}\frac{\partial \bar{E}_3}{\partial t}+i\frac{\partial H_3}{\partial x}+\frac{4\pi e n_0}{c}(\bar{v}_{i3}-\bar{v}_{e3}) = -\frac{4\pi e}{c}(n_{i2}\bar{v}_{i1}-n_{e2}\bar{v}_{e1}) \tag{4.6.8}$$

where, the right-hand side of (4.6.7) represents the sum of plasma currents of electrons and ions whereas their complex conjugates are represented by the right-hand side of (4.6.8).

From equations (4.6.5) and (4.6.7) we get

$$\frac{\partial^2 E_3}{\partial t^2}-c^2\frac{\partial^2 E_3}{\partial x^2}+4\pi e n_0\frac{\partial}{\partial t}(v_{i3}-v_{e3}) = NR_3 \tag{4.6.9}$$

where, $NR_3 = -\frac{4\pi e}{c}\frac{\partial}{\partial t}(n_{i2}v_{i1}-n_{e2}v_{e1})$

and from equations (4.6.6) and (4.6.8) we have

$$\frac{\partial^2 \bar{E}_3}{\partial t^2}-c^2\frac{\partial^2 \bar{E}_3}{\partial x^2}+4\pi e n_0\frac{\partial}{\partial t}(\bar{v}_{i3}-\bar{v}_{e3}) = \overline{NR_3} \tag{4.6.10}$$

where, $\overline{NR_3} = -\frac{4\pi e}{c}\frac{\partial}{\partial t}(n_{i2}\bar{v}_{i1}-n_{e2}\bar{v}_{e1})$

Let

$$NE_3 = -v_{e2x}\frac{\partial v_{e1}}{\partial x} - i\frac{e}{m_e c}v_{e2x}H_1 - \frac{\partial}{\partial t}\frac{v_{e1}^2\bar{v}_{e1}}{2c^2},$$

$$\overline{NE}_3 = -v_{e2x}\frac{\partial \bar{v}_{e1}}{\partial x} - i\frac{e}{m_e c}v_{e2x}\bar{H}_1 - \frac{\partial}{\partial t}\frac{\bar{v}_{e1}^2 v_{e1}}{2c^2},$$

$$NI_3 = -v_{i2x}\frac{\partial v_{i1}}{\partial x} + i\frac{e}{m_i c}v_{i2x}H_1 - \frac{\partial}{\partial t}\frac{v_{i1}^2\bar{v}_{i1}}{2c^2},$$

$$\overline{NI}_3 = -v_{i2x}\frac{\partial \bar{v}_{i1}}{\partial x} - i\frac{e}{m_i c}v_{i2x}\bar{H}_1 - \frac{\partial}{\partial t}\frac{\bar{v}_{i1}^2 v_{i1}}{2c^2}.$$

Eliminating v_{e3} and v_{i3} from the equations (4.6.1), (4.6.3) and (4.6.9), we get an equation of third order electric field as

$$[(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2})\{(\frac{\partial}{\partial t}-i\Omega_e+\nu_e)(\frac{\partial}{\partial t}+i\Omega_i+\nu_i)-\nu_e\nu_i\}+(\omega'_{pe}+\omega_{pi})\frac{\partial^2}{\partial t^2}]E_3$$

$$=[(\frac{\partial}{\partial t}-i\Omega_e+\nu_e)(\frac{\partial}{\partial t}+i\Omega_i+\nu_i)-\nu_e\nu_i]NR_3+-4\pi e n_0\frac{\partial}{\partial t}\{(\frac{\partial}{\partial t}+i\Omega_i)NE_3-(\frac{\partial}{\partial t}-i\Omega_e)NI_3\}$$

(4.6.11)

Similarly, eliminating $\overline{v_{e3}}$ and $\overline{v_{i3}}$ from the equations (4.6.2), (4.6.4) and (4.6.10), we get the equation of $\overline{E_3}$ i.e. the complex conjugate of order E_3 as

$$[(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2})\{(\frac{\partial}{\partial t}+i\Omega_e+\nu_e)(\frac{\partial}{\partial t}-i\Omega_i+\nu_i)-\nu_e\nu_i\}+(\omega_{pe}+\omega_{pi})\frac{\partial^2}{\partial t^2}]\overline{E_3}$$

$$=[(\frac{\partial}{\partial t}+i\Omega_e+\nu_e)(\frac{\partial}{\partial t}-i\Omega_i+\nu_i)-\nu_e\nu_i]\overline{NR_3}-4\pi e n_0\frac{\partial}{\partial t}\{(\frac{\partial}{\partial t}-i\Omega_i)NE_3-(\frac{\partial}{\partial t}+i\Omega_e)\overline{NI_3}\}$$

(4.6.12)

It is evident from equations (4.6.11) and (4.6.12) that the nonlinear sources contain two parts, one of which is relativistic and the other is non-relativistic. The non-relativistic part contains Lorentz force, plasma current and convective derivative terms, whereas, relativistic part contains only relativistic effects.

Now, we have to find out the nonlinear secular free solutions [Bellman (1994)] of first harmonic fields correct upto third order [Chakraborty etal., (1984)]. The nonlinear secular free solutions of first harmonic fields correct upto third order means that only first harmonic terms are to be retained and terms containing higher harmonics are to be neglected because the nuclear responses at these higher harmonics are extremely small [Chaio and Godine (1969)].

Using the nonlinear phases $\theta_l = k_{l+}x-\omega t$ and $\overline{\theta}_r = \overline{k}_{r-}x-\omega t$ in (4.6.11) and then retaining only first harmonic fields correct upto third order, we have the following two relations

$$(n_{l+}^2-1)\{(1+Y_e+iZ_e)(1-Y_i+iZ_i)+Z_eZ_i\}+(X_e+X_i)$$

$$= - X[\frac{1}{8}(C_{22}\bar{C}_3 + C_{11}\bar{C}_1)(n_r + n_l)\{(1+Y_e+iZ_e)(1-Y_i+iZ_i) + Z_eZ_i\}]\alpha\bar{\alpha}e^{i(\theta_r-\bar{\theta}_r)}$$

$$-\frac{X}{4}\{(C_{11}\bar{n}_r(\bar{C}_1 + M_e)(1-Y_i)+C_{22}\bar{n}_r(C_3 + M_i)\}(1+Y_e)]\bar{\alpha}\alpha e^{i(\theta_r-\bar{\theta}_r)}$$

$$+\frac{X}{2}[2\{\bar{C}_1C_1C_2(1-Y_i)+\bar{C}_3C_3C_4(1+Y_e)\}\alpha\bar{\alpha}e^{i(\theta_r-\bar{\theta}_r)}$$

$$+\{\bar{C}_2^2\bar{C}_2(1-Y_i)+C_4^2C_4(1+Y_e)\}\beta\bar{\beta}e^{i(\theta_l-\bar{\theta}_l)}] \tag{4.6.13}$$

and $(\bar{n}_{r-}^2-1)\{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i\} + (X_e + X_i)$

$$= X[-\frac{1}{8}(\bar{C}_{22}C_4 + \bar{C}_{11}C_2)(n_r + n_l)\{(1-Y_e-iZ_e)(1+Y_e-iZ_e) + Z_eZ_i\}]\bar{\beta}\beta e^{i(\theta_l-\bar{\theta}_l)}$$

$$+\frac{1}{4}\{(\bar{C}_{11}n_l(C_2 - M_e)(1+Y_i)+\bar{C}_{22}n_l(C_4 - M_i)(1-Y_e)\}\bar{\beta}\beta e^{i(\theta_l-\bar{\theta}_l)}$$

$$-\frac{1}{2}[2\{\bar{C}_1\bar{C}_2C_2(1+Y_i)+\bar{C}_3\bar{C}_4C_4(1-Y_e)\}\beta\bar{\beta}e^{i(\theta_l-\bar{\theta}_l)}$$

$$+\{\bar{C}_1^2\bar{C}_1(1+Y_i)+C_3^2C_3(1-Y_e)\}\alpha\bar{\alpha}e^{i(\theta_r-\bar{\theta}_r)}] \tag{4.6.14}$$

where, $n_{l+} = \frac{k_{l+}c}{\omega}$ and $\bar{n}_{r-} = \frac{\bar{k}_{r-}c}{\omega}$, and, more precisely, we have $n_{l+} = n_l + nd_l$ and $\bar{n}_{r-} = \bar{n}_r + n\bar{d}_r$, where nd_l and $n\bar{d}_r$ are the nonlinear corrections of n_l and $\bar{n}_r$ respectively.

In fact, the first two terms of the right-hand sides of (4.6.13) and (4.6.14) come from the non-relativistic corrections, whereas, the relativistic corrections represent the last two terms of (4.6.13) and (4.6.14).

Similarly, Using the nonlinear phases $\theta_r = k_{r+}x - \omega t$ and $\bar{\theta}_l = \bar{k}_{l+}x - \omega t$ in (4.6.12) and retaining also the first harmonic terms correct upto third order from, we have two more relations as

$(n_{r+}^2-1)\{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i\} + (X_e + X_i)$

$$= X[-\frac{1}{8}(\bar{C}_{22}C_4+\bar{C}_{11}C_2)(n_r+n_l)\{(1-Y_e-iZ_e)(1+Y_e-iZ_e)+Z_eZ_i\}]\bar{\beta}\beta e^{i(\theta_l-\bar{\theta}_l)}$$

$$+\frac{1}{4}\{(\bar{C}_{11}n_l(C_2-M_e)(1+Y_i)+\bar{C}_{22}n_l(C_4-M_i)(1-Y_e)\}\bar{\beta}\beta e^{i(\theta_l-\bar{\theta}_l)}$$

$$-\frac{1}{2}[\{2\bar{C}_1\bar{C}_2C_2(1+Y_i)+\bar{C}_3\bar{C}_4C_4(1-Y_e)\}\beta\bar{\beta}e^{i(\theta_l-\bar{\theta}_l)}$$

$$+\{\bar{C}_1^2\bar{C}_1(1+Y_i)+C_3^2C_3(1-Y_e)\}\alpha\bar{\alpha}e^{i(\theta_r-\bar{\theta}_r)}] \tag{4.6.15}$$

and $(\bar{n}_{l-}^2-1)\{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i\}+(X_e+X_i)$

$$= X[-\frac{1}{8}(C_{22}\bar{C}_4+C_{11}\bar{C}_2)(n_r+n_l)\{(1-Y_e-iZ_e)(1+Y_i-iZ_i)+Z_eZ_i\}]\beta\bar{\beta}e^{i(\theta_l-\bar{\theta}_l)}$$

$$+\frac{1}{4}\{(\bar{C}_{11}n_l(C_2-M_e)(1+Y_i)+\bar{C}_{22}n_l(C_4-M_i)(1-Y_e)\}\beta\bar{\beta}e^{i(\theta_l-\bar{\theta}_l)}$$

$$-\frac{1}{2}[\{2\bar{C}_1\bar{C}_2C_2(1+Y_i)+\bar{C}_3\bar{C}_4C_4(1-Y_e)\}\beta\bar{\beta}e^{-(i\theta_l-\bar{\theta}_l)}$$

$$+\{C_1^2C_1(1+Y_i)+\bar{C}_3^2C_3(1-Y_e)\}\alpha\bar{\alpha}e^{i(\theta_r-\bar{\theta}_r)}] \tag{4.6.16}$$

where, $n_{r+}=\frac{k_{r+}c}{\omega}$ and $\bar{n}_{l-}=\frac{\bar{k}_{l-}c}{\omega}$, we also have $n_{r+}=n_r+nd_r$ and $\bar{n}_{l-}=\bar{n}_l+\bar{nd}_l$ and the terms nd_r and $\bar{nd}_l$ are the nonlinear corrections of n_r and $\bar{n}_l$ respectively.

The relations (4.6.13) and (4.615) are known as nonlinear dispersion relation of left and right circularly polarized electromagnetic waves in a magnetized dissipative plasma and their complex conjugates are (4.6.16) and (4.6.14) respectively. They are amplitude dependent dispersion relations. When an ambient magnetic field is present in a plasma, the plane polarized wave can be assumed to be consisting of left and right circularly polarized waves. On the other hand, the wave comprises of elliptically polarized waves in an unmagnetized plasma.

Neglecting dissipative terms (i.e., $Z_e = 0$ and $Z_i = 0$) and retaining only the nonlinear correction terms nd_l and nd_r in the equations (4.6.13) and (4.615), we have, after a little algebraic calculations, the intensity dependent nonlinear dispersion relations as

$$nd_l = n_{l(nl)} + n_{l(rl)} \tag{4.6.17}$$

and

$$nd_r = n_{r(nl)} + n_{r(rl)} \tag{4.6.18}$$

where $n_{r(rl)}$ and $n_{l(rl)}$ are nonlinear terms due to relativistic effects for right and left circularly polarized waves respectively. The nonlinear non-relativistic terms for the same are $n_{r(nl)}$ and $n_{l(nl)}$ respectively. Their expressions are the following.

$$n_{r(nl)} = -\frac{X}{16n_r\xi_+}[(C_{22}\bar{C}_4 + C_{11}\bar{C}2)(n_r+n_l)\xi_+ - 2C_{11}\bar{n}_l(\bar{c}_2 - M_e)(1+Y_i) - 2C_{22}\bar{n}_l(\bar{c}_4 - M_i)(1-Y_e)]\beta\bar{\beta}e^{i(\theta_l - \bar{\theta}_l)} \tag{4.6.19}$$

$$n_{r(rl)} = -\frac{X}{8n_r\xi_+}[\{C_1\bar{C}_2C_2(1+Y_i) + C_3\bar{C}_4C_4(1-Y_e)\}\beta\bar{\beta}e^{i(\theta_l - \bar{\theta}_l)} + 0.5\{C_1^2\bar{C}_1(1+Y_i) + C_3^2\bar{C}_3(1-Y_e)\}\alpha\bar{\alpha}e^{i(\theta_l - \bar{\theta}_l)}] \tag{4.6.20}$$

$$n_{l(nl)} = \frac{X}{16n_l\xi_-}[(C_{22}\bar{C}_3 + C_{11}\bar{C}1)(n_r+n_l)\xi_+ - 2C_{11}\bar{n}_r(\bar{c}_1 + M_e)(1-Y_i) - 2C_{22}\bar{n}_r(\bar{c}_3 + M_i)(1+Y_e)]\alpha\bar{\alpha}e^{i(\theta_r - \bar{\theta}_r)} \tag{4.6.21}$$

$$n_{l(rl)} = \frac{X}{8n_r\xi_-}[\{C_1\bar{C}_1C_2(1-Y_i) + C_3\bar{C}_3C_4(1+Y_e)\}\alpha\bar{\alpha}e^{i(\theta_r - \bar{\theta}_r)} + 0.5\{C_2^2\bar{C}_2(1-Y_i) + C_4^2\bar{C}_4(1+Y_e)\}\beta\bar{\beta}e^{i(\theta_l - \bar{\theta}_l)}] \tag{4.6.22}$$

where $\xi_\pm = (1 \mp Y_e)(1+Y_i)$, $X = \omega_p^2/\omega^2$, $\omega_p^2 = 4\pi e^2 n_0/m$.

Similarly, two other nonlinear dispersion relations for $\bar{nd}_l$ and $\bar{nd}_r$ can also be derived when the bar quantities are used. Since the plasma is undamped, i.e., collision frequencies are ignored, we can drop the bar from all quantities and may treat them as real.

It is evident that the expressions nd_l and nd_r are coupled by the nonlinear sources i.e., convective derivative, Lorentz force, plasma current and relativistic momentum of charged particles. Moreover, They are intensity dependent. Therefore, there exists a mutual exchange of energy between two circularly polarized waves in presence of a magnetic field, and an elliptically polarized wave in absence of a magnetic field, with plasma nonlinearities.

The Faraday rotation(FR) angle Φ can be defined as

$$\Phi = \frac{\omega}{2\pi c}(n_l - n_r)L \tag{4.6.23}$$

where n_l and n_r are refractive indices of the polarized wave or waves, L is the characteristic gradient scale length of plasma and other quantities have their usual meanings.

It is evident that if the refractive indices n_l and n_r are linear and the relation (4.6.23) is used to measure the FR angle Φ, then the corresponding angle would be the linear FR angle. Subsequently, the amount of magnetic field can be estimated easily. In linear case, the magnitude of such magnetic field would be exactly equal to what was supplied from outside during the experiment. However, if there is no supplied magnetic field at the beginning, then it would be observed that in linear approximation n_l and n_r, given in equations (4.4.10) and (4.4.9) respectively, are equal which turns out to be zero FR angle. On the other hand, if the refractive indices are nonlinear, then the net FR angle is the sum of linear (Φ_{linear}) and nonlinear ($\Phi_{nonlinear}$) FR angles. Its concomitant magnetic field should be the combination of the ambient magnetic field (which is equal to the magnitude of the magnetic field taken during experiment) plus the induced magnetic field (which is spontaneous and a consequence of the inverse Faraday effect, IFE) [Stiger and Woods (1972), Talin etal., (1975)]).

To understand the linear and nonlinear induced birefringence, to calculate the linear (Φ_{linear}) and nonlinear ($\Phi_{nonlinear}$) FR angles, and also to obtain the induced magnetic field for the interaction of high frequency laser fields with a magnetized two-component nondissipative plasma, we have done a systematic study starting from the linear dispersion relations of (4.4.9) and (4.4.10) to the nonlinear dispersion relations of (4.6.17) and (4.6.18) respectively. Subsequently, it has been shown that even in the absence of magnetic field the FR angle exists in some high frequency nonlinear plasma phenomena. High frequency means that higher powers of $X_{e,i}$ and $Y_{e,i}$ can be neglected.

Case-I. In the linear limit for high frequency laser (i.e., $X_{e,i} << 1$ and $Y_{e,i} << 1$) in magnetized ($H_0 \neq 0$) and unmagnetized ($H_0 = 0$) plasmas; simplifying relations (4.4.9) and (4.4.10), we have the linear dispersion relations of left and right circularly polarized waves in the magnetized plasma as

$$n_l = 1 - \{(X_e + X_i)/2\} + \{(X_e + X_i)/2\}(Y_e - Y_i) \tag{4.6.24}$$

and

$$n_r = 1 - \{(X_e + X_i)/2\} - \{(X_e + X_i)/2\}(Y_e - Y_i) \tag{4.6.25}$$

and for the unmagnetized plasma (4.6.24) and (4.6.25) reduce to the form

$$n_l = 1 - \{(X_e + X_i)/2\} \tag{4.6.26}$$

$$n_r = 1 - \{(X_e + X_i)/2\} \tag{4.6.27}$$

Using the equations (4.6.24) and (4.6.25) in relation (4.6.23), the linear FR angle (Φ_{linear}) is

$$\Phi_{linear} = (\omega X_e Y_e / 2\pi c) L \tag{4.6.28}$$

from which follows [Krall and Trivelpiece (1973)]$

$$\Phi_{linear} = V_{ct} H_0 L \tag{4.6.29}$$

where $V_{ct} = 2\pi e^3 n_0 / m^2{}_e c^2 \omega^2$ is known as the Verdet constant of the medium [Krall and Trivelpiece (1973)].

It is evident that Φ_{linear} mainly depends on the behaviour of electrons of the plasma due to the fact that it varies with the electron plasma frequency ω_{pe} and the electron cyclotron frequency Ω_e. Moreover, it is independent of the intensity of the waves. On the other hand, if we take equations (4.6.26) for n_l and (4.6.27) for n_r to study FR it is obvious that Φ_{linear} =0 holds because the refractive indices of n_l and n_r in (4.6.26) and (4.6.27), respectively, are exactly equal, i.e., the dispersion rates of the given polarized waves are the same. It follows that the linear FR (Φ_{linear}) angle in the absence of a magnetic field will not exist, in agreement with the FR phenomenon.

We are interested in studying the unmagnetized plasma behaviour and so, our next analysis in this chapter will be confined to that aspect only.

Case-II. In a non-relativistic limit, for high frequency ($X_{e,i}$<<1 and $Y_{e,i}$ <<1) in an unmagnetized ($H_0 = 0$) plasma; simplifying equations (4.6.19) and (4.6.21) we have the non-relativistic dispersion relations of the polarized waves as

$$n_{l(nl)} = (XM_eM_i/16)\,(M_eX_i + M_iX_e)\alpha^2 \tag{4.6.30}$$

$$n_{r(nl)} = (XM_eM_i/16)\,(M_eX_i + M_iX_e)\beta^2 \tag{4.6.31}$$

Using equations (4.6.30), (4.6.31) and (4.6.23), we find the nonlinear non-relativistic Faraday rotation angle is found out to be

$$\phi_{non\text{-}rel} = (\omega/2\pi c)\,(XM_eM_i/16)\,(M_eX_i + M_iX_e)(\alpha^2 - \beta^2) \tag{4.6.32}$$

From the above expression, it is clear that even in the absence of a dc magnetic field, a finite FR angle exists for an elliptically polarized wave but it will disappear when circularly polarized waves are considered, i.e., $\alpha = \beta$. It may also be noted that it is intensity dependent and both, electrons and ions, are dominating with the equal order of magnitude because the ratio of M_eX_i and M_iX_e is unity.

Case-III. In the relativistic limit, for high frequency laser fields (X_{ei} « 1 and Y_{ei}) in an unmagnetized plasma (H_0=0); the dispersion relations (4.6.20) and (4.6.22) may be written as

$$n_{l(rl)} = \left(\frac{X}{8}\right)\left[(Me^3 + M_i^3)(2\alpha^2 + \beta^2)\{1 + 0.5(X_e + X_i)\}\alpha^2 - 4(M_e^3Y_e - M_i^3Y_i)\beta^2\right] \tag{4.6.33}$$

$$n_{r(rl)} = -\left(\frac{X}{8}\right)\left[(Me^3 + M_i^3)(\alpha^2 + 2\beta^2)\{1 + 0.5(X_e + X_i)\}\beta^2 - 4(M_e^3Y_e - M_i^3Y_i)\alpha^2\right] \tag{4.6.34}$$

From the above two relations (4.6.33) and (4.6.34) we may write the nonlinear relativistic FR angle Φ_{rel-FR} as

$$\Phi_{rel-FR} = \left(\frac{\omega L X}{16\pi c}\right)\left[3(Me^3 + M_i^3)(\alpha^2 + \beta^2)\{1 + 0.5(X_e + X_i)\} - 4(M_e^3Y_e - M_i^3Y_i)(\alpha^2 - \beta^2)\right] \tag{4.6.35}$$

It is evident from relation (4.6.35) that in relativistic limit the FR angle exists in the absence of magnetic fields. It is dominated by electron motion over an order of magnitude $(M_e / M_i)^3$ that is equivalent to $(m_i / m_e)^3$. Moreover, it persists even for circularly polarized waves (i.e., for $\alpha = \beta$) and can be written in simplified form from the relation (4.6.35) as

$$\Phi_{rel-FR} = (\frac{3X\omega}{8\pi c})(Me^3 + M_i^3)\{1 + 0.5(X_e + X_i)\}\alpha^2 L \qquad (4.6.36)$$

For numerical results, in context of our weak relativistic model [Max (1982)], we may assume that laser has an energy level of 100 J with 100ps (full width and half maximum) pulse focussed on a target of spot radius 40 μm. It produces the laser irradiance $I \simeq 2\times10^{16}$ W/cm^2, which gives $\alpha^2 \simeq 0.097$. We also assume that the laser has a 1 μm wavelength which yields the frequency $\omega \simeq 1.886\times10^{15}$/s. Further, we take the plasma density in such a way that $X_e \simeq 0.01$, and also choose the characteristic length L which equals to twice the spot radius. Then the relation (4.6.36) gives approximately the relativistic FR angle (Φ_{rel-FR}) as 0.27 radians which turns out to be the angle of rotation $\simeq$15.5 degrees. It is expected that such angle of rotation can be measured in the laboratory in future.

In conclusion, nonlinearly induced birefringence corresponds to a nonlinear FR angle, which enforces one to estimate the order of induced magnetization (i.e., IFE effect) for the propagation of the polarized waves in an unmagnetized plasma. An ambient magnetic field may help to enhance such magnetization. Parametric and resonance effects may also play an important role for enhancing Faraday rotation angles and hence magnetic fields. All these will be studied elsewhere.

$ Relation (4.10.6) and (4.10.9) of Krall and Trivelpiece (1973) correspond to equations (4.3.11) and (4.6.29) respectively.