2 Equations under study: Origin & Motivation

2.1 The nonlinear equations under study :

2.1.1 Generalized Yang equations :

$$\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44} =$$

$$k'[(1/\phi)(\phi_1^2 + \phi_2^2 + \phi_3^2 + \varepsilon\,\phi_4^2) - (1/\phi)(\psi_1^2 + \psi_2^2 + \psi_3^2 + \varepsilon\psi_4^2)$$

$$- (1/\phi)(\chi_1^2 + \chi_2^2 + \chi_3^2 + \varepsilon\,\chi_4^2) - (2/\phi)(\psi_1\chi_2 - \psi_2\chi_1 - \psi_4\chi_3 - \psi_3\chi_4)] \qquad (2.1a)$$

$$\psi_{11} + \psi_{22} + \psi_{33} + \varepsilon\,\psi_{44} =$$

$$k'[(2/\phi)(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \varepsilon\phi_4\psi_4) + (2/\phi)(\phi\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4)] \qquad (2.1b)$$

$$\chi_{11} + \chi_{22} + \chi_{33} + \varepsilon\chi_{44} =$$

$$k'[(2/\phi)(\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4) + (2/\phi)(\phi_2\psi_1-\phi_1\psi_2+\phi_3\psi_4-\phi_4\psi_3)] \qquad (2.1c)$$

where $\varepsilon = \pm 1$, k' are arbitrary constants.

The equations with $k' = 1$ and $\varepsilon = 1$ represent the celebrated Yang's equations[12]. 'Yang's equation' was written by Yang in relation to the condition of self duality for a SU(2) R-gauge field on Euclidean four dimensional flat space.

The equations were originally written as

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \phi_y\,\phi_{\bar{y}} - \phi_z\,\phi_{\bar{z}} + \rho_y\,\bar{\rho}_{\bar{y}} + \rho_z\,\bar{\rho}_{\bar{z}} = 0 \qquad (2.2a)$$

$$\phi(\rho_{y\bar{y}} + \rho_{z\bar{z}}) - 2\,\rho_y\,\phi_{\bar{y}} - 2\,\rho_z\,\phi_{\bar{z}} = 0 \qquad (2.2b)$$

where an over bar denotes the complex conjugate, ϕ and ρ are functions of $y, \bar{y}, z, \bar{z}$, ϕ is real, ρ is complex and $\sqrt{2}\,y = x^1 + i\,x^2$, $\sqrt{2}z = x^3 - i\,x^4$, x^1, x^2, x^3, x^4 are real.

Once one has found ρ and ϕ, the corresponding R-gauge potentials are given by

$$\phi\vec{b}_y = (i\rho_y, \rho_y, -i\phi_y),\ \phi\vec{b}_{\bar{y}} = (-i\bar{\rho}_{\bar{y}}, \bar{\rho}_y, i\phi_y) \tag{2.3a}$$

$$\phi\vec{b}_z = (i\rho_z, \rho_z, -i\phi_z),\ \phi\vec{b}_{\bar{z}} = (-i\bar{\rho}_{\bar{z}}, \bar{\rho}_{\bar{z}}, i\phi_z) \tag{2.3b}$$

and R-gauge field strengths $F_{\mu\upsilon}$ are given by

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu \tag{2.4a}$$

$$B_\mu = b^i_\mu X_i \tag{2.4b}$$

and $X_i = -\left({}^1/_2\right) i\sigma_i$ (2.4c)

where σ_i are 2×2 Pauli matrices.

The condition of self duality will be represented by all such such solution except when ϕ is zero. Because when ϕ is zero $F_{\mu\nu}$ becomes singular and the solutions can only be regarded as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like $F'_{\mu\nu} \rightarrow U^{-1} F_{\mu\nu} U$ eliminates the singularities.

When written in real form the equations reduce to (2.1) with $k' = 1$ and $\varepsilon = 1$.

2.1.2 Generalized Charap equations :

$$\phi_{11}+\phi_{22}+\phi_{33}+\varepsilon\phi_{44}=k''\{2\phi[\exp(-\beta)]\left(\phi_1^2+\phi_2^2+\phi_3^2+\varepsilon\phi_4^2\right)$$

$$+2\psi[\exp(-\beta)](\phi_1\psi_1+\phi_2\psi_2+\phi_3\psi_3+\varepsilon\phi_4\psi_4)$$

$$+2\chi[\exp(-\beta)](\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4)\} \qquad (2.5a)$$

$$\psi_{11}+\psi_{22}+\psi_{33}+\varepsilon\psi_{44}=k''\{2\psi[\exp(-\beta)]\left(\psi_1^2+\psi_2^2+\psi_3^2+\varepsilon\psi_4^2\right)$$

$$+2\phi[\exp(-\beta)](\phi_1\psi_1+\phi_2\psi_2+\phi_3\psi_3+\varepsilon\phi_4\psi_4)$$

$$+2\chi[\exp(-\beta)](\chi_1\psi_1+\chi_2\psi_2+\chi_3\psi_3+\varepsilon\chi_4\psi_4)\} \qquad (2.5b)$$

$$\chi_{11}+\chi_{22}+\chi_{33}+\varepsilon\chi_{44}=k''\{2\chi[\exp(-\beta)]\left(\chi_1^2+\chi_2^2+\chi_3^2+\varepsilon\chi_4^2\right)$$

$$+2\phi[\exp(-\beta)](\chi_1\phi_1+\chi_2\phi_2+\chi_3\phi_3+\varepsilon\chi_4\phi_4)$$

$$+2\psi[\exp(-\beta)](\chi_1\psi_1+\chi_2\psi_2+\chi_3\psi_3+\varepsilon\chi_4\psi_4)\} \qquad (2.5c)$$

where $\beta=\ln\left(f_\pi^2+\phi^2+\psi^2+\chi^2\right)$, k'' is an arbitrary constant, $\varepsilon=\pm1$

The equations with $k''=1$ and $\varepsilon=-1$ represent the celebrated Charap's equation. The 'Charap's Equation' was witten by Charap to describe a Chiral field[13]. The Charap's equation was originally written as

$$\Box'\phi = k''\eta^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{2.6a}$$

$$\Box'\psi = k''\eta^{\mu\nu}\frac{\partial\psi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{2.6b}$$

$$\Box'\chi = k''\eta^{\mu\nu}\frac{\partial\chi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{2.6c}$$

where

$$\Box'\phi = \phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44}$$

$$\phi_1 = \frac{\partial\phi}{\partial x^1}, \phi_{11} = \frac{\partial^2\phi}{\partial x^{1^2}}$$

where

$$\eta^{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

$$= 1 \text{ for } \mu = \nu \neq 4$$

$$= \varepsilon \text{ for } \mu = \nu = 4$$

$$\varepsilon = +1 \text{ or } -1$$

$$\beta = \ln\left(f_{\pi}^2 + \phi^2 + \psi^2 + \chi^2\right)$$

$$f_{\pi} = \text{constant}$$

The Lagrangian is given by

$$L = \frac{1}{2}(g_{11}\partial_{\mu}\phi\partial^{\mu}\phi + g_{22}\partial_{\mu}\psi\partial^{\mu}\psi + g_{33}\partial_{\mu}\chi\partial^{\mu}\chi + 2g_{12}\partial_{\mu}\phi\partial^{\mu}\phi$$

$$+2g_{13}\partial_\mu\phi\partial^\mu\chi + 2g_{23}\partial_\mu\psi\partial^\mu\chi) \tag{2.7a}$$

where g_{ij} are such that Γ^1_{ij}, the Christopher symbols, take the form

$$\Gamma^1_{ij} = -\left(f_\pi^2 + \phi^2 + \psi^2 + \chi^2\right)^{-1}\left(\partial^1_i\phi_i + \partial^1_i\phi_j\right) \tag{2.7b}$$

In (5.2a) ϕ_1, ϕ_2 and ϕ_3 represent ϕ, ψ and χ respectively.

When written explicitly the equations take the form (2.5) with $k'' = 1$ and $\varepsilon = -1$

2.2 Origin of the equations under study :

2.2.1 Yang Euclidean SU(2) R-gauge field equations :

At the classical level, the mathematical expaination of Spin-1 particle originated with Maxwell's famous equations for electromagnetism, and their generalization in 1954 due to Yang and Mills. The symmetry group related with electromagnetism is Abelian. In an Abelian group one can apply transformation in any order with no change in result. On, the other respect, the symmetry group related with the corresponding generalization due to Yang and Mills is non-Abelian. In a non-Abelian group the same two transformations done in two different orders give different results. Combining, all these theories are sometimes called 'Gauge Theories'. According to Yang and Mill "A change in gauge means a change of phase factor $\psi \to \psi', \psi' = (\exp\ i\alpha)\psi$, a change that is devoid of any physical consequences, Since ψ may depend on x, y, z and t, the relative phase factor of ψ at two different space-time points is therefore completely arbitrary. In other words, the arbitrariness in choosing the phase factor is local in character".

The corresponding quantum theory was constructed for electromagnetism by Feyman, Schwinger and Tomonaga in 1940's and for the Yang-Mills generalization by 't Hoot and Veltman in the 1970's. Feyman et. al. was honoured with the Nobel prize in October 1965,

while ‘t Hoot and Veltman were honoured with the same in 1970’s. Quantum electromagnetism relates the photon and its interaction with charged particles, while quantum Yang-Mills theory relates W and Z bosons and gluons and their interaction. The combination of all these theories sums up a single large theory called the ‘Standard Model’ of particle interactions, which is a quantum gauge theory.

In this perspective we describe the equations due to Yang [12].

Many years ago Weyl advocated that electromagnetic field can be formulated in terms of an Abelian gauge transformation. Then the idea was also adapted to non-abelian transformation. One can call such formulations as differential formulations. C. N. Yang is the pioneer to formulate the gauge field in an integral formalism, which is superior to the differential formalism as it allows for natural developments of extra concepts. There is a scope of a mathematical and physical discussion of the gravitational field as a gauge field, resulting in equations related but very much different from Einsteins.

In recent years there has been a great interest in sourceless gauge fields. The self-dual gauge field is sourceless. One important characteristics of this field of study is the chaotic behavior of field theories. Early investigation leads to understanding the structure of the field theoretic vacuum and asymptotic states of theory with particular reference to strong interaction physics. This study led to a belief that there was a connection between color confinement and chaos in quantum dynamics. Gauge theories support solitons. If the systems are considered to be conservative and described by Hamiltonians, non integrability of the the evolution equation indicates chaos. Under this circumstances Yang [12] used the equation given below. They are Laplace-like equation for three real variables or for variables one real and one complex and are obtained when the condition of selfduality for a SU(2) gauge field on Euclidean four-dimensional flat space is integrated once(equations(2.1) with $k' = 1$ and $\varepsilon = 1$).

2.2.2 Charap Chiral field equations of pion dynamics :

John M Charap had taken a simple model analogous to the Chiral invariant dynamics of Zero mass mesons. A system of massless particles could obey laws such as 'the number of left hand and right hand particles are each 'conserved' or ' the iso-spins carried by the left hand hand particles and by the right hand particles are each conserved'. The category 'left hand particles' includes their charge conjugates, namely , right hand antiparticles. Such conservation laws originates from symmetries of the form of a product of an internal symmetry group for the left-hand particles and another for the right hand particles. This kind of symmetry group is called Chiral. In the model due to Charap instead of considering a multiplet of fields, with the appropriate gradient self-couplings of a Chiral invariant theory, work has been done with a system having only a finite number of degrees of freedom. For a particular choice of chiral transformation, called tangential parametrization(Charap[13]) the field equations for the chiral invariant model of pion dynamics take the form(Charap, [13])((2.5) with $k'' = 1$ and $\varepsilon = -1$)

2.3 Motivation for the investigations of the equations under study:

Apart from the physical significance described above the equations((2.1) with $k' = 1$ and $\varepsilon = 1$) due to Yang and equations((2.5) with $k'' = 1$ and $\varepsilon = -1$) due to Charap have some mathematically fascinating features. There is a considerable similarity between these two sets of equations. Firstly, the two sets of equations are similar in form. Secondly, both of them allow (i) reduction to equations in two independent variables which are conformally invariant equations permitting one to obtain infinitely many other solutions from any solution of these conformally invariant equations and (ii) those reduced equations closely resemble the Lund-Regge equations[[14],[15]].

The generalized Lund-Regge equations are,

$$\theta_{11} + \theta_{22} - 4g(\theta)\left(\lambda_1^2 + \lambda_2^2\right) = 0 \tag{2.8a}$$

$$\left[\lambda_1 \exp\left(-\int p(\theta)\,d\theta\right)\right]_1 + \left[\lambda_2 \exp\left(-\int p(\theta)\,d\theta\right)\right]_2 = 0 \tag{2.8b}$$

where $\theta = \theta\left(x^1, x^2\right), \lambda = \lambda\left(x^1, x^2\right), \theta_1 = \dfrac{\partial\theta}{\partial x^1}$ and so on.

With $g = 0$, the equations (2.8) reduce to a conformally invariant set of equations, an important example of which is the physically interesting equations of two dimensional Heisenberg Ferro magnets (discussed in [[16], [17]]) . The reduced form of the equations (2.8) mentioned above which originate from Yang equations ((2.1) with $k' = 1$ and $\varepsilon = 1$)closely resemble this situation. This is, however, at least a contrast that there are two equations for the Heisenberg Ferro magnets, whereas the equations((2.1) with $k' = 1$ and $\varepsilon = 1$ obtained from those due to Yang [12] have three equations. On the other hand, all of the reduced equations mentioned above [[14],[15]] which originate from the Charap equations (2.5) with $k'' = 1$ and $\varepsilon = -1$ are of the same from as one of the two equations in (2.8). However, the resemblance between the solutions of the non-linear sigma model and self-dual gauge fields is well known and, for example, forms the basis for the Atiyah-Manton [18] approach to the construction of approximate solutions to the Skyrme model [[19],[20]]. This has its applications to the study of nucleon-nucleon interactions in the model. For, a recent example, see Leese, Manton & Schroers [21].

These characteristics have motivated us to study the general set of equations having Yang equations ((2.1) with $k' = 1$ and $\varepsilon = 1$) which were obtained at the time of discussing the condition of self-duality for SU(2) gauge fields on Euclidean space and Charap equations ((2.5) with $k'' = 1$ and $\varepsilon = -1$) for chiral invariant pion dynamics under tangential parametrizations as the particular cases.

Since the solutions to some of the particular forms of the generalized equations under study have been demonstrated to be physical in nature they may be useful in a huge area of physical research for example field theories and particle physics with reference to Chiral models and Skyrme models in relation to soliton solutions andspatio-temporal chaos (see references [[19], [20]])

It is quite fascinating to note that:

For $\varepsilon = +1$, the three equations (i.e. eqns (2.1),(2.5)&(4.78) are Laplace-like . Since for $\varepsilon = +1$,

ϕ, ψ, χ can ultimately be written in terms of ζ which satisfy

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0$$

For $\varepsilon = -1$, the three above stated equations are Wave like. Since for $\varepsilon = -1$,

ϕ, ψ, χ can ultimately be written in terms of ζ which satisfy

$$\zeta_{11} + \zeta_{22} + \zeta_{33} - \zeta_{44} = 0$$

Laplace equation and Wave equations holds very significance in physical research. Some examples are given below:

- **Laplace Equation :**

The Laplace equation is

$$\nabla^2 \psi = 0$$

which is the elliptic equation occuring very often in physical problems. This equation is also called the potential function.

<u>Some Examples of Laplace's equation in Physics</u>

(a) Gravitation :

(i) Both inside and outside the attracting matter the force of attraction $\vec{F}$ can be explained in terms of a gravitational potential ψ by the equation

$$\vec{F} = grad\ \psi$$

(ii) In empty space ψ obeys the Laplace's equation $\nabla^2 \psi = 0$.

(iii) At any point at which the density of gravitating matter is ρ the potential ψ satisfies Poisson's equation $\nabla^2 \psi = -4\pi\rho$.

(iv) When there is matter distributed over a surface, the potential function ψ takes different forms ψ_1, ψ_2 on opposite sides of the surface, and on the surface these two functions satisfy the conditions

$$\psi_1 = \psi_2 , \qquad \frac{\partial \psi_2}{\partial n} - \frac{\partial \psi_1}{\partial n} = -4\pi\sigma$$

where σ is the surface density of the matter and n is normal to the surface pointing from the region 1 into the region 2.

(v) There can be no singularities in ψ except at isolated masses.

(b) Irrotational Motion of a Perfect Fluid:

(i) The velocity $\vec{q}$ of a perfect fluid in irrotational motion can be written in terms of a velocity potential ψ by the equation

$$\vec{q} = -grad\ \psi$$

(ii) At all points of the fluid where there are no sources or sinks the function ψ obeys the Laplace's equation $\nabla^2 \psi = 0$.

(iii) When the fluid is in contact with a rigid surface which is moving so that a typical point p of it has velocity $\vec{U}$, then $\left(\vec{q} - \vec{U}\right).\vec{n} = 0$, where $\vec{n}$ is therefore that

$$\frac{\partial \psi}{\partial n} = -\vec{U}.\vec{n}$$

At all points on the surface.

(iv) If the fluid is at rest at infinity, $\psi \to 0$, but if there is a uniform velocity V in the z direction, this condition is replaced by the condition $\psi \approx -Vz$ as $z \to \infty$.

(v) The function ψ has no singularities with the exception of sources or sinks.

(c) Dielectrics

In the presence of dielectrics the electrostatic potential ψ satisfies the following conditions:

(i) In the presence of charges $div\left(\kappa grad\ \ \psi\right) = -4\pi\rho$, where κ is the dielectric constant.

(ii) If two media are in contact, one has two forms ψ_1, ψ_2 for the potential on opposite sides of the surface, but on the surface on one has

$$\psi_1 = \psi_2 , \qquad \kappa_1 \frac{\partial \psi_1}{\partial n} = \kappa_2 \frac{\partial \psi_2}{\partial n}$$

(iii) At the surface of a conductor $c(v)$ is replaced by the equation

$$\kappa \frac{\partial \psi}{\partial n} = -4\pi\sigma$$

- **Wave Equation :**

The wave equation is:

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

which is a typical hyperbolic equation

If one assume a solution of the wave equation or the form

$$\psi = \Psi\left(x, y, z\right) e^{\pm ikct}$$

Then the function Ψ must satisfy the equation

$$\left(\nabla^2 + k^2\right)\Psi = 0$$

Which is called the space form of the wave equation or Helmholtz's equation.

(a) A string with transverse vibration:

If a string of uniform linear density ρ is stretched to a uniform tension T , and if, in the equilibrium position, the string coincides with the x axis, then when the string is disturbed

slightly from its equilibrium position, the transverse displacement $y(x,t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where $c^2 = \frac{T}{\rho}$. At any point $x = a$ of the string which is fixed $y(a,t) = 0$ for all values of t.

(b) A bar with Longitudinal vibration:

If a uniform bar of elastic material of uniform cross section whose axis rests on Ox is stressed in such a way that each point of a typical cross section of the bar takes the same displacement $\xi(x,t)$, then

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \tag{2.9}$$

where $c^2 = \frac{E}{\rho}$, E being the Young's modulus and ρ the density of the material of the b are determined by the ar.

At any point on the bar the stress is given by

$$\sigma = E\frac{\partial \xi}{\partial x}$$

For, example suppose that the velocity of the end $x = 0$ of the bar $0 \le x \le a$ is prescribed to be $v(t)$, say, and the other end $x = a$ is free from stress. Suppose at time $t = 0$ the bar is at rest. Then the longitudinal displcements of sections of the bar are expressed by the partial differential equations (2.9) with the boundary and initial conditions

(i) $\frac{\partial \xi}{\partial t} = v(t)$ for $x = 0$

(ii) $\frac{\partial \xi}{\partial x} = 0$ for $x = a$

(iii) $\xi = \frac{\partial \xi}{\partial t} = 0$ at $t = 0$ for $0 \le x \le a$