Chapter-I

1 Introduction

1.1 Nonlinearity and partial differential equations:

The orthodox practice is to formulate the laws of physics in terms of differential equations and then to solve the differential equations in different physical situations. Though the procedure of finding the solutions of differential equations is just the inverse of the process of the formation of these equations, often it is found out that it is much more difficult to get the solutions. So, to get the solutions of differential equations has become a pivotal problem of theoretical physics. The problem has become still more difficult with the fact that differential equations formed from physical situations are often nonlinear.

Few decades back the nonlinear partial differential equations (nPDE ’s) were considered a closed chapter. The fact was that such equations are very difficult to study. Linear differential equations have the advantage that the principle of (linear) superposition holds in their cases, i.e. adding two or more solutions, one can get a new solution and the general solution could be expressed as a linear combination of the particular solutions.

Principle of superposition holds for linear differential equations only i.e. principle of linear superposition is not satisfied by non-linear partial differential equations. This is a severe difficulty for non-linear differential equations and to obtain general exact solutions for the non-linear differential equations become more difficult. However, there exist family of non-linear (and even linear) equations which satisfy non-linear superposition principles. But there is no universal law in this case (see [1]). Eminent physicist Richard Feynman[2] remarked that: “The next great era of awakening of human intellect may well produce a method of understanding the qualitative content of equation.” The philosophy behind this would be honoured from the observation of another eminent personality of physical science. Eugene Wigner remarked that the main role of mathematics in physics consists not in its being an instrument (i.e. computations) but in being the language of physics (see [3]). This role of mathematics is being executed for about last few centuries chiefly by differential equations.

1.2 Solitary solutions

The interest for the study of nonlinear partial differential equations(nPDE's) and the mathematical study of solitary waves was revolutionized with the work of Zabusky and Kruskal [4] in the year 1965. Their work originated from a physical problem and is also a novel example of how computational results may lead to the development of new mathematics, just as observational and experimental results have done since the time of Archimedes(see [5]).

Observing the Fermi-Pasta-Ulm[6] model of photons in an anharmonic lattice Zabusky and Kruskal [4] were focussed to the work on Korteweg de Vries(KdV) equation. They considered the following initial-value problem in a periodic domain:

$$u_t + u\ u_{xx} + \delta u_{xxx} = 0 \qquad (1.1a)$$

where $u_t = \dfrac{\partial u}{\partial t}, \quad u_x = \dfrac{\partial u}{\partial x}$ etc.

$$u(2,t) = u(0,t)\ , \quad u_x(2,t) = u_x(0,t)\ , \quad u_{xx}(2,t) = u_{xx}(0,t)\ , \text{ for all } t, \qquad (1.1b)$$

and $u(x,0) = \cos \pi x$ for $0 \le x \le 2$ (1.1c)

The interest for choosing the boundary conditions stated above is that they fit numerical integration of the system. Zabusky and Kruskal observed that the solution breaks up into a train of eight solitary waves, each like sech – squared solution, that these waves move through one another as the faster ones catch up the slower ones,and that finally the initial state (or something very similar) occurs. The word 'Soliton' was termed by Zabusky and Kruskal [4] after 'Photon'. etc., to point that a soliton is a localized entity which may keep its identity after an interaction.

In course of time mathematicians have found a more general term than the so called sech-squared solution of the KdV equation. They have propogated the idea of solitary waves. It is a solution of a non-linear system, which is shown by a hump-shaped wave of permanent form, whether it is a soliton, or not.

A simple analytical approach is as follows [5]

$$u(x,t) = f(X), X = x - ct \tag{1.2}$$

Putting (1.2) in (1.1) and integrating once one gets

$$-cf + \frac{1}{2} f^2 + \delta f_{XX} - A = 0 \tag{1.3}$$

where A is an arbitrary constant of integration.

Integrating (1.3) once again we get

$$-3cf^2 + f^3 + 3\delta f_X^2 - 6Af - 6B = 0 \tag{1.4}$$

where B is arbitrary constant of integration.

Since one seeks a solitary wave, one can add the boundary conditions that

$$f, f_X, f_{XX} \to 0 \text{ as } X \to \pm\infty \tag{1.5}$$

Immediately we get $A, B = 0$ and

$$3\delta f_X^2 = 3cf^2 - f^3 \tag{1.6}$$

Equation (1.6) can be rewritten as

$$X = \sqrt{3\delta} \int \frac{df}{f\sqrt{3c - f}} \tag{1.7}$$

The substitution

$$f = 3c \operatorname{sech}^2 \theta f \tag{1.8}$$

then will give the solution

$$u = f(X) = 3c\,\mathrm{sech}^2\left\{\frac{1}{2}\sqrt{\frac{c}{\delta}}(X - X_0)\right\} \qquad (1.9)$$

For any constants $c \geq 0$ and X_0.Obviously, δ must also be greater than zero.

If we plot (1.9), we can get the solitary wave.

However, if we take the equation (1) as

$$u_t - u\ u_{xx} + \delta u_{xxx} = 0$$

we will still get a solitary wave but an inverted one.

1.2.1 Balance between self-focussing and dispersion :

The KdV equation can be regarded as the superimposition of two waves given by:

$$u_t + u\ u_x = 0 \qquad (1.10a)$$

$$u_t + \delta u_{xxx} = 0 \qquad (1.10b)$$

Solution of (1.10b) is dispersive one.

The solution of (1.10a) is a shock wave. The wave will become self-focussed, i.e. the profile of a particular wave become more and more sharpened.

In the KdV (1.1) these two types of solutions which are just reverse in behavior seem to balance each other and will generate a single wave with permanent shape and size, i.e. the so called solitary wave.the KdV (1.1) these two types of solutions which are opposite in behavior.

Although, such kind of simple example is not available for complicated cases like ours.

1.3 Painleve' analysis:

The Painleve' analysis for integrability originates from the contribution of the great mathematician Sofya Kovalevskaya[[7],[8]] in relation to the equation of two groups defining the motion of a heavy rigid body about a fixed point which are given by

$$A\frac{dp}{dt} + (C - B)qr = Mg\left(y_0\gamma'' - z_0\gamma'\right) \qquad (1.11a)$$

$$B\frac{dp}{dt}+(A-C)rp=Mg\left(z_0\gamma-z_0\gamma^{''}\right) \qquad (1.11b)$$

$$C\frac{dp}{dt}+(B-A)pq=Mg\left(x_0\gamma^{'}-z_0\gamma\right) \qquad (1.11c)$$

and

$$\frac{d\gamma}{dt}=r\gamma^{'}-q\gamma^{''} \qquad (1.12a)$$

$$\frac{d\gamma^{'}}{dt}=r\gamma^{''}-q\gamma \qquad (1.12b)$$

$$\frac{d\gamma^{''}}{dt}=r\gamma-q\gamma^{'} \qquad (1.12c)$$

Two particular cases were known before her which were possible i.e.

(i)Euler case, when $x_0=y_0=z_0=0$,

(ii)Lagrange case, for which $A=B,\ x_0=y_0=0$.

Kovaleskaya applied new technique to the problem. She regarded time t to be a complex variable. She tried to find a solution by assuming that the functions $p,q,r,\gamma,\gamma^{'},\gamma^{''}$ have poles in the complex plane of the variable t.

She tried to find a solution by assuming that the functions have poles in the complex plane of the variable . If one of the poles is $t=t_1$, then one can try to find a solution in the form of the series

$$p=\tau^{-n_1}\left(p_0+p_1\tau+p_2\tau^2+...........\right), \qquad (1.13a)$$

$$q=\tau^{-n_1}\left(q_0+q_1\tau+q_2\tau^2+...........\right), \qquad (1.13b)$$

$$r=\tau^{-n_1}\left(r_0+r_1\tau+r_2\tau^2+...........\right), \qquad (1.13c)$$

$$\gamma=\tau^{-n_1}\left(f_0+f_1\tau+f_2\tau^2+...........\right), \qquad (1.13d)$$

$$\gamma^{'}=\tau^{-m_2}\left(g_0+g_1\tau+g_2\tau^2+...........\right), \qquad (1.13e)$$

$$\gamma^{''}=\tau^{-m_3}\left(h_0+h_1\tau+h_2\tau^2+...........\right), \qquad (1.13f)$$

Substituting these series into the equations (1.11) and (1.12), Kovalevskaya found the order of the possible poles:

$$m_1 = m_2 = m_3 = 2, \qquad\qquad n_1 = n_2 = n_3 = 1$$

and the existence conditions for solutions in the form (1.13). It turned out that they holds both for the known two cases ((i) Euler and (ii) Lagrange) mentioned above and for another case. This new example is a discovery of Kovalevskaya and is given by

$$A = B = 2C, \qquad\qquad z_0 = 0$$

After that the method was examined by numerous mathematicians. But all of them were related to nonlinear Ordinary differential equations (ODE's) only.

The first application of this technique was applied to nonlinear PDE's by Ablowitz, Ramani and Segur[9]. They conjectured that every nonlinear ODE obtained by an exact reduction (i.e. through analogous transformation) of an integrable nonlinear PDE has Painleve' property.

They also advocated an algorithm for inspecting whether a particular nonlinear ODE has the Painleve property or not.

The so called ARS-conjecture [i.e. conjecture advocated by Ablowitz, Ramani and Segur[[9],[10]] is not complete in the regard that it is not possible to know exactly how many similarity reductions a particular nonlinear PDE permits.

The technique of Weiss, Tabor and Carnavale [11] generalized the approach of Ablowitz, Ramani and Segur [9],[10] and applied the technique of the ARS conjecture to a nonlinear PDE.

The approach of Weiss, Tabor and Carnavale [11] is considered a highly successful conjecture. An important fact is that for all important nPDE's with single dependent variable the Painleve' analysis according to Weiss et. al. itself could provide Lax-pair , the basic condition for integrability of nPDE's. The approach of Weiss et. al. can provide important exact solution although it establishes an association of the existence of Painleve' property and integrability.

The results of Painleve analysis establishes a strong correlation with the existence or absence of chaos [11]. For, ODE’s by chaos we associate a sensitive dependence on initial conditions.