Appendix A

Cosmic-ray air shower theory under approximations A and B

A.1 Cosmic-ray air shower theory under approximations A and B

The one dimensional (1D) and the three dimensional (3D) theories are distinguished based on whether the theory addresses just longitudinal or both lateral and longitudinal shower development. In the analytical approach, the time distribution can be derived from the solution of the 3D model providing the densities of the particles and their energy distributions. The four dimensional (4D) simulation is more generally reserved to the Monte Carlo approach where electrons and photons are followed simultaneously in space-time coordinates [1].

A.1.1 Approximations in the theoretical model of the cascade diffusion equations

The approximation A neglects the ionization losses taking into account the radiation process and the *pair creation* process. It allows to establish from the elementary gains and losses in particles and energy. In the case of the 1D theory, the most simple system of transport equations which are perfectly symmetric [2], is given by

$$\frac{\delta \boldsymbol{\pi}}{\delta X} = -\mathrm{A}'\boldsymbol{\pi} + \mathrm{B}'\boldsymbol{\gamma} \tag{A.1}$$

$$\frac{\delta \boldsymbol{\gamma}}{\delta X} = -\sigma_0\boldsymbol{\gamma} + \mathrm{C}'\boldsymbol{\pi} \tag{A.2}$$

We use here the notations of Nishimura in [2], $\boldsymbol{\pi}(E,X)dE$ and $\boldsymbol{\gamma}(E,X)dE$ representing respectively the average numbers of electrons and photons with energy between E and $E+dE$ at a depth X in radiation lengths. The integral operators A' and B' correspond respectively to the losses in electron number by *bremsstrahlung* and gains by pair productions, whereas σ_0 represents the cross section for pair production (but, $-\sigma_0\gamma$ accounts the loss of photons through pair production). The integral operator C' corresponds the gain in number of photons from electron *bremsstrahlung*.

The approximation A is better adapted to the growing phase of the cascade and to high energy photons and electrons. The approximation B is more realistic one. In addition to the approximations stated under approximation A, it also incorporates the effect of the ionization loss $\epsilon\frac{\delta\pi}{\delta E}$, thus replacing the first term of the system (A.1) by

$$\frac{\delta\boldsymbol{\pi}}{\delta X} = -\mathrm{A}'\boldsymbol{\pi} + \mathrm{B}'\boldsymbol{\gamma} + \epsilon\frac{\delta\boldsymbol{\pi}}{\delta E} \tag{A.3}$$

In parallel, the 3D diffusion equations can be inferred after introducing the functions $\boldsymbol{\pi}(E,\boldsymbol{r},\boldsymbol{\theta})$ and $\boldsymbol{\gamma}(E,r,\boldsymbol{\theta})$. For instance, in the case of the approximation B they take the simple form in the so called Landau approximation:

$$\frac{\delta\boldsymbol{\pi}}{\delta E} + \boldsymbol{\theta}\frac{\delta\boldsymbol{\pi}}{\delta\boldsymbol{r}} = -\mathrm{A}'\boldsymbol{\pi} + \mathrm{B}'\boldsymbol{\gamma} + \epsilon\frac{\delta\boldsymbol{\pi}}{\delta E} + \frac{E_s^2}{4E^2}\left(\frac{\delta^2\boldsymbol{\pi}}{\delta\theta_1^2} + \frac{\delta^2\boldsymbol{\pi}}{\delta\theta_2^2}\right) \tag{A.4}$$

$$\frac{\delta\boldsymbol{\gamma}}{\delta X} + \boldsymbol{\theta}\frac{\delta\boldsymbol{\gamma}}{\delta\boldsymbol{r}} = -\sigma_0\boldsymbol{\gamma} + \mathrm{C}'\boldsymbol{\pi} \tag{A.5}$$

The more general differential description in Approximation B without Landau simplification is obtained by replacing in this last system equation A.5 by the expression

$$\frac{\delta\boldsymbol{\pi}}{\delta E} + \boldsymbol{\theta}\frac{\delta\boldsymbol{\pi}}{\delta\boldsymbol{r}} = -\mathrm{A}'\boldsymbol{\pi} + \mathrm{B}'\gamma + \epsilon\frac{\delta\boldsymbol{\pi}}{\delta E} + \int[\boldsymbol{\pi}(\boldsymbol{\theta}-\boldsymbol{\theta'}) - \boldsymbol{\pi}(\boldsymbol{\theta'})]\sigma(\boldsymbol{\theta'})\mathrm{d}\boldsymbol{\theta'} \tag{A.6}$$

The multiple coulomb scattering governs the lateral deflections submitted by the electrons passing through the atmosphere. The small angle approximation owing a simple expression of the mean scattering angle when the electron passes through an elementary thickness dX is an important step to express the equilibrium described by the equations equations A.4, A.5 and A.6, especially A.4 and A.5 with the Landau approximation.

A.1.2 Approximations in the numerical treatment

The differential equations A.4 and A.5 were solved after application of the Hankel transform to r, Mellin transform to E and Laplace transform to X [2, 3]. Retransformations from the solutions expressed in terms of complex functions required further approximations, the most important being the saddle point method. The final results expressed in numerical densities were again fitted with a tolerable agreement with the so called NKG formula (2) following an earliest Nishimura formula.

Another approach to solve the system of equations A.4 and A.5 was performed by the method of adjoint equations [4, 5] and the resulting structure functions were found steeper. Here also several numerical approximations had to make to get the solution.

Surprisingly, ρ_{EM}(r) which have not been checked above 10^{17} eV was used *in extenso* to calculate the radio synchrotron emission in giant extensive air showers [6]: recalculating those densities with CORSIKA-EGS4, we observed that the discrepancies remain small for axis distances lower than $3r_{\mathrm{m}}$ containing fortunately the largest part of the source of radio emission or fluorescence in EAS.

A.1.3 Shower age parameter in longitudinal and lateral developments

The age parameter s was first derived from the solution of equations A.1 and A.2 using the Mellin transformation, s being the variable in the complex plane. The total number of electrons (obtained by the inverse Mellin transformation) is obtained for $s = \overline{s}$ following the relation

$$\lambda_1'(\overline{s})X + \log\frac{E_0}{E} - \frac{1}{\overline{s}} = 0. \tag{A.7}$$

Taking into account the elementary solutions (where $\lambda_1'(\bar{s})$ is a function varying slowly), the relation between s and X was derived from the approximation at cascade maximum

$$\frac{\delta\bar{s}}{\delta X}(\lambda_1'(\bar{s})X + \log\frac{E_0}{E} - \frac{1}{\bar{s}}) + \lambda_1(\bar{s}) = 0 \tag{A.8}$$

and the general properties of s were established by the fact that the maximum of the cascade is at $s = 1$, the cascade is developing (growth) when $s \leq 1$, whereas it is decaying (absorption) if $s \geq 1$.

Those considerations in approximation B are brought through the relation 3.8 for $s_{\|}$, the so called Greisen formula for longitudinal development, along with the relation 3.5. One of the most clear presentation of the qualitative relation between $s_{\|}$ and the lateral profile of the cascade under approximation B was demonstrated by Cocconi [7] showing that a lateral distribution is becoming flatter when $s \geq 1$, of course for $s_{\|} = s_{\perp}$. The case of relation 3.4 for $s_{\|}$ was also considered by Cocconi and Nishimura to take into account some effects of density resulting of the atmospheric inhomogeneity. Relation 3.5 has several advantages passing from an asymptotic tendency near axis when $r \to 0$ following r^{s-2} to a steeper power law when $r \to \infty$. The NKG function is based on the Eulerian Beta function, $\mathrm{B(u,v)}$, taken here in the case of cylindrical symmetry, from :

$$N_{\mathrm{e}} = \int_0^{\infty} 2\pi r \rho_{\mathrm{NKG}}(r)\mathrm{d}r \tag{A.9}$$

$$= 2\pi\mathrm{C}(s_{\perp})\int_0^{\infty}(\frac{r}{r_{\mathrm{m}}})^{s_{\perp}-1}(\frac{r}{r_{\mathrm{m}}}+1)^{s_{\perp}-4.5}\mathrm{d}(\frac{r}{r_{\mathrm{m}}}), \tag{A.10}$$

where appears the classical form :

$$\mathrm{B(u,v)} = \int_0^{\infty}\frac{\mathrm{y^{u+1}}}{(1+\mathrm{y})^{\mathrm{u+v}}}\mathrm{dy}, \tag{A.11}$$

for $y = \frac{r}{r_0}$, u = $s_{\perp} - 2$, v = $6.5 - 2s_{\perp}$.

This normalization via A.10 gives the opportunity to link one single density to N_{e} as

$$\rho_{\mathrm{NKG}}(r) = \frac{N_{\mathrm{e}}}{r_{\mathrm{m}}^2}\mathrm{f_{NKG}}(r) \tag{A.12}$$

where $s_\perp$ must be a constant (versus r) corresponding to a fixed value of X in relation 3.5, otherwise expressing the integrations in term of Euler Beta function is not valid in general.