Chapter - V A Modified Method for Solving Non-linear Problems using "Constant Deflection Contour" Method.

During the process of investigation while using two sets of governing equations i.e. equations (3.11)and (3.12) or equations (3.12) and (3.13) both have been utilized. Since the first set (3.11) and (3.12) though simplifies the computational hazards yields not very satisfactory results. This prompts the present investigator to have a little more careful examination and application of the second set of equations.

In the present chapter the investigator used the second set of equations (3.12) and (3.13) for all illustration. In order to make a comparative study some of the problems treated in the previous chapter have been reinvestigated and some of new problems have also been treated in the present chapter. The second set of a equations appears to be more effective in the vibrational analysis. Obvious reason is that it involves a fourth order differential equations whereas the first one involves a third order differential equation. To verify the application of the " Constant deflection Contour " method with equations (3.12) and (3.13), four specific cases will be considered.

1) Circular plate with built in edge (considered immovable)
2) Annular plate with outer boundary clamped and inner boundary free
3) Annular plate with outer boundary simply supported and inner boundary free.
4) Elliptic plate with clamped edge

5.1

Problem - 1

Non-Linear Vibration of a Clamped Rigid Circular Plate :

A rigid circular plate which is clamped along its boundary is considered. The family of isodeflection curves are concentric circles represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{a^2} \qquad [5.1.1]$$

Clearly u = 0 defines the boundary and u = 1 is the centre of the plate where the deflection is maximum under an uniform load p .
The deflection and stress functions are assumed to be

$$w = W(u)\Psi(t)h$$

$$F = F'(u)\Psi^2(t)h \qquad [5.1.2]$$

For Such variable u equations(3.12) and (3.13) as deduced by Banerjee and Rogerson [122]
in chapter III will reduce to

$$D\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]h\psi(t)$$

$$+\frac{h^3}{2}\left[(1-u)\frac{d}{du}\left(\frac{dW}{du}\frac{dF'}{du}\right) - \left(\frac{dW}{du}\frac{dF'}{du}\right)\right]\psi^3(t)$$

$$= \frac{pa^4}{16} + \rho\frac{h^2a^4}{16}W\psi_{,tt}(t) \qquad [5.1.3]$$

$$(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2} = \frac{E}{4}(1-u)\left(\frac{dW}{du}\right)^2 \qquad [5.1.4]$$

The first step of the method of solution as explained in Chapter- III, W can be assumed as $W = \sum_i A_i u^{2i}$
compatible with the boundary conditions for a clamped boundary, where A_i 's are constants which may be evaluated while applying Galerkin procedure due to orthogonal property of the error function. However, since we are concerned more with the applicability rather then exact solution, we may try with a rough approximation by considering a first term only $W \cong u^2$

The first integral of equation (5.1.4) yields

$$(1-u)\frac{d^2F'}{du^2} - \frac{dF'}{du} = \frac{Eu^3}{3} + B_1 \qquad [5.1.5]$$

While the second integral becomes

$$(1-u)\frac{dF'}{du} = \frac{Eu^2}{12} + B_1u + B_2 \qquad [5.1.6]$$

B_1 and B_2 are constants subject to immovable condition

$$\left[2(1-u)\frac{d^2F'}{du^2} - (1-\nu)\frac{dF'}{du}\right]_{u=0} = 0 \qquad [5.1.7]$$

Further equations (5.1.5) and (5.1.6) are valid for the whole domain bounded by C_u, then for u = 0

$$\left.\frac{d^2F'}{du^2}\right|_{u=0} - \left.\frac{dF'}{du}\right|_{u=0} = B_1, \quad \left.\frac{dF'}{du}\right|_{u=0} = B_2 \qquad [5.1.8]$$

Also when $u = 1$, equation (5.1.6) reduces to

$$B_1 + B_2 + \frac{E}{12} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.1.9)$$

Solving equations(5.1.7), (5.1.8) and (5.1.9) one gets B_1 and B_2 and equation (.5.1.6) reduces to

$$(1-u)\frac{dF'}{du} = \frac{Eh}{12(1-\nu)}\left[-2+(1+\nu)u+(1-\nu)u^4\right] \qquad [5.1.10]$$

Combining equations (5.1.3), (5.1.6) and (5.1.10) one can get the error function

$$\epsilon_1 = 4h\psi(t) + \frac{Eh^4}{12D(1-\nu)} + \left[-2+2(1+\nu)u+5(1-\nu)u^4\right]\psi^3(t) + \rho\frac{ha^4}{16D}\psi_{,tt}(t)\,u^2 - \frac{pa^4}{16D} \qquad [5.1.11]$$

Minimizing the error function by application of Galerkin procedure

$$\iint_{\Omega_u} \epsilon_1 u^2 du = 0$$

which on evaluation yields the following time differential equations as

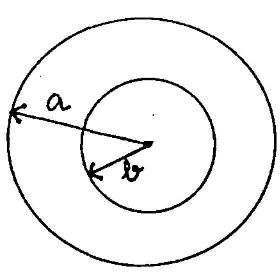
$$\rho h\psi_{,tt}(t) + \frac{80}{9}\frac{Eh^4\psi(t)}{(1-\nu^2)a^4} + \frac{Eh^4}{a^4}\frac{(230-90\nu)}{63(1-\nu)}\psi^3(t) = \frac{5}{3}p \qquad [5.1.12]$$

The above equation is in exact agreement with the result of Yamaki [22] for vibration of a circular plate with clamped immovable edge.

5.2 Non-Linear Vibration of Annular Plates

An Annular plate vibrating at moderately large amplitude is considered. The geometry of the structure has been shown in the adjoining figure.

The region occupied by the plate lies between two concentric circles of radii a and b respectively (a>b).



The isodeflection curves are given by

$$u = 1 - \frac{x^2+y^2}{a^2} \qquad [5.2.1]$$

Where the contour C_o defines the outer boundary for $u = 0$ and C_u for $u = u_1$ (say), for the inner boundary, i.e.

$$u_1 = 1 - \frac{x^2+y^2}{b^2} \qquad [5.2.2]$$

In the present case evaluating the integrals (3.9a) and (3.9b) the limits of u will be u_1 to u instead of ' 1 to u. The nature of the basic equations remains the same except for some constant term dependent on u_1. For example the first term of $\iint_\Omega \nabla^4 W d\Omega$

may be evaluated as

$$\int_{u_1}^{u} (1-u)^2 \frac{d^4W}{du^4} du \oint \frac{ds}{\sqrt{u_{,x}^2+u_{,y}^2}}$$

$$= (1-u)^2 \frac{d^3W}{du^3} - (1-u_1)^2 \left(\frac{d^3W}{du^3}\right)_{u=u_1} +$$

$$+2\int_{u_1}^{u} (1-u) \frac{d^3W}{du^3} du \oint \frac{ds}{\sqrt{u_{,x}^2+u_{,y}^2}} \qquad [5.2.3]$$

In the case for a rigid circular plate $u_1 = 1$, making the second term to vanish, so, the representative of equation (3.11) for the present case of annular circular plate reduces to

$$D\left[(1-u)^2 \frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2}\right] h\psi(t) + \frac{h^3}{2}(1-u)\frac{dF'}{du}\frac{dW}{du}\psi^3(t)$$

$$+ \rho \frac{h^2 a^4}{16} \psi_{,tt}(t) \int_{u_1}^{u} W(u) du + \frac{pa^4}{16}(1-u)$$

$$+ MN(t) = 0 \qquad [5.2.4]$$

Where M is a constant term involving parameters u_1, $\left(\frac{dW}{du}\right)_{u_1}$ and N (t) is dependent on t only. But when equation (5.2.4) differentiated with respect to u it reduces to

$$D\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]h\psi(t) + \frac{h^3}{2}\frac{d}{du}\left[(1-u)\frac{dF'}{du}\frac{dW}{du}\right]\psi^3(t)$$
$$+ \rho\frac{h^2a^4}{16}W\psi_{,tt}(t) = \frac{pa^4}{16} \qquad [5.2.5]$$

However the transformed equation equivalent to equation (5.1.4) differs from the present one by a constant

$$(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2} = \frac{E}{4}(1-u)\left(\frac{dW}{du}\right)^2$$
$$+ B_1 \qquad [5.2.6]$$

Hence the two governing equations for the present problem are equations (5.2.5) and (5.2.6) Two specific cases will be considered.
Circular plate with mixed boundary condition namely annular plate

a) Outer boundary clamped and inner boundary free.

b) Outer boundary simply supported and inner boundary free.

5.2 a) Problem - 2 a

<u>Non- Linear vibration of a annular plate with immovable outer boundary and free inner boundary :</u>

It can be verified that the expression like

$$W(u) \cong u^2 + 16\cdot 888u^3 \cong u^2 + 0\cdot 8042u^4$$
$$\cong u^2 - 1\cdot 37766u^3 + 0\cdot 8698u^4 \qquad [5.2a.1]$$

all satisfy the given condition [as deduced from (3.1.2.) and (3.1.6) mathematically]

$$W\Big|_{u=0} = \frac{dW}{du}\Big|_{u=0} = 0$$

$$\left[2(1-u)\frac{d^3W}{du^3} - (5-2\nu)\frac{d^2W}{du^2}\right]_{u=u_1} = 0$$

$$\left[2(1-u)\frac{d^2W}{du^2} - (1+2\nu)\frac{dW}{du}\right]_{u=u_1} = 0 \qquad [5.2a.2]$$

depending on assumption
$W = Au^2 + Bu^3$, $= Au^2 + Bu^4$ or $W = Au^2 + Bu^3 + Cu^4$
But it will be proper to assume
$W = u^2 - 1.3776u^3 + 0.8698u^4$ ....(5.2a.3)
With the above value of W, the third integral of equation (5.2.6) will yield

$$(1-u)\frac{dF'}{du} = E\left[0.0833u^4 - 0.2066u^5 + 0.2583u^6 - 0.1712u^7 + 0.063u^8\right] - B_1\left[(1-u)\log(1-u)\right] + B_2 u + B_3 \qquad [5.2a.4]$$

The stress conditions, for the inner boundary being free and outer boundary being clamped immovable are not sufficient to evaluate the constants B_1, B_2, B_3. So one must therefore impose certain legitimate condition further. In general $\frac{d^2F'}{du^2}$ or $\frac{dF'}{du}$ may or may not be zero on the simply supported boundary [106]. However , here it has been observed that vanishing of any one of them implies vanishing of the other. So, we are to choose either of the following conditions

a) if $\frac{d^2F'}{du^2}$ or $\frac{dF'}{du} = 0$ on $u = 0$, then $B_2 = B_3 = 0$

b) if $\frac{d^3F'}{du^3} = 0$, then none of B_i's are zero.

Hence we assume that F' is such a function of u which makes its third derivative to vanish from the outer boundary. We then reject the condition(a) and accept (b) as it is more probable; in which case equation (.2a.4) becomes

$$(1-u)\frac{dF'}{du} = E\left[0.8033u^4 - 0.2066u^5 + 0.2583u^6 - 0.1712u^7 + 0.063u^8\right] + 0.017842\left[(1-u)\log(1-u) + u\right] - 0.01657u + 0.029373 \qquad [5.2a.5]$$

The time differential equation likewise as derived in sec (5.1), for the present problem becomes

$$\rho h^2 \psi_{,tt}(t) + \frac{Eh^4}{a^4}\left[28.6676\psi(t) + 4.7355\psi^3(t)\right] = 6.3302\,p \qquad [5.2a.6]$$

5.2 b Problem - 2 b

<u>Non-Liner (a) Vibration of Annular Plate with Simply Supported Outer Boundary and Free Inner Boundary :</u>

It can be verified that that the expression like

$W = u + 0.325\, u^2 - 0.1632\, u^3$ ....(5.2 b.1)
satisfies the boundary conditions [as deduced from (3.1.3), (3.1.4), (3.1.5) and (3.1.6) mathematically] for b/a = ½

$$2(1-u)\frac{d^2W}{du^2} = (1+\nu)\frac{dW}{du} \quad \text{for } u=0$$

$$\left.\begin{aligned}(1-u)\frac{d^2W}{du^2} &= 2(1+\nu)\frac{dW}{du}\\ (1-u)\frac{d^3W}{du^3} &= 2(5-\nu)\frac{d^2W}{du^2}\end{aligned}\right\} \text{for } u=u_1$$

depending on assumption $W = Au + Bu^2 + Cu^3$ with above value of W, the third integral of equation (5.2.6) is given by

$$(1-u)\frac{dF'}{du} = E\left[0\cdot125u^2 + 0\cdot05416u^3 - 0\cdot01223u^4 - 0\cdot0082u^5 + 0\cdot0021u^6\right] + B_1\left[(1-u)\log(1-u) + u\right] + B_2u + B_1 \quad [5.2b.2]$$

Stress conditions are given by

$$\frac{d^2F'}{du^2} = \frac{dF'}{du} = 0 \quad \text{for } u=0$$

Published in the Journal of Vibration Problem [ICOVP-99, Vol B] and $\frac{dF}{du} = 0$ for $u = u_1$

which leads to

$B_2 = B_3 = 0$, and $B_1 = - 0.1463E$
Then equation (5.2b.2) becomes

$$(1-u)\frac{dF'}{du} = E\left[0.125u^2 + 0.05416u^3 - 0.01223u^4 - 0.0082u^5 + 0.0021u^6\right] - 0.1463\left[(1-u)\log(1-u) + u\right] \qquad [5.2b.3]$$

The time differential equation likewise as derived in sec. (5.1)

$$\rho h^2 \psi_{,tt}(t) + \frac{Eh^4}{a^4}\left[6.9715\psi^3(t) + 2.4471\psi(t)\right] = 1.3028\,p \qquad [5.2b.4]$$

5. 2 C

Numerical Results and Discussion :
A close look into the final equations for non liner vibrations of plates considered here [viz. equation (5.1.12),(5.2a.6), (5. 2b.4)] they may be represented in general by

$$\rho h^2 \psi_{,tt}(t) + C_1\psi(t) + C_3\psi^3(t) = Q$$

Where C_1, C_3 are the coefficients of $\psi(t)$ and $\psi^3(t)$ equations (5. 1.12), (5.2a.6), (5.2b.4) as the case may be and Q is the load parameter. We will now discuss different aspect of the analysis.

5.2c.1] Free - Liner Vibrations :

If the non-liner term and the load parameter be set to zero, the liner frequency parameter may be easily obtained

$$\Omega^* = \omega a^2\sqrt{\frac{\rho h}{D}}$$

= 10.3279 (circular plate)
= 17.693 (annular plate with clamped outer boundary and free inner boundary)
= 5.169 (annular plate with simply supported outer boundary and free inner boundary)

being the linear frequency, with $\nu = 0.3$ and $\frac{b}{a} = \frac{1}{2}$

$\frac{b}{a} = \frac{1}{2}, \nu = 0.3$

$$\Omega^* = \omega a^2 \sqrt{\frac{\rho h}{D}}$$

Table : 36

Rigid Circular Plate	10.3279 [present]
Annular plate clamped outer boundary and free inner boundary	17.693 [present], 17.70 [111], 17.85 FEM[120], 17.747 Finite strip[116]
Annular plate simply supported outer boundary free inner boundary	5.169 [present], 5.138 [116],

5.2 C. 2.

Non-Linear free vibration of Annular Plates :

For non-linear free vibration one sets p = 0 to get

$$\rho h \psi_{,tt}(t) + C_1 \psi(t) + C_3 \psi^3(t) = 0$$

Which is the Duffing type equation and its solution is well known. If T * and T be the time periods of non-linear and linear vibrations, respectively, then the relative time period may be expressed as [22]

$$\frac{T^*}{T} = \left[1 + \frac{3}{4}\xi^2\right]^{-1/2} \qquad [5.2c.1]$$

ξ is the non- dimensional relative amplitude of vibration representing W_{max}/h in [22]

Equation (4.2 c.1) may be recast for the three problems with ν = 0.3

$$\frac{T^*}{T} = \left[1 + \frac{3}{4} \times 0.47125\,\xi^2\right]^{-1/2}$$

For rigid circular plate

$$= \left[1 + \frac{3}{4} \times 2.51055\,\xi^2\right]^{-1/2}$$

For annular plate with outer boundary clamped and inner boundary free

$$= \left[1 + \frac{3}{4} \times 3.8228\,\xi^2\right]^{-1/2}$$

For annular plate with outer boundary simply supported and inner boundary free

The numerical results showing the variation of $\frac{T^*}{T}$ with relative amplitude $\left(\frac{W_{max}}{h}\right)$ for rigid circular plate, annular plate with outer boundary clamped and inner boundary free and annular plate with simply -supported outer boundary and free inner boundary have been depicted in table : $\nu = 0.3$

Table 37: T*/T

Relative Amplitude ξ	Rigid Circular Plate and [22]	Annular Plate = 0.5 Clamped outer edge and free inner edge (present)	Simply Supported outer edge and free inner edge (present)
0	1.000	1.000	1.000
0.25	0.9891	0.9459	0.92086
0.5	0.9585	0.8246	0.7632
0.75	0.9133	0.6969	0.5392
1.0	0.8596	0.5890	0.5085
1.25	0.8026	0.5037	0.4276
1.5	0.7463	0.4370	0.4276
1.75	0.6930	0.3844	0.3667
2.0	0.6437	0.3424	0.2832

5. 2 C. 3]

Static deflection for the same problems may be evaluated from equations (5.1.11), (5.2a.6), (5.2b.4) after rejecting the inertial, terms

$$\frac{pa^4}{Eh^4} = \left.\begin{array}{l} 5.861\left(\frac{W_m}{h}\right) + 2.762\left(\frac{W_m}{h}\right)^3 \text{ [present]} \\ = 5.848\left(\frac{W_m}{h}\right) + 2.754\left(\frac{W_m}{h}\right)^3 \text{ Ref [22]} \end{array}\right\} \text{For rigid circular plate}$$

$$= 17.655\left(\frac{W_m}{h}\right) + 44.3237\left(\frac{W_m}{h}\right)^3 \quad \text{for outer boundary clamped and inner boundary free}$$

$$= 2.1715\left(\frac{W_m}{h}\right) + 8.3010\left(\frac{W_m}{h}\right)^3 \quad \text{For outer boundary Simply supported and inner boundary free}$$

Annular plate

Table 38: The Numerical results showing the static deflection shown in Table : pa^4/Eh^4

Wm / h	Rigid Circular Plate	Annular Plate with outer boundary clamped inner boundary free	Annular Plate with outer boundary simply supported and inner boundary free
0	0	0	0
0.25	1.5084	5.1063	0.6724
0.5	3.2758	14.3680	2.12275
0.75	5.5610	31.9400	5.128
1	8.6230 8.6020[22] 9.000[117]	61.9787	10.47
1.25	12.7208	106.6385	18.924
1.50	18.1133	176.0750	31.2672
1.75	25.0593	268.4430	48.2881
2.00	33.818	389.9000	68.751

The values of the load parameter for rigid circular plate as deduced in the present analysis are more close to the values of Way [117] even slightly better than those of Yamaki [22]. . Also the result are in excellent agreement with those of Ref [118].

Discussion and Conclusion :

(for the Problem 5.1, 5.2 a and 5.2 b)

From the results given here for static and dynamic analysis of plates vibrating at large amplitude, it appears that the application of the present method is justified.
The present results are in exact agreement with those obtained by Banerjee and Rogerson [122].
For the linear analysis the results obtained by this method are more close to exact results in comparison with those obtained by other method. For static deflection of annular plates the results cannot be compared for non-availability of such studies.

In conclusion it may be accepted that the application of " Constant Deflection Contour " method is justified and appears to be easier than the other existing methods. The most important point is that the method can be applied to study static and dynamic behaviour of structures having uncommon or complex boundaries for which other methods may fail to analyse. The application of polynomial expressions for the deflection function and stress function in conjunction with Galerkin procedure appears to produce excellent results. In case of free non-linear vibration for plates considered here, the results are compared very well with those previously obtained [22, 72]. Additionally the load deflection relation for rigid circular plate coincides very close with that of Way [117]

5.4 Problem - 4

Non-Liner Vibrations of Elliptic Plate Clamped along its Boundary :

During the process of investigation it has already been observed in chapter IV (problem -1) that the result for vibration of elliptic plates using the equations (3.11) and (3.12) though simplifies the computational hazards, yields not very satisfactory results.

This prompts the present investigator to reinvestigate the same problem using the second set of equations (3.12) and (3.13) for a comparative study .

For an elliptic plate, clamped along its boundary, the family of isodeflection curves are represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [5.4.1]$$

a and b are the semi-major and semi-minor axes. u = o defines the boundary. u = 1 is the centre of the plate where the deflection is maximum under an uniform load p

Deflection and stress functions are given by

$$w = W(u)\psi(t)$$
$$F = F'(u)\psi^2(t) \qquad [5.4.2]$$

For such variable u equation (3.12) and (3.13) in Chapter III will reduce to

$$\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\left[(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2}\right] = 2E(1-u)\left(\frac{dW}{du}\right)^2 \qquad [5.4.3]$$

$$2D\,\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]\psi(t)$$

$$+\frac{8h}{a^2b^2}\frac{d}{du}\left[(1-u)\frac{dF'}{du}\frac{dW}{du}\right]\psi^3(t) - p + \rho h W\psi(t)_{,tt} = 0 \qquad [5.4.4]$$

$$\text{Let } W(u) = \sum_{i=1}^{n} A_i u^{2i} \qquad [5.4.5]$$

Compatible with the boundary conditions for a clamped boundary A_i's are evaluated. Since we are concerned more with the applicability rather than exact solution we may try with a rough approximation by considering the first term only i.e.

$W = u^2$...... (5.4.6)

Since equation (5.4.6) does not represent the exact solution, Galerkin procedure may be applied to minimize the error

We first solve the equation (5.4.3) with $W=u^2$,

The first integral of equation (5.4.3.) yields

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)\frac{d^2F'}{du^2}-\frac{dF'}{du}\right]=\frac{8E}{3a^2b^2}u^3+B_1 \qquad [5.4.7]$$

While the second integral becomes

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)\frac{dF'}{du}\right]=\frac{2}{3}\frac{E}{a^2b^2}+B_1u+B_2 \qquad [5.4.8]$$

B_1 and B_2 are constants subject to immovable condition

$$\left[2(1-u)\frac{d^2F'}{du^2}-(1-2\nu)\frac{dF'}{du}\right]_{u=0}=0 \qquad [5.4.9]$$

Further (5.4.7) and (5.4.8) are valid for the whole domain bounded by C_u then for $u=0$

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left.\frac{dF'}{du}\right|_{u=0}=B_2 \qquad [5.4.10]$$

$$\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\left[\frac{d^2F'}{du^2}-\frac{dF'}{du}\right]_{u=0}=B_1 \qquad [5.4.11]$$

When u = 1 equation (5.4.8) reduces to

$$B_1+B_2=-\frac{2}{3}\frac{E}{a^2b^2} \qquad [5.4.12]$$

Solving equations (5.4.10), (5.4.11) and (5.4.12)

We get

$$B_1=\frac{2E}{3a^2b^2}\frac{(1+2\nu)}{(1-2\nu)}$$

$$B_2=\frac{-4E}{3a^2b^2(1-2\nu)} \qquad [5.4.13]$$

Then equation (5.4.8) reduces to

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}(1-u)\frac{dF'}{du} = \frac{2E\left[(1-\nu)u^4+(1+\nu)u-2\right]}{3a^2b^2(1-\nu)} \qquad [5.4.14]$$

Combing equations (5.4.4), (5.4.6) and (5.4.14) we get error function.

$$\begin{aligned}\epsilon_1 = & -\frac{8D(3a^4+3b^4+2a^2b^2)}{a^4b^4}(1-u)A\psi(t) \\ & + \frac{16Eh\left[(1-\nu)u^4+(1+\nu)u-2\right]}{3(1-\nu)(3a^4+3b^4+2a^2b^2)}A^3\psi^3(t) \\ & + p(1-u)+\rho hA\psi_{,tt}(t)\left[\frac{u^3}{3}-\frac{1}{3}\right]\end{aligned}$$

Minimizing the error function by application of Galerkin procedure
Which on evaluation yields the following time differential equation for $\nu = 0\cdot 3$

$$\frac{2}{3}D\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}A\psi(t) + \frac{1\cdot 84\,Eh}{(3a^4+3b^4+2a^2b^2)}A^3\psi^3(t) + \frac{1}{18}\rho hA\psi_{,tt}(t) = \frac{p}{12} \qquad [5.4.15]$$

Equn (5.4.15) may be put in a simpler form

$$\psi_{,tt}(t) + C_1\psi(t) + C_3\psi^3(t) = Cp \qquad [5.4.16]$$

where $C_1 = \dfrac{12D}{\rho h}\dfrac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}$

$C_3 = 33\cdot 12\,\dfrac{E}{\rho}\dfrac{A^2}{(3a^4+3b^4+2a^2b^2)}$

$C = \dfrac{3}{2A\rho h}$

a) For free liner vibration $p = 0$, $C_3 = 0$

$$\psi_{,tt}(t) + C_1\psi(t) = 0$$

The linear frequency parameter

$$C_1^{1/2} = \left[\frac{12D}{\rho h} \frac{(3a^4+3b^4+2a^2b^2)}{a^4 b^4} \right]^{1/2}$$

For circular plate b = a

Linear frequency for a circular plate = $9 \cdot 797 \sqrt{\dfrac{D}{\rho h a^4}}$ where a is the radius of the plate

b) Non linear free vibration

For Non-linear free vibration one puts p = 0 in the equation (5.4.16)

$$\psi_{,tt}(t) + C_1 \psi(t) + C_3 \psi^3(t) = 0$$

If one designates T* and T as the time periods of non- linear and linear vibrations respectively then the relative period is given by

$$\frac{T^*}{T} = \left[1 + \frac{3}{4} \frac{C_3}{C_1} \xi^2 \right]^{-1/2}$$

ξ is the non-dimensional relative amplitude

$$\frac{T^*}{T} = \left[1 + \frac{3}{4} \times \frac{33 \cdot 12 (1-\nu^2) m^4}{(3m^4+2m^2+3)^2} \left(\frac{A_0}{h} \right)^2 \right]^{-1/2}$$

Where m = a/b.

For circular plate m = 1 and for ν = $0 \cdot 3$ one gets

$$\frac{T^*}{T} = \left[1 + \frac{3}{4} \times 0 \cdot 4709 \, \xi^2 \right]^{-1/2}$$

The numerical results showing the variation of T*/T with relative amplitude ξ (= $\frac{A_0}{h}$) for rigid elliptical plate have been depicted in the Table - (39)

C) Static deflection for the same problem may be evaluated from equation (5.4.16) after rejecting the inertial term

$$\frac{pa^4}{Eh^4} = \frac{2}{3}\frac{(3m^4+2m^2+3)}{(1-\nu^2)}\left(\frac{W_0}{h}\right) + \frac{22.08\,m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3$$

Which for circular plate and for $\nu = 0.3$ reduces to

$$\frac{pa^4}{Eh^4} = \left\{5.8608\left(\frac{W_0}{h}\right) + 2.76\left(\frac{W_0}{h}\right)^3\right\} \quad \text{present study}$$

$$= \left\{5.848\left(\frac{W_0}{h}\right) + 2.754\left(\frac{W_0}{h}\right)^3\right\} \quad \text{Ref (22)}$$

Numerical results showing the dependence of central deflection $\frac{W_0}{h}$ on load parameter $\frac{pa^4}{Eh^4}$ are shown in table (40)

Table [3 9]
Dependence of relative time period T*/T of non-liner and liner vibrations on relative amplitudes for different values of aspect ratio (m) , $\nu = 0.3$

$\frac{W_0}{h}$	T*/T		
	m= 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.2	0.9930	0.9955	0.9979
0.4	0.9728	0.9826	0.9917
0.6	0.9419	0.9622	0.9818
0.8	0.9031	0.9356	0.9683
1.0	0.8596	0.9054	0.9517
1.2	0.8416	0.8703	0.9386
1.4	0.7708	0.8345	0.9114
1.6	0.7246	0.7983	0.8887
1.8	0.6828	0.7624	0.8649
2.0	0.6437	0.7275	0.8404
2.2	0.6075	0.69405	0.8157
2.5	0.5752	0.6469	0.7786

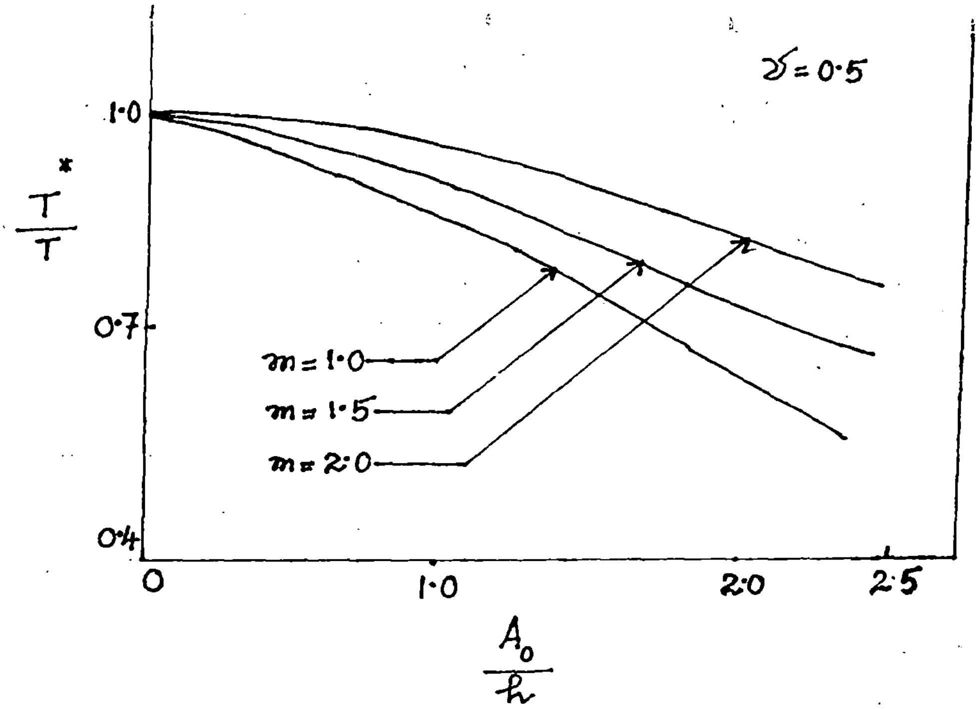


Figure $\overline{\text{XIII}}$: Time Period Ratio $\frac{T^*}{T}$ against Relative Amplitude for Elliptic Plates

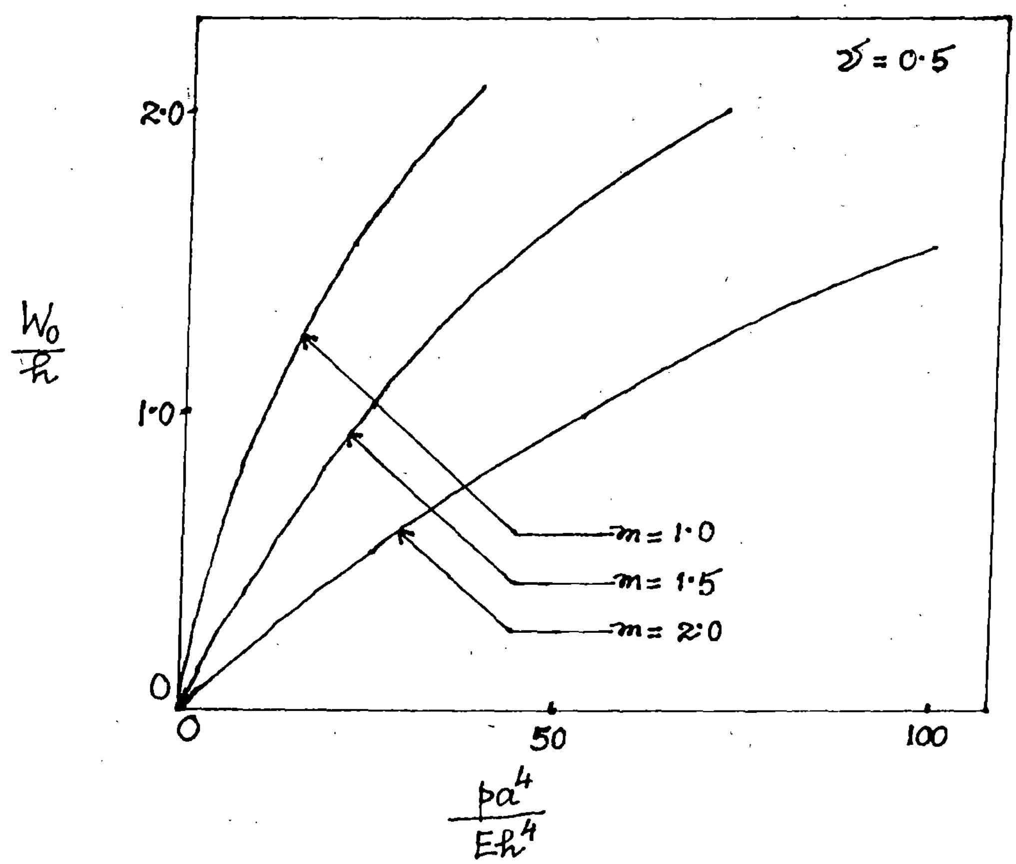


Figure $\overline{\text{XIV}}$: Load Deflection Curve for Elliptic Plates

Table [40]

Dependence of Central Deflection $\frac{W_0}{h}$ on Load Parameter $\frac{pa^4}{Eh^4}$ for elliptical plate $\nu = 0.3$, for different aspect ratio, m

W_0/h	pa^4/Eh^4		
	m = 1	m = 1.5	m= 2.0
0	0	0	0
0.2	1.1742	3.3599	8.6918
0.4	2.4619	6.9636	17.6711
0.6	4.1725	11.03668	27.2251
0.8	6.1017	15.8.92	37.6413
1.0	8.6206	21.5478	49.207
1.2	11.802	28.4587	60.4852
1.6	20.6818	46.7742	93.6747
1.8	26.64508	58.6516	112.712
2.0	33.818	72.6576	134.427

It has already been studied that the result for vibration of elliptic plates using the first set (3.11) and (3.12) is not satisfactory. But in this chapter utility of the second set of equations (3.12) and (3.13) has been observed. Hence in conclusion it may be said that the second set of equations should preferably be used to investigate large amplitude, vibration problems. However for rough approximation the first set may be used in cases where the structures have complicated boundaries or the governing differential equations become too much complicated in nature.