Chapter - II

NON LINEAR THEORY OF ELASTICITY

When an elastic body undergoes deformation under the action of external forces stresses and strains are developed within the body. The state of stress at a point within the body is specified, at most by nine components of stress.In the linear theory the strains in the middle surface are neglected in which the deflections are small compared with the thickness of the plate. If one wishes to study the exact analysis of the non linear theory of plates he may be refered to the Donnel's work [110]. The relations between the strains and displacement may be put as [110].

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \left[-\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] z$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \left[-\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}\right] z$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left[-2\frac{\partial^2 w}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial w}{\partial x}\right.$$

$$+ 2\frac{\partial^2 v}{\partial x \partial y}\cdot\frac{\partial w}{\partial y} + 2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$\left. + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\nabla^2 w\right] z \quad \text{.............} [2\cdot1]$$

in which higher powers and products of the displacements involving z^2 have been omitted. We will confine our attention to cases where both the strains and deflections slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are small compared to unity; and in general, for plates used in mechanics and structures the allowable strains and deflection slopes are very small compared to unity. It is important to note that for the membrane part of the strain terms like $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial x}{\partial y}$ and $\frac{\partial v}{\partial x}$ are important, but the terms involving the squares or product of themselves will be negligible, while the flexural strain terms like $\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)$ are very small compared to the principal flexural terms like $\frac{\partial^2 w}{\partial x^2}$. Similar arguments may be presented for the ommission of terms like $\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial x}\right)$ and ter-m containing z^2. Moreover the derivation of the strain-displacement relations is based on Love-Kirchhoff hypothesis (i.e., the linear filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed middle surface and suffer no extensions).With these assumption and aproximations, the total strains in the layer of the plate parallel to and a distance z from the middle surface, they can be written as

$$\epsilon_x = \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - z\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = \left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] - z\frac{\partial^2 w}{\partial y^2}$$

$$\epsilon_{xy} = \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] - 2z\frac{\partial^2 w}{\partial x \partial y} \quad \text{-----------} [2 \cdot 2]$$

where the terms within third bracket being constants with respect to 'z' are membrane strain and the last terms containing the factor 'z' are flexural strains.

Stress - Strain Relations :

For isotropic materials the elastic stress-strain relations or Hook's Law have been found from experiments to be, in general, as :

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_z - \nu\sigma_x\right)$$

$$\epsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_x - \nu\sigma_y\right)$$

$$\epsilon_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

$$\epsilon_{yz} = \frac{1}{G}\sigma_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz}$$

$$\epsilon_{xz} = \frac{1}{G}\sigma_{xz} = \frac{2(1+\nu)}{E}\sigma_{xz} \quad \text{------} [2 \cdot 3]$$

But for plane stress, i.e., when σ_x, σ_y and σ_{xy} are assumed to be uniform over the thickness and σ_z, σ_{xz}, σ_{yz} are every where zero, the relations given by [2.3] will then reduce to

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \sigma_x = \frac{E}{(1-\nu^2)}(\epsilon_x + \nu\epsilon_y)$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \sigma_y = \frac{E}{(1-\nu^2)}(\epsilon_y + \nu\epsilon_x)$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\epsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} \qquad \sigma_{xy} = \frac{E}{2(1+\nu)}\epsilon_{xy} \qquad [2.4]$$

For the case of plane strain, i.e., when $\epsilon_z = 0$, equation (2.3) will then become

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\epsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right)$$

$$\epsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right)$$

$$\epsilon_{xy} = \frac{2\left[1+\frac{\nu}{(1-\nu)}\right]}{E/(1-\nu^2)}\sigma_{xy} \qquad [2.5]$$

Principle of Virtual Work :

In obtaining the solutions of elastic problems energy principles and variational methods play an important role. It will be seen later that the governing differential equations are direct consequences of the minimisation of the energy expressions associated with the structure concerned. The method is termed as 'variational method' since it is based on calculus of variation. The basis of 'calculus of variation' is the 'principle of virtual work' enuncited by the great mathematician John Bernouli in 1717, - "If under the action of a certain force a particle undergoes an arbitrary small displacement, called a virtual displacement and if the particle retains its condition of equilibrium then the total workdone by the force is zero."

Since the principle of virtul work is very common in every sphere of mathematics we would better leave it here and carry out the required mathematical operations without giving much emphasis on the theory of '' . ' 'the principle.

Let us denote the total workdone against the mutual actions between the particles in an elastic body due to the virtual displacements $\delta u, \delta v, \delta w$, by δV, where $\delta u, \delta v, \delta w$ are the displacements parallel to the axes of a Cartesian system of coordinates wih respect to a certain origin; then the total work done by the mutual actions is $-\delta V$.

If there be forces applied at the boundary of the body and if X, Y, Z be the components of the body forces along the x,y,z directions respectively, and $\bar{X}, \bar{Y}, \bar{Z}$ be the component of the boundary forces per unit area then the work done by them

$$W = \iiint (X\,\delta u + Y\,\delta v + Z\,\delta w)\,dx\,dy\,dz + \iint (\bar{X}\,\delta u + \bar{Y}\,\delta v + \bar{Z}\,\delta w)\,dA \qquad [2\cdot 6]$$

dA being the elementary area and the integration being taken over the part of the boundary surface of the body, on which displacements are not prescribed. It is important to note here that the part of the surface where forces are prescribed is the same as the part where displacements are not prescribed. We may assume further that the external forces are constants during the virtual displacement, when we put

$\delta(V-W) = 0$ ------------------------ [2.7]

One can interpret the result given by the equation (2.7), in comparing various values of the displacements u,v, and w, the displacements which actually occur in an elastic system under the given external forces are those which lead to zero variation of the total energy (potential energy) of the system for any virtual displacement from the position of equilibrium.

Principle of Minimum Potential Energy and Principle of Complementary Energy :

The expression (V-W), consisting of V, the potential energy of deformation, and -W, the potential energy of the external forces is called the 'Potential Energy' of the system. For stable equilibrium it can be shown that the total potential energy of the system is positive, hence in this case the total potential energy of the system is a minimum. This is the principle of minimum potential energy.

In case of a vibrating plate there is an additional energy, the Kinetic Energy. If we denote it by T then we can form the Lagrangian

$L = T - U$, U is the potential energy ------------------------ [2.8].

Applying Hamilton's principle we can further show that the Hamiltonian

$H = T + U$ = the total energy

If the potential energy is independent of velocities (i.e., independent of u,v,w) remains positive, since by defination, T, the kinetic energy is positive definite.

Instead of varying the displacements from those at equilibrium, one may want to vary the stress components. If $\delta\sigma_x, \delta\sigma_y, \delta\sigma_{xy}$ be the small variations in the stress components $\sigma_x, \sigma_y, \sigma_{xy}$ respectively, then the change in the strain energy per unit thickness of the plate may be written as

$$\delta\Pi = \iint \left[\frac{1}{E}\left(\sigma_x\,\delta\sigma_x + \sigma_y\,\delta\sigma_y - \nu\sigma_x\,\delta\sigma_y - \nu\sigma_y\,\delta\sigma_x\right) + \frac{1}{G}\,\sigma_{xy}\,\delta\sigma_{xy}\right] dA \qquad [2\cdot 9]$$

The body forces being given external forces, remain unchanged but on that part of the boundary where surface forces are not prescribed, corresponding to the variation of stress components, there will be some variation in the boundary surface forces. In this case also it can be seen that the variation of the total energy 'Π' is zero, i.e.,

$$\delta \Pi = \delta(V^* - W^*) = 0 \qquad [2\cdot10]$$

Where V* isthe strain energy per unit thickness of the plate and W* is the work done by the boundary-surface forces. The expression Π is called the Complementary Energy of the system. We have seen in the case of potential energy, the same deduction may be made and we may conclude that "for all stress satisfying the equilibrium conditions in the interior and on the part of the boundary surface where the surface forces are prescribed, the stresses which satisfy the compatibility equations, are such that the complementary energy assumes a satisfactory value". Note that we have imposed the restriction on the stresses to satisfy the compatibility equations for the deduction of equation (2.10) depends on this restriction.

Deduction of Equation of Equilibrium and Boundary Conditions :

Let us consider a plate of thickness 'h'. The mid-plane of the plate is given by z = 0 and the two surfaces are denoted by S_1 and S_2, defined by $z = \frac{h}{2}$ and $z = -\frac{h}{2}$, respectively.
Hence forth we shall be mainly concerned with plane-stress only. Accordingly, we consider the total strain energy

$$V = \iiint_{-h/2}^{h/2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_{xy} \epsilon_{xy})\, dz\, dA \qquad [2\cdot11]$$

combining equation (2.9) and (2.4). Using the stress-strain relations given by (2.4) to express the strains in terms of stresses, or the stresses in terms of strains equation (2.11) may be written as

$$V = \frac{1}{2E} \iint_R \int_{-h/2}^{h/2} [\sigma_x^2 - 2\nu\sigma_x\sigma_y + \sigma_y^2 + 2(1+\nu)\sigma_{xy}^2]\, dz\, dA \qquad [2\cdot12]$$

or,

$$V = \frac{E}{2(1-\nu^2)} \iint_R \int_{-h/2}^{h/2} \left[\epsilon_x^2 + 2\nu\epsilon_x\epsilon_y + \frac{(1-\nu)}{2}\epsilon_{xy}^2 + \epsilon_y^2\right] dz\, dA \qquad [2\cdot13]$$

Replacing the strains in the expression (2.11) and taking the first variation of the strain energy one can write

$$\delta V^{(1)} = \iint_R \int_{-h/2}^{h/2} \left[\sigma_x \left\{ \frac{\partial}{\partial x}\delta u + \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\delta w - z\frac{\partial^2}{\partial x^2}\delta w \right\} \right.$$

$$+ \sigma_{xy}\left\{ \frac{\partial}{\partial y}\delta u + \frac{\partial}{\partial x}\delta v - 2z\frac{\partial^2 \delta w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\delta w + \frac{\partial w}{\partial y}\frac{\partial}{\partial x}\delta w \right\}$$

$$\left. + \sigma_y \left\{ \frac{\partial}{\partial y}\delta v + \frac{\partial w}{\partial y}\frac{\partial}{\partial y}\delta w - z\frac{\partial^2 \delta w}{\partial y^2} \right\} \right] dz\, dA \qquad [2{\cdot}14]$$

We shall now preform the integration with respect to z first and introduce the stress resultants N_x, N_y, N_{xy} and moment intensity M_x, M_y, M_{xy} respectively, defined by

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz = h\sigma_{xm} \qquad M_x = \int_{-h/2}^{h/2} \sigma_x \, z\, dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, dz = h\sigma_{ym} \qquad M_y = \int_{-h/2}^{h/2} \sigma_y \, z\, dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, dz = h\sigma_{xym} \qquad M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, z\, dz \qquad [2{\cdot}15]$$

Where the symbol N_x represents forces per unit length in the x-direction, N_y in the y-direction and $N_{xy} = N_{yx}$ is a force in the xy-direction. Similarly M_x, M_y and M_{xy} represent the moments per unit length in the respective directions as shown in the **figure 1**

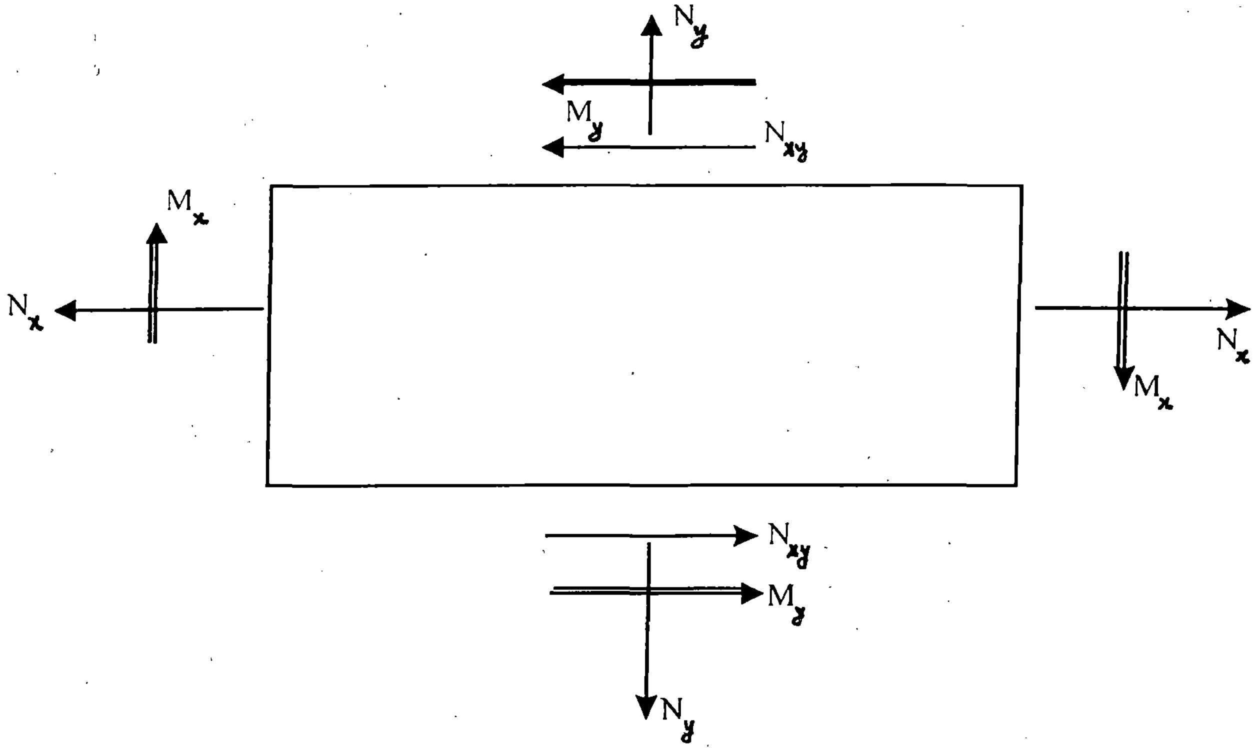

Figure1 : Stress and moment resultants for a rectangular plate.

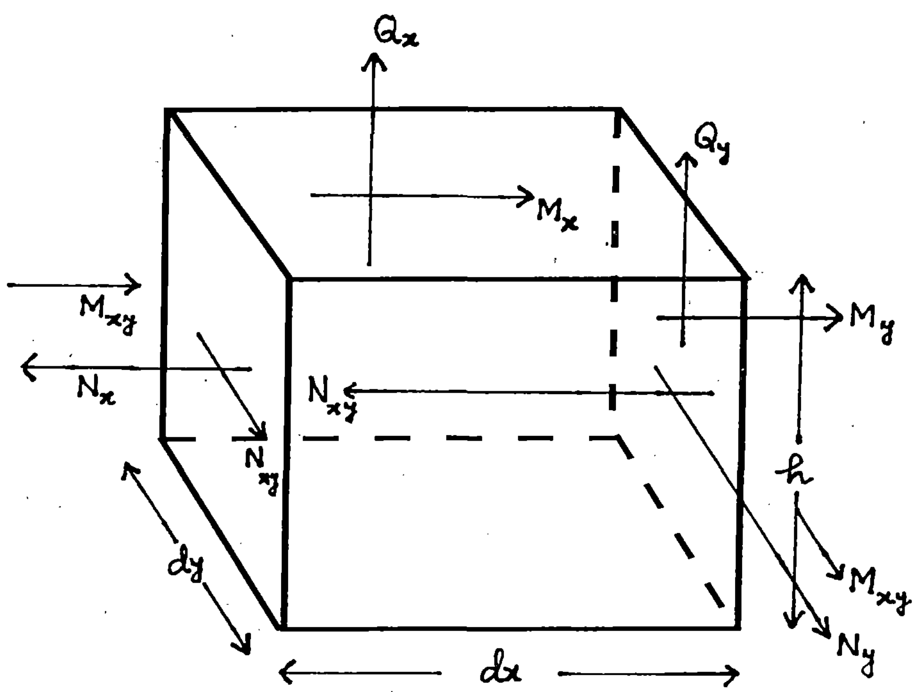

Figure2 : Stress resultants and moments

For practical purpose we may write $N_{xy} = N_{yx}$ and equation (2.9) may be replaced by

$$\delta V^{(1)} = \iint \Big\{ N_x \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) - M_x \frac{\partial^2 \delta w}{\partial x^2}$$

$$+ N_{xy} \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} \right)$$

$$- 2M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} + N_y \left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right)$$

$$- M_y \frac{\partial^2 \delta w}{\partial y^2} \Big\} dx dy \qquad [2.16]$$

If $\bar{F}_x$, $\bar{F}_y$ and $\bar{F}_z$ be the components of the external forces acting in the directions of x, y and z axes respectively then the virtual workdone is given by

$$\delta^{(1)} W^* = -\iint_{S_1} (\bar{F}_x \delta U + \bar{F}_y \delta V + \bar{F}_z \delta W) \, ds dz \qquad [2.17]$$

where S_1 is that part of the side boundary where the external forces are prescribed

$$U = u - z\frac{\partial w}{\partial x}, \quad V = v - z\frac{\partial w}{\partial y}, \quad W = w \qquad [2.18]$$

u,v,w being the displacement components of the middle surface of the plate.

Considering the load intensity 'p' the principle of virtual work for the present problem (for simplicity we are, for the time being, avoiding the expression for the kinetic energy associated with the motion of the plate) may be put in the following form

$$\overset{(1)}{\delta}(\Phi) = \iiint (\sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \sigma_{xy} \delta\epsilon_{xy})\, dx\, dy\, dz$$

$$- \iint_{S_1} (\bar{F}_x \delta U + \bar{F}_y \delta V + \bar{F}_z \delta W)\, ds\, dz$$

$$- \iint_{S_m} p\, \delta w\, dx\, dy \qquad [2{\cdot}19]$$

where S_m denote the mid-surface region of the plate.
Let us introduce the following integrals.

$$\bar{N}_{xs} = \int_{-h/2}^{h/2} \bar{F}_x\, dz \qquad \bar{N}_{ys} = \int_{-h/2}^{h/2} F_y\, dz$$

$$\bar{N}_z = \int_{-h/2}^{h/2} \bar{F}_z\, dz$$

$$\bar{M}_{xs} = \int_{-h/2}^{h/2} \bar{F}_x\, z\, dz \qquad \bar{M}_{ys} = \int_{-h/2}^{h/2} \bar{F}_y\, z\, dz \qquad [2{\cdot}20]$$

The first variation of the potential of the applied forces, including the load intensity 'p', corresponding to equation (2.17) may be written as

$$\delta^{(1)} W^{*} = -\int_{S_m} p\, \delta w\, dx\, dy + \iint_{S_1} \bar{N}_{\nu}\, \delta u_{\nu}\, ds + \iint_{S_1} N_{\nu s}\, \delta u_s\, ds \qquad [2.21]$$

where u_ν and u_s are the in-plane displacements of the boundary of the plate in directions normal and tangential respectively to the boundary (Figure 2.3). N_ν is taken as positive in compression as shown in Fig-**3**

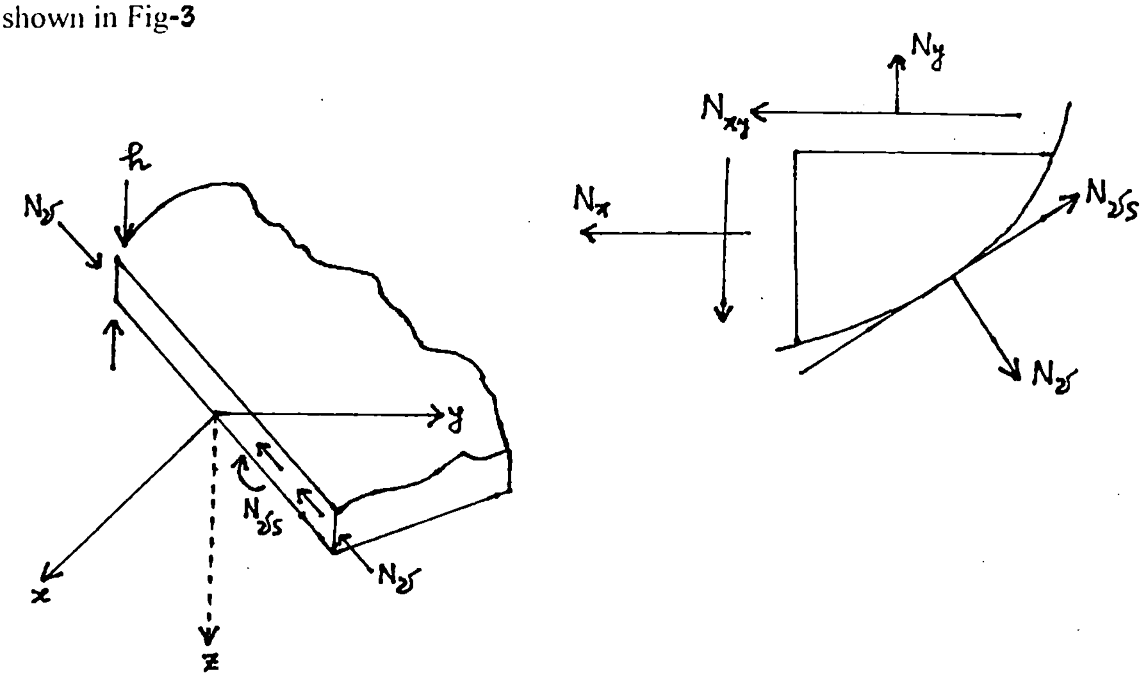

Fig 3 : Boundary loading on the plate segment

If the direction cosines of the normal (drawn outward) be (l,m,o) i.e., $l = \cos(x,\nu)$ and $m = \cos(y,\nu)$, we can write

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial \nu} - m\frac{\partial}{\partial s} \quad , \quad \frac{\partial}{\partial y} = m\frac{\partial}{\partial \nu} + l\frac{\partial}{\partial s} \qquad [2.22]$$

The displacements along the normal and tangential directions may be put

$u_\nu = lu + mv$, $u_s = -mu + lv$ -------------------- (2.23)

With the above notations we can express

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$$N_{\nu} = l^2 N_x + 2lm N_{xy} + m^2 N_y$$

$$N_{\nu s} = (N_y - N_x) lm + N_{xy}(l^2 - m^2) \qquad [2.24]$$

Also $N_{x\nu} = (l N_{\nu} - m N_{\nu s}) = l N_x + m N_{xy}$

$$N_{y\nu} = (m N_{\nu} + l N_{\nu s})$$

or, $N_{y\nu} = (l N_{xy} + N_y m)$ $\qquad [2.25]$

$$M_{\nu} = l^2 N_x + 2lm M_{xy} + M_y m^2$$

$$M_{\nu s} = (M_y - M_x) lm + M_{xy}(l^2 - m^2) \qquad [2.26]$$

$$\bar{M}_{\nu} = \bar{M}_{x\nu} l + \bar{M}_{y\nu} m,$$

$$\bar{M}_{\nu s} = -\bar{M}_{x\nu} m + \bar{M}_{y\nu} l \qquad [2.27]$$

$$M_{x\nu} = l M_x + m M_{xy}$$

$$M_{y\nu} = M_{xy} l + M_y m$$

Finally we introduce the transverse shear forces of plate theory

$$Q_{\nu} = Q_x l + Q_y m$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \qquad [2.28]$$

We are now in a position to utilise the extermization process and to apply Green's Theorem so long as the du and dv are involved in the integrals in the following equation obtained from relations given by equations (2.16) - (2.28).

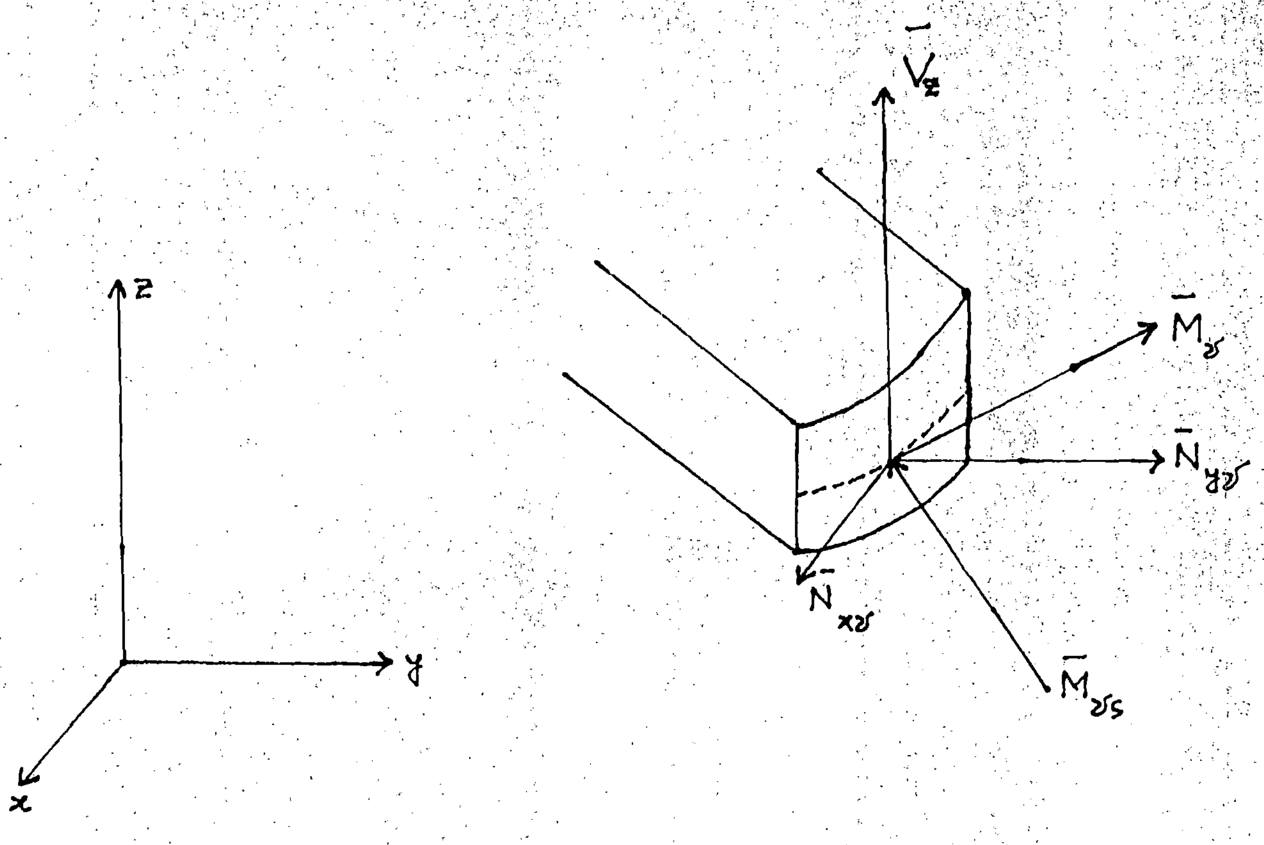

Figure :- 4 Forces on the Plate Filament

Combining equations (2.16) and (2.21) we may rewrite for the total potential energy variation :

$$\delta\Pi_s^{(1)} = \iint\Big[N_x\Big(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\Big) - M_x\frac{\partial^2 \delta w}{\partial x^2}$$

$$+ N_{xy}\Big(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\Big)$$

$$- 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} + N_y\Big(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\Big) - M_y\frac{\partial^2}{\partial y^2}\delta w\Big]dx\,dy$$

$$-\iint p\,\delta w\,dx\,dy + \int \bar{N}_\nu\,\delta u_\nu\,ds + \int N_{\nu s}\,\delta u_s\,ds = 0 \qquad [2{\cdot}29]$$

We now perform the line integrals considering the equilibrium of the plate element and the notations defined earlier.

$$\delta\Pi_s^{(1)} = -\iint\Big[\Big(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\Big)\delta u + \Big(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\Big)\delta v$$

$$+\Big\{\frac{\partial^2 M_x}{\partial x^2} + \frac{2\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}\Big(N_x\frac{\partial w}{\partial x}\Big)$$

$$+ \frac{\partial}{\partial y}\Big(N_{xy}\frac{\partial w}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(N_{xy}\frac{\partial w}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(N_y\frac{\partial w}{\partial y}\Big) + p\Big\}\delta w\Big]dx\,dy$$

$$+ \int_\Gamma (N_\nu + \bar{N}_\nu)\,\delta u_\nu\,ds + \int_\Gamma (N_{\nu s} + \bar{N}_{\nu s})\,\delta u_s\,ds$$

$$- \int_\Gamma M_\nu \frac{\partial \delta w}{\partial \nu}\,ds + \int_\Gamma \Big(Q_\nu + \frac{\partial}{\partial s}M_{\nu s} + N_\nu\frac{\partial w}{\partial \nu} + N_{\nu s}\frac{\partial w}{\partial s}\Big)\delta w\,ds$$

$$- \big(M_{\nu s}\,\delta w\big) = 0 \qquad [2{\cdot}30]$$

The last expression accounts for corners in the boundary. Considering the condition of equilibrium, one may deduce from equation [2.30]

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad [2.31]$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad [2.32]$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{2\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w}{\partial x}\right)$$

$$+ \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right) + p = 0 \qquad [2.33]$$

The mechanical boundary conditions are obtained form the remaining line integrals on the boundary

Either $N_\upsilon = - N_\upsilon$ or u is specified -------------------- (2.34)

Either $N_{\upsilon s} = - N_{\upsilon s}$ or u_s is specified ------------------- (2.35)

Either $M_\upsilon = 0$ or $\frac{\partial w}{\partial \upsilon}$ is specified -------------------- (2.36)

Either

$$Q_\upsilon + \frac{\partial M_{\upsilon s}}{\partial s} + N_\upsilon \frac{\partial w}{\partial \upsilon} + N_{\upsilon s} \frac{\partial w}{\partial s} = 0$$

or w is specified --------------------------- (2.37)

The last term in equation (2.30) which accounts for the corners, indicates

At discontinuties [$M_{\upsilon s} \delta w$] = 0 ---------------------- (2.38)

Equation (2.31 - 2.33) together with the geometrical boundary conditions constitute the problem for a flat plate in large deflection.

With proper transformations, we can get the mechanical boundary conditions otherwise

On the boundary, c_1 :

$$N_{x\nu} = \bar{N}_{x\nu} \quad ; \quad N_{y\nu} = \bar{N}_{y\nu}$$

$$\left(Q_x l + Q_y m + N_{x\nu}\frac{\partial w}{\partial x} + N_{y\nu}\frac{\partial w}{\partial y} \right) + \frac{M_s}{s}\left[= \left(V_z + \frac{\partial M_{\nu s}}{\partial s} \right) \right]$$

$$= \bar{V}_z + \frac{\partial \bar{M}_{\nu s}}{\partial s}$$

$$M_{\nu} = \bar{M}_{\nu} \qquad [2{\cdot}39]$$

We shall now establish the stress resultant displacement relations from equation (2.15) after performing the necessary integrations and expressing them in terms of partial derivatives of the three displacements u, v, w as :

$$N_x = h\sigma_{xm} = C\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \nu\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}\right] \qquad [2{\cdot}40]$$

$$N_y = h\sigma_{ym} = C\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \nu\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}\right] \qquad [2{\cdot}41]$$

$$N_{xy} = h\sigma_{xym} = \frac{C(1-\nu)}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \qquad [2{\cdot}42]$$

$$N_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad [2{\cdot}43]$$

$$N_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \qquad [2{\cdot}44]$$

$$M_{xy} = -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y} \qquad [2{\cdot}45]$$

where $D = Ch^2 = \frac{Eh^3}{12(1-\nu^2)}$, the flexural rigidity

and $C = \frac{Eh}{12(1-\nu^2)}$, the extensional rigidity

Now introducing the Airy's stress function F defined by

$$N_x = h\sigma_{xm} = h\frac{\partial^2 F}{\partial y^2}$$

$$N_y = h\sigma_{ym} = h\frac{\partial^2 F}{\partial x^2}$$

$$N_{xy} = h\sigma_{xym} = -h\frac{\partial^2 F}{\partial x \partial y} \qquad [2{\cdot}46]$$

compatible with equations (2.30) and (2.31).

We can now express the membrane strains $[\epsilon_{xm}, \epsilon_{ym}, \epsilon_{xym}]$ in terms of membrane stress (σ_{xm}, σ_{ym}, σ_{xym}) and equate the same in terms of displacements and Airy's stress function in the following form :

$$\epsilon_{xm} = \frac{1}{E}\left(\sigma_{xm} - \nu\sigma_{ym}\right) = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right]$$

$$= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_{ym} = \frac{1}{E}\left(\sigma_{ym} - \nu\sigma_{xm}\right) = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right]$$

$$= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\epsilon_{xym} = \frac{2(1+\nu)}{E}\sigma_{xym} = \frac{-2(1+\nu)}{Eh}\frac{\partial^2 F}{\partial x \partial y}$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad [2{\cdot}47]$$

Applying the operators $\frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x^2}$; , $\frac{-\partial^2}{\partial x \partial y}$ to the first, second and third of the above equations, respectively and adding them together, in order to eleminate u and v one obtains

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \qquad [2{\cdot}48]$$

Further assuming that the thickness of the plate is constant, and combining equation (2.33) with equations (2.40) - (2.45) one can write

$$D\nabla^4 w = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + p \qquad [2{\cdot}49]$$

Equation (2.48) and (2.49) are the well known Von Karmañ plate equations.
The equation (2.48) is known as the 'compatibility equation' and the equation (2.49) is the equation of equilibrium in the direction of the z axis. These equations henceforth will be termed as Karmañ equations. These Karmañ equations may be written in a simplified form by introducing the non linear operator α , defined by

$$\mathcal{L}(a,b) = \frac{\partial^2 a}{\partial x^2}\frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 a}{\partial y^2}\frac{\partial^2 b}{\partial x^2} - 2\frac{\partial^2 a}{\partial x \partial y}\frac{\partial^2 b}{\partial x \partial y} \qquad [2 \cdot 50]$$

$$\nabla^4 F = -\frac{Eh}{2}\mathcal{L}(w, w) \qquad [2 \cdot 51]$$

$$D\nabla^4 w = \mathcal{L}(F, w) + p \qquad [2 \cdot 52]$$

The above two equations are the governing equations for thin plates at large deflections under a static load. However, they may be extended to a dynamic case by changing p by $\left(p - \rho h \frac{\partial^2 w}{\partial t^2}\right)$

While deducing the governing differential equations for a dynamic case, we shall consider the kinetic energy of the plate, given by

$$T_e = \iiint \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dx\,dy\,dz \qquad [2 \cdot 53]$$

In this case we shall have to minimize the integral

$$\delta \Pi^* = \int_{t_1}^{t_2} \left(\Pi_s - T_e\right) dt \qquad [2 \cdot 54]$$

when the integral is obtained by combining equations (2 .29) and (2.53). The equations of equilibrium will thus be transformed to

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho h \frac{\partial^2 u}{\partial t^2} \qquad [2.55]$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = \rho h \frac{\partial^2 v}{\partial t^2} \qquad [2.56]$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right)$$

$$+ \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w}{\partial x}\right)$$

$$+ p = \rho h \frac{\partial^2 w}{\partial t^2} \qquad [2.57]$$

These equations are equivalent to equations (2.31) - (2.33) for the static case differing only by the inertia terms on the right hand side. If we neglect the inertia in the plane of the plate i.e., if we set the right hand side of equations (2.55) and (2.56) to zero, the resulting equations will then be transformed to

$$\nabla^4 F = -\frac{Eh}{2} \mathcal{L}(w, w) \qquad [2.58]$$

$$D \nabla^4 w = \mathcal{L}(F, w) + p - \rho h \frac{\partial^2 w}{\partial t^2} \qquad [2.59]$$

These are the governing differential equations for thin plates at large amplitudes. The deflection function w is dependent on the space coordinates as well as on the time. It is important to note that the load 'p' may be uniform, concentrated at a point or distributed over a segment of the plate; it may be dependent or independent of time, as for example, in case of forced vibration, p becomes function of time.