Chapter - VI

The Non-liner Vibration Analysis of Elastic Shells.

The main purpose of the thesis is to make an extensive study on the application of "Constant Deflection Contour Method". In previous Chapters it has been shown that the application of this method, on isotropic elastic plate problems is effective and the analysis appears to be easier than the other existing methods.

To make further investigation the application of this method to shell structures will be made in this chapter. However the projection of the curve of intersection of the plane z = 0 with the structure must be known in all cases, how much complicated the structure may be.

Though several studies [8, 9,38,39,40,54,59,109,115] have been made on vibration analysis of shell structures in which most of the problems deal with linear analysis. The study of the non-liner analysis is very restrictive in nature and it appears that more detailed study on large amplitude vibration of shell structures should be made.

Regarding the application of " Constant deflection contour Method " on shell structures Mazumdar and Jones[115] had made some useful studies on shell structures. However, all of them concerned with the linear approach. Recently Banerjee [109] has made an attempt to extend the "Constant Deflection Contour" method to shell problems. But no specific illustration has been made in support of the theoretical study. This prompts the present investigator to make an attempt to do some work in this sphere in support of the applicability of " Constant Deflection Contour" method to shell problems involving statical or dynamical cases.

Let an elastic isotropic shallow shell be considered. The equation of its middle surface referred to a system of orthogonal Co-ordinates (x, y, z) represented by

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} + \frac{xy}{R_{12}}$$

For a shallow shell $r = (x^2+y^2)^{1/2}$ is small compared to the radii of curvature R_1, R_2, aid R_{12} everywhere in the region. R_1, R_2, aid R_{12} may be taken to be constant.

The well-known Von Karman equation, extended to a dynamic case for shallow shell may be written as [109]

$$D\nabla^4 w = h\mathcal{L}(F, w) + p + \rho h w_{,tt} - K_1 h F_{,yy} - K_2 h F_{,xx} \qquad [6\cdot1]$$

$$\nabla^4 F = -\frac{E}{2}\mathcal{L}(w, w) + EK_1 w_{,yy} + EK_2 w_{,xx} \qquad [6\cdot2]$$

K_1, and K_2 are principal curvatures.

If $u = u(x,y)$ defines the lines of equal deflection equations (6.1) and (6.2) becomes

$$D\left[A_1\frac{d^4w}{du^4} + A_2\frac{d^3w}{du^3} + A_3\frac{d^2w}{du^2} + A_4\frac{dw}{du}\right] = h\left[A_5\frac{dw}{du}\frac{dF}{du} + A_6\frac{d}{du}\left(\frac{dw}{du}\frac{dF}{du}\right)\right] - h\left[A_7\frac{dF}{du} + A_8\frac{d^2F}{du^2}\right] + p - \rho h w_{,tt} \qquad [6.3]$$

$$A_1\frac{d^4F}{du^4} + A_2\frac{d^3F}{du^3} + A_3\frac{d^2F}{du^2} + A_4\frac{dF}{du} = -\frac{E}{2}\left[A_5\left(\frac{dw}{du}\right)^2 + A_6\frac{d}{du}\left(\frac{dw}{du}\right)^2\right] + EA_7\frac{dw}{du} + EA_8\frac{d^2w}{du^2} \qquad [6.4]$$

Where

$$A_1 = \left(u_{,x}^2 + u_{,y}^2\right)^2$$

$$A_2 = 6\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy}\right) + 2\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx}\right) + 8u_{,x}u_{,y}u_{,xy}$$

$$A_3 = 4\left(u_{,x}u_{,xxx} + u_{,y}u_{,yyy}\right) + 4\left(u_{,x}u_{,xyy} + u_{,y}u_{,xxy}\right) + 2u_{,xx}u_{,yy} + 4u_{,xy}^2 + 3\left(u_{,xx}^2 + u_{,yy}^2\right)$$

$$A_4 = u_{,xxxx} + u_{,yyyy} + 2u_{,xxyy}$$

$$A_5 = 2u_{,xx}u_{,yy} - 2u_{,xy}^2$$

$$A_6 = \left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx} - 2u_{,x}u_{,y}u_{,xy}\right)$$

$$A_7 = K_1 u_{,yy} + K_2 u_{,xx}$$

$$A_8 = K_1 u_{,y}^2 + K_2 u_{,x}^2$$

Integrating equation (6.3) and (6.4) over area

$$\iint_{\Omega} D\left[A_1\frac{d^4w}{du^4} + A_2\frac{d^3w}{du^3} + A_3\frac{d^2w}{du^2} + A_4\frac{dw}{du}\right]d\Omega$$

$$= h\iint_{\Omega}\left[A_5\frac{dw}{du}\frac{dF}{du} + A_6\frac{d}{du}\left(\frac{dw}{du}\frac{dF}{du}\right)\right]d\Omega$$

$$- h\iint_{\Omega}\left[A_7\frac{dF}{du} + A_8\frac{d^2F}{du^2}\right]d\Omega - \iint_{\Omega}\rho h w_{,tt}\, d\Omega$$

$$+ \iint_{\Omega} p\, d\Omega \qquad [6\cdot5]$$

$$\iint_{\Omega}\left[A_1\frac{d^4F}{du^4} + A_2\frac{d^3F}{du^3} + A_3\frac{d^2F}{du^2} + A_4\frac{dF}{du}\right]d\Omega$$

$$= -\frac{E}{2}\iint_{\Omega}\left[A_5\left(\frac{dw}{du}\right)^2 + A_6\frac{d}{du}\left(\frac{dw}{du}\right)^2\right]d\Omega$$

$$+ E\iint_{\Omega}\left[A_7\frac{dw}{du} + A_8\frac{d^2w}{du^2}\right]d\Omega \qquad [6\cdot6]$$

For an elliptical shell the lines of equal deflection are represented by

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Where u=o defines the boundary and u=1 is the centre of concentric ellipses, where the deflection is maximum under an uniform load p

Deflection and stress functions are assumed to be

$$\begin{aligned} w &= AW(u)\psi(t) \\ F &= AF'(u)\psi^2(t) \end{aligned} \qquad [6\cdot7]$$

For such a variable u, equations (6.5) and (6.6), on integration reduce to

$$\begin{aligned} &2DP\left[(1-u)^2\frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2}\right]A\psi(t) \\ &+ \frac{8h}{a^2b^2}(1-u)\frac{dF'}{du}\frac{dW}{du}A^2\psi^3(t) \\ &+ 2h\left(\frac{K_1}{b^2}+\frac{K_2}{a^2}\right)(1-u)\frac{dF'}{du}A\psi^2(t) \\ &+ \rho h A\psi_{,tt}(t)\int_1^u W\,du + p(1-u) = 0 \end{aligned} \qquad [6\cdot8]$$

$$\begin{aligned} &P\left[(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2}\right]A\psi^2(t) \\ &= \frac{2E}{a^2b^2}(1-u)\left(\frac{dW}{du}\right)^2A^2\psi^2(t) \\ &+ E\left(\frac{K_1}{b^2}+\frac{K_2}{a^2}\right)(1-u)\frac{dW}{du}A\psi(t) \end{aligned} \qquad [6\cdot9]$$

$$\text{where } P = \frac{3a^4+3b^4+2a^2b^2}{a^4b^4}$$

$$\text{Let } W(u) = \sum_{i=2}^{\alpha} A_i u^i$$

Compatible with the boundary conditions for a clamped boundary, A_i's are evaluated. Let a rough approximation be considered with the first term only.

$W = Au^2$

With $W = Au^2$, the first integral of equation (6.9)

$$\left[(1-u)\frac{d^2F'}{du^2} - \frac{dF'}{du}\right]A\psi^2(t) = \frac{8E}{3a^2b^2P}u^3A^2\psi^2(t) + E\Upsilon u^2A\psi(t) + B_1 \qquad [6.10]$$

where $\Upsilon = \frac{1}{P}\left[\frac{K_1}{b^2} + \frac{K_2}{a^2}\right]$

While the second integral becomes

$$(1-u)\frac{dF'}{du} A\psi^2(t) = \frac{2E}{3a^2b^2P} u^4 A^2 \psi^2(t)$$
$$+ \frac{1}{3}\Upsilon E u^3 A\psi(t) + B_1 u + B_2 \qquad [6.11]$$

B_1 and B_2 are constant subject to immovable condition

$$\left[2(1-u)\frac{d^2F'}{du^2} - (1-\nu)\frac{dF'}{du}\right]_{u=0} = 0 \qquad [6.12]$$

Further equations (6.10) and (6.11) are valid for whole domain bounded by C_u then for u = 0

$$A\psi^2(t)\left[\frac{d^2F'}{du^2}\bigg|_{u=0} - \frac{dF'}{du}\bigg|_{u=0}\right] = B_1, \qquad [6{\cdot}13]$$

$$\frac{dF'}{du}\bigg|_{u=0} A\psi^2(t) = B_2 \qquad [6{\cdot}14]$$

For u = 1 , equation (6.11) becomes

$$\frac{2E}{3a^2b^2P} A^2\psi^2(t) + \frac{1}{3}\Upsilon E A\psi(t) + B_1 + B_2 = 0 \qquad [6{\cdot}15]$$

Solving equations (6.12) , (6.13), 6.14 and (6.15) one gets

$$B_1 = \frac{2E(1+\nu)}{3a^2b^2P(1-\nu)} A^2\psi^2(t) + \frac{1}{3}\Upsilon E \frac{(1+\nu)}{(1-\nu)} A\psi(t)$$

$$B_2 = -\frac{4}{3}\frac{E A^2\psi^2(t)}{a^2b^2P(1-\nu)} - \frac{2}{3}\frac{\Upsilon E}{(1-\nu)} A\psi(t)$$

and equation (6.11) reduces to

$$(1-u)\frac{dF'}{du} A\psi^2(t) = \frac{2E}{3a^2b^2P(1-\nu)}\left[(1-\nu)u^4 + (1+\nu)u - 2\right] A^2\psi^2(t)$$
$$+ \frac{1}{3}\frac{\Upsilon E}{(1-\nu)}\left[(1-\nu)u^3 + (1+\nu)u - 2\right] A\psi(t) \qquad [6.16]$$

Taking the first derivative of equation (3.8) in agreement with the proposed theory explained in chapter - V

$$2DP\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]A\psi(t)$$

$$+\frac{8h}{a^2b^2}\frac{d}{du}\left[(1-u)\frac{dF'}{du}\frac{dW}{du}\right]A^2\psi^3(t) + 2\Upsilon hP\frac{d}{du}\left[(1-u)\frac{dF'}{du}\right]A\psi^2(t)$$

$$+\rho hAW\psi_{,tt}(t) - p = 0 \qquad [6\cdot17]$$

Substituting equation (6.16) into equation (6.17) and using Galerkin procedure to minimize the error one gets.

$$\left[\frac{8}{3}DP + 0\cdot8162\Upsilon^2 EhP\right]A\psi(t) + 5\cdot142\frac{\Upsilon Eh}{a^2b^2}A^2\psi^2(t)$$

$$+ 7\cdot3651\frac{Eh}{a^4b^4P}A^3\psi^3(t) + \frac{1}{5}\rho hA\psi_{,tt}(t) = \frac{p}{3} \qquad [6\cdot18]$$

Equation (6.18) can be put in a simpler form

$$\psi_{,tt}(t) + C_1\psi(t) + C_2\psi^2(t) + C_3\psi^3(t) = C\cdot p \qquad [6\cdot19]$$

$$C_1 = \frac{5P}{\rho h}\left[\frac{8}{3}D + 0\cdot8126\Upsilon^2 Eh\right]$$

$$C_2 = 25\cdot71\frac{\Upsilon E}{\rho a^2b^2}A$$

$$C_3 = 36\cdot8255\frac{EA^2}{\rho a^4b^4P}$$

$$C = \frac{5}{3}\frac{p}{\rho hA}$$

Free linear vibration ;
For free vibration putting p = 0 in equation (6.19)

$$\psi_{,tt}(t) + C_1\psi(t) + C_2\psi^2(t) + C_3\psi^3(t) = 0$$

The linear frequency parameter is given by

$$\omega = C_1^{1/2} = \left[\frac{5P}{\rho h}\left(\frac{8}{3}p + 0.8126\, r^2 E h\right)\right]^{1/2}$$

$$\omega a^2\sqrt{\rho/Eh^2} = (3m^4+2m^2+3)^{1/2}\left[\frac{1.111}{(1-\nu^2)} + 1.0155\left(\frac{2r}{h}\right)^2\right]^{1/2}$$

The variation of $\omega a^2\sqrt{\rho/Eh^2}$ for elliptic circular shell are shown in Table (41)

b) Non-linear free vibration

For non-linear free vibration $p = 0$

$$\psi_{,tt}(t) + C_1\psi(t) + C_2\psi^2(t) + C_3\psi^3(t) = 0$$

If we designate ω^* and ω as the frequencies of non-linear and linear vibration respectively

$$\frac{\omega^*}{\omega} = \left[1 + \frac{3}{4}\frac{C_3}{C_1} - \frac{5}{6}\left(\frac{C_2}{C_1}\right)^2\right]^{1/2}$$

ξ is the non-dimensional relative amplitude

$$\frac{\omega^*}{\omega} = \left[1 + \frac{5.5238\, m^4\xi^2}{(3m^4+2m^2+3)^2\left[0.2442+0.2031\left(\frac{2r}{h}\right)^2\right]} - \frac{5.508\left(\frac{2r}{h}\right)^2 m^4\xi^2}{(3m^4+2m^2+3)^2\left\{0.2442+0.2031\left(\frac{2r}{h}\right)^2\right\}^2}\right]^{1/2}$$

for $\nu = 0.3$

Numerical results showing the variation of $\frac{\omega^*}{\omega}$ with relative amplitude $\xi = \frac{A_0}{h}$ for elliptical shell have been shown in the table (42–46)

C The static deflection of the same problem may be evaluated from equation (6.18) neglecting the inertial term as

$$\left[0.7326 + 0.6093\left(\frac{2r}{h}\right)^2\right](3m^4+2m^2+3)\left(\frac{W_0}{h}\right) + 7.713\left(\frac{2r}{h}\right)m^2\left(\frac{W_0}{h}\right)^2 + \frac{22.09\, m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3 = \frac{p a^4}{E h^4}$$

The numerical results showing the dependence of central deflection on load parameter are shown in tables [47 - 51]

Table : 41 Variation of linear frequency in units of corresponding flat-plate frequency for a complete spectrum of aspect ratio and ($\frac{2r}{h}$)

	m=1		m=1.5		m=2	
	Present Study	[115]	Present Study	[115]	Present Study	[115]
0	1.	1.05	1.68	1.77	2.71	2.85
0.5	1.099	1.15	1.88	1.93	2.98	3.12
1.0	1.35	1.40	2.279	2.36	3.67	3.80
1.5	1.694	1.75	2.85	2.95	4.60	4.75
2.0	2.08	2.16	3.50	3.64	5.64	5.87
3.0	2.91	2.99	4.90	5.73	7.91	8.12
5.0	4.66	4.51	7.68	7.59	12.69	12.25
10	9.17	7.62	15.44	12.83	24.91	20.69
20	18.26	14.1	30.21	23.74	49.61	38.29

Table: 42. Variation of $\frac{\omega^*}{\omega}$ on relative amplitudes $\frac{A_0}{h}$ for different values of m.

$\nu = 0.3$, $\frac{2r}{h} = 0$

	ω^*/ω		
A_0/h	$m = 1$	$m = 1.5$	$m = 2$
0	1.000	1.000	1.000
0.2	1.007	1.0044	1.0020
0.4	1.0272	1.0176	1.0082
0.6	1.0617	1.0392	1.0185
0.8	1.1073	1.0688	1.0327
1.0	1.1633	1.1056	1.0506
1.2	1.2283	1.1490	1.0722
1.4	1.3010	1.1982	1.0971
1.6	1.3801	1.2527	1.1251
1.8	1.4645	1.3117	1.1561
2.0	1.5535	1.3746	1.1897

Table : 43 Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3, \frac{2r}{h} = 1$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2.0
0	1.000	1.000	1.000
	0.9925	0.9953	0.9978
0.5	0.9699	0.9811	0.991
1.0	0.8734	0.9224	0.9648
1.5	0.6831	0.8153	0.9191
2.0	0.228	0.6356	0.8508
2.5		0.2622	0.7541

Table : (44) Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3, \frac{2r}{h} = 5$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2
0	1.000	1.000	1.000
.5	0.9925	0.995	0.9977
1.0	0.9700	0.9813	0.9914
1.5	0.8876	0.9562	0.9806
2.0	0.8740	0.9230	0.9654
2.5	0.7945	0.87678	0.9453

Table : 45 Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3, \frac{2r}{h} = 10$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2
0	1.000	1.000	1.000
.5	0.9979	0.9983	0.9994
1.0	0.9919	0.9935	0.9977
1.5	0.9818	0.9854	0.9946
2.0	0.9674	0.9740	0.9905
2.5	0.9486	0.9591	0.9852

Table : 46 Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3$, $\frac{2r}{h} = 20$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2
0	1.000	1.000	1.000
.5	0.9994	0.9996	0.9998
1.0	0.9979	0.9986	0.9993
1.5	0.9953	0.9970	0.9986
2.0	0.9917	0.9947	0.9975
2.5	0.9871	0.9981	0.9962

Table : 47 Dependence of Central Deflection $\frac{W_0}{h}$ on Load Parameter for different values of m

$\nu = 0.3$, $\frac{2r}{h} = 0$

	pa^4/Eh^4		
W_0/h	m=1	$m=1.5$	$m=2$
0	0	0	0
0.2	1.1742	3.3599	8.6918
0.4	2.4619	6.9636	17.6711
0.6	4.1725	11.03668	27.2251
0.8	6.1017	15.8192	37.6413
1.0	8.6206	21.5478	49.207
1.2	11.802	28.4587	60.4852
1.6	20.6818	46.7742	93.6747
2.0	33.818	72.6576	134.427

Table : 48 Dependence of Central Deflection on Load Parameter

$\nu = 0.3$ $\frac{2r}{h} = 1$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	2.4765	6.5565	16.4539
0.4	5.5716	14.3336	37.0076
0.6	9.804	23.478	59.905
0.8	14.922	34.248	86.147
1.0	21.203	42.974	116.0125
1.2	28.751	68.681	149.774
1.6	48.18	98.468	230.182
1.8	60.40	120.974	277.396
2.0	74.392	146.54	329.668

Table - 49 Dependence of Central Deflection on Load Parameter

$\nu = 0.3 \quad \frac{2r}{h} = 5$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	27.106	86.585	194.597
0.4	54.43	180.347	401.823
0.6	91.108	281.529	605.763
0.8	128.256	390.354	833.706
1.0	169.04	507.08	1102.15
1.2	213.563	613.906	1362.76
1.6	314.372	906.928	1926.47
2.0	431.78	1217.2	2494.76

Table - 50 Dependence of Central deflection on load parameter pa^4/Eh^4 for different values of m

$\nu = 0.3 \quad \frac{2r}{h} = 10$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	101.762	286.679	739.987
0.4	209.836	587.273	1504.943
0.6	324.336	902.61	2295.15
0.8	445.413	1232.30	3110.91
1.0	573.19	1576.9	3952.51
1.2	707.789	1935.11	4820.21
1.6	998.03	2495.9	6635.14
2.0	1317.2	3530.2	8557.92

Table - 51

Dependence of Central deflection $\left(\frac{W_0}{h}\right)$ on load parameter for different values m

$\nu = 0.3 \qquad \frac{2r}{h} = 20$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	397.19	569.91	2909.12
0.4	806.856	1167.83	5867.89
0.6	1229.126	1793.97	8876.59
0.8	1664.133	2448.5	11935.98
1.0	2112.02	3131.9	15045.6
1.2	2572.86	3844.147	18205.07
1.6	3534.2	5356.37	24666.8666
2.0	4549.12	6986.4	31360.6

Table (41) shows that the results tally well with those of Jones & Mazumdar [115] for lower values of measure of shallowness ($2r/h$). The present values of the liner frequency are little grater than these obtained in Reference [115] and the difference increases for higher values of ($2r/h$). This may be due to the rough approximation made for deflection function.

Table - (42) shows the values of relative frequency for different values of m (= aspect ratio) for ν = 0.3 and ($2r/h$) = 0 i.e. when the structure assumes that of a plate . From table (42) the frequency ratio (non-liner : liner) increases with the increase of the value of $\left(\frac{A_0}{h}\right)$ (relative amplitude) for all values of m. However the variation of (ω^*/ω) is not so significant for higher values of m which is a obvious expectation as m increases the structure behaves like a beam whereas the results of tables (43 – 46) in which the variation of the measure of shallowness has been considered and the effect is just the reverse what has been observed in the previous case (Table - 42). In these tables the value of relative frequency increases with the increase of $\frac{A_0}{h}$ and the effect is most significant for m = 1 and the effect of non-linearity decreases with the increase of $2r/h$ and m.

As it has always been treated to find the static behaviour of the structure in all the previous problems as a by-product of the ongoing analysis, tables (47- 51) show the dependence of central deflection W_0/h. on the load parameter pa^4/Eh^4. For $2r/h = 0$ the results for m = 1 are in excellent agreement with those of Yamaki [22]. The effect of the dependence become more significant with the increasing values of the aspect ratio m. However as the measure of shallowness ($2r/h$) increases the deflections are more significant for smaller values of m and it is a maximum for m = 1 when a particular value of the load parameter is concerned.