INTRODUCTION

The theory of thermo-elasticity is concerned with the influence of the thermal state upon the distribution of stress and strain and with the inverse effect, that of deformation upon the thermal state of an elastic medium , Duhamel [71] in 1838, initiated the subject deriving equation for the distribution of strain in an elastic medium containing temperature gradients. Subsequently, these results were rediscovered by several authors but Neumann gave the present form known as Duhamel-Neumann relations. The basic theory was applied by Duhamel to a number of problems and later used by Neumann and other authors as the basis of the detailed study. These investigations were instrumental in developing techniques for the solution of thermo-elastic problems but not until the present century did the subject received the practical stimulus. There have been a rapid development of thermo-elasticity stimulated by various engineering sciences in the post war years. A considerable progress in the field of air-craft and machine structures, mainly with gas and steam turbines, highway engineering especially in the preparation of air base, and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stress play an important role and frequently even a primary role.

For most practical problems the effect of the stresses and deformations upon the temperature distribution is quite small and can be neglected. The procedure allows the determination of the temperature distribution in the solid resulting from prescribed thermal condition to become first, and independent step of a thermal stress analysis; the second step is then the determination of the stresses and deformation of the body due to this temperature distribution. Before proceeding further, it will be worthwhile mentioning briefly equation of heat conduction in steady state and dynamic state of thermo-elasticity.

EQUATION OF HEAT CONDUCTION

Let in the space (X_r) a solid body B be bounded by the surface S and $T(X_r, t)$ denote the temperature at point (X_r) and at the time t. Then temperature differences between the points of the region B results in a flow of heat. Across a surface element $d\sigma$ at the point (X_r) the quantity of heat flowing in the time interval Δt is

$$\nabla Q = -\lambda T_{,n} \, d\sigma \, \Delta t$$

where λ is the coefficient of internal heat conduction, $T_{,n} = \dfrac{\partial T}{\partial n}$ is the normal derivative of the temperature at the point (X_r) of the surface element, in the direction of heat flow.

Now we investigate the equilibrium due to heat in a region B_1 bounded by S_1 constituting a part of B. The quantity of heat flowing into the region B_1 across the boundary S_1 in the time Δt is given by.

$$\Delta Q' = \lambda \int_{S_1} T_{,n} \, d\sigma \, \Delta t$$

If W denotes the quantity of heat generated in unit volume in unit time, then the quantity of heat generated inside the region under consideration then

$$\Delta Q'' = \int_{B_1} W \, d\sigma \, \Delta t$$

On the other hand, $\Delta Q = \Delta Q' + \Delta Q''$ can be determined from

$$\Delta Q = \int_{B_1} c \rho \dot{T} \, dv \, \Delta t$$

where ρ is the density and c is the specific heat of the body. The condition $\Delta Q = \Delta Q' + \Delta Q''$ implies the equation

$$\int_{B_1} (c\rho\dot{T} - W)\, dv - \lambda \int_{S_1} T_{,n}\, d\sigma = 0$$

which by divergence theorem becomes

$$\int_{B_1} (c\rho\dot{T} - W - \lambda T_{,kk})\, dv = 0$$

Since this is true for all arbitrary region B_1, hence

$$T_{,kk} - \frac{\dot{T}}{s} = -\frac{Q}{s} \tag{1}$$

where $s = \frac{\lambda}{\rho c}$, $W = Qc\rho$

We have used tensor notation, i.e.

$$T_{,i} = \frac{\partial T}{\partial X_i},\ T_{,kk} = \nabla^2 T$$

in a Cartesian coordinate system. Dots represent derivatives with respect to time.

Solution of equations (1) determine temperature as a function of position and time. If the temperature is independent of time and if there are no heat sources inside the region B, then (1) can be by Laplace equation

$$T_{,kk} = 0 \tag{2}$$

and hence in this case, temperature function is a potential function.

EQUATIONS OF THERM-ELASTICITY

Generation of stress and strain in a body take place due to non-uniform distribution of temperature. The temperature T represents the increment of the temperature from the initial stressless state. We assume that the change in temperature is small and therefore it has no influence on the mechanical and thermal properties of the body.

We shall confine ourselves to an isotropic homogeneous body with respect to both its mechanical and thermal properties. Let u_i $(i = 1, 2, 3)$ be the components of displacement vector $\vec{u}$, e_{ij} $(i,j = 1, 2, 3)$ be the components of strain tensor and σ_{ij} $(i, j = 1, 2, 3)$, the components of stress tensor.

In the linear theory of elasticity, the strain tensor e_{ij} is considered with the displacement vector by the relation

$$e_{ij} = \frac{(u_{i,j} + u_{j,i})}{2}, \quad i,j = 1, 2, 3 \tag{3}$$

The strain tensor is symmetric, *i.e.* $e_{ij} = e_{ji}$. The components of strain tensor can not be arbitrary, since they should have the following six relations- the so called compatibility conditions:

$$e_{ij,kl} + e_{kl,ij} - e_{jl,ik} - e_{ik,jl} = 0 \qquad i,j,k,l = 1, 2, 3 \tag{4}$$

which are satisfied identically if e_{ij} is expressed by u_i in accordance with (3) when the displacement field is continuous.

In therm-elasticity strain tensors are made up of two parts. The first part e^0_{ij} is a uniform expansion proportional to the temperature rise T. Since this expansion is the same in all directions for an isotropic body , only normal strains and no shearing strains arise in this manner. If α_t is the coefficient of linear expansion and δ_{ij} is the Kronecker's symbol, then

$$e^0_{ij} = \alpha_t \, T \, \delta_{ij}, \qquad i,j = 1, 2, 3 \tag{5}$$

The second part $e^{/}_{ij}$ comprises the strains required to maintain the continuity of the body as well as those arising because of external loads. These strains are related to the stresses by means of the Hooke's law of linear isothermal elasticity .

Hence

$$e'_{ij} = \frac{\left[\sigma_{ij} - \frac{\nu}{1+\nu}\Theta\,\delta_{ij}\right]}{2\mu_1}, \qquad i,j = 1, 2, 3. \tag{6}$$

where μ_1 is the shear modulus, ν is the Poission's ratio and $\Theta = \sigma_{kk}$ is the sum of the normal stresses. Hence finally we have

$$e_{ij} = e^0_{ij} + e'_{ij} = \alpha_t\, T\,\delta_{ij} + \frac{\left[\sigma_{ij} - \frac{\nu}{1+\nu}\Theta\,\delta_{ij}\right]}{2\mu_1}, \tag{7}$$

the so called Duhamel-Neumann relations.

Denoting $\theta = e_{kk}$, we have from (7)

$$\theta - 3\alpha_t T = \frac{1-2\nu}{E}\,\theta, \qquad E = 2\mu_1(1+\nu) \tag{8}$$

where E is the Young's modulus.

Solving (7) for stresses, we have

$$\sigma_{ij} = 2\mu_1\, e_{ij} + (\lambda\theta - \gamma T)\,\delta_{ij}, \qquad i,j = 1, 2\ 3 \tag{9}$$

where λ, γ are Lame's elastic constants given by the relations

$$\nu = \frac{\lambda}{2(\lambda + \mu_1)}, \quad \gamma = (3\lambda + 2\mu_1)\alpha_t$$

Now, it order to find the equations of elastic equilibrium, let us consider a body B with boundary S loaded in an arbitrary way and placed in a stationary temperature field . Let us consider the equilibrium of a sub-domain B_1 with boundary S_1. If F_i denotes the components of the body force per unit volume and P_i, the components of surface traction acting on the surface S_1 then from the condition of equilibrium we obtain the following three equations for the region B_1:

$$\int_{B_1} F_i\, dv + \int_{S_1} P_i\, d\sigma = 0, \qquad i = 1, 2, 3$$

Taking into account that $P_i = \sigma_{ij} n_j$, where n_j denotes the components of unit normal vector of surface S_1, we get, on making use of divergence theorem

$$\int_{B_1} (F_i + \sigma_{ij,j})\, dv = 0$$

Since this is true for an arbitrary region B_1, the equilibrium equations take the form

$$\sigma_{ij,j} + F_i = 0, \qquad i = 1,2,3 \tag{10}$$

If in these equilibrium equations, we express stresses by strains and then by displacements, we obtain a system of three equations in which the unknown functions are the components of displacement vector :

$$\mu_1\, u_{i,kk} + (\lambda + \mu)\, u_{k,ki} + F_i - \gamma T_{,i} = 0, \qquad i,k = 1,2,3 \tag{11}$$

In cylindrical coordinates (r, θ, z), let u, v, w represent the components of displacement vector $\vec{u}$; $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ represent normal stresses and $\tau_{rz}, \tau_{\theta z}, \tau_{r\theta}$ represent shear stresses. In the case of axial symmetry about the z-axis, equations (11) reduce to two equations

$$\nabla^2 u - r^{-2} u + \frac{1}{(1-2\nu)}\, \theta_{,r} - \frac{2(\nu+1)}{(1-2\nu)}\, \alpha_t\, T_{,r} = 0$$

$$\nabla^2 w + \frac{1}{(1-2\nu)}\, \theta_{,z} - \frac{2(\nu+1)}{(1-2\nu)}\, \alpha_t\, T_{,z} = 0 \tag{12a, b}$$

where $\theta = \dfrac{\partial u}{\partial r} + \dfrac{u}{r} + \dfrac{\partial w}{\partial z}$, $\qquad \nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$

To solve the equations (11) in the absence of body forces *i.e.* $F_i = 0$ Goodier [80] introduced a thermoelastic potential ϕ in terms of which the displacement vector is defined by the relation

$$u_i = \frac{\partial \phi}{\partial x_i} \tag{13}$$

and ϕ is a particular solution of the Poission's equation

$$\nabla^2 \phi = T(x_r) \tag{14}$$

A well known particular integral of (14) is

$$\phi(x_r) = -\frac{1}{4\pi} \int_v \frac{T(\xi_r)\, dV(\xi_r)}{R(x_r, \xi_r)} \tag{15}$$

where $R(x_r, \xi_r)$ is the distance between the points (x_r) and (ξ_r).

Integrals of the type (15) were employed by Borchardt [49] in a general discussion of the theory of thermoelasticity and also to solve certain special problems involving asymmetric distribution of temperature in solids with spherical and circular boundaries. Problems concerning spheres and cylinders are dealt in (PP.362-67) [37]. The problems of thin elastic plates, under fairly general distribution of temperatures have been considered by Galerkin [84], Nadai [29] Marguerre [100], Sokolnokoff [142] and Pell [124]. Several approximate solutions of the engineering problems concerned with thermal stresses in plates and rods are discussed in chapter-14 of Timoshenko and Goodier's "Theory of Elasticity" [37].

The calculation of the steady-state thermal stresses in an isotropic elastic half-space or slab with traction free faces has been the subject of several investigations. The distribution of thermal stress due to special temperature distribution in infinite and semi-infinite solids have been discussed by a number of authors, *viz.* Mindlin and Cheng [108], Myklested [110], Sternberg and Mc'Dowel [146] using an extension of Boussinesq - Papkowitch method of isothermal elasticity solved the problem of half-space. The basis of the method is that the solution of the equation of equilibrium (11) may be expressed in terms of the four Boussinesq-Papkowitch functions, one of which is the solution of Poisson's equation and remaining three are of Laplace equation. These equations have been studied extensively, particularly in potential theory, and general procedures of their solutions are known. Sneddon and Locket [139] approached this class of problems by direct solution of the equations of thermo-elasticity using a double Fourier integral transform method, the results being transformed to Hankel type integral in the case of axial symmetry. A further approach due to Nowinski [120] exploits the fact that in steady-state therm-elasticity each component of the displacement vector is a bi-harmonic function which can be expressed as a combination of harmonics. Possibly the most economical method of solutions of the type of problems is that of Williams [152] who expressed the displacement vector in terms of two scalar potential functions, one of which is directly related to the temperature field. Further, Muki [25] has introduced the displacement and stress components in the form of Hankel transform for the particular solution of the therm-elastic equations.

It is to note that Nowacki [28] has made thorough survey of the problems of both elasto-static and elasto-dynamic in presence of the temperature excellently.

ABOUT THE THESIS

In the present age of science and technology it is inevitable to have a study on the problems of thermo-elasticity because of the increasing range of applications of the theory and analysis of the thermal stresses in industry, and especially in advanced technologies such as Aerospace Engineering, Laser Engineering, Design of Turbines, Micro Electronics Industry. The subject has tremendous importance in compliance with its application in Geophysical and Seismological problems. The interest in this field of science has been increasing among mechanical engineers, semi-conductor engineers and chemical engineers.

The works of this thesis involve some important and interesting problems of thermoelasticity keeping in view of their desirability and applicability in the development of the modern technology. It is largely involved in the determination of basic and fundamental objectives *viz.* displacement, stresses and waves due to disturbances of temperature and their impact on the elastic body with endever to obtain results which will be important in applications to applied mathematics, engineering and technology. Here, both statical and dynamical problems of thermoelasticity have been dealt with. The complete work is divided into five Chapters. Problems in each chapter are relevant to each other.

In this dissertation work, Chapter-I contains two very useful problems of thermoelasticity. First one is an isotropic problem and the second one is anisotropic. In both the cases, thermal stresses have been derived in the elastic semi-space subject to heat exposure on the bounding surface of Isotropic and anisotropic media assuming that there are no heat sources inside the semi-space. If $T(x_1, x_2, 0) = f(x_1, x_2)$ is prescribed then the determination of the state of stress due to heating of the plane $x_2 = 0$ has been the subject of many investigations. E. Melan and H. Parkus [104] investigated the action of a concentrated heat source situated in the plane $x_3 = 0$ of a thermally insulated semi-space and proved that in this case a plane state of stress exists . The same conclusion was obtained by A. I. Lurye [99] who applied a different method, with respect to both semi-space and a layer. E. Sternberg and E. L. McDowell [146] in their investigation presented a solution to the problem by means of a method which was an extension of the Boussinseq-Papkowitch method [51] to thermal problems. A different procedure employing the Fourier Integral Transform was chosen by I. N. Sneddon and F. J. Lockett [139] . The solution can also be derived by introducing the themoelastic displacement potential and satisfying the boundary condition by means of the Love or Galarkin function [20].

Very few authors pay their attentions on the problems of superimposition of small deformation on large deformation due to temperature distribution on a surface of elastic body. Green, Rivlin and Shield [81], Green and Zerna [15] considered small deformation superposed on a large deformation of constant temperature. England and Green [76] extended to work to obtain general solution of the equations for the small superposed deformation and steady state temperature distribution in a compressible and incompressible body in terms of three stress functions. P. Choudhuri [56] considered on ideal elastic body deformed from a state of zero stress and strain and uniform temperature, subjected to small displacements and steady state temperature distribution with small deformation to be superposed on large deformation.

The general theory is also applied to the infinitesimal deformation of a thin sheet of incompressible isotropic elastic material which is first subjected to a finite pure homogeneous deformation by forces in its plane. A differential equation is obtained for the small defluxion of the sheet due to small forces acting normally to its face. This equation is solved completely in the case of a clamped circular sheet subjected to a pure homogeneous deformation having equal extension ratios in the place of the sheet, the small bending force being uniformly distributed over a face of the sheet. Finally, equations are obtained for the homogeneously deformed sheet subjected to infinitesimal generalized plane stress, and a method of solution by complex variable technique is indicated.

This paper deals with the body to be incompressible and small deformations have been superimposed on initial large deformations of the initially isotropic body. Assuming existence of temperature distribution within a circular region on the surface of the semi-infinite solid, the radial and cross-radial stress components are obtained when the distributed temperature is hemispherical or paraboloidal. Numerical results for the variation of $\tau_{\theta\theta}$ on the surface $z = 0$ have been deduced for Mooney type material. Further, the case when stresses become unstable due to two equal extension ratios exceeds the value 0.3 is discussed.

The second problem is a two dimensional thermoelastic steady-state boundary value problem. The objective of this paper is to derive stresses in a transversely isotropic semi-infinite elastic strip, temperature distribution due to the blow of jet flame being prescribed on the straight bounding edge while two parallel edges are in contact with rigid insulated surfaces. Basic equations are followed from Hearmon [17]. Numerical evaluation for the normal stress on the insulated edges have been found for a particular material Mapple wood [12] collecting experimental results from Smithsonain physical table [12].

Chapter - II Consists of two important problems of thermoelasticity concerning the determination of temperature distribution and thermal stresses in a semi-infinite medium on inertia effects due to time-dependent temperature. In a paper, Sternberg and Chakraborty [147] have discussed a case in inertia effects in a transient thermoelastic problem.

The first problem determines the distribution of temperature and stresses in a semi-infinite thin elastic rod when its free end is subjected to (a) an instantaneous rise in temperature (b) a transient temperature distribution.

The second problem also determines the distribution of temperature and stresses in a thin semi-infinite elastic plate when its boundary is subjected to (a) an instantaneous rise in temperature and (b) transient temperature distribution.

In those cases methods of operational calculus have been employed and exact solutions in terms of error functions have been arrived at.

Chapter - III includes two heat-flux boundary value problems of thermoelasticity. The thermal stress problem of a circular plate at zero temperature except for heated region on the plane face was considered by Nowacki [115]. The solution obtained by him satisfied the boundary conditions concerned with the stress on the edge surface in an approximate manner only. Das et. al [68] solved a problem of a thick plate of infinite radius of an isotropic material with stress-free edges subject to variable temperature distribution . The object of the first problem is to find the exact solution of the thermoelastic problem of a thick plate of infinite radius of an isotropic material, with stress- free edges subjected to two different temperature distributions. In the first case, we assume a constant flux of heat within a circular region of exposure, the exterior of the circular region being free from any flux of heat. Secondly, a paraboloidal distribution of temperature within the circular region is assumed, the exterior being insulated. Numerical calculation for the variation of resultant stress $(\widehat{rr} + \widehat{\theta\theta})$ on the free surface have been obtained in the second case. The second problem deals with the case of semi-infinite transversely isotropic solid. The problem of determining the steady state thermal stresses and displacements in a semi-infinite elastic medium bounded by a plane was treated by Sternberg and McDowell [146] by the use of Green's Functions. They proved that the stress field induced by an arbitrary distribution of surface temperature is plane and parallel to the boundary and obtained the solutions in closed forms for a circular region of exposure with uniform or hemispherical distribution of temperature. Similar problem was discussed by Sharma [133] by using integral transform methods. He discussed the same problem in case of transversely isotropic material. In a paper Sneddon and Lockett [139] discussed the same problem by using double Fourier transform methods and arrived at the same results. This paper deals with the determination of the steady-state thermal stresses and displacements in a semi-infinite elastic medium of transversely isotropic material. A general solution, corresponding to an arbitrary flux of heat has been obtained first. In particular, circular and parabolic flux of heat has been considered. Hankel transform and its inversion [35] are applied to determine temperature distribution. Numerical results of the variation of $[\sigma_\theta]$ on the free surface have been found for a particular material Magnesium [82].

Chapter - IV deals with two contact problems of thermoelasticity subject to heating effect on aelotropic bodies. Most of the investigations employ the classical equation of heat conduction which assumes that the flow of heat in an elastic material is independent of the variation of strain. Since this assumption is not concordant with the laws of thermodynamics, the effect of rate of strain on the flow of heat is to be taken into account . A modified heat conduction equation has now been considered by several workers in a number of problems of which the major portion is in three dimensions [146]. First problem is concerned with the determination temperature distribution and stresses in an infinite aelotropic thin plate with a circular hole due to sudden heating of the boundary of the hole. Stress-strain relations are written due to Sharma [134]. Modified heat conduction equation is used due to Nariboli [111]. The differential equations of heat conduction coupled with elastic deformation are solved simultaneously as they do not admit of independent-solution. Laplace transform method is applied as a mathematical tool. In the second problem, thermal stresses have been calculated when a long aelotropic cylinder is in contact with a hot ring on its outer surface. The investigation of the steady state of the problems has been made by Das [65]. When the material of the cylinder is isotropic stress-strain relations in the presence of temperature for long aelotropic cylinder are taken according to Sharma [134]. Numerical results for the stress on the boundary of the cylinder have been obtained when the material of the cylinder is made of Magnesium [82].

Chapter - V is concerned with two dynamical problems of therm-elasticity. In the first problem, components of displacement and stress are determined due to disturbance produced by a periodic heat nucleus and the second problem is on the generation of waves produced by an impulsive heat nucleus.

The problem of calculating the components of stress at a point in an elastic solid when it is deformed by the application of surface traction which vary with time is of considerable interest in soil mechanics, in the theory of foundations and in the branch of applied mathematics. There has been extensive discussion of the corresponding statical problems. But Sneddon [136] in his palerno lecture discusses dynamical problems of this type in a systematic way. Special problems have been solved by Lamb [92].

In a problem, Eason, Fulton and Sneddon [72] have dealt with the determination of distribution of stresses in an infinite elastic solid when the time dependent body force act upon certain region of the solid. Assuming strains to be small, the general solution of the equation of motion for any distribution of body forces is derived by the four-dimensional Fourier transforms [31] and from that general solution is derived for the isotropic solid. The solution of the equation of motion in the case in which the distribution of the body force is symmetrical about an axis is also derived . The solutions of some typical two dimensional and three dimensional problems are considered and exact analytical expressions are found for the components of displacement and stress. In the present discussion, at first detailed solution of the two dimensional therm-elastic problem is obtained and the distribution of displacement and stress have been derived when the time dependent body force and temperature act on certain region of the solid. Then the problem consists of deducing the displacements and stresses due to the disturbance produced by the insertion of a periodic heat nucleus in the solid. Weber's Bessels function [39] of the second kind is ultimately realized for those cases. Capitalising the above procedure another interesting two dimensional problem is taken into account to determine components of displacement and stress when an impulsive heat nucleus act in the solid. Dirac Delta functions [72] are utilized in the time dependent body force and temperature acting at the origin in order to obtain the solution of the equation of motion . In each case , strains are assumed to be infinitesimal so that the equations of the classical theory of elasticity [20] are applicable. Fourier transform technique [9] is applied in both the cases separately.