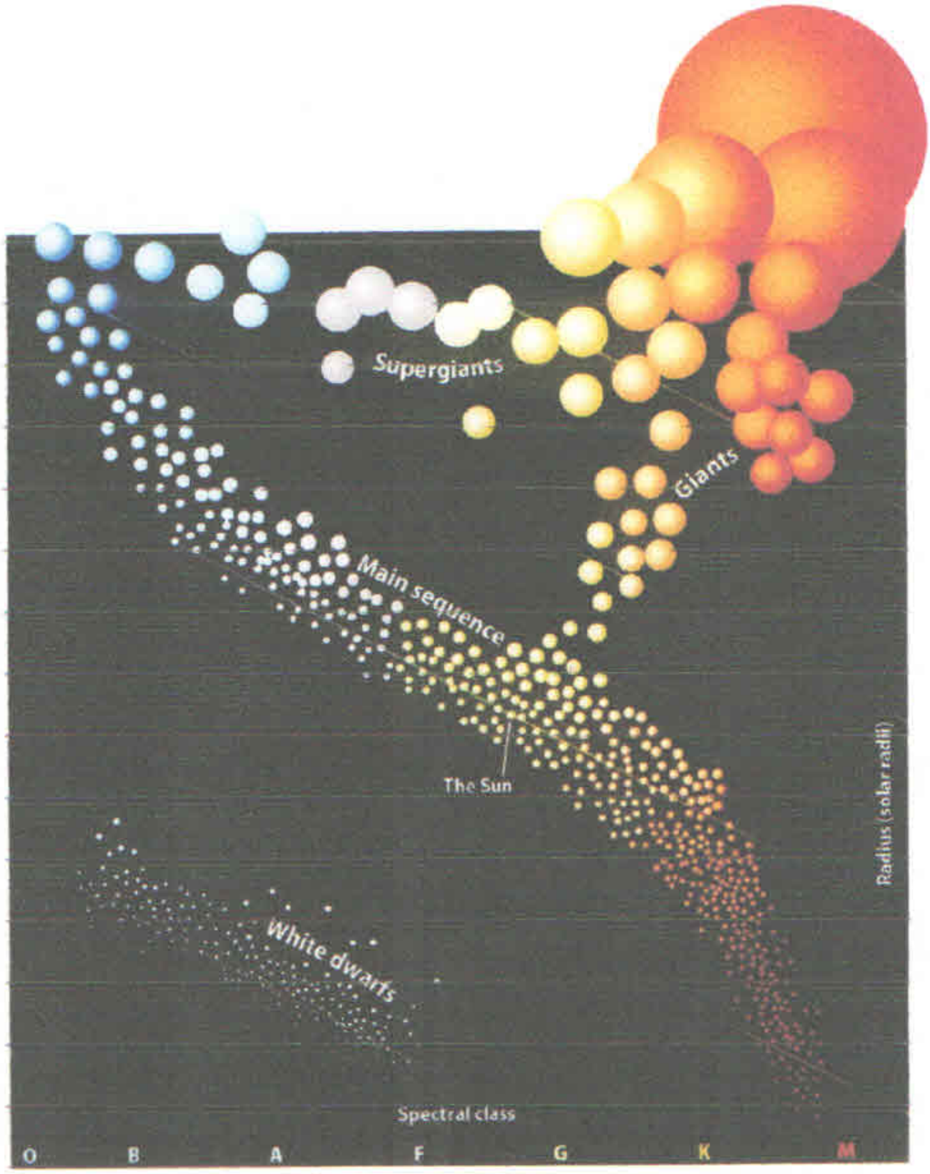
Chapter 1

Introduction

1.1 Compact Objects: Historical Background and recent developments

Normal Stars are gravitationally bound state of hydrogen gas where gravitational pull is balanced by the outward push developed due to thermonuclear fusion. When most of the fuel of a normal star is exhausted, it starts collapsing as it cannot withstand the gravitational pull. In 1910, a Danish chemist and astronomer Ejnar Hertzsprung (1873-1967) and American astronomer Henry Norris Russel (1877-1957) created the most famous graph in astronomy which plots stars brightness on the Y-axis against the spectral types on the X-axis which is known as H-R diagram. It allows us to learn a great deal about stellar evolution. Modern version of the H-R diagram replace the spectral type of the stars with a precise colour index which is known as colour magnitude diagram (CMD) shown in fig. (I). The patterns of stars on the



Supergiants
Giants
Main sequence
The Sun
White dwarfs
Radius (solar radii)
Spectral class
O
B
A
F
G
K
M

fig-I

H-R diagram provides the vital clue in understanding the stellar structure and their evolution. A normal star contracts appreciably from its original size thereby increasing the density of the star and at a sufficiently high density, a non-thermal pressure originates via degenerate fermions and particle interaction. Thus at the end of its evolution when nuclear fuel exhausts, a massive star collapses due to its own gravity. Depending on its mass the final outcome can be a white dwarf or a neutron star or a black hole [1, 2]. Under this circumstances the mass of a star is ($M < 1.24M_{\odot}$) (Chandrasekhar mass limit) [3], a stable star is formed where gravitational pull is balanced by the electron degeneracy pressure and is called white dwarf. Landau [4] anticipated the existence of a star with mass greater than $1.5M_{\odot}$ with density of matter so large that atomic nuclei come in close contact forming one gigantic nucleus made up of only neutrons with mass number $A > 10^{56}$. In 1932, discovery of neutron by Chadwick [5] and the interpretation of Heisenberg that neutrons are spin-$\frac{1}{2}$ particle lead to the extension of the idea of degenerate electron gas by Chandrasekhar, to that of neutron. If the mass of a star is greater than Chandrasekhar mass limit ($M > 1.24M_{\odot}$), the neutron degeneracy pressure then prevents further collapse of a star giving birth to a new star called neutron star. Baade and Zwicky in 1933 predicted the existence of neutron stars [6]. It is known that supernova represent the transition from ordinary stars to neutron stars, which in their final stages led to formation of an extremely closely packed neutrons, with very small radius and extremely high density nuclear matter. In 1939, Tolman, Oppenheimer and Volkoff [10] investigated spherically symmetric compact star with hydrostatic equilibrium and obtained a governing equation for understanding neutron stars. In radio observations, Hewish in 1962 observed pulsar in the Crab Nebula but it was unexplained. Later in 1967, Bell and Hewish discovered pulsars from observations which are actually a rotating neutron stars thus hypothesis turned into reality. In astrophysics, the term compact objects is used to refer collectively white dwarfs, neutron stars, other exotic dense stars, and black holes. The typical mass of a neutron star is $\sim$ $(1-3)$ $\mathrm{M}_{\odot}, radius$ $\sim$10 $\quad km$ with mean density $\rho{\sim}10^{14} gm/cc$. The compact objects belongs to a class of stars having extreme conditions of density and temperature which cannot be attained in the terrestrial laboratory. Due to enormous density range spanned by compact objects, their analysis involved a deep physical understanding of the structure of matter and the nature of fundamental interactions. Thus the structure, material compositions and the equation of state (EoS) for the internal matter of a compact star including its stability studies are subjects of fundamental importance in Astronomy and Astrophysics. In compact objects, however a very strong gravitational field may originate due to highly condensed state of matter present inside it. In such stars simple Newtonian description is not enough to investigate the properties of these objects. The criteria is determined by the factor $\frac{GM}{c^2}$, when the factor is $\sim$ $O(1)$ that appreciably large, Newtonian mechanics is not enough. The general theory of relativity is thus an important tool to study compact objects. The compact objects are essentially static over the life time of the universe which emerged out of their final stage of stellar evolution. Although EoS of neutron star matter below the saturation density of nuclear matter is relatively known, the EoS of matter at still higher densities are yet to be determined.

It is believed that the core of some compact objects might comprise of exotic states of matter such as condensate pions and kaons, and transition from hadronic matter to quark matter. Recent observations predicted that there are some pulsars whose estimated mass and radii are not compatible with the EoS of neutron stars. Consequently it is believed that there exist more compact stars composed of quark matter only which are called strange stars. To name a few candidates for such classes are X-ray pulsars namely, Her X-1, X-ray burster 4U1820-30, millisecond pulsar SAX J 1808.4-3658, X-ray sources 4U 1728-34, PSR 0943+10 and RX J 185635-3754. Witten [7], Farhi and Jaffe [14] theoretically shown the possibility of the existence of compact star composed of quark matters (strange quark) as the possible true ground state of hadrons. The matter density of such compact objects are normally above the nuclear density which have maximum mass and radius both less than that of neutron stars. On the basis of compactification factor ($u = \frac{M}{b}$, where M is the mass and b is the radius), the stars are classified as follows : (i) Normal stars: $u \sim 10^{-5}$ (ii) White dwarfs: $u \sim 10^{-3}$ (iii) Neutron stars: $u \sim 0.1 \; to \; 0.2$ (iv) Strange stars: $u \sim 0.2 \; to < 0.5$ (v) Black holes: $u = 0.5$

Relativistic models of stars in hydrostatic equilibrium have been proposed ever since the vacuum solution of Einstein field equations obtained by Schwarzchild in 1916. The discovery of pulsars which are known as rotating neutron stars, are investigated theoretically in the framework of general relativity to study the structure and properties of matter in terms of parameters of the theory. It may be pointed out here that the conventional approach for obtaining stellar models in GR is to prescribe an EoS *i.e.* $p = p(\rho)$ where p and ρ represent pressure and density respectively to solve the general relativistic field equations. Tolman [9], Oppenheimer and Volkoff

[10] (TOV equation) obtained relativistic solutions for a static fluid distribution of Neutron star in gravitational equilibrium. Buchdhal [11] and Bondi [12] generalized the interior Schwarzchild solution of a general static fluid sphere making use of mass concentration, central density and pressure at the surface. Further relativistic solutions are used to explore suitable EoS for the interior matter of neutron stars. The follow up work of Bodmer [13] and Witten [14] on compact objects having ultra nuclear densities with a possibility of strange quark inside indicates that they may exist in nature. In the case of superdense stars apart from conventional approach based on TOV equations, it may be worthwhile to explore an alternative approach which does not require prior information of EoS of matter present inside the star. In the literature a number of species of compact objects with very high densities are discovered in recent times [15]. The theoretical investigation on compact astrophysical object in relativistic astrophysics spans over a few decades, till today we do not have a complete understanding on these objects in nature. Ivanov [16] has shown that pressure inside the compact objects is isotropic to begin with if one considers perfect fluid. In general a polytropic EoS is also used widely to describe the structure of a white dwarf [17] or a less compact stars. However theoretical understanding with observations made it clear that a deviation from local isotropy for the fluid pressure inside the star may exist. It has been shown [18, 19, 20] that at very high enough density with small size, anisotropic pressure plays an important role in determining stellar properties at various physical conditions [21, 22]. In a star with anisotropic pressure the radial pressure (p_r) is different from that of the tangential pressure ($p_\perp$). It has been noted [23, 24] that anisotropy may arise from different kinds of phase transitions or pion

condensation. Chan, Herrera and Santos [26] have studied the role played by the local pressure aniosotropy on the onset of instabilities and showed that a small anisotropy might in principle drastically change the stability of the system. Dev and Gleiser [27] obtained exact relativistic solution in an anisotropic star and found that $p_\perp > p_r$. It is shown [28] that an anisotropic star become more stable compared to an isotropic star. It is also shown that the transverse pressure accommodates a star with higher compactness than that obtained by Buchdahl-Bondi bound with perfect fluid [11, 12]. An exact solution of Einstein-Maxwell equations is obtained [29] where anisotropic pressure is shown due to the consequence of charge fluid distribution. The role of charge in a relativistic quark star was considered by Mak and Harko [25]. Ivanov [16] obtained a bound on surface red-shift parameter for a realistic anisotropic star. A number of papers appeared where the role of pressure anisotropy on anisotripic stellar structure have been investigated [20, 21, 30]. Sharma and Maharaj [31] investigated role of anisotropy on the surface tension of strange star.

The pulsars are rotating neutron stars and it behaves like a dipole emitting electromagnetic radiation. The anisotropy within a highly compact stars may be due to the presence of strong electric field [32]. One might argue for the occurance of stable charged astrophysical compact object in nature [33] as a result of all macroscopic bodies with charge neutrality or with a small amount of charge so that structure of star is not much affected in the later stages of this evolutions. In the early phases of evolution of compact stars, *e.g.,* right at the birth from the core collapse supernova, a charge neutrality may not be attained immediately and the presence of electromagnetic field has been shown to leave huge effect in the star. Ray *et al.* [34, 35]

shown that although huge charge which disrupt the structure of the star due to the balance of forces and the strength of their coupling virtually has no effect in the EoS of matter. Although the mass of many of the compact stars are determined with a fair precision, the accurate determination of radius is the main problem. As an alternative to neutron star models, strange stars have been considered in the studies of compact relativistic astrophysical bodies. Similar to neutron stars, strange stars are also considered likely to emerge from the core-collapse of a massive star during a supernova explosion, or during a primordial phase transition where quarks clump together. Another hypothesis is that an accreting neutron star in a binary system, can accrete enough mass to induce a phase transition at the centre or the core, to become a strange star. In the case of a nucleus with quarks the MIT Bag model is proposed which can be extended in the case of a compact object. However to study strange stars making use of the MIT Bag model at high densities is still an open issue. Chodos and Detweiler [36] used the phenomenological MIT Bag model, where they assumed that the quark confinement is caused by a universal Bag pressure at the boundary of any region containing quarks, namely the hadrons. The equation of state describing the strange matter in the bag model was obtained by Witten [14]. Weber [37] has shown that for a stable quark matter, the Bag constant is restricted to a particular range. It is remarkable to note that a strange matter equation of state seems to explain the observed compactness of many astrophysical bodies such as HER X-1, 4U 1820-30, SAX J1808.4-3658, 4U 1728-34, PSR 0943+10, and RX J185635 as pointed out by Paul *et al.,* [38], Rahaman *et al.* [39, 40]. Dey *et al.* [41] studied strange stars by assuming an interquark vector potential originating from gluon exchange and a density dependent scalar potential which restores chiral symmetry at a high density. In the new approach of Dey *et al.* [41] formulation, the equation of state can be approximated to a linear form. If pulsars are modelled as strange stars, the linear equation of state might also be a possibility as established in the analysis of Sharma and Maharaj [31]. For a given central density or pressure, the conservation equation can be integrated to compute the macroscopic features such as mass and radius of a star. In some stars where the densities inside the star are beyond nuclear matter density, the matter anisotropy may play a crucial role, as the conservation equations are modified. In Ref. [32] modelling of strange star in the presence of strong electric field has been considered which represents anisotropic pressure distribution.

The search for the exact solutions describing static isotropic or anisotropic stellar type configurations has continuously attracted the interest of physicists. A number of exact interior solutions (for isotropic or anisotropic) of the gravitational field equations that satisfy the required general physical conditions inside the star have been reported in the literature. Out of 127 published solutions, Delgaty and Lake [42] analyzed them and found only 16 solutions satisfy all the conditions necessary for a physically viable stellar model. The study of the interior of general relativistic stars *via* finding exact solutions of the field equations is still an open active field of research in Astrophysics.

Since the pioneering work of Bowers and Liang [22] there is an extensive literature devoted in the study of anisotropic spherically symmetric static general relativistic configurations. The theoretical investigations of Ruderman [18] about more realistic stellar models show that the nuclear matter may be anisotropic at least in high density ranges ($> 10^{15}$ g/cm^3), where it is important to treat the nuclear interactions relativistically. Recent developments due to observational astrophysical data, dense matter and its impact on global properties are studied in the literature . Gangopadhyay *et al.* [43] studied the density dependent quark masses using modified Richardson potential in the presence of chiral symmetry restoration and found that the observed mass of the pulsars can be used to predict the radii by fitting the strange star EoS. It is expected that in the near future accurate estimation of mass and radii of compact objects from observations will help us to explore matter composition and the EoS exatly. A number of observed space mission that are coming up to study astrophysical objects are expected to enrich the knowledge in this direction. Thus compact stars can be recognised as promissing laboratories for studying matter under extreme conditions. Theoretical modelling of compact objects in the framework of general relativity is thus useful to study the properties and structures of these objects.

1.2 Methodology

The compact stars are relativistic objects, so we consider Einstein Field equations which is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4}\, T_{\mu\nu} \tag{1.1}$$

where $g_{\mu\nu}$, G, c, $R_{\mu\nu}$, R and $T_{\mu\nu}$ are metric tensor, Newton's gravitational constant, velocity of light, Ricci tensor, Ricci scalar and energy-momentum tensor respectively. For a static spherically symmetric configuration we consider the metric which is given by

$$ds^2 = A^2(r)c^2dt^2 - B^2(r)dr^2 - r^2(d\theta^2 + sin^2\theta\; d\phi^2) \tag{1.2}$$

where A and B are arbitrary functions dependent only on r. The Einstein field equations for metric eq.(1.2) are reduced to the following system

$$\frac{8\pi G}{c^4}T_0^0 = -\frac{1}{B}\left[\frac{1}{r^2} - \frac{B'}{rB}\right] + \frac{1}{r^2}, \tag{1.3}$$

$$\frac{8\pi G}{c^4}T_1^1 = -\frac{1}{B}\left[\frac{A'}{rA} + \frac{1}{r^2}\right] + \frac{1}{r^2} \tag{1.4}$$

$$\frac{8\pi\ G}{c^4}T_2^2 = \frac{8\pi G}{c^4}T_3^3 = -\frac{1}{2B}\left[\frac{A''}{A} - \left(\frac{A'}{A}\right)^2 + \frac{1}{2}\left(\frac{A'}{A}\right)^2 + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{2}\frac{A'B'}{AB}\right] \tag{1.5}$$

where A' denotes differentiation with respect to r.

In the other case for a spherically symmetric configuration of a star in isotropic coordinate system, the metric is given by

$$ds^2 = A^2(r,t)\ \ c^2\ \ dt^2 - B^2(r,t)(dr^2 + r^2(d\theta^2 + sin^2\theta\ d\phi^2)) \tag{1.6}$$

The Einstein field equations corresponding to the line element eq.(1.6) are obtained as

$$\frac{8\pi G}{c^4}T_0^0 = -\frac{1}{B^2}\left(\frac{2B''}{B} + \frac{4B'}{rB} - \frac{B'^2}{B^2}\right) + \frac{3\dot{B}^2}{A^2B^2}, \tag{1.7}$$

$$\frac{8\pi G}{c^4}T_1^1 = \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2B'}{rB} + \frac{2A'B'}{AB} + \frac{2A'}{rA}\right) + \frac{1}{A^2}\left(-\frac{2\ddot{B}}{B} - \frac{2\dot{A}\dot{B}}{AB}\right) \tag{1.8}$$

$$\frac{8\pi G}{c^4}T_2^2 = \frac{8\pi G}{c^4}T_3^3 = -\frac{2\ddot{B}}{A^2B} + \left(\frac{2\dot{A}\dot{B}}{A^3B} - \frac{\dot{B}^2}{A^2B^2} + \frac{A'}{rAB^2} + \frac{B'}{rB^3}\right) + \frac{A''}{AB^2} - \frac{B'^2}{B^4} + \frac{B'''}{B^3}. \tag{1.9}$$

$$\frac{8\pi G}{c^4}T_0^1 = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{\dot{B}B'}{B^2} + \frac{A'\dot{B}}{AB}\right). \tag{1.10}$$

The general form of energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p_\perp)u_\mu u_\nu + p_\perp g_{\mu\nu} + (p_r - p_\perp)X_\mu X_\nu + q_\mu u_\nu + q_\nu u_\mu - \zeta\Theta(g_{\mu\nu} + u_\mu u_\nu). \tag{1.11}$$

where ρ is the energy density of the fluid, p_r is the radial pressure, $p_\perp$ is the tangential pressure, $q^\mu_\nu = q\ \ \delta^\mu_\nu$ is the radial heat flux vector, X^μ is an unit-four vector along the radial direction and u^μ is the 4-velocity of the fluid and $u^\mu q_\mu = 0, X_\mu X^\mu = 1, X_\mu u^\mu = 0$.

- Here perfect fluid condition can be recovered for $p_r = p_\perp$, $q_{\mu\nu} = 0$, $X^\mu = 0$ and $\zeta = 0$.
- For anisotropic conditions $p_r \neq p_\perp$ $q_{\mu\nu} = 0$ $X^\mu = 0$ and $\zeta = 0$.
- For viscous fluid with anisotropic conditions $p_r \neq p_\perp$ $q_{\mu\nu} = 0$ $X^\mu = 0$ and $\zeta \neq 0$.

In eq.(1.11), the coefficient of viscosity is taken positive *i.e.,* $\zeta > 0$.

1.2.1 Anisotropic Fluid

The energy momentum tensor for an anisotropic fluid is given by

$$T_{\mu\nu} = (\rho + p_\perp)\ u_\mu\ u_\nu + p_\perp\ g_{\mu\nu} + (p_r - p_\perp)\ X_\mu\ X_\nu \tag{1.12}$$

where ρ represents the energy-density and p_r is radial pressure and $p_\perp$ is transverse pressure.

1.2.2 Perfect Fluid

The Energy momentum tensor of the perfect fluid is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + g_{\mu\nu} p. \tag{1.13}$$

where ρ is the density and p is the pressure.

1.2.3 Maxwell's Electromagnetic field

In the case of charged star, the electromagnetic field $F^{\mu\nu}$ satisfies Maxwell equations :

$$\nabla_{\mu}F^{\mu\nu} = 4\pi J^{\nu} \tag{1.14}$$

and

$$F_{\alpha\beta,\gamma} = 0 \tag{1.15}$$

where J^{ν} represents four current. To describe a charged compact object we consider an electric charge at rest in comoving coordinates of the fluid. The four potential and the current are given by

$$A_{\mu}(t,r) = A(t,r)\delta^{0}_{\mu} \qquad J^{\nu} = \rho_{EM}(t,r)u^{\nu} \tag{1.16}$$

The covariant form of the electric field is

$$E_{\mu} = F_{\mu\nu}u^{\nu} \tag{1.17}$$

The corresponding magnetic feild is

$$B_{\rho} = \frac{1}{2}\epsilon_{\rho\mu\nu\sigma}u^{\mu}F^{\nu\sigma} \tag{1.18}$$

where $\epsilon_{\rho\mu\nu\sigma}$ represents four dimensional totally antisymmetric volume element. In the thesis we consider electromagnetic field with $B_{\rho} = 0$. Because of spherical symmetry the only non-vanishing component of electromagnetic field is $F^{01} = -F^{10}$.

1.2.4 Stellar Model

Usually to solve Einstein field equation given by eq.(1.1) one must know the EoS for determining the geometry in addition to the usual approach by TOV equation where EoS is to be required apriori. In relativistic astrophysics the EoS of matter in a compact star at extreme condition is not yet known clearly. In such case one can study the properties of compact star for a given known geometry. In the thesis the relativistic stellar models are obtained by the former approach. In this case the metric potentials A and B are determined using eqs.(1.3)- (1.5) or eqs.(1.7)- (1.10) . To obtain stellar model we begin with an ansatz for one of the metric functions $B(r)$ and determine the other metric function making use of the Einstein field equation. For a static compact star with isotropic as well as anisotropic matter distributions the relativistic solution for the interior geometry is matched with the Schwarzschild line element at the boundary $r = b$.

(i) The Schwarzschild metric in anisotropic form is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.19}$$

where $d\Omega^2 = (d\theta^2 + Sin^2\theta d\phi^2)$ and M represents mass of the star $(c = 1)$.

(ii) The Schwarzschild metric in isotropic form $(c = 1)$ is given by

$$ds^2 = \left(\frac{1 - \frac{M}{2b}}{1 + \frac{M}{2b}}\right)^2 dt^2 - \left(1 + \frac{M}{2b}\right)^4 (dr^2 + r^2 d\Omega^2) \tag{1.20}$$

(iii) For a static compact star with electromagnetic field the relativistic solution for the interior geometry is matched with the first fundamentals of the Reissner-Nordström line element at the boundary $r = b$ which is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + Sin^2\theta d\phi^2). \tag{1.21}$$

for $c = 1$ and q represents charge.

A number of relativistic solutions that are obtained from Einstein field equations may not be relevant for constructing physically viable stellar model. For this we test the model to determine the square of the sound speed *i.e.*, $\frac{dp}{d\rho} < 1$. This condition is used to probe whether the model is causally viable or not.

1.3 Vaidya Tikekar Formalism

Vaidya and Tikekar assumed an ansatz for a spheroidal geometry given by the metric function $B(r)$ in eq.(1.2) which is given by

$$B^2(r) = \frac{1 + \lambda \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \tag{1.22}$$

where λ and R are two arbitrary parameters. For an anisotropic matter distribution of a static compact object the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p_\perp)\;\; u_\mu\;\; u_\nu + p_\perp\;\; g_{\mu\nu} + (p_r - p_\perp)\;\; X_\mu\;\; X_\nu \tag{1.23}$$

where u_μ is the fluid four-velocity, n_μ is radially directed unit space-like vector, ρ is the energy-density , p_r is the radial pressure and $p_\perp$ is the tangential pressure. Now Einstein field equation for the metric given by eq.(1.2) can be rewritten as

$$\rho = -\frac{1}{B}\left(\frac{1}{r^2} - \frac{B'}{rB}\right) + \frac{1}{r^2} \tag{1.24}$$

$$p_r = -\frac{1}{B}\left(\frac{A'}{rA} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{1.25}$$

$$p_\perp = -\frac{1}{2B}\left[\frac{A''}{A} - \left(\frac{A'}{A}\right)^2 + \frac{1}{2}\left(\frac{A'}{A}\right)^2 + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{2}\frac{A'B'}{AB}\right] \tag{1.26}$$

where we use Gravitational unit $G = \frac{1}{8\pi}$ and $c = 1$. Now using eq.(1.25) and (1.26) one gets

$$2B\Delta = \frac{A''}{A} - \frac{1}{2}\frac{A'^2}{A^2} - \frac{A'}{rA} - \frac{B'}{rB} - \frac{A'B'}{2AB} - \frac{2}{r^2} - \frac{2}{r^2}(1-B) \tag{1.27}$$

where we denote the measure of pressure anisotropy $\Delta = p_\perp - p_r$. Using the anstaz eq.(1.22) it is possible to solve eq.(1.27) to determine the potential $A(r)$. It may be pointed out here that for an isotropic fluid distribution $\Delta = 0$ and the metric potentials are then determined accordingly which corresponds to isotropic fluid distribution which will be also taken up here. We consider two categories of geometries for the interior space time in the next section.

Category I: Spheroidal Geometry

The ansatz of a spheroidal geometry is given by

$$B^2(r) = \frac{1 + \lambda\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \tag{1.28}$$

where λ is a spheroidicity parameter and R is curvature parameter related with the configuration of a star model. It corresponds to a spheroidal geometry. A 3-spheroid immersed in a four dimensional Euclidean space is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{1.29}$$

where it satisfies the Cartesian equation given by

$$\frac{x_4^2}{H^2} + \frac{(x_1^2 + x_2^2 + x_3^2)}{R^2} = 1 \tag{1.30}$$

where H and R are constants. The section $x_4 = constant$ leads to a sphere. On the other hand $x_1 = constant$, $x_2 = constant$ and $x_3 = constant$ leads to hyperboloid of two sheets. Considering the following parametrization

$$x_1 = rsin\theta cos\phi, \qquad x_2 = rsin\theta sin\phi, \quad x_3 = rcos\theta, \qquad x_4 = H\sqrt{1 - \frac{r^2}{R^2}}$$

in the metric given by eq.(1.29) , the metric on the 3-spheroid takes the form

$$ds^2 = \left(\frac{1 + \lambda \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \right) dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.31}$$

where $\lambda = 1 - \left(\frac{H}{R}\right)^2$. The above spheroidal 3-space is spherically symmetric and regular everywhere. In this case $\lambda = -1$ corresponds to flat space and $\lambda = 0$ corresponds to hyperboloid.

Category II: Pseudo-Spheroidal Geometry

The ansatz of a pseudo-spheroidal geometry is given by

$$B^2(r) = \frac{1 + \lambda \frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}} \tag{1.32}$$

where λ is a spheroidicity parameter and R is curvature parameter related with the configuration of a star model. A 3-spheroid immersed in a four dimensional Euclidean space is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{1.33}$$

where it satisfies the Cartesian equation given by

$$\frac{x_4^2}{H^2} - \frac{(x_1^2 + x_2^2 + x_3^2)}{R^2} = 1 \tag{1.34}$$

where H and R are constants. The section $x_4 = constant$ leads to a sphere. On the other hand $x_1 = constant$, $x_2 = constant$ and $x_3 = constant$ leads to hyperboloid of two sheets. Considering the following parametrization

$$x_1 = rsin\theta cos\phi, \qquad x_2 = rsin\theta sin\phi, \quad x_3 = rcos\theta, \qquad x_4 = H\sqrt{1 + \frac{r^2}{R^2}}$$

in the metric given by eq. (1.33) , the metric on the 3-pseudo-spheroid takes the form

$$ds^2 = \frac{1 + \lambda \frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.35}$$

where $\lambda = 1 + \left(\frac{H}{R}\right)^2$. The above pseudo-spheroidal 3-space is spherically symmetric and regular everywhere. In this case $\lambda = -1$ corresponds to flat space and $\lambda = 0$ corresponds to a hyperboloid.

1.3.1 Tolman-Oppenheimer-Volkoff (TOV) model

Relativistic stellar models are obtained by using the conventional approach prescribing EoS *i.e.*, $p = p(\rho)$ first with a given central density to solve general relativistic TOV equation. The TOV equation is given by

$$-\frac{dp}{dr} = \frac{1}{r^2}(\rho + p)\left(2M(r) + pr^3\right)\left(1 - \frac{2M(r)}{r}\right)^{-1} \tag{1.36}$$

$$\frac{dM(r)}{dr} = \frac{1}{2}\rho(r)r^2 \tag{1.37}$$

where we use gravitational unit $G = \frac{1}{8\pi}$, $c = 1$. TOV solution may be obtained for a given EoS $p = p(\rho)$. Using the initial condition that $M(r = 0) = 0$ for a given central density and the radius of the star is determined from the conditions that the pressure at the boundary vanishes. The pressure is non zero at the center which decreases and finally tends to zero at the surface. Thereby mass to radius ratio of the star can be studied for various central densities. The TOV approach is very useful for studying the properties of compact stars having nuclear densities. A number of papers came up with considerable results for neutron stars. However for the stars having densities more than nuclear density, the composition as well as EoS becomes very uncertain. Construction of such stellar models with exotic composition have already been considered. However to solve TOV equation for a given compact object prior knowledge of EoS is required, which make the exercise difficult and atmost tentative. In the case of superdense star, it may be worthwhile to explore an alternative approach in addition to conventional approach which does not require prior knowledge of EoS [11, 44, 45, 46]. Vaidya-Tikekar gave a prescription to explore stellar models for compact objects in this direction. Hence the equation of state can be computed from the resulting metric. As expected with Tolman method, unphysical pressure-density configurations are found more frequently than physical ones. Hence a new solution which should be regular, well behaved, and can reasonably model a compact astrophysical stellar object is always acceptable.

1.3.2 Vaidya-Tikekar Model: (An alternative approach)

In Vaidya-Tikekar model, for a given ansatz for the geometry of the 3-space (or higher dimensional space) one can determine an appropriate EoS for the matter in the cold compact star using Einstein field equations. This approach is very convenient when the EoS of the matter inside the compact object is unclear. The method is an elegant and simple by the choice of 3-geometry which permits the exact solution of Einstein field equations. The shape of the compact object may be different from the spherical shape which has been used here to determine the EoS. In this prescription one may consider a given ansatz. In the case of spheroidal geometry

$$B^2(r) = \frac{1 + \lambda \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \tag{1.38}$$

where λ and R are two parameters which specify the 3-geometry or (D-1)-dimensional geometry at the $t = constant$ hyper surface representing a spheroidal geometry embedded in a 4-dimensional (D-dimensional) Euclidean space. The parameters will eventually determines the properties of the constituent matter. Since one of the metric function is assigned using eq.(1.27) for isotropic pressure, one obtains a second order differential equation for $A(r) = \Psi(r)$, which is given by

$$(1 + \lambda - \lambda x^2)\Psi_{xx} + \lambda x \Psi_x + \lambda(\lambda + 1)\Psi = 0 \tag{1.39}$$

where $x^2 = 1 - \frac{r^2}{R^2}$ in 4-dimensions and Ψ_x represents the derivative of Ψ with respect to x. The solution of the above eq.(1.39) was obtained by Vaidya and Tikekar for $\lambda = 2$, by Tikekar for $\lambda = 7$. Maharaj and Leach [47] obtained solutions for a set of values of λ. Eq. (1.39) further can be reduced to a simple second order differential equation using the transformation $z^2 = \frac{\lambda}{\lambda+1}\left(1 - \frac{r^2}{R^2}\right)$ which is given by

$$(1 - z^2)\Psi_{zz} + z \;\; \Psi_z + (\lambda + 1)\Psi = 0 \tag{1.40}$$

where Ψ_z is the derivative of Ψ *w.r.t* z and Ψ_{zz} is the differentiation of Ψ_z *w.r.t* z once again. Mukherjee, Paul and Dadhich [48] obtained a general solution to the above eq.(1.40) which is given by

$$\Psi = \Psi_0 \left[\frac{cos[(n_0 + 1)\xi + \delta]}{n_0 + 1} - \frac{cos[(n_0 - 1)\xi + \delta]}{n_0 - 1}\right] \tag{1.41}$$

where $\xi = cos^{-1}z$, $n_0^2 = \lambda + 2$, ψ_0 and δ are constants to be determined by matching the solution at the boudary $r = b$ with the exterior vaccuum solution of Schwarzschild. In the case of a compact star we impose the following conditions:

- At the boundary of the star the interior solution is matched with the exterior solution, *i.e.,*

$$A^2|_{r=b} = B^{-2}|_{r=b} = \left(1 - \frac{2M}{b}\right) \tag{1.42}$$

- At the boundary of the star p_r should vanish.

The physical properties namely, energy density (ρ), pressure (p) of a general relativistic star are given by,

$$\rho = \frac{1}{R^2(1-z^2)}\left[1 + \frac{2}{(\lambda+1)(1-z^2)}\right] \tag{1.43}$$

$$p = -\frac{1}{R^2(1-z^2)}\left[1 + \frac{2z}{(\lambda+1)}\frac{\psi_z}{\psi}\right] \tag{1.44}$$

- The mass of the star is given by

$$M(r) = 4\pi \int_0^b \rho(r) r^2 dr. \tag{1.45}$$

In this case the mass is given by

$$M = \frac{(1+\lambda)\frac{b^3}{R^2}}{2(1+\lambda\frac{b^2}{R^2})} \tag{1.46}$$

There are four parameters in the solutions, namely λ, R, ψ_0 and δ. Two of the parameters are matched with exterior Schwarzschild metric, one of them can be fixed with given input radius or central density or surface density of the star. The only left out parameter λ may be used to characterize the relevant EoS. Although a number of EoS for a given M and b are possible with one parameter family of EoS, not all the EoS are acceptable, which however can be understood by precession determination of mass and radius of a star.

The method can be extended to accomodate a compact object with interior space describe by pseudo-spheroidal geometry. The ansatz for a pseudo-spheroidal geometry is given by

$$B^2(r) = \frac{1+\lambda\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}} \tag{1.47}$$

where λ and R are arbitrary constants. The model is also found useful to construct stellar models in many cases [30, 49]. The above relativistic solutions can be generalized in higher dimensional space-time geometry also. It is found that for a given A and B stellar models can be constructed with three arbitrary parameters describing geometries [50] considering a metric given by eq.(1.6) corresponding to a static metric. The metric function $A(r)$ given by eq.(1.41) also contains two unknowns Ψ_0 and δ. Thus there are four unknowns namely λ, R, Ψ_0 and δ. The parameter R can be determined by knowledge of central density. The parameters Ψ_0 and δ are determined by the boundary conditions for a given mass-radius of a star. Hence there is one parameter λ left to be determined. Thus Vaidya-Tikekar model describes a one parameter family of EoS. The energy-density and pressure are determined by the spheroidicity parameter λ, however, the pressure is a function density which is highly non-linear and are determined numerically.

1.4 Objective of the present work

The objective of the proposed research work is to obtain relativistic solutions of spherically symmetric star in hydrostatic equilibrium. The relativistic solutions will be used to construct stellar models for different geometries and investigate the properties of compact objects namely energy density, pressure and other physical features inside the stars, in the usual four and in higher dimensions. Stellar models are constructed in isotropic, anisotropic cases. Einstein Maxwell equations are also solved to construct stellar models in the presence of electric charge. The importance of the solutions obtained in the above cases will be employed for different known astrophysical objects. Finally, collapse of anisotropic compact object is studied considering the new relativistic solutions obtained by us in the framework of the geometry describe by an isotropic metric.