# Chapter 4

# PERFECT FLUID DARK MATTER MODEL AND CAUTIONS REGARDING THE DETERMINATION OF HALO MASS.

## 4.1 Introduction

A fluid is said to be perfect when heat conduction, viscosity or other transport or dissipative process are negligible. Perfect fluid has the simple form of stress-energy tensor $T^{\alpha\beta}$. In an inertial frame a prefect fluid which is at rest, is characterized by its energy density($\rho$) and pressure($p$). Its stress energy tensor is given by

$$T^{\alpha\beta} = diag(\rho,\, p,\, p,\, p). \tag{4.1}$$

In general a typical fluid may not be at rest rather it may flow with a four velocity $u(x)$ which depends on the position of the point. Therefore, stress-energy tensor will be a function of $u(x)$, $\rho(x)$ and $p(x)$. The most general admissible form of the stress-energy tensor that can be made of by $u(x)$ and $g^{\alpha\beta}$ is

$$T^{\alpha\beta} = Au^{\alpha}u^{\beta} + Bg^{\alpha\beta}. \tag{4.2}$$

The coefficients A and B can be determined by the requirement that the stress-energy tensor given by (4.2) must reduces to (4.1) in the frame of an observer at rest with respect to the fluid where $u^{\alpha} = (1, 0, 0, 0)$. With this restriction, expression (4.2) reduces to [1]

$$T^{\alpha\beta} = (p + \rho)u^{\alpha}u^{\beta} + pg^{\alpha\beta}. \tag{4.3}$$

If the pressure of a perfect fluid vanishes, it is called 'dust' and (4.3) will, then, reduce to $T^{\alpha\beta} = \rho u^{\alpha}u^{\beta}$.

Here, we are interested to study the general features of dark matter such as its equation of state with the input of flat rotational curve condition and the assumption that dark matter can be described as a perfect fluid. Stress tensor controls the gravitational influence of an arbitrary dark matter component. There are many models in the literature that deal with the prediction of anisotropic dark matter fluid stress tensor while, on the other hand, there are neither physical mechanism nor observational evidence explaining why a spherical distribution should have such anisotropy. Therefore, it should be more reliable to consider an isotropic perfect fluid distribution for dark matter because predictions from such model at stellar and cosmic scales have been observationally supported without any uncertainty.

As we are searching for the unknown nature of the DM which is one of the most relevant question that modern scientist would like to solve, this is particularly important to know what exactly is determined by the observations and what comes as extra assumptions. Any extra assumption on the nature of dark matter, in order to obtain informations about it, will bias the problem. Matos et al.(2011) [2] have shown that how an extra assumption on the nature of dark matter will skew the problem, with a simple model for the DM halo.

## 4.2 Perfect Fluid solution of the Dark Matter

We shall treat the matter compositions in the galactic halo region as a perfect fluid defined by stresses $T_r^r = T_\theta^\theta = T_\phi^\phi = p$ where $T_\nu^\mu$ is the matter energy momentum tensor. The general static spherically symmetric space-time is given by the metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.4}$$

$\nu(r)$ and $\lambda(r)$ are the metric potentials. With the flat rotational curve condition for a circular stable geodesic motion in the equatorial plane, the metric potential $e^{\nu(r)}$ takes the form $e^{\nu(r)} = B_0 r^l$ with $l = 2(v_{tg})^2$ and $B_0$ is an integration constant [3]. For a typical galaxy, the tangential velocity($v_{tg}$) becomes approximately a constant with $v_{tg} \sim 10^{-3}$ km/s and therefore $l \approx 10^{-6}$. Einstein field equations [2] with the help of chosen stress tensor will give [3]

$$(e^{-\lambda})' + \frac{ae^{-\lambda}}{r} = \frac{c}{r}, \tag{4.5}$$

where

$$
\begin{aligned}
a &= -\frac{4(1+l)-l^2}{2+l}, \\
c &= -\frac{4}{2+l}.
\end{aligned}
\tag{4.6}
$$

A solution of Eq.(4.5) is given by [3]

$$
e^{-\lambda} = \frac{c}{a} + \frac{D}{r^a}, \tag{4.7}
$$

with $D$ as an integration constant. The space time metric given by (4.4) with its coefficients $e^{\nu(r)}(= B_0 r^l)$ and $e^{\lambda}$ (given by Eq.(4.7)) is an interior solution describing the region with constant tangential velocity and it has to be joined with the exterior region having other types of space time. Though this space-time metric is not asymptotically flat, the extent of $v_{tg} = constant$ ends at some larger distance where the region becomes asymptotically flat. Even now, the galactic boundary is not defined observationally. If we can do so properly, the junction conditions will provide us the value of the constant '$D$'.

### 4.2.1 Features of the Perfect Fluid solution

The pressure($p$) and density($\rho$) can be found from Einstein field equations as [3]

$$
\rho = \frac{1}{8\pi G}\left[\frac{l(4-l)}{4+4l-l^2}\frac{1}{r^2} - \frac{D(6-l)(1+l)}{2+l} r^{\frac{l(2-l)}{2+l}}\right], \tag{4.8}
$$

$$
p = \frac{1}{8\pi G}\left[\frac{l^2}{4+4l-l^2}\frac{1}{r^2} + D(1+l) r^{\frac{l(2-l)}{2+l}}\right]. \tag{4.9}
$$

The last term in the expressions of both energy density and pressure given by Eqs.(4.8) and (4.9), respectively, increases with radial distance. This terms are the general relativistic corrections to Newtonian expressions $\rho_{Newton} = \frac{1}{8\pi G}\frac{1}{r^2}$. The first term on the right hand side of Eq.(4.8) corresponds to the expected Newtonian term in the leading order with a general relativistic correction term $\frac{1}{8\pi G}\frac{5l^2}{4r^2}$ which is very small. Due to the second term in Eq.(4.8), corresponding mass increases non-linearly with radial distance $r$ showing a completely different customary feature of dark matter energy density. This contribution will vanish if $D$ vanishes.

The equation of state parameter $\omega$ is given by

$$\begin{aligned} \frac{p}{\rho} &= \omega = \frac{l^2 r^a + D(1+l)(4+4l-l^2)}{l(4-l)r^a - D(6-l)(1+l)(4+4l-l^2)/(2+l)} \\ &\approx \frac{l^2 r^a + 4D}{4lr^a - 12D} \end{aligned} \tag{4.10}$$

(as $l$ is small). Total matter in the flat rotational curve region of galaxies would be non-exotic if $\omega$ is positive. Thus, the dark matter will satisfy the known energy conditions if [3]

$$-\frac{l^2 r^a}{4} < D < \frac{lr^a}{3}. \tag{4.11}$$

As $r$ is typically between ten kpc to few hundred kpc and $l = 10^{-6}$, $D$ would be very small in order to maintain the non-exoticness of $\omega$. The equation of state of the dark matter component for $D = 0$ is

$$\omega = \frac{l}{4-l} = 2.5 \times 10^{-7}. \tag{4.12}$$

Thus, $p \ll \rho$ indicating it's Newtonian nature. Also, as $\omega$ is positive, the fluid is non-exotic in nature. But, the recommended value of D for the existence of non-exotic matter in the halo is that $D \leq 10^{-11}$ [3].

Now, from the metric function (4.7), with $D = 0$, we can infer that

$$e^{\lambda} = \frac{a}{c} = 1 + \frac{4l - l^2}{4} > 1. \tag{4.13}$$

Thus, we reproduce the crucial condition $e^{\lambda} > 1$ that has to satisfy by any valid metric and is essential for signature protection. As the halo matter is not exotic in nature (follows from Eq.(4.10)), so we expect attractive gravity in the halo. However, a positive energy density does not always lead to attractive gravity [4, 5]. Hence we have to calculate the total gravitational energy $E_G$ as proposed by Misner, Thorne & Wheeler 1973; Lynden-Bell, Katz & Bicak, 2007 [6]. It can be shown that the total gravitational energy is small but negative for arbitrary $D$ (non-zero or zero) with $r_2 > r_1 > 0$. So, the gravity in the halo is attractive in nature. The study of the geodesic equations for a test particle also show that particles are attracted towards the center i.e. gravity on the galactic scale is attractive.

More over, the circular orbits in which test particles are moving around galaxies are stable whatever the value of $D$ may be. Thus, the solution given by Eq.(4.7) satisfies some crucial physical requirements like stability of circular orbits, attractive gravity in the halo region. Also the matter in the galactic halo is non-exotic. Again, if the matter in the flat rotational curve region be non-exotic then the universe should be nearly flat (if not exactly so) which is consistent with modern cosmological observations.

## 4.3 Einstein's equations for Perfect Fluid and Scalar Field model

Let us recall the static and spherically symmetric space-time metric described by the line element

$$ds^2 = -e^{2\Phi(r)/c^2}dt^2 + \frac{1}{1 - \frac{2m(r)}{c^2 r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \tag{4.14}$$

Due to the symmetries of this space-time, all physical quantities depend only on radial coordinate $r$ as the gravitational potential $\Phi(r)$ and the mass function $m(r)$ does. Einstein's equations for the given metric are [7],

$$m' = -\frac{4\pi r^2}{c^2}T^t_t, \tag{4.15}$$

$$(1 - \frac{2mG}{c^2 r})\frac{\Phi'}{c^2} - \frac{mG}{c^2 r^2} = \frac{4\pi G r}{c^4}T^r_r, \tag{4.16}$$

where the prime $' \equiv \partial/\partial r$ . In order to make a comparison between two hypothesis on the nature of DM, Matos et al. have chosen two types of dark matter model namely perfect fluid dark matter model and scalar field dark matter model. For the different nature of the fluids under consideration, the conservation equations of the matter energy generating the curvature of the space-time will be treated separately for each fluid.

Consider the stress-energy tensors

$$T_{\mu\nu} = (p + \rho c^2)u_\mu u_\nu + p g_{\mu\nu}, \tag{4.17}$$

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + 2V(\phi)), \tag{4.18}$$

for the perfect fluid and scalar field respectively where $\rho = (1+\varepsilon)\rho_0$ is the energy density of the fluid with $\rho_0$ is the rest mass energy density, $\varepsilon$ is the internal energy per unit mass, $u^\mu$ is the co-moving four velocity normalized by $u_\mu u^\mu = -c^2$, $p$ being the pressure of the fluid, $\phi_{,\alpha} = \partial\phi/\partial x^\alpha$ and $V(\phi)$ is the scalar potential.

The field equations that can be deduced from the conservation equations $T^\mu_{\nu;\mu} = 0$, for both the perfect fluid and scalar field, can surprisingly be expressed as a single equation i.e. one have to consider a single field equation for both types of matter, which is [3]

$$T_r^{r'} + (T_r^r - T_t^t)(\frac{\Phi'}{c^2} + \frac{2a}{r}) = 0, \tag{4.19}$$

with the specification that $a = 0$ for the perfect fluid and $a = 1$ for the scalar field.

In order to determine what kind of model suit the present dark matter scenarios best, we are to solve two types of system of equations namely, the Einstein's equations (Eqs.(4.15, 4.16)) and the field Eq.(4.19). In short, we have to find out $m$, $\Phi$, $p$ and $\rho$ for the case when the space-time is curved due to the perfect fluid or $m$, $\Phi$, $\phi$ and $V(\phi)$ for the case where the curvature of the space-time is due to the scalar field. Thus, we have to deal with three equations corresponding to four unknown functions. Therefore the unknown functions can not uniquely be determined from the system of equations and we can make one and only one extra assumption and no more. It is important to stress on the fact that once the extra assumption is made i.e. additional data are given, there is no more room left for any other extra assumption and the rest of the functions can automatically be determined by the Eqs.(4.15), (4.16) and (4.19). As for example, if we choose an equation of state for the perfect fluid $p = p(\rho)$ as an extra assumption or an explicit form for the potential $V(\phi)$ in the case of the scalar field, the form of the rest of the functions will automatically be determined by the system of equations not by any supposition on them. To solve the system of equations, Matos et al. have chosen the observations on rotation curves in spirals galaxies to complement the above field equations and used the light deflection by lensing to discriminate between different halo type models.

### 4.3.1 Discrepancy of mass functions

The gravitational potential ($\Phi$) and the tangential velocity of a test particles, $v_{tg}$, in circular motion is related by

$$\frac{\Phi'}{c^2} = \frac{\beta^2}{r}, \tag{4.20}$$

where $\beta^2 = v_{tg}^2/c^2$. With a given tangential velocity, $v_{tg}$, measured by observations of rotational curves in galaxies help us to determine the gravitational potential through Eq.(4.20) letting no room for an equation of state for the perfect fluid or for a given scalar field potential. Therefore, the approximations $2Gm/c^2r \ll 1$ and especially $p << \rho$ are, in general, extra hypothesis which will strongly bias the DM problem.

The gravity equations, Eqs.(4.15, 4.16), with the help of the Eq.(4.20) reduced to an equation (with no approximations) with mass function as the only free function

[3]

$$m^{'} + P(r)m = Q(r), \tag{4.21}$$

where

$$P(r) = \frac{2r\beta^{2'} - (1+2\beta^2)(3-2a-\beta^2)}{(1-2a+\beta^2)r}, \tag{4.22}$$

$$Q(r) = \frac{c^2}{G}\frac{r\beta^{2'} - \beta^2(2-2a-\beta^2)}{(1-2a+\beta^2)}. \tag{4.23}$$

The functions $P(r)$ and $Q(r)$ are the function of the type of the fluid that we are dealing with i.e. a function of "$a$" and of the rotation curves profile. Solving the differential Eq.(4.21), the mass function can be obtained in terms of the gravitational potential($\Phi$) through the integral

$$m = \frac{\int e^{\int^r P(r')dr'} Q(r)dr + C}{e^{\int^r P(r')dr'}}, \tag{4.24}$$

C is an arbitrary constant of integration. For the case of the perfect fluid, other functions, namely, the density and pressure can directly be computed from Eqs.(4.15) and (4.16), respectively.

The gravitational mass inferred by the same velocity profile is sufficiently different for the perfect fluid and scalar field cases, as we will see in the next. When the velocity function is a constant, say $\beta_0$, the gravitational potential and mass function are respectively given by [3]

$$\Phi = c^2 \ln(\frac{r}{r_0})^{\beta_0^2}, \tag{4.25}$$

$$m_{\beta_0} = \frac{c^2}{G}(\frac{\beta_0^2(2(1-a)-\beta_0^2)}{2(1+2(1-a)\beta_0^2-\beta_0^4)}r + Cr^{\frac{(1+2\beta_0^2)(3-2a-\beta_0^2)}{1-2a+\beta_0^2}}), \tag{4.26}$$

where $C$ is the integration constant of Eq.(4.24). A non zero value of $C$ may cause the change of the singularity of the line element (4.14). The mass function in the case of the perfect fluid with $a = 0$, is given by [3]

$$m_{pf} = \frac{c^2}{2G}\frac{\beta_0^2(2-\beta_0^2)}{1+2\beta_0^2-\beta_0^4}r. \tag{4.27}$$

Considering scalar field as the DM candidates, the mass function is (from Eq.(4.26)) given by [3]

$$m_{sf} = \frac{c^2}{G}(-\frac{\beta_0^4}{2(1-\beta_0^4)}r + Cr^{-(1+2\beta_0^2)}). \tag{4.28}$$

As $\beta_0^2 > 0$, the second term which goes as $r^{-(1+2\beta_0^2)}$ will diverge at the origin. To get ride off from this problem, we had to take non-static space-times, like the oscillations [8]. It can be mentioned that some features for the case of scalar field with a non zero constant $C$ in the mass function, have been discussed in [5]. Taking $C = 0$, one can have [3]

$$m_{sf} = -\frac{c^2}{2G}\frac{\beta_0^4}{(1-\beta_0^4)}r. \tag{4.29}$$

The effective mass of the scalar field $m_{eff} \sim \frac{2}{r_0\sqrt{1-\beta_0^2}}$, is a function of the characteristic distance of the halo which is of the order of kilo-parsecs. For a typical galaxy with $\beta_0 \sim 10^{-3}$, this will correspond to a very light boson mass $\sim 10^{-23}\ eV/c^2$ which is in a good concordance with the one found in [9]. Thus, looking at the mass functions given by Eqs.(4.27) and (4.29), we can infer that how remarkably different are the mass expressions derived from each type of the ingredient of the DM halo. In other way, we are able to show how the single observation of the rotational velocities in halos decides the salient features of the perfect fluid model or of the scalar field. As the mass (as in Eq.(4.28)) associated to the scalar field may be negative with a proper choice of $C$, we have to be cautious with the assumption on static metrics which are very limited for the scalar field. Although, a negative mass is not too new to us [10].

### 4.3.2 Measurement of discrepancies via deflection angle

We have previously shown that the DM halo models are discriminateable through the measurement of mass function. Another effective tool, in this regard, is the measurements of the deflection angle which not only strongly depends on the type of matter but also on the specific characteristics of the type of matter considered.

The deflection angle ($\Delta\varphi$) of the light ray by the gravitational lensing for the line element (4.14), is given by [11]

$$\Delta\varphi = -\int_{\infty}^{r_m} \frac{r_m dr}{r^2\sqrt{(1-\frac{2Gm}{rc^2})[e^{-2\Phi(r)/c^2}e^{-2\Phi(r_m)/c^2}-\frac{r_m^2}{r^2}]}}, \tag{4.30}$$

where $r_m$ is the radius of maximal approach. For our present models with constant rotational velocity, the deflection angle with $C=0$ reduce to the following expressions

$$\Delta\varphi_{pf} = \int_0^1 \sqrt{\frac{1-\beta_0^2(\beta_0^2-2)}{x^{2\beta_0^2}-x^2}}dx, \tag{4.31}$$

$$\Delta\varphi_{sf} = \int_0^1 \sqrt{\frac{1-\beta_0^4}{x^{2\beta_0^2}-x^2}}dx, \tag{4.32}$$

with $x = \frac{r_m}{r}$. Since the deflection angle is independent of the radius of maximal approach ($r_m$), it can be measured for a typical value of the velocity. Taking $v_{tg} = 250$

$km/s$, that corresponds to $\beta_0 = 1/1200$, the deflection angle can be evaluated as [2]

$$\begin{aligned} \Delta\varphi_{pf} &= 0.899547, \\ \Delta\varphi_{sf} &= 0.449546, \end{aligned} \tag{4.33}$$

where the deflection angles are expressed in arc second units. Notice that the difference between these two deflection angles is of almost half arc second. Therefore, the rotational velocity together with the deflection of light produced by the galactic halo can uncover the true nature of the DM.

Matos et al.[2] have shown that how the graph of the mass functions of the dark matter models can be an effective tool to discriminate them, by considering a velocity profile from well known NFW [12, 13] model. They have shown that (FIGURE 1. of [2]) the behavior of both the halo masses (perfect fluid and scalar field), near the origin and far away from it, are significantly different. They have also mentioned that the observations of the deflection angle can be a efficient tool to determine which type of matter is actually composing the DM halo. These examples show how two different types of matter can be consistent with the observation of rotation curves of DM halos, though they lead to different conclusions to the inferred mass function.

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