Chapter 6

ON GENERALIZED WORMHOLE IN THE EDDINGTON-INSPIRED BORN-INFELD (EiBI) GRAVITY

6.1 Introduction

One of the fundamental discoveries in astrophysics in recent times is that the universe is currently accelerating [1,2]. A possible explanation for the late-time cosmic acceleration could be due to the infra-red modifications [3] of Einstein's General Relativity (GR). Such alternative theories of gravity involve more general combinations of curvature invariants than the pure Einstein-Hilbert term. One such modified theory is the Eddington-inspired Born-Infeld (EiBI) gravity. What is this EiBI gravity? It is a prototype of theories that could be termed as the "gravitational avatar of non-linear electrodynamics" [4].

To be more specific, note that Eddington's original gravitational action is incomplete in the sense that it does not contain matter. Bañados and Ferreira [5] resurrected Eddington's proposal for the gravitational action in the presence of cosmological constant extending it to include matter fields in the form of a Born-Infeld like structure [6] of non-linear electrodynamics. The outcome is the modern form of EiBI gravity, which provides an alternative theory of the Big Bang with a novel, non-singular description of the Universe. The EiBI model is currently extensively applied in the literature to many other astrophysical scenarios such as the solar system, structure of neutron stars or dark matter etc [7-14]. Astrophysical scenarios today also include wormholes as an integral part, and we shall be dealing with one such solution here.

The solutions of the EiBI theory contain an extra parameter κ having the inverse dimension of the cosmological constant Λ. The theory is ideologically relatively new and very different from GR, except in the limit $\kappa \to 0$. Thus, the true EiBI theory must always have $\kappa \neq 0$, and this parameter is expected to modify different GR physical observables. In the same spirit, we wish to investigate the effect of κ on the observable quantities associated with a wormhole in EiBI theory. Such a wormhole has in fact been recently derived by Harko *et al.* [15], which could be regarded as a $\kappa \neq 0$ generalization of the original "zero total mass" Ellis-Bronnikov (EB) wormhole of the Einstein minimally coupled scalar field theory with a negative kinetic term[1]. Assuming that the EB wormhole has a standard coordinate throat radius r_0, what we mean by zero total mass here is that the individual masses in suitable units of the two mouths ($+r_0/2$ and $-r_0/2$) add exactly to zero, when $\kappa = 0$. The new generalized wormhole ($\kappa \neq 0$) derived by Harko *et al.* [15] is being extensively cited in the literature [18]. Thus, it is of interest to find out what corrections κ contribute to the observables of the zero mass general relativistic EB wormhole.

The purpose of this chapter is to derive several useful results that can be stated as follows: (i) The zero total mass behavior is preserved even when $\kappa \neq 0$. (ii) A non-zero κ prevents the tidal forces in the geodesic orthonormal frame from becoming arbitrarily large near $r_0 \sim 0$, *contrary* to what happens near a small Schwarzschild horizon radius, $M \sim 0$. (iii) A non-zero κ also influences light bending, which provides a possibility to estimate κ through gravitational lensing observations. (iv) A non-zero κ has a role in the flare-out and energy conditions. (v) Finally, in the Appendix, we point out the reasons why one cannot interpret the EiBI exotic matter either as phantom or as ghost field considered in GR.

[1] Recall that the 1973 Ellis "drainhole" solution [16] was independently discovered also by Bronnikov [17]. The term "wormhole" was seemingly not in vogue in 1973. Hence our current nomenclature EB wormhole, which belongs to general relativity, hence corresponds to $\kappa = 0$. The EiBI wormhole derived in [15] can be called its EiBI generalization due to the presence of the parameter $\kappa \neq 0$.

In Sec.6.2, we give a brief outline of the EiBI gravity to make the topic more transparent. Then, in Sec.6.3, we briefly describe the wormhole under investigation and calculate the masses of its two mouths. After a brief review of tidal forces in a Lorentz boosted frame in Sec.6.4, we calculate in Sec.6.5 the excess tidal forces experienced by a traveler in geodesic motion near the throat of the wormhole. We devote Sec.6.6 to a discussion of the role of κ in the flare-out and energy conditions. In Sec.6.7, we calculate the effect of κ on the bending of light passing by the positive mass mouth. The final section (Sec.6.8) concludes the paper, followed by an Appendix.

6.2 EiBI field equations

In 1924, Eddington [19] suggested that at least in free, de-Sitter space, the fundamental dynamical variable should be the affine connection Γ and proposed a gravitational action $S_{\mathrm{Edd}} = 2\kappa \int d^4x \sqrt{\det |R_{\mu\nu}(\Gamma)|}$, where κ is a constant with inverse dimension of Λ. Varying Γ, one obtains the field equations $\nabla_\alpha \left(2\kappa\sqrt{|R|}R^{\mu\nu}\right) = 0$, where ∇_α is the covariant derivative defined by Γ and $R^{\mu\nu}$ is the inverse of $R_{\mu\nu}$. Eddington's field equations can be solved if we define a new tensor $q_{\mu\nu}$ such that $\nabla_\alpha \left(\sqrt{|q|}q^{\mu\nu}\right) = 0$. The field equations then become $2\kappa\sqrt{|R|}R^{\mu\nu} = \sqrt{|q|}q^{\mu\nu}$, which reduce to Einstein-de Sitter field equations if $g_{\mu\nu} = q_{\mu\nu}$ and $\kappa = \Lambda^{-1}$. Thus Eddington's proposal is a good motivation for building a more general action alternative to Einstein's gravity. However, Eddington's theory does not include matter. Therefore, Bañados and Ferreira [5] included matter, a metric $g_{\mu\nu}$, a Born-Infeld [6] like structure replacing the pure affine Eddington action by a new action that gave birth to what is now called EiBI theory in the literature (for details, see [5]).

We shall focus on the EiBI theory embodied in the Bañados-Ferreira action [5] given by[2]

$$S_{\mathrm{EiBI}} = \frac{1}{16\pi}\frac{2}{\kappa}\int d^4x \left[\sqrt{\det |g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|} - \lambda\sqrt{\det |g_{\mu\nu}|}\right] + S_{\mathrm{matter}}\left[g, \Psi_{\mathrm{matter}}\right], \tag{6.1}$$

where Ψ_{matter} is a generic matter field, $\sqrt{\det |g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|}$ is a Born-Infeld like structure [6], λ is a dimensionless parameter, $g_{\mu\nu}$ is the physical metric tensor, $R_{\mu\nu}(\Gamma)$ is the symmetric part of the Ricci tensor built solely from the connection Γ, yet

[2] The action was first proposed by Vollick [20], but the matter fields were introduced in a non-conventional way inside the square root, unlike in (1).

undefined. For small values of κR, the action (6.1) reproduces the Einstein-Hilbert action with a constant $\dfrac{\lambda-1}{\kappa}$, identified as the cosmological constant (This will be more transparent from the expansion of the field equations below):

$$\Lambda = \frac{\lambda - 1}{\kappa}. \tag{6.2}$$

For large values of κR, the action approximates to matter-free Eddington action S_{Edd}. To ensure asymptotic flatness of solutions in the EiBI theory ($\kappa \neq 0$), one must have $\Lambda = 0$, which in turn would entail $\lambda = 1$ from Eq.(6.2).

The field equations are based on a Palatini-type formulation, where the metric tensor $g_{\mu\nu}$ and the connection Γ are the two independent dynamical variables that are varied in the action (6.1). Varying with respect to $g_{\mu\nu}$, one obtains ($|X|$ denotes det $|X_{\mu\nu}|$):

$$\frac{\sqrt{|g + \kappa R|}}{\sqrt{|g|}} \left[(g + \kappa R)^{-1}\right]^{\mu\nu} - \lambda g^{\mu\nu} = -8\pi\kappa T^{\mu\nu}, \tag{6.3}$$

where the usual stress tensor $T^{\mu\nu}$ is raised or lowered with $g_{\mu\nu}$. The field Eq.(6.3) expands as [5]: $R_{\mu\nu} \simeq \left(\dfrac{\lambda-1}{\kappa}\right) g_{\mu\nu} + T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T + \kappa \left[S_{\mu\nu} - \frac{1}{4} g_{\mu\nu} S\right]$, where $S_{\mu\nu} = T^{\alpha}_{\mu} T_{\alpha\nu} - \frac{1}{2} T T_{\mu\nu}$ and $\dfrac{\lambda-1}{\kappa}$ can be identified with Λ. Note that Einstein's GR is recovered as $\kappa \to 0$.

The variation with respect to Γ can be simplified by introducing an auxiliary metric $q_{\mu\nu}$ compatible with Γ defined by $\Gamma^{\alpha}_{\beta\gamma} \equiv \dfrac{1}{2} q^{\alpha\sigma} \left[\partial_\gamma q_{\sigma\beta} + \partial_\beta q_{\sigma\gamma} - \partial_\sigma q_{\beta\gamma}\right]$ so that the equation of motion becomes[3]

$$q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}. \tag{6.4}$$

Bañados and Ferreira [5] restricted their analysis to cases, where matter couples *only* to the metric $g_{\mu\nu}$ determining the geodesic equation $\nabla_\mu T^{\mu\nu}_{(g)} = 0$ but coupling to $\Gamma(q)$ may arise due to quantum gravitational corrections. Since the auxiliary metric $q_{\mu\nu}$ is connected $g_{\mu\nu}$, EiBI does not introduce any extra degree of freedom.

[3] Alternatively, variation with respect to the connection Γ leads to the corresponding field equations. By defining $q_{\mu\nu} := g_{\mu\nu} + \kappa R_{\mu\nu}$ [Eq.(4)], after some manipulations, the field equations take the form $\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} q^{\alpha\sigma} \left[\partial_\gamma q_{\sigma\beta} + \partial_\beta q_{\sigma\gamma} - \partial_\sigma q_{\beta\gamma}\right]$ (see Ref.[12] for details). So, Eq.(4) and this form are exactly equivalent field equations. This explains the genesis of the auxiliary metric $q_{\mu\nu}$: It is coming from the variation of the dynamical variable Γ. Of course, Eq.(4) is more illuminating, and we take it. Only in vacuum $q_{\mu\nu} = g_{\mu\nu}$, but *inside* matter they are different. This is the essence of the EiBI theory.

Thus, only the metric $g_{\mu\nu}$ is of physical interest as far as gravitational observables are concerned. Combining (6.3) and (6.4), one finds

$$\sqrt{|q|}q^{\mu\nu} = \lambda\sqrt{|g|}g^{\mu\nu} - 8\pi\kappa\sqrt{|g|}T^{\mu\nu}, \tag{6.5}$$

where $q^{\mu\nu}$and $g^{\mu\nu}$ are the matrix inverses of $q_{\mu\nu}$ and $g_{\mu\nu}$ respectively. (6.4) and (6.5) provide the complete set of general EiBI field equations for arbitrary λ. In vacuum ($T_{\mu\nu} = 0$), $g_{\mu\nu} = q_{\mu\nu}$, $\Gamma = \Gamma(g)$ and hence EiBI and GR are completely equivalent.

For the specific case of asymptotically flat solutions, $\lambda = 1$, and hence (6.5) can be rewritten as,

$$q^{\mu\nu} = \tau\left(g^{\mu\nu} - 8\pi\kappa T^{\mu\nu}\right), \tag{6.6}$$

where

$$\tau = \sqrt{\frac{|g|}{|q|}}. \tag{6.7}$$

Harko *et al.* [15] further simplified the Eqs.(6.4 and 6.6) combining them into a form that looks much more familiar:

$$R^{\mu}_{\nu} = 8\pi S^{\mu}_{\nu}, \tag{6.8}$$

$$S^{\mu}_{\nu} = \tau T^{\mu}_{\nu} - \left(\frac{1-\tau}{8\pi\kappa} + \frac{\tau}{2}T\right)\delta^{\mu}_{\nu}, \tag{6.9}$$

where $R^{\mu}_{\nu} = q^{\mu\sigma}R_{\sigma\nu}$, $R = R^{\mu}_{\mu}$, and $T^{\mu}_{\nu} = T^{\mu\sigma}g_{\sigma\nu}$, $T = T^{\mu}_{\mu}$ (note the roles of q and g metrics).

6.3 Wormhole solution : Masses of the two mouths

The wormhole solution is derived in [15] by solving Eq.(6.8) under certain restrictive conditions such as spherical symmetry and asymptotic flatness, the latter requiring $\lambda = 1$. These assumptions of course limit the applicability of EiBI theory but make the problem at hand much simpler to handle. One spin-off is that the description of the physical behavior of the wormhole is now controlled by the only remaining parameter κ. The physical metric $g_{\mu\nu}$ and the auxiliary metric $q_{\mu\nu}$ respectively are taken as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -e^{\nu(r)}dt^2 + e^{\sigma(r)}dr^2 + f(r)d\Omega^2, \tag{6.10}$$

$$q_{\mu\nu}dx^{\mu}dx^{\nu} = -e^{\beta(r)}dt^2 + e^{\alpha(r)}dr^2 + r^2d\Omega^2. \tag{6.11}$$

The wormhole is assumed to be threaded by anisotropic matter described by the stress tensor $T^{\mu\nu} = p_t g^{\mu\nu} + (p_t + \rho)U^\mu U^\nu + (p_r - p_t)\chi^\mu \chi^\nu$, where χ^μ is the unique spacelike vector in the radial direction, $\chi^\mu = e^{-\sigma(r)/2}\delta^\mu_r$, p_r is the radial pressure, p_t is the tranverse pressure, ρ is the energy density, U^μ is the four velocity such that $g_{\mu\nu}U^\mu U^\nu = -1$. Since geodesics are determined by the metric $g_{\mu\nu}$, all observable effects connected to geodesics such as light deflection or tidal forces should be calculated only in the physical metric $g_{\mu\nu}$.

Note that τ of (6.7) can be obtained from T^μ_ν through the expression $\tau = |\delta^\mu_\nu - 8\pi\kappa T^\mu_\nu|^{-1/2}$, which in turn can be expressed in terms of stress quantities

$$\tau = \left[(1 + 8\pi\kappa\rho)\,(1 - 8\pi\kappa p_r)(1 - 8\pi\kappa p_t)^2\right]^{-1/2}. \tag{6.12}$$

The above form suggests arbitrary functions a, b and c defined by

$$\begin{aligned} a(r) &= \sqrt{1 + 8\pi\kappa\rho}, && (6.13)\\ b(r) &= \sqrt{1 - 8\pi\kappa p_r}, && (6.14)\\ c(r) &= \sqrt{1 - 8\pi\kappa p_t}, && (6.15) \end{aligned}$$

that help one write the components of the field Eqs.(6.8) in manageable forms that finally yield

$$e^\beta = e^\nu \frac{c^2}{a^2},\ e^\alpha = e^\sigma a^2 c^2,\ f = \frac{r^2}{ab}. \tag{6.16}$$

The specific wormhole solution obtained by Harko *et al.* [15] is based on simplifying assumptions that

$$a(r)b(r) = 1, \beta = 0. \tag{6.17}$$

Then the reduced system of field Eqs.(6.8) yield

$$e^\alpha = 1 - \frac{r_0^2}{r^2},\ a^4 = \frac{1}{1 + 2\kappa r_0^2/r^4},\ c^2 = a^2. \tag{6.18}$$

These, together with (6.16), lead to

$$\begin{aligned} q_{\mu\nu} &: \ e^{\beta(r)} = 1,\ e^{\alpha(r)} = \frac{1}{1 - r_0^2/r^2},\\ g_{\mu\nu} &: \ e^{\nu(r)} = 1,\ e^{\sigma(r)} = \frac{1 + 2\kappa r_0^2/r^4}{1 - r_0^2/r^2}, \end{aligned} \tag{6.19}$$

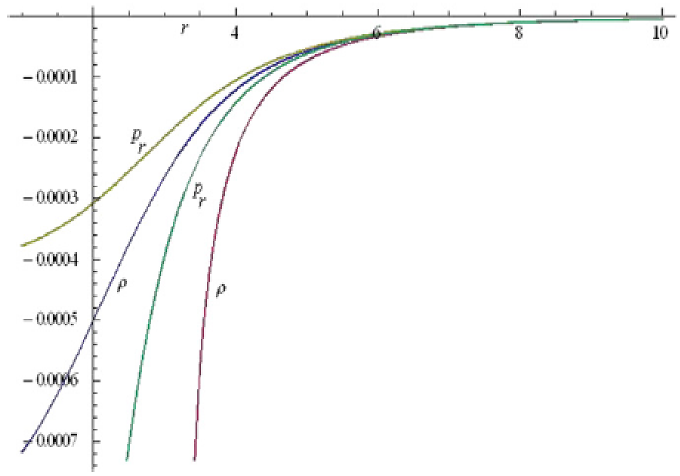

Fig.6.1. Plot of ρ and p_r vs r of EiBI equations (6.22) and (6.23) at $r_0 = 1$. The red and blue curves for ρ correspond to $\kappa = -50$ and 50 respectively, while the green and grey curves for p_r correspond to $\kappa = -50$ and 50 respectively. The values of ρ and p_r are always negative for arbitrary values of r_0 and κ. Only certain representatives plots are displayed.

where r_0 is an arbitrary constant. Hence we have the metric $g_{\mu\nu}$ given by (6.19), viz.,

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \left(\frac{1 + 2\kappa r_0^2/r^4}{1 - r_0^2/r^2}\right) dr^2 + r^2[d\theta^2 + \sin^2\theta d\varphi^2], \tag{6.20}$$

The metric (6.20) is a symmetric, twice asymptotically flat regular wormhole having asymptotic masses on either side of the throat, where r_0 has the meaning that it is the standard coordinate throat radius $r_{\rm th} = r_0$, $r_0 < r < +\infty$. In the limit $\kappa \to 0$, one recovers the massless EB wormhole of GR [16,17].

To obtain the asymptotic masses, one needs to cover *both* sides of the wormhole by a single regular chart defined by

$$r^2 = \ell^2 + r_0^2, \tag{6.21}$$

which is dictated by dimensional considerations, so the throat is now appearing at $\ell_{\rm th} = 0$. Then the charts covering individual sides respectively are $-\infty < \ell \leq 0$ and

$0 \le \ell < +\infty$, both meeting at the throat. Further, the structure of EiBI theory provides an energy density that can be obtained from (6.13) and (6.18) as

$$\rho(r) = \frac{1}{8\pi\kappa}\left[\frac{1}{\sqrt{1+2\kappa r_0^2/r^4}} - 1\right], \tag{6.22}$$

and the pressures from (6.14), (6.15) and (6.18)

$$p_r(r) = \frac{\rho(r)}{1+8\pi\kappa\rho(r)},\ p_t = -\rho. \tag{6.23}$$

Fig.6.1 shows that $\rho(r) < 0$, $p_r(r) < 0$ for all values of r and for all values of κ positive or negative. From (6.22), we can obtain masses on individual sides using the prescription[4]:

$$M^+ = 4\pi\int_0^\infty \rho r^2 \frac{dr}{d\ell} d\ell, \tag{6.24}$$

$$M^- = 4\pi\int_{-\infty}^0 \rho r^2 \frac{dr}{d\ell} d\ell. \tag{6.25}$$

As such, the integrals cannot be evaluated in a closed form although the integrand is continuous everywhere including at $\ell = 0$ and vanishing at $\ell \to \pm\infty$. Further, the density function $\rho \to -\frac{r_0^2}{8\pi r^4}$ as $\kappa \to 0$ but it's no surprise since at this limit the EiBI theory reduces to Einstein's theory. Also, note that $\rho \to 0$ as $\kappa \to \infty$. This is in perfect accordance with the pure Eddington theory ($\kappa R \to \infty$) without matter. Thus, the behavior of ρ shows no pathology anywhere and we can legitimately expand it in powers of κ, which yields

$$\rho = -\frac{r_0^2}{8\pi r^4} + \frac{3\kappa r_0^4}{16\pi r^8} - \frac{5\kappa^2 r_0^6}{16\pi r^{12}} + ... \tag{6.26}$$

The limit $\kappa \to 0$ yields the first term that is just the familiar exotic scalar field density $\rho^\phi = -\frac{r_0^2}{8\pi r^4}$ in the massless EB wormhole of GR. The masses can be found by term by term integration

$$M^+ = +\frac{r_0}{2} + \frac{3\kappa}{20 r_0} - \frac{5\kappa^2}{36 r_0^3} + ... \tag{6.27}$$

$$M^- = -\frac{r_0}{2} - \frac{3\kappa}{20 r_0} + \frac{5\kappa^2}{36 r_0^3} - ... \tag{6.28}$$

[4]The standard coordinate volume "$4\pi r^2 dr$" is the most appropriate volume measure for assessing the amount of matter, exotic or otherwise, see [21].

Note the correction terms due to κ. It is evident that the masses are of equal value but of opposite signs. Though either mouth of the wormhole can exhibit gravitational effects such as lensing [22] (caused either by attractive M^+, or by repulsive M^-), the total mass of the whole configuration adds exactly to zero, $M^+ + M^- = 0$, even when $\kappa \neq 0$. Hence, the massless character of the general relativistic EB wormhole is preserved also in the case of its EiBI counterpart (6.20). In the limit $\kappa \to 0$, one recovers the usual EB masses $+\frac{r_0}{2}$ and $-\frac{r_0}{2}$, which add to zero, that are made purely of the ghost scalar field ϕ of GR defined by the stress $T_{\mu\nu} = \epsilon \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\mu}$, with $\epsilon = -1$.

The masses in (6.24, 6.25) are "bare masses", while the Schwarzschild masses are zero. This conclusion follows from the twice asymptotically flat regular massive EB wormhole [16,17,23-28], sometimes also called the anti-Fisher solution, given by

$$d\tau_{\mathrm{EB}}^2 = -F dt^2 + F^{-1}[d\ell^2 + (\ell^2 + r_0^2)(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{6.29}$$

$$F = \exp\left[-\pi\gamma + 2\gamma \tan^{-1}\left(\frac{\ell}{r_0}\right)\right], \tag{6.30}$$

$$\phi = \lambda \left[\pi + 2\tan^{-1}\left(\frac{\ell}{r_0}\right)\right], \tag{6.31}$$

with the constraint $2\lambda^2 = 1 + \gamma^2$. The Schwarzschild masses on either side of the EB wormhole (6.29)-(6.31) are γr_0 and $-\gamma r_0 e^{\pi\gamma}$ as can be seen by expanding the metric tensor [16,17]. Thus, when $r_0 \neq 0$, $\gamma = 0$, these masses vanish and the solution reduces to massless EB wormhole

$$d\tau_{\mathrm{EB}}^2 = -dt^2 + d\ell^2 + (\ell^2 + r_0^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{6.32}$$

$$\phi = \frac{1}{\sqrt{2}}\left[\pi + 2\tan^{-1}\left(\frac{\ell}{r_0}\right)\right]. \tag{6.33}$$

Under the transformation $r^2 = \ell^2 + r_0^2$, one obtains in standard coordinates

$$d\tau_{\mathrm{EB}}^2 = -dt^2 + \frac{dr^2}{1 - r_0^2/r^2} + r^2[d\theta^2 + \sin^2\theta d\varphi^2], \tag{6.34}$$

$$\phi = \frac{1}{\sqrt{2}}\left[\pi + 2\tan^{-1}\left(\frac{\sqrt{r^2 - r_0^2}}{r_0}\right)\right]. \tag{6.35}$$

The metric part (*sans* scalar field ϕ) of the above solution is the EiBI metric (6.20) with $\kappa = 0$. As we see, it is just a special case ($r_0 \neq 0$, $\gamma = 0$) of the metric part of the massive EB wormhole (6.29)-(6.31).

This situation leads to a natural enquiry[5]: Just as the metric (6.20) is the EiBI generalization of the massless EB metric (6.34), does there exist a similar EiBI generalization of the massive EB metric (6.29)? We are not aware of such generalization as yet, but the possibility is certainly not ruled out if, instead of the anisotropic source tensor $T^{\mu\nu}$ used by Harko *et al.* [15], one uses a ghost or some other kind of scalar field and solve the EiBI field Eqs.(6.8) to find a solution. This would be a rewarding task by itself but we do not attempt it here.

6.4 Tidal forces in a Lorentz boosted frame

We start with the general form of a static spherically symmetric physical metric:

$$d\tau^2 = -\frac{F(r)}{G(r)}dt^2 + \frac{dr^2}{F(r)} + R^2(r)[d\theta^2 + \sin^2\theta d\varphi^2]. \tag{6.36}$$

For a traveler in a static orthonormal basis, we shall denote the only nonvanishing components of the Riemann curvature tensor as $\mathbf{R}_{0101}$, $\mathbf{R}_{0202}$, $\mathbf{R}_{0303}$, $\mathbf{R}_{1212}$, $\mathbf{R}_{1313}$, and $\mathbf{R}_{2323}$. Radially freely falling observers with conserved energy E are connected to the static orthonormal frame by a local Lorentz boost with an instantaneous velocity given by

$$\mathbf{v} = \left[1 - \frac{F}{GE^2}\right]^{1/2}. \tag{6.37}$$

Then the nonvanishing Riemann curvature components in the Lorentz boosted frame ($\hat{\ }$) with velocity $\mathbf{v}$ are ($k = 2, 3$):

$$\mathbf{R}_{\widehat{0}\widehat{1}\widehat{0}\widehat{1}} = \mathbf{R}_{0101} \tag{6.38}$$

$$\mathbf{R}_{\widehat{0}\widehat{k}\widehat{0}\widehat{k}} = \mathbf{R}_{0k0k} + (\mathbf{R}_{0k0k} + \mathbf{R}_{1k1k}) \sinh^2\alpha \tag{6.39}$$

$$\mathbf{R}_{\widehat{1}\widehat{k}\widehat{1}\widehat{k}} = \mathbf{R}_{1k1k} + (\mathbf{R}_{0k0k} + \mathbf{R}_{1k1k}) \sinh^2\alpha \tag{6.40}$$

$$\mathbf{R}_{\widehat{0}\widehat{k}\widehat{1}\widehat{k}} = (\mathbf{R}_{0k0k} + \mathbf{R}_{1k1k}) \sinh\alpha \cosh\alpha, \tag{6.41}$$

where $\sinh\alpha = \mathbf{v}/\sqrt{1-\mathbf{v}^2}$. The relative tidal acceleration $\Delta a_{\widehat{j}}$ between two parts of the traveler's body in his orthonormal basis is given by

$$\Delta a_{\widehat{j}} = -\mathbf{R}_{\widehat{0}\widehat{j}\widehat{0}\widehat{p}}\xi^{\widehat{p}}, \tag{6.42}$$

[5]We thank an anonymous referee for raising this query.

where $\overrightarrow{\xi}$ is the vector separation between the two parts [29]. Thus the curvature components contributing to tidal force on the traveler in the Lorentz boosted frame are $\mathbf{R}_{\widehat{0}\widehat{1}\widehat{0}\widehat{1}}$, $\mathbf{R}_{\widehat{0}\widehat{2}\widehat{0}\widehat{2}}$, and $\mathbf{R}_{\widehat{0}\widehat{3}\widehat{0}\widehat{3}}$. (Components in the coordinate basis are not required here). In the case of charged Reissner-Nordström black hole, there occurs a remarkable cancellation, viz., $\mathbf{R}_{0k0k} + \mathbf{R}_{1k1k} = 0$ such that the tidal accelerations in the static and moving frame are the same! The same cancellation of course occurs in the Schwarzschild spaceime too, which is only an uncharged special case.

For a Schwarzschild mass M, the curvature components of interest are

$$\mathbf{R}_{\widehat{0}\widehat{1}\widehat{0}\widehat{1}} = \mathbf{R}_{0101} = -\frac{2M}{r^3},\, \mathbf{R}_{\widehat{0}\widehat{2}\widehat{0}\widehat{2}} = \mathbf{R}_{\widehat{0}\widehat{3}\widehat{0}\widehat{3}} = \frac{M}{r^3},\ \text{etc} \tag{6.43}$$

Thus, at the horizon of the Schwarzschild black hole, $r_{\mathrm{h}} = 2M$, the curvature tensor $\mathbf{R}_{\widehat{0}\widehat{j}\widehat{0}\widehat{p}} \propto \frac{1}{M^2} \to \infty$ as $M \to 0$. So the smaller the black hole, the larger are the tidal forces near the horizon. We wish to examine a similar situation near the throat of a wormhole though the throat is physically entirely different from a black hole horizon.

6.5 Effect of κ on tidal forces

We want to calculate the effect of geodesic motion on the tidal forces experienced by a freely falling observer. In this direction, we first note that, in the Lorentz boosted frame, $\mathbf{R}_{\widehat{0}\widehat{1}\widehat{0}\widehat{1}} = \mathbf{R}_{0101}$, hence it is unaffected by geodesic motion. Second, because of spherical symmetry, we note that $\mathbf{R}_{\widehat{0}\widehat{2}\widehat{0}\widehat{2}} = \mathbf{R}_{\widehat{0}\widehat{3}\widehat{0}\widehat{3}}$, so it is enough to calculate only $\mathbf{R}_{\widehat{0}\widehat{2}\widehat{0}\widehat{2}}$. And finally, with $k = 2$, we can rewrite Eq.(6.39) for the generic metric (6.36) as:

$$\mathbf{R}_{\widehat{0}\widehat{2}\widehat{0}\widehat{2}} = -\frac{1}{R}\left[R''\left(E^2G - F\right) + \frac{R'}{2}(E^2G' - F')\right], \tag{6.44}$$

where primes denote derivatives with respect to r. The conserved energy E of the falling observer can be decomposed as

$$E^2 = \frac{F}{G} + \frac{F}{G}\left(\frac{\mathbf{v}^2}{1-\mathbf{v}^2}\right) = E_{\mathrm{s}}^2 + E_{\mathrm{ex}}^2, \tag{6.45}$$

where E_{s}^2 represents the value of E^2 in the static frame and E_{ex}^2 represents the enhancement in E_{s}^2 due to geodesic motion. We can now decompose $\mathbf{R}_{\widehat{0}\widehat{2}\widehat{0}\widehat{2}}$ as follows

[30]:

$$\begin{aligned} \mathbf{R}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}} &= -\frac{1}{R}\left[\frac{R'}{2}\left(E_{\mathrm{s}}^2 G' - F'\right)\right] - \frac{1}{R}\left(R''G + \frac{R'G'}{2}\right)E_{\mathrm{ex}}^2 \\ &= \mathbf{R}^{(\mathrm{s})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}} + \mathbf{R}^{(\mathrm{ex})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}} \end{aligned}$$

The first term represents the curvature component in the static frame, while the term $\mathbf{R}^{(\mathrm{ex})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}}$ represents overall enhancement in curvature in the Lorentz-boosted frame over that in the static frame. It is this part that needs to be examined as the observer approaches the throat.

Applying the above to the generalized EB wormhole metric (6.20), we see that

$$F(r) = G(r) = \frac{1 - r_0^2/r^2}{1 + 2\kappa r_0^2/r^4},\ R(r) = r. \tag{6.46}$$

A little algebra will show that $\mathbf{R}^{(\mathrm{s})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}} = 0$ and

$$\left|\mathbf{R}^{(\mathrm{ex})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}}\right| = \left[\frac{r_0^2(r^4 + 4\kappa r^2 - 2\kappa r_0^2)}{(r^4 + 2\kappa r_0^2)^2}\right]\left(\frac{\mathbf{v}^2}{1 - \mathbf{v}^2}\right), \tag{6.47}$$

which, in the limit $r \to r_0$, gives

$$\left|\mathbf{R}^{(\mathrm{ex})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}}\right| = \left(\frac{1}{2\kappa + r_0^2}\right)\left(\frac{\mathbf{v}^2}{1 - \mathbf{v}^2}\right). \tag{6.48}$$

Now suppose that $\kappa \to 0$ (general relativity) and of course $\mathbf{v} \neq 0$. Then, as $r_0 \to 0$, the excess tidal force in the geodesic frame near the throat becomes *arbitrarily large,* $\left|\mathbf{R}^{(\mathrm{ex})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}}\right| \to \infty$. This behavior is very similar to the case of small mass Schwarzschild black hole, as explained in Sec.6.4. The only physical difference is that here we are near a narrow throat instead of a small black hole horizon. In contrast, however, depending on the values of non-zero κ, the tidal forces may become *arbitrarily small*, $\left|\mathbf{R}^{(\mathrm{ex})}_{\widetilde{\hat{0}\hat{2}\hat{0}\hat{2}}}\right| \to 0$, even when $r_0 \to 0$. This is the novelty of the generalized wormhole (6.20) brought about by the presence of the parameter κ.

6.6 Flare-out and energy conditions

Defining $e^{-\sigma(r)} = 1 - \frac{m(r)}{r}$, where $m(r)$ is the Morris-Thorne (MT) [29] shape function, and assuming that the shape of the axially symmetric embedding surface is $z = z(r)$, the requirement that the wormhole flares out to two asymptotically flat space times is that the geometric condition $\frac{d^2r}{dz^2} = \frac{m - m'r}{2m^2} > 0$ be satisfied at or near the throat. This inequality imposes a constraint on the type of source stress tensor $T^{\mu\nu}$, that can be nicely rephrased in terms of the MT dimensionless flare-out function defined by: $\zeta := -\frac{\rho + p_r}{|\rho|} = \frac{2m^2}{r\,|m'|}\frac{d^2r}{dz^2} > 0 \Rightarrow \rho + p_r < 0$.

Harko *et al.* [15] defined an alternative flare-out condition that imposes a constraint on the shape function such that $H := \sigma' e^{-\sigma} = \frac{m'r - m}{r^2} < 0$ and using it obtained the generic inequality

$$8\pi\kappa(\rho + p_r) < \frac{\kappa b^2}{r}\frac{(c')^2}{c^2}e^{-\sigma(r)}. \tag{6.49}$$

At the throat $r_0 = m(r_0)$, $e^{-\sigma(r)} = 0$ and so $(\rho + p_r) < 0$. Thus, the Null Energy Condition (NEC) is violated showing that this violation is a necessary condition for the flare-out. But if it so happens that $\frac{(c')^2}{c^2}e^{-\sigma(r)} \to K$ as $r \to r_0$, then $0 < \rho + p_r < K$, and NEC need not be violated, hence no flare-out. Most importantly, note that ρ and p_r here are *not* derived from the Einstein field equations using MT metric form $e^{-\sigma(r)} = 1 - \frac{m(r)}{r}$, the form being used here only for notational convenience. Instead, ρ and p_r are obtained in Eqs.(6.22, 6.23) using only the EiBI equations. Likewise, H and the left hand side of the inequality (6.49) are expressed in terms of the true EiBI functions given in Sec.6.3.

The flare-out condition for the present wormhole (6.20) turns out to be

$$H = \sigma' e^{-\sigma} = -\frac{2rr_0^2(r^4 + 4\kappa r^2 - 2\kappa r_0^2)}{(r^4 + 2\kappa r_0^2)^2}, \tag{6.50}$$

which, at the throat r_0, yields

$$H_0 = \sigma' e^{-\sigma}\Big|_{r_0} = -\frac{2r_0}{2\kappa + r_0^2} < 0, \tag{6.51}$$

implying that the NEC is violated: $\rho + p_r < 0$. We can explicitly see from Eqs.(6.22) and (6.23) that

$$\rho + p_r = -\frac{r_0^2}{4\pi r^2\sqrt{r^4 + 2\kappa r_0^2}} < 0, \tag{6.52}$$

showing that the NEC is violated for all positive κ. The Weak Energy Condition (WEC) is also violated for all r including at the throat. As follows from Eq.(6.22)

$$\rho(r) = \frac{1}{8\pi\kappa}\left[\frac{1}{\sqrt{1 + 2\kappa r_0^2/r^4}} - 1\right] < 0, \tag{6.53}$$

for all positive κ. *It is thus clear that the source of (6.20) does not respect the WEC and NEC, implying that the wormhole is threaded by exotic matter.*

It is to be noted[6] that κ can also be negative [8,12], say $\kappa = -\kappa'$, $\kappa' > 0$. Then

$$H_0 = -\frac{2r_0}{r_0^2 - 2\kappa'}, \tag{6.54}$$

which implies that $H_0 < 0$ imposes a condition on the throat radius: $r_0^2 > 2\kappa'$. Precisely the same condition is required for the WEC and NEC violations as well. From Eqs.(6.52) and (6.53), we have at the throat

$$\rho|_{r_0} = -\frac{1}{8\pi\kappa'}\left[-1 + \frac{r_0}{\sqrt{r_0^2 - 2\kappa'}}\right] < 0, \quad (\rho + p_r)|_{r_0} = -\frac{1}{4\pi r_0\sqrt{r_0^2 - 2\kappa'}} < 0, \tag{6.55}$$

both hold only if the reality condition $r_0^2 > 2\kappa'$ holds. This suggests that the value $\sqrt{2\,|\kappa|}$ provides a lower bound on the size of the throat r_0, when $\kappa < 0$.

Note that $H \sim (\text{length})^{-1}$, while $\rho + p_r \sim (\text{length})^{-2}$, by definition. Hence we find a difference between the Eq.(6.54) and the second of Eqs.(6.55), but they qualitatively mean the same physical behavior $-$ flare-out and the concomitant NEC violation respectively. The influence of κ on the energy conditions and the flare-out condition is evident from the above Eqs.(6.50)-(6.55). The individual plots of $\rho(r)$ and $p_r(r)$ exhibit similar behavior to that of $\rho + p_r$ and hence only the representative plots of $\rho + p_r$ are given in Fig.6.2 for several values of κ.

Since the wormhole (6.20) is threaded by exotic matter (WEC and NEC both violated), it would be quite reasonable to enquire if EiBI exotic matter could somehow be connected to phantom energy or ghost scalar field ϕ within the framework of GR. Unfortunately, this connection seems unlikely at the level of either field equations or solutions, when $\kappa \neq 0$ (see Appendix). The reason is that the EiBI paradigm ($\kappa \neq 0$) is very different from that of GR ($\kappa \to 0$). Specifically, in the EiBI field Eq.(6.8), the left hand side is made entirely of the auxiliary metric $q_{\mu\nu}$, while the right hand source term S_ν^μ is a combination of $g_{\mu\nu}$ and $q_{\mu\nu}$ (via $\tau = \sqrt{|g|\,/\,|q|}$). GR limit implies through Eqs.(6.6) and (6.7) that $\tau = 1$, when $g_{\mu\nu}$ and $q_{\mu\nu}$ become identical, and only then from Eq.(6.8) we end up with Einstein's field equations.

The above notwithstanding, one might be curious to try, at the solution level, to imbed $e^{-\sigma(r)} = 1 - \frac{m(r)}{r}$ and $\nu = 0$ ($\Rightarrow$ redshift function $\Phi = 0$) from (6.19) into

[6]We thank another anonymous referee for pointing it out.

the *Einstein field equations*, and use the reverse technique of MT [29] to find the GR version of the EiBI exotic matter:

$$\rho_{\mathrm{GR}} = \frac{1}{8\pi r^2}\frac{dm}{dr} = \frac{r_0^2\left(6\kappa r^2 r_0^2 + 4\kappa^2 r_0^2 - 6\kappa r^4 - r^6\right)}{8\pi(r^5 + 2\kappa r r_0^2)^2}, \tag{6.56}$$

$$(\rho + p_r)_{\mathrm{GR}} = \frac{1}{8\pi}\left(\frac{1}{r^2}\frac{dm}{dr} - \frac{m}{r^3}\right) = \frac{r_0^2\left(2\kappa r_0^2 - 4\kappa r^2 - r^4\right)}{4\pi(r^4 + 2\kappa r_0^2)^2}. \tag{6.57}$$

These are evidently very different from the corresponding EiBI Eqs.(6.52, 6.53). However, when $\kappa \to 0$, both EiBI Eq.(6.52) and the GR Eq.(6.57) converge to the same EB value at the throat as expected, viz., $(\rho + p_r)|_{r_0} = -1/4\pi r_0^2$. The plots of Eq.(6.57) in Fig.6.3 are given for $r_0 = 1$ and several values of κ [that is, fixing the values of masses, see Eqs.(6.27, 6.28)]. For values of $\kappa \neq 0$, Figs.6.2 and 6.3 exhibit different behavior. The difference is pronounced for large negative values of κ. As an example, for $\kappa = -4$, Eqs.(6.56, 6.57) give $\rho_{\mathrm{GR}} > 0$, $(\rho + p_r)_{\mathrm{GR}} > 0$ in the neighborhood of the throat $r \sim r_0 = 1$, i.e., no violation of WEC and NEC, which is in contradistinction to EiBI plots in Fig.6.2. Nonetheless, values of r_0 and κ may be suitably adjusted so that $\rho_{\mathrm{GR}} < 0$, $(\rho + p_r)_{\mathrm{GR}} < 0$ can also be achieved (lower plots in Fig.6.3). But this GR version of EiBI exotic matter corresponds to neither phantom nor ghost scalar field, as will be shown in the appendix.

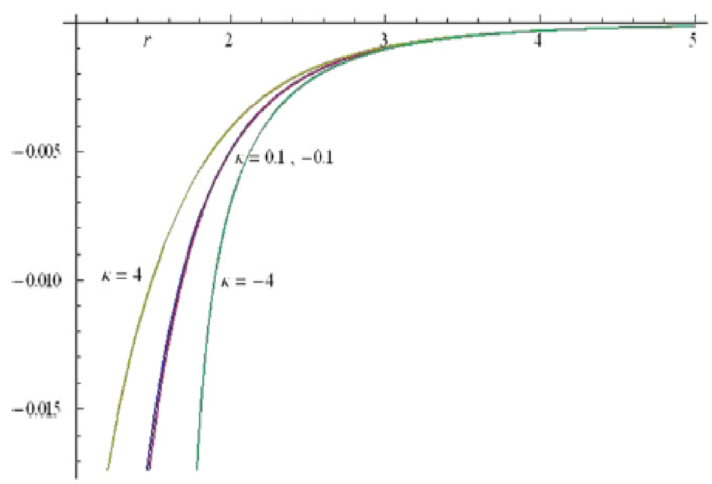

Fig.6.2. Plot of $\rho + p_r$ versus r of EiBI equation (52) at $r_0 = 1$. It shows that the NEC is violated for positive and negative values of κ. Similar curves follow for arbitrary values of r_0 and κ. Only certain representative plots are displayed.

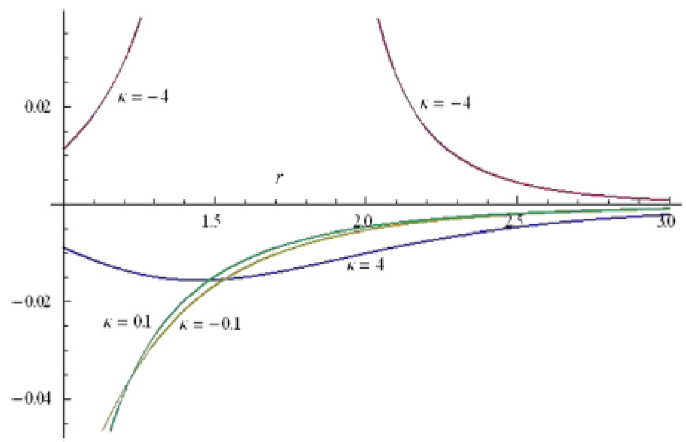

Fig.6.3. Plot of $\rho + p_r$ versus r of GR equation (57) at $r_0 = 1$. For relativelylarge negative κ, say, $\kappa = -4$, NEC is not violated. Values of r_0 and κ can be adjusted to have NEC violation. Only certain representative plots are displayed.

6.7 Light deflection

Light path equation in the equatorial plane, to second order in r_0^2, where κ appears first, is given by $(u = 1/r)$:

$$\frac{d^2u}{d\varphi^2} + u = -\left[\frac{u}{b^2} + 2\left(1 - \frac{2\kappa}{b^2}\right)u^3 + 6\kappa u^5\right] r_0^2, \tag{6.58}$$

where b is the impact parameter. The exact light path equation for the $\kappa = 0$ case, derived earlier by Bhattacharya and Potapov [31], can be recovered from the above. The minimum of r, or the maximum of u, denoted u_m, is the turning point of the motion. This occurs where $du/d\varphi = 0$ giving

$$b = 1/u_{\max} = R_0. \tag{6.59}$$

The perturbative solution is taken as

$$u = u_0 + u_1 \tag{6.60}$$

so that the linearized equations are

$$\frac{d^2u_0}{d\varphi^2} + u_0 = 0 \Rightarrow u_0 = \frac{\cos\varphi}{R}, \tag{6.61}$$

$$\frac{d^2u_1}{d\varphi^2} + u_1 = -\left[\frac{u_0}{b^2} + 2\left(1 - \frac{2\kappa}{b^2}\right)u_0^3 + 6\kappa u_0^5\right] r_0^2, \tag{6.62}$$

where R is a constant. The remaining equation (6.62) can be integrated so that the solution u becomes:

$$\begin{aligned} u = {} & \frac{\cos\varphi}{R} - \frac{r_0^2}{64b^2R^5}[\{56\kappa R^2 + 16R^4 - 2b^2(33\kappa + 14R^2)\}\cos\varphi \\ & +\{b^2(15\kappa + 4R^2) - 8\kappa R^2\}\cos 3\varphi + b^2\kappa\cos 5\varphi \\ & -(120b^2\kappa + 48b^2R^2 - 96\kappa R^2 - 32R^4)\varphi\sin\varphi]. \end{aligned} \tag{6.63}$$

After changing $\varphi \to \pi/2 + \delta$ in Eq.(6.63), and assuming small δ such that $\sin\delta \simeq \delta$, $\cos\delta \simeq 1$, and expanding to order r_0^2, we find, following Bodenner and Will [32], that

$$\delta \simeq \pi r_0^2\left(\frac{3}{8R^2} - \frac{1}{4b^2} + \frac{15\kappa}{16R^4} - \frac{3\kappa}{4b^2R^2}\right). \tag{6.64}$$

We now have to find the minimum value of R, which is the closest approach distance R_0. The minimum of R is the maximum of u_m, which can be shown by differentiation to occur at $\varphi = 0$. Putting $\varphi = 0$ in Eq.(6.63), setting $u_{\max} = 1/R_0$, and inverting, we get,

$$\frac{1}{R} \simeq \frac{1}{R_0} + O\left(\frac{1}{R_0^3}\right) \Rightarrow R \simeq R_0 = b. \tag{6.65}$$

Using this in Eq.(6.64), we get the two-way deflection ϵ as

$$\epsilon = 2\delta \simeq \frac{\pi r_0^2}{4R_0^2} + \frac{3\pi\kappa r_0^2}{8R_0^4}. \tag{6.66}$$

The first term exactly coincides with that obtained in Ref.[31], while the second term explicitly reveals the effect of κ.

6.8 Conclusions

The work reported here is an extension of the work by Harko *et al.* [15], wherein they derived a wormhole solution that could be described either as an EiBI wormhole or as generalized massless EB wormhole of GR. We attempted to present the

EiBI basics maintaining clarity and brevity, leading the readers from the motivation all the way to the EiBI wormhole (6.20) that contains a crucial parameter κ. The value of κ away from zero signifies departure from general relativistic effects and has been shown in the literature to depend on the chosen astrophysical scenarios [7-14,33,34]. In the same spirit, we have found in the foregoing the correction terms due to κ contributing to various observables in the massless EB wormhole.

We showed in Sec.6.3 that the massless character is preserved also in the generalized EB wormhole (6.20), where $\kappa \neq 0$. In Sec.6.5, we found a remarkable result is that the tidal forces can be arbitrarily small or finite even at a small throat radius ($r_0 \sim 0$) for *non-zero* values of κ. This result is in contradiction to that in general relativity, where the tidal forces become arbitrarily large in the limit of small Schwarzschild horizon radius ($M \sim 0$), as argued in the previous Sec.6.4. Then we discussed in Sec.6.6 the inter-relations among κ, the flare-out and energy conditions in EiBI showing that the source of (6.20) does not respect the WEC and NEC for $\kappa > 0$. For $\kappa < 0$, the throat radius has a lower bound $2\sqrt{|\kappa|}$ for ρ and $\rho + p_r$ to be real, but the energy conditions are still not respected. Posing the EiBI wormhole as a general relativistic one, we find that energy conditions may or may not be respected depending on the choices of r_0 and κ. This is more of a curious GR exercise as we show in the Appendix that the EiBI wormhole cannot be fitted into the GR framework with a phantom or ghost source scalar field ϕ even with a potential $V(\phi)$. In Sec.6.7, we have shown that the two-way light deflection on the positive side of the mouth has a correction term proportional to κ.

Some immediate tasks remain: The gravitational lensing by the general relativistic ($\kappa = 0$) EB wormhole has been already investigated by Abe [22]. Hence it would be of interest to study the influence of $\kappa \neq 0$ on the lensing observables in the generalized metric (6.20) taking into account our correction term to light deflection obtained in Eq.(6.66). Another important question is the issue of stability. It is already shown within the framework of GR that the $\kappa = 0$ case is unstable both under linear and non-linear perturbations [23,24,25]. Similarly, stability of the wormhole (6.20) has to be studied within the framework of EiBI theory for which $\kappa \neq 0$. It is yet to be determined whether the presence of non-zero κ can allow stability.

6.9 Appendix

We shall show that the exotic matter threading the EiBI or generalized EB wormhole (6.20) ($\kappa \neq 0$) is neither phantom nor ghost in the GR framework. For phantom matter, the equation of state parameter should be $\omega = \frac{p_r}{\rho} < -1$. On the other hand, we have from Eqs.(6.22, 6.23)

$$\omega = \frac{p_r}{\rho} = \sqrt{1 + 2\kappa r_0^2/r^4} > 0, \forall \kappa, r \tag{A1}$$

including at the throat $r = r_0$. The EiBI exotic matter therefore cannot be phantom anywhere in the spacetime regardless of whether κ is positive or negative.

However, for the $\kappa = 0$, it is well known that the EB wormhole (6.34) is threaded by matter made purely of a minimally coupled ghost scalar field in GR. The question then we ask is: Can we find in GR a similar minimally coupled scalar field ϕ with an arbitrary potential $V(\phi)$ for the $\kappa \neq 0$ EiBI wormhole (6.20)? The answer, unfortunately, seems to be in the negative.

Consider the action with a minimally coupled scalar field ϕ and a potential $V(\phi)$ given by

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[R - \epsilon (\nabla\phi)^2 - 2V(\phi) \right], \tag{A2}$$

where, notationally, $(\nabla\phi)^2 \equiv g^{\mu\nu}\phi_\mu \phi_\nu$, $\phi_\mu \equiv \partial\phi/\partial x^\mu$ and $\epsilon = \pm 1$. Variation with respect to the metric $g_{\mu\nu}$ and ϕ gives respectively the field equations

$$R_{\mu\nu} = \epsilon \phi_\mu \phi_\nu + g_{\mu\nu} V, \tag{A3}$$

$$\phi^{;\alpha}_{;\alpha} = -\frac{\partial V}{\partial \phi}. \tag{A4}$$

The value $\epsilon = -1$ corresponds to what is called a ghost scalar field ϕ. We choose the metric ansatz

$$d\tau^2 = -B(r)dt^2 + A(r)dr^2 + r^2[d\theta^2 + \sin^2\theta d\varphi^2]. \tag{A5}$$

From the left hand side of the field equation (A3), since $\dot{A} = 0$, it follows that $R_{tr} = R_{rt} = \dfrac{\dot{A}}{2A} = 0$, so we get $\dot{\phi}\phi' = 0$, where prime denotes differentiation with respect to r and dot denotes differentiation with respect to t. So we can either have $\phi' = 0$ or $\dot{\phi} = 0$. We choose the latter and assume $\phi = \phi(r)$ so that we get from the

Eqs.(A3):

$$\frac{B''}{2A}-\frac{B'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{B'}{rA} = V, \tag{A6}$$

$$-\frac{B''}{2B}+\frac{B'}{4B}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{A'}{rA} = \epsilon\phi'^2+AV, \tag{A7}$$

$$1-\frac{1}{A}+\frac{rA'}{2A^2}-\frac{rB'}{2AB} = r^2V. \tag{A8}$$

For the EiBI metric (6.20), we have

$$B(r)=1,\ A(r)=\frac{1+2\kappa r_0^2/r^4}{1-r_0^2/r^2}. \tag{A9}$$

Putting them in (A6), we have $V=0$ but the difficulty is that the field equation (A8), viz.,

$$1-\frac{1}{A}+\frac{rA'}{2A^2}=0 \tag{A10}$$

is *not* satisfied by the function $A(r)$ unless $\kappa=0$. This lack of self-consistency indicates that the exotic source matter in (6.20) is unlikely to be represented by a GR ghost scalar field. Note that although the GR Eqs.(6.56, 6.57) yield (for suitable values of r_0 and κ) exotic source matter obtained via the reverse MT [29] method, unless we are able to derive them from some kind of exotic scalar field via action of the type (A2), we cannot connect the solution (6.20) with $\kappa\neq 0$ to a GR solution with a coupled scalar field ϕ typical of the EB solutions.

However, there is always the possibility to introduce ghost or phantom or some other scalar field into the EiBI theory *itself* by including them in the action (6.1) from the start and analyze the corresponding solutions, if any. That would be a separate task by itself and is left for the future. Having said that, we point out that Deser and Gibbons [4] considered the EiBI type of Lagrangian and took the usual Christoffel connection $\Gamma(g)$ [instead of $\Gamma(q)$] and treated $g_{\mu\nu}$ as the only dynamical variable. The resulting field equations were fourth order with ghosts [20]. But the EB solutions result from second order field equations with ghost source, and thus different from the one considered in [4].

6.10 References

[1] A. Riess *et al.*, Astron. J. **116**, 1009 (1998)

[2] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999)

[3] S. M. Carroll *et al.*, Phys. Rev. D **70**, 043528 (2004)

[4] S. Deser and G.W. Gibbons, Class. Quant. Grav.**15**, L35 (1998)

[5] M. Bañados and P. G. Ferreira, Phys. Rev. Lett. **105**, 011101 (2010)

[6] M. Born and L. Infeld, Proc. Roy. Soc. A **144**, 425 (1934)

[7] J. Casanellas, P. Pani, I. Lopes and V. Cardoso, Astrophys. J. **745**, 15 (2012)

[8] P. Pani, V. Cardoso and T. Delsate, Phys. Rev. Lett. **107**, 031101 (2011)

[9] T. Harko *et al.*, Phys. Rev. D **88**, 044032 (2013)

[10] J. H. C. Scargill, M. Bañados, and P. G. Ferreira, Phys. Rev. D **86**, 103533 (2012)

[11] P. P. Avelino and R. Z. Ferreira, Phys. Rev. D **86**, 041501(R) (2012)

[12] P. Pani and T.P. Sotiriou, Phys. Rev. Lett. **109**, 251102 (2012)

[13] Q.-M. Fu *et al.*, Phys.Rev. D **90**, 104007 (2014)

[14] X.-L. Du *et al.*, Phys. Rev. D **90**, 044054 (2014)

[15] T. Harko *et al.*, arXiv:1307.1883 [gr-qc]

[16] H.G. Ellis, J. Math. Phys. **14**, 104 (1973); *Errata*: J. Math. Phys. **15**, 520 (1974)

[17] K.A. Bronnikov, Acta Phys. Polon. B **4**, 251 (1973)

[18] S.D. Odintsov *et al.*, Phys. Rev. D **90**, 044003 (2014); M. Lagos *et al.*, Phys. Rev. D **89**, 024034 (2014); T. Harko *et al.*, Galaxies **2**, 496 (2014); A.N. Makarenko *et al.*, Phys. Rev. **90**, 024066 (2014); M. Bouhmadi-López *et al.*, JCAP 11 (2014) 007; M. Bouhmadi-López *et al.*, Phys. Rev. D **90**, 123518 (2014); M. Bouhmadi-López *et al.*, Eur. Phys. J. C **75**, 90 (2015); S.-W. Wei *et al.*, arXiv:1405.2178 [gr-qc]; R. Shaikh, arXiv:1505.01314 [gr-qc]; E. Berti *et al.*, arXiv:1501.07274 [gr-qc]

[19] A.S. Eddington, *The Mathematical Theory of Relativity*, Cambridge University Press (1924)

[20] D. N. Vollick, Phys. Rev. D **69** (2004) 064030; *ibid.* D **72**, 084026 (2005)

[21] M. Visser, S. Kar and N. Dadhich, Phys. Rev. Lett. **90**, 201102 (2003); S. Kar, N. Dadhich and M. Visser, Pramana, **63**, 859 (2004)

[22] F. Abe, Astrophys. J. **725**, 787 (2010)

[23] J.A. González, F.S. Guzmán and O. Sarbach, Class. Quant. Grav. **26**, 015010 (2009); *ibid.* **26**, 015011 (2009)

[24] K.A. Bronnikov, J.C. Fabris and A. Zhidenko, Euro. Phys. J. C **71**, 1791 (2011)

[25] K. A. Bronnikov, R. A. Konoplya and A. Zhidenko, Phys. Rev. D **86**, 024028 (2012)

[26] K.A. Bronnikov and S. Grinyok, Grav. Cosmol. **10**, 237 (2004)

[28] K.A. Bronnikov and A.A. Starobinsky, JETP Lett. **85**,1 (2007)

[29] M.S. Morris and K.S. Thorne, Am. J. Phys. **56**, 395 (1988)

[30] G.T. Horowitz and S.F. Ross, Phys. Rev. D **56**, 2180 (1997)

[31] A. Bhattacharya and A. A. Potapov, Mod. Phys. Lett. A **29**, 2399 (2010)

[32] J. Bodenner and C.M. Will, Am. J. Phys. **71**, 770 (2003)

[33] R. Izmailov *et al.*, Mod. Phys. Lett. A **30**, 1550056 (2015)

[34] A.A. Potapov *et al.*, JCAP 07 (2015) 018