Differ Equ Dyn Syst
DOI 10.1007/s12591-015-0253-y

ORIGINAL RESEARCH

Comparative Study of Homotopy Analysis and Renormalization Group Methods on Rayleigh and Van der Pol Equations

Aniruddha Palit[1] · **Dhurjati Prasad Datta**[2]

Abstract A comparative study of the homotopy analysis method and an improved renormalization group method is presented in the context of the Rayleigh and the Van der Pol equations. Efficient approximate formulae as functions of the nonlinearity parameter ε for the amplitudes $a(\varepsilon)$ of the limit cycles for both these oscillators are derived. The improvement in the renormalization group analysis is achieved by invoking the idea of nonlinear time that should have significance in a nonlinear system. Good approximate plots of limit cycles of the concerned oscillators are also presented within this framework.

Keywords Rayleigh Van der Pol equation · Homotopy analysis method · Renormalization group

Mathematics Subject Classification 34C07 · 34C26 · 34E15

Introduction

The study of non-perturbative methods for nonlinear differential equations is of considerable recent interest. Among the various well known singular perturbation techniques such as multiple scale analysis, method of boundary layers, WKB method and so on [1,2], the recently developed homotopy analysis method (HAM) [3,4] and the renormalization group method (RGM) [5–8] appear to be very attractive. The aim of these new improved methods is to derive in an unified manner uniformly valid asymptotic quantities of interest for a given nonlinear dynamical problem. Although formulated almost parallely over the past decades

✉ Dhurjati Prasad Datta
dp_datta@yahoo.com

Aniruddha Palit
mail2apalit@gmail.com

1 Surya Sen Mahavidyalaya, Siliguri, West Bengal 734004, India

2 Department of Mathematics, University of North Bengal, Siliguri, West Bengal 734013, India

Published online: 26 July 2015

or so, relative strength and weakness of these two approaches have yet to be investigated systematically. The purpose of this paper is to undertake a comparative study of HAM and RGM in the context of the Rayleigh equation

$$\ddot{y} + \varepsilon\left(\frac{1}{3}\dot{y}^3 - \dot{y}\right) + y = 0 \tag{1}$$

and the Van der Pol equation

$$\ddot{x} + \varepsilon\, \dot{x}\left(x^2 - 1\right) + x = 0, \tag{2}$$

where the dots are used to designate the derivative with respect to time t. The Rayleigh and the Van der Pol (VdP) equations represent two closely related nonlinear systems and have found significant applications in the study of self excited oscillations arising in biology, acoustics, robotics, engineering etc. [9,10]. It is easy to observe that differentiating (1) with respect to time t and putting $\dot{y}\,(t) = x\,(t)$ we obtain (2). Both these systems have unique isolated periodic orbit (limit cycle). The amplitude of a periodic oscillation $y(t)$ (or $x\,(t)$) is generally defined by $\max|y\,(t)|$ (or $\max|x\,(t)|$) over the entire cycle. It is well known that the naive perturbative solutions of these equations are useful when $0 < \varepsilon \ll 1$ and yields the asymptotic value $a(\varepsilon) \approx 2$ of the amplitude for the limit cycle correctly. For $\varepsilon \gg 1$, simple analysis based on singular perturbation theory also yields the asymptotic amplitude for the relaxation oscillation as $a(\varepsilon) \approx 2$ for the VdP equation. However, the conventional perturbative approaches fail when ε is finite. One of the aim of this paper is to determine efficient approximate formulae for the amplitude of the limit cycle for the above systems by both HAM and RGM. Lopez et al. [4] have reported an efficient formula for the amplitude of the VdP limit cycle by HAM. We note here that a key difference in Rayleigh and VdP oscillators is the fact that with increase in input energy (voltage), the amplitude of the Rayleigh periodic oscillation increases, when that of the VdP oscillator remains almost constant at the value 2, with possible increase in the corresponding frequency only. For large ε ($\geq$1) relaxation oscillations, on the other hand, the Rayleigh system shows up a rather fast building up and slow subsequent release of internal energy, when the VdP models the reverse behaviour, with slow rise and fast drop in the accumulated energy.

As remarked above, HAM and RGM are formulated to determine the uniformly valid global asymptotic behaviours of relevant dynamical quantities like amplitude, period, frequency etc. related to periodic solutions of these equations for finite values of ε, by devising efficient methods in eliminating divergent secular terms of the naive perturbation theory. HAM seems to have the advantage of yielding uniformly convergent solutions of very high order in the nonlinearity parameter ε utilizing a freedom in the choice of a free parameter h. The computation of higher order term could be facilitated by symbolic computational algorithms. This method is used to obtain good approximate solutions for the VdP equation by a number of authors [4,11]. Lopez et al. [4] derived efficient formulae for estimating the amplitude of the limit cycle of the VdP equation for all values of $\varepsilon > 0$. Although, HAM is now considered to be an efficient method in the study of non-perturbative asymptotic analysis, it is recently pointed out [12] that this method might fail even in some innocent looking nonlinear problems.

The RGM, on the other hand, has a rich history, being originally formulated for managing divergences in the quantum field theory and later having deep applications in phase transitions and critical phenomena in statistical mechanics. Subsequently, Chen et al. [5,6] successfully translated the RG formalism into the study of nonlinear differential equations. It is noted that RGM is more efficient and accurate than conventional singular perturbative approaches in obtaining global informations from a naive perturbation series in ε. It is also recognized that

RGM generated expansions yield ε-dependent space/time scales naturally, when conventional approaches normally require invoking such scales in an ad hoc manner. The pertubative RGM, however, appears to have the limitation that the computation of higher order renormalized solutions could be quite involved and tedious. More serious is the inability of assuring the convergence of the renormalized expansions for large nonlinearity parameter. Further, there is still no evidence in the literature that RGM could be employed successfully to asymptotic estimation of the amplitude of an isolated periodic orbit for all values of ε as was reported for HAM [4].

Here we report analytic expressions of the amplitude of the periodic solutions of both the Rayleigh equation (1) and the VdP equation (2) as functions of ε. We have made a comparative study of these two sets of formulae using both HAM and RGM. The HAM contains a control parameter $h = h(\varepsilon)$ which controls the convergence of the approximation to the numerically computed exact value of the amplitude for all values of ε. Suitable choice of h can control the relative percentage error.

The original RGM gives an approximation to the exact solution for small values of ε. We report here the RG solution upto order 3. To the authors' knowledge this seems to be the first higher order computation other than second order computations reported so far by various authors [6,8]. A comparison of the amplitude of the periodic cycle with the exact computations reveals that even the present higher order perturbative approximations fails to give accurate estimation for moderate values of ε. As the higher order RG computations are quite laborious and inefficient, it is very unlikely that higher order computations of amplitude would improve the quality of the estimated amplitude of the limit cycle. Further, in RGM one does not have the resource of a free parameter equivalent to $h(\varepsilon)$ of HAM to improve the convergence of the RG expansions.

A major contribution in the present study is to propose an *improved* RGM (IRGM). In IRGM, we advocate the concept of *nonlinear time* [13–15] that extends the original RG idea of eliminating the divergent secular term of the form $(t - t_0)\sin t$, where t_0 is the initial time, of the naive perturbation series for the solution of the nonlinear problem, by exploiting the arbitrariness in fixing the initial moment t_0. The original prescription rests on introducing new initial time τ in the form $(t - \tau + \tau - t_0)\sin t$ and to allow the renormalized amplitude $R = R(\tau)$ and phase $\theta = \theta(\tau)$ of the renormalized solution to depend on the new parameter, viz., $\tau - t_0$ so that the original naive perturbative, constant values of amplitude R_0 and phase θ_0 (=0) (say) 'flow' following the RG flow equations of the form

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = f(R, \varepsilon), \quad \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = g(R, \varepsilon)\,. \tag{3}$$

The functions in the right hand sides of the RG flow equations, in general, should depend both on R and θ, besides the explicit ε dependence. We suppress the θ dependence for simplicity that should suffice for our present analysis of the Rayleigh and VdP equations [c.f. Eqs. (30), (31)]. The flow equations are derived from the consistency condition that the actual renormalized solution $y(t, \tau)$ should be independent of the arbitrary initial adjustment τ: $\frac{\partial y}{\partial \tau} = 0$. The final form of the uniformly valid RG solution $y_R(t)$ is obtained by setting $\tau = t$ that eliminates the secular terms. Let us remark here that the actual convergence of the RG expansions is not well addressed and should require further investigations. Moreover, estimation of asymptotic amplitude for a limit cycle as $t \to \infty$, for instance, from the perturbation expansion of f is expected to fail for $\varepsilon >\approx O(1)$.

In the framework of nonlinear time, we suppose the arbitrary initial time τ to depend explicitly on the nonlinearity parameter (coupling strength) ε of the nonlinear equation, so that one can write $\tau/\varepsilon = \varepsilon^h$ where $h = h(\varepsilon t)$, $\varepsilon t > 1$ is a slowly varying (almost constant), free (asymptotic) control parameter for $t \to \infty$ and $\varepsilon \to$ either to 0 or ∞, to be utilized judiciously to improve the convergence and non-perturbative global asymptotic behaviour of the original RG proposal ($h < 0$ for $0 < \varepsilon < 1$). In Appendix, we give an overview, in brief, of an extended analytic framework that naturally supports nontrivial existence of such an asymptotic scaling parameter $h(\tilde{\tau})$ as a function of the rescaled $O(1)$ variable $\tilde{\tau} = \varepsilon t \sim O(1)$, satisfying what we call the *principle of duality structure*. The secular terms in the naive perturbation series would now be altered instead as $(t - \tau/\varepsilon + \tau/\varepsilon - t_0)\sin t$ and we obtain the new RG flow equations in the form

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = f_0(R)\left(1 + O\left(\varepsilon^2\right)\right), \quad \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \varepsilon g_1(R)\left(1 + O\left(\varepsilon^3\right)\right), \tag{4}$$

where $f_0(R)$ and $g_1(R)$ are nonzero, minimal order R dependent terms in the respective perturbation series. Following the analogy of RG prescription in annulling secular divergence through corresponding 'flowing' of the renormalized perturbative amplitude and phase, we next make *the key assumption that there exists, for a given nonlinear oscillation, a set of right control parameters h^i that would absorb any possible secular or other kind of divergence in the higher order perturbation series*, (see Appendix for justification), so that in the asymptotic limit $t \to \infty$, one obtains the finite, non-perturbative flow equations directly for the periodic orbit of the nonlinear system

$$\frac{\mathrm{d}a}{\mathrm{d}\tau_1} = f_0(a), \quad \frac{\mathrm{d}\theta}{\mathrm{d}\tau_2} = g_1(a), \tag{5}$$

where $\tau_i = \varepsilon^{i h^i_{RG}(\tilde{\tau})}$, and $h^i_{RG}(\tilde{\tau})$ is a finite *scale* independent control parameter in the rescaled variable $\tilde{\tau} \sim O(1)$ and $a(\varepsilon) = \lim_{t\to\infty} R(\varepsilon, t)$ is the ε- dependent amplitude of the limit cycle. A simple quadrature formula should then relate the control parameter $h_{RG} = h^1_{RG}$ with the amplitude $a(\varepsilon)$. As a consequence, adjusting the control parameter h_{RG} suitably, one can generate an efficient algorithm to estimate the amplitude $a(\varepsilon)$ that would compare well with the exact values, upto any desired accuracy. It will transpire that the control parameter $h_{RG}(\varepsilon)$ must respect some asymptotic conditions depending on the characteristic features of a particular relaxation oscillation (c.f. "Improved RG Method: Nonlinear Time" section).

Exploiting the rescaling symmetry, one may as well rewrite the above non-perturbative flow Eq. (5) in the equivalent $\tilde{\tau} \sim O(1)$-dependent scaling variable $\tau = \tilde{\tau}^{H_{RG}(\tilde{\tau})}$, $\left(H_{RG}(\tilde{\tau}) = h_{RG}(\tilde{\tau})\frac{\log\tilde{\tau}}{\log\varepsilon}\right)$, for each fixed value of the nonlinearity parameter ε that should expose small scale $\tilde{\tau} \sim O(1)$-dependent variation of the amplitude. As a biproduct that would allow one to retrieve an efficient approximation of the limit cycle orbit for the nonlinear oscillator. It turns out that the general framework of IRGM is quite successful in obtaining excellent fits for the limit cycle orbit even for relaxation oscillation corresponding to nonlinearity parameters $\varepsilon \geq 1$.

It follows that the application of the of idea of nonlinear time in RG formalism offers one with a robust formalism for global asymptotic analysis for a general nonlinear system that might even be advantageous in many respects compared to HAM. The application of nonlinear time in HAM will be considered separately.

The paper is organized as follows. In "Computation of Amplitude by HAM" section we have deduced the solution to Eq. (1) by HAM. In "Computation of Amplitude by RG Method" section we compute the classical RG solution upto $O(\varepsilon^3)$ order and compare estimated values of the limit cycle amplitude with the exact values. The improved RG method is presented in "Improved RG Method: Nonlinear Time" section. This introduces a control parameter h_{RG} in the RG analysis. In "Approximate Formula for Amplitude" section approximate analytic formulae are deduced for the amplitudes of the limit cycles of the Rayleigh and the VdP equations. Efficient match with the exact values can be obtained by appropriate choice of h_{RG}. In "Approximate Limit Cycle" section we present efficient approximations of the limit cycle orbits for the Rayleigh and VdP oscillators for $\varepsilon = 5$. We close our discussions in "Concluding Remarks" section. In Appendix 1 we present a brief outline of the formal structure of the analytic formalism presented here. In Appendix 2 an alternative approach in the derivation of non-perturbative flow equations is presented.

Computation of Amplitude by HAM

The homotopy analysis method proposed by Liao [3,11] is used to obtain the solution of non-linear equation even if the problem does not contain a small or large parameter. HAM always gives a family of functions at any given order of approximation. An auxiliary parameter h is introduced in HAM to control the convergence region of approximating series involved in this method to the exact solution. HAM is based on the idea of homotopy in topology. In simple language, it involves continuous deformation of the solution of a linear ordinary differential equation (ODE) to that of desired nonlinear ODE. The solution of linear ODE gives a set of functions called *base functions*. One advantage of HAM is that it can be used to approximate a nonlinear problem by efficient choice of different sets of base functions. A suitable choice of the set of base functions and the convergence control parameter can speed up the convergence process.

In this paper we consider the self-excited system (1), which can be rewritten in the form

$$\ddot{U}(t) + \varepsilon \left(\frac{1}{3} \dot{U}^3(t) - \dot{U}(t) \right) + U(t) = 0, \quad t \geq 0, \tag{6}$$

where the dot denotes the derivative with respect to the time t. A limit cycle represents an isolated periodic motion of a self-excited system. This is an isolated closed curve Γ (say) in the phase plane so that any path in its suitable small neighbourhood starting from a point, specified by some given initial condition, ultimately converges to (or diverge from) Γ. Consequently, this periodic motion represented by limit cycle is independent of initial conditions. It, however, involves the frequency ω and the amplitude a of the oscillation. Therefore, without loss of generality, we consider an initial condition

$$U(0) = a, \qquad \dot{U}(0) = 0. \tag{7}$$

In [6], an alternative initial condition i.e. $U(0) = 0,\ \dot{U}(0) = a$ was considered. Let, with slight abuse of notations,

$$\tau = \omega t \quad \text{and} \quad U(t) = a\, u(\tau)$$

so that (6) and (7) respectively become

$$\omega^2 u''(\tau) + \varepsilon \left(\frac{1}{3} a^2 \omega^2 u'^2(\tau) - 1 \right) \omega u'(\tau) + u(\tau) = 0 \tag{8}$$

and

$$u(0) = 1, \quad u'(0) = 0. \tag{9}$$

Since the limit cycle represents a periodic motion, so we suppose that the initial approximation to the solution $u(\tau)$ to (8) can be taken as

$$u_0(\tau) = \cos \tau.$$

Let ω_0 and a_0 respectively denote the initial approximations of the frequency ω and the amplitude a.

We consider a linear operator

$$\mathcal{L}[\phi(\tau, p)] = \omega_0^2 \left[\frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \phi(\tau, p) \right] \tag{10}$$

so that for the coefficients C_1 and C_2,

$$\mathcal{L}(C_1 \sin \tau + C_2 \cos \tau) = 0. \tag{11}$$

We further consider a nonlinear operator

$$\begin{aligned} \mathcal{N}[\phi(\tau, p), \Omega(p), A(p)] &= \Omega^2(p) \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} \\ &+ \varepsilon \left[\frac{1}{3} A^2(p) \Omega^3(p) \left(\frac{\partial \phi(\tau, p)}{\partial \tau} \right)^3 - \Omega(p) \left(\frac{\partial \phi(\tau, p)}{\partial \tau} \right) \right] + \phi(\tau, p). \end{aligned} \tag{12}$$

Next, we construct a homotopy as

$$\mathcal{H}[\phi(\tau, p), h, p] = (1-p)\mathcal{L}[\phi(\tau, p) - u_0(\tau)] - h\, p\, \mathcal{N}[\phi(\tau, p), \Omega(p), A(p)] \tag{13}$$

where $p \in [0, 1]$ is the embedding parameter and h a non-zero auxiliary (control) parameter used to improve the convergence of series expansions. Setting $\mathcal{H}[\phi(\tau, p), h, p] = 0$ we obtain zero-th order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau, p) - u_0(\tau)] - h\, p\, \mathcal{N}[\phi(\tau, p), \Omega(p), A(p)] = 0 \tag{14}$$

subject to the initial conditions

$$\phi(0, p) = 1, \quad \left. \frac{\partial \phi(\tau, p)}{\partial \tau} \right|_{\tau=0} = 0. \tag{15}$$

Clearly, as p increases from $p = 0$ to $p = 1$, (14) changes from $\mathcal{L}[\phi(\tau, p) - u_0(\tau)] = 0$ to $\mathcal{N}[\phi(\tau, p), \Omega(p), A(p)] = 0$ and as a consequence $\phi(\tau, p)$ varies from the initial guess $\phi(\tau, 0) = u_0(\tau) = \cos \tau$ to the exact solution $\phi(\tau, 1) = u(\tau)$, so does $\Omega(p)$ from ω_0 to exact frequency ω and $A(p)$ from a_0 to the exact amplitude a. It can be shown that assuming $\phi(\tau, p)$, $\Omega(p)$, $A(p)$ analytic in $p \in [0, 1]$ so that

$$u_k(\tau) = \frac{1}{k!} \left. \frac{\partial^k}{\partial p^k} \phi(\tau, p) \right|_{p=0}, \quad \omega_k = \frac{1}{k!} \left. \frac{\partial^k}{\partial p^k} \Omega(p) \right|_{p=0}, \quad a_k = \frac{1}{k!} \left. \frac{\partial^k}{\partial p^k} A(p) \right|_{p=0} \tag{16}$$

we have

$$u(\tau) = \sum_{k=0}^{\infty} u_k(\tau), \tag{17}$$

$$\omega = \sum_{k=0}^{\infty} \omega_k, \tag{18}$$

$$a = \sum_{k=0}^{\infty} a_k, \tag{19}$$

where $u_k(\tau)$ are solutions of the k-th order deformation equation

$$\mathcal{L}\left[u_k(\tau) - \chi_k u_{k-1}(\tau)\right] = h\, R_k(\tau) \tag{20}$$

subject to the initial conditions

$$u_k(0) = 0, \qquad u'_k(0) = 0 \tag{21}$$

in which

$$\begin{aligned} R_k(\tau) &= \frac{1}{(k-1)!}\left.\frac{\partial^{k-1}}{\partial p^{k-1}}\mathcal{N}\left[\phi(\tau,p),\Omega(p),A(p)\right]\right|_{p=0} \\ &= \sum_{n=0}^{k-1} u''_{k-1-n}(\tau)\sum_{j=0}^{n}\omega_j\omega_{n-j} + u_{k-1}(\tau) \\ &\quad + \frac{\varepsilon}{3}\sum_{n=0}^{k-1}\sum_{i=0}^{n}\left(\sum_{r=0}^{i} a_r a_{i-r}\right)\times\left(\sum_{s=0}^{n-i}\omega_s\sum_{h=0}^{n-i-s}\omega_h\omega_{n-i-s-h}\right) \\ &\quad \times\left(\sum_{j=0}^{k-1-n} u'_j(\tau)\sum_{m=0}^{k-1-n-j} u'_m(\tau)\, u'_{k-1-n-j-m}(\tau)\right) - \varepsilon\sum_{n=0}^{k-1}\omega_n u'_{k-1-n}(\tau) \end{aligned} \tag{22}$$

and

$$\chi_k = \begin{cases} 0, & k \le 1, \\ 1, & k > 1. \end{cases} \tag{23}$$

To ensure that the solution to the k-th order deformation equation (20) do not contain the secular terms $\tau \sin\tau$ and $\tau\cos\tau$ the coefficients of $\sin\tau$ and $\cos\tau$ in the expressions of R_k in (22) must vanish giving successive values of ω_k and a_k.

The linear equation $\mathcal{L}(\phi(\tau, p)) = 0$ represents a simple harmonic motion with frequency 1. So, we choose the initial guess of ω as $\omega_0 = 1$. Again, by perturbation method [1] we find $a \to 2$ as $\varepsilon \to 0$. So, we choose the initial guess of a as $a_0 = 2$. Solving the differential equations given by (14), (15), (20), (21) and avoiding the generation of secular terms in each iteration we obtain

$$\begin{aligned} u_1(\tau) &= -\frac{1}{24}h\varepsilon\sin 3\tau + \frac{1}{8}h\varepsilon\sin\tau, \quad \omega_1 = -\frac{1}{16}h\varepsilon^2, \quad a_1 = \frac{1}{8}h\varepsilon^2, \\ u_2(\tau) &= \left(\frac{1}{384}h^2\varepsilon^3 - \frac{1}{24}h^2\varepsilon - \frac{1}{24}h\varepsilon\right)\sin 3\tau - \frac{1}{64}h^2\varepsilon^2\cos 3\tau \\ &\qquad + \frac{1}{64}h^2\varepsilon^2\cos\tau + \left(\frac{1}{8}h^2\varepsilon - \frac{1}{128}h^2\varepsilon^3 + \frac{1}{8}h\varepsilon\right)\sin\tau \end{aligned}$$

so that

$$\begin{aligned} R_1 &= \frac{1}{3}\varepsilon\sin 3\tau, \\ R_2 &= \frac{1}{24}\left[3h\varepsilon^2\cos 3\tau + \left(8h\varepsilon - \frac{1}{2}h\varepsilon^3\right)\sin 3\tau\right]. \end{aligned}$$

Computing R_k successively, we can find the successive expressions of $u_k(\tau)$, ω_k and a_k. The first order approximation to the amplitude in (19) is

$$a \approx a_0 + a_1 = 2 + \frac{1}{8}h\varepsilon^2 = a_E(\varepsilon)\ (\text{say}). \tag{24}$$

The above first order expression for the amplitude involves as yet arbitrary control parameter h. Lopez et al. [4] proposed specific ε-dependent expressions for h to obtain an efficient

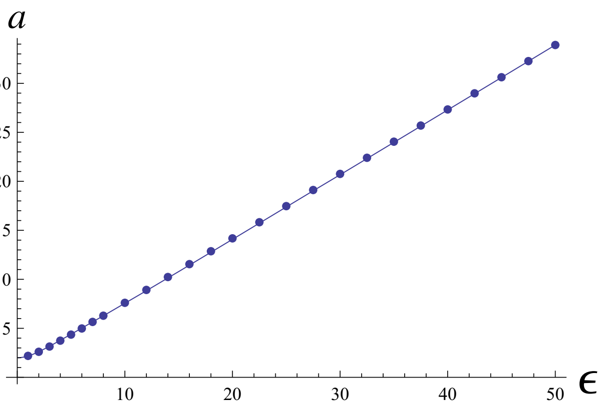

Fig. 1 The exact amplitude of Rayleigh Equation (by *solid line*) and its approximation $a_E(\varepsilon)$ given by (27) (by *bold points*) for $0 < \varepsilon \leq 50$

formula for the VdP limit cycle amplitude. They made the proposal that h, besides being continuous, must also vanish in the limits of $\varepsilon \to 0$ and $\varepsilon \to \infty$ to reproduce the zeroth order perturbative solutions. In our application of HAM for the Rayleigh limit cycle amplitude, we have chosen a different set of base functions and so can weaken the condition considerably, both on the continuity and the asymptotic limit $\varepsilon \to \infty$. From careful inspections of the graph of the exact amplitude (Fig. 1), it turns out that an appropriate ansatz for the control parameter h is given by

$$h = \frac{1}{0.5 + \varepsilon\, b(\varepsilon)}, \tag{25}$$

where $b(\varepsilon)$ is taken as the step function in the domain $0 < \varepsilon \leq 50$ as follows:

ε :	$0 < \varepsilon \leq 4$	$4 < \varepsilon \leq 5$	$5 < \varepsilon \leq 7$	$7 < \varepsilon \leq 8$	$8 < \varepsilon \leq 9$
$b(\varepsilon)$:	0.162	0.165	0.168	0.171	0.174
ε :	$9 < \varepsilon \leq 11$	$11 < \varepsilon \leq 15$	$15 < \varepsilon \leq 20$	$20 < \varepsilon \leq 30$	$30 < \varepsilon \leq 50$
$b(\varepsilon)$:	0.176	0.179	0.181	0.183	0.185

With this particular form of h, we are able to find an analytic approximation $a_E(\varepsilon)$ to the numerically computed exact value $a = a(\varepsilon)$ in the domain $0 < \varepsilon \leq 50$ with maximum relative percentage error

$$\left| \frac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100 \right|$$

less than 1 %. Obviously, better accuracy fit can be obtained by considering finer subdivisions in the definition of $b(\varepsilon)$. We remark that a piece-wise continuous ε dependence of h as above is admissible in the framework of HAM.

Since the exact graph of $a(\varepsilon)$ is almost a straight line for sufficiently large ε $(7 < \varepsilon \leq 50)$, we can reduce the number of steps to four only. Let us choose

$$h = \frac{8m}{\varepsilon} - \frac{56m}{\varepsilon^2} + \frac{8c}{\varepsilon^2} - \frac{16}{\varepsilon^2}, \qquad 7 < \varepsilon \leq 50 \tag{26}$$

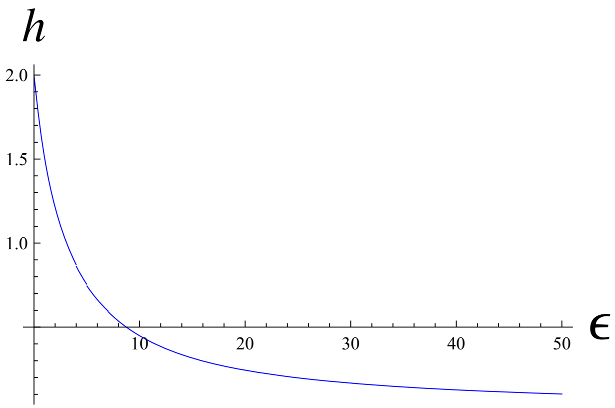

Fig. 2 The graph of $h(\varepsilon)$ used for approximation of the amplitude by HAM given by (27) for $0 < \varepsilon \leq 50$

so that (24) becomes

$$a_E(\varepsilon) = \begin{cases} 2 + \frac{1}{8}\left(\frac{1}{0.5+0.162\,\varepsilon}\right)\varepsilon^2 & 0 < \varepsilon \leq 4, \\ 2 + \frac{1}{8}\left(\frac{1}{0.5+0.165\,\varepsilon}\right)\varepsilon^2 & 4 < \varepsilon \leq 5, \\ 2 + \frac{1}{8}\left(\frac{1}{0.5+0.168\,\varepsilon}\right)\varepsilon^2 & 5 < \varepsilon \leq 7, \\ m(\varepsilon - 7) + c, & 7 < \varepsilon \leq 50, \end{cases} \tag{27}$$

where m and c are computed from the exact solution as

$$m = \frac{a(50) - a(7)}{50 - 7} = 0.657692 \quad \text{and} \quad c = a(7) = 5.63108$$

keeping the maximum relative percentage error

$$\left| \frac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100 \right|$$

less than 1 %. The plot of $a_E(\varepsilon)$ given by (27) is shown by bold points in Fig. 1 (explicit discontinuities of h at $\varepsilon = 4$, 5 and 7 are not visible at the resolution of the plotted figure) .

As remarked above, Lopez et. al. [4] proposed that a reasonable property for h would be to vanish in the limits as $\varepsilon \to 0$ and $\varepsilon \to \infty$. However, from (25) and (26) we observe that a suitable approximation to the amplitude of Rayleigh equation can be obtained even if h do not follow this property. The graph of $h(\varepsilon)$ is given in Fig. 2 for $0 < \varepsilon \leq 50$ (discontinuity in h is not visible at the level of resolution in the figure).

To summarize, one can obtain more accurate approximate formula by suitable choices of the control parameter $h(\varepsilon)$ upto any desired level of accuracy. We also note that a piecewise continuous control parameter h enables us to obtain good approximation by solving only the first order deformation equation. However, the first order HAM estimated amplitude $a(\varepsilon)$ is $O(\varepsilon^2)$.

We report the estimation of the amplitude of the limit cycle for the Rayleigh and VdP equations by the improved RG method in "Approximate Formula for Amplitude" section. We do not undertake the computation of the VdP amplitude by HAM separately, as that was already reported by Lopez et al. [4].

Computation of Amplitude by RG Method

The RGM introduced by Chen, Goldenfeld and Oono (CGO) [5,6] gives a unified formal approach to derive asymptotic expansions for the solutions of a large class of nonlinear ODEs. The RG method is used in solid state physics, quantum field theory and other areas of physics. One advantage of RGM is that it starts from naive perturbation expansion of a problem and is expected to yield automatically the gauge functions such as fractional powers of ε and logarithmic terms in ε in the renormalized expansion. One does not require to have any prior knowledge to prescribe these unexpected gauge functions in an ad hoc manner. DeVille et. al. [7] have introduced an algorithmic approach for RGM which we adopt for the following application. As it will transpire the RGM appears to be deficient in estimating amplitude of a periodic orbit because of the absence of any free control parameter. In a latter section we have improved this RGM to incorporate a control parameter similar to HAM and derive efficient estimations of amplitudes of both the Rayleigh and VdP equations. However, before the introduction of the improved RG method (IRGM), we first discuss the *conventional* RG method, given by DeVille et. al. and use it to obtain amplitude and phase equations for the Rayleigh equation (1). These equations are already obtained in [6,7] to the order $O\left(\varepsilon^3\right)$ which agree with the experimental values as $\varepsilon \to 0$ only. We have extended these results to the order $O\left(\varepsilon^4\right)$ and notice that higher order perturbative computations of the RG flow equations would fail to obtain good estimation of the amplitude of the periodic cycle for all values of ε.

Substituting the naive expansion

$$y\left(t\right) = y_0\left(t\right) + \varepsilon y_1\left(t\right) + \varepsilon^2 y_2\left(t\right) + \varepsilon^3 y_3\left(t\right) + \cdots$$

in (1), we find at each order

$$\begin{aligned}
O\left(1\right): &\quad \ddot{y}_0 + y_0 = 0,\\
O\left(\varepsilon\right): &\quad \ddot{y}_1 + y_1 = \dot{y}_0 - \frac{1}{3}\dot{y}_0^3,\\
O\left(\varepsilon^2\right): &\quad \ddot{y}_2 + y_2 = \dot{y}_1 - \dot{y}_0^2\dot{y}_1,\\
O\left(\varepsilon^3\right): &\quad \ddot{y}_3 + y_3 = \dot{y}_2 - \dot{y}_0^2\dot{y}_2 - \dot{y}_1^2\dot{y}_0.
\end{aligned}$$

The solutions are

$$\begin{aligned}
y_0\left(t\right) &= Ae^{i\left(t-t_0\right)} + c.c.,\\
y_1\left(t\right) &= \frac{1}{24}iA^3e^{i\left(t-t_0\right)} + \frac{1}{2}A\left(1-AA^*\right)\left(t-t_0\right)e^{i\left(t-t_0\right)} - \frac{1}{24}iA^3e^{3i\left(t-t_0\right)} + c.c,\\
y_2\left(t\right) &= \left(\frac{1}{32}A^3 - \frac{3}{64}A^4A^*\right)e^{i\left(t-t_0\right)}\\
&\quad + \left(-\frac{1}{24}iA^4A^* + \frac{1}{16}iA^3(A^*)^2 + \frac{1}{48}iA^3 + \frac{1}{48}iA^2(A^*)^3 - \frac{1}{8}iA\right)\left(t-t_0\right)e^{i\left(t-t_0\right)}\\
&\quad + \left(\frac{3}{8}A^3(A^*)^2 - \frac{1}{2}A^2A^* + \frac{1}{8}A\right)\left(t-t_0\right)^2e^{i\left(t-t_0\right)}\\
&\quad + \left(\frac{3}{64}A^4A^* - \frac{1}{32}A^3 + \frac{1}{192}A^5\right)e^{3i\left(t-t_0\right)}\\
&\quad - \frac{1}{16}iA^3\left(1-AA^*\right)\left(t-t_0\right)e^{3i\left(t-t_0\right)} - \frac{1}{192}A^5e^{5i\left(t-t_0\right)} + c.c.,
\end{aligned}$$

$$
\begin{aligned}
y_3(t) = &\left(-\frac{1}{384} i A^6 A^* + \frac{37}{1536} i A^5 (A^*)^2 + \frac{1}{2304} i A^5\right.\\
&\left.+\frac{1}{512} i A^4 (A^*)^3 - \frac{7}{256} i A^4 A^* - \frac{1}{128} i A^3\right) e^{i(t-t_0)}\\
&+\begin{pmatrix} +\frac{1}{1152} A^6 A^* + \frac{5}{128} A^5 (A^*)^2 - \frac{119}{1152} A^4 (A^*)^3 + \frac{11}{384} A^3 (A^*)^4 \\ -\frac{7}{128} A^4 A^* + \frac{1}{48} A^3 + \frac{11}{64} A^3 (A^*)^2 - \frac{1}{64} A^2 (A^*)^3 \end{pmatrix} (t-t_0)\, e^{i(t-t_0)}\\
&+\begin{pmatrix} +\frac{3}{64} i A^5 (A^*)^2 - \frac{3}{32} i A^4 (A^*)^3 - \frac{1}{24} i A^4 A^* - \frac{1}{32} i A^3 (A^*)^4 \\ +\frac{3}{32} i A^3 (A^*)^2 + \frac{1}{192} i A^3 + \frac{1}{16} i A^2 A^* + \frac{1}{48} i A^2 (A^*)^3 - \frac{1}{16} i A \end{pmatrix} (t-t_0)^2\, e^{i(t-t_0)}\\
&+\left(-\frac{5}{16} A^4 (A^*)^3 + \frac{9}{16} A^3 (A^*)^2 - \frac{13}{48} A^2 A^* + \frac{1}{48} A\right)(t-t_0)^3\, e^{i(t-t_0)}\\
&+\begin{pmatrix} +\frac{1}{4608} i A^7 + \frac{7}{512} i A^6 A^* - \frac{37}{1536} i A^5 (A^*)^2 - \frac{1}{128} i A^5 \\ -\frac{1}{512} i A^4 (A^*)^3 + \frac{1}{128} i A^3 + \frac{7}{256} i A^4 A^* \end{pmatrix} e^{3i(t-t_0)}\\
&+\begin{pmatrix} -\frac{1}{96} A^6 A^* - \frac{7}{64} A^5 (A^*)^2 + \frac{1}{128} A^5 + \frac{1}{384} A^4 (A^*)^3 \\ +\frac{21}{128} A^4 A^* - \frac{1}{16} A^3 \end{pmatrix} (t-t_0)\, e^{3i(t-t_0)}\\
&+\left(-\frac{5}{64} i A^5 (A^*)^2 + \frac{1}{8} i A^4 A^* - \frac{3}{64} i A^3\right)(t-t_0)^2\, e^{3i(t-t_0)}\\
&+\left(\frac{17}{2304} i A^5 - \frac{17}{1536} i A^6 A^* - \frac{5}{4608} i A^7\right) e^{5i(t-t_0)}\\
&+\left(\frac{5}{384} A^6 A^* - \frac{5}{384} A^5\right)(t-t_0)\, e^{5i(t-t_0)}\\
&+\frac{1}{1152} i A^7 e^{7i(t-t_0)} + c.c.
\end{aligned}
$$

We choose the homogeneous parts to the solutions y_1, y_2 and y_3 in such a manner that the solutions vanish at the initial time t_0, i.e. $y_1(t_0) = y_2(t_0) = y_3(t_0) = 0$. Next, we renormalize the integration constant A and create a new renormalized quantity $\mathcal{A}$ as

$$A = \mathcal{A} + a_1 \varepsilon + a_2 \varepsilon^2 + a_3 \varepsilon^3 + O\left(\varepsilon^4\right),$$

where the coefficients $a_1, a_2, a_3, \ldots$ are chosen to absorb the homogeneous parts of the solutions $y_1, y_2, \ldots$. Choosing

$$
\begin{aligned}
a_1 &= -\frac{i}{24}\mathcal{A}^3,\ a_2 = -\frac{\mathcal{A}^3}{32}\left(1 - \frac{3}{2}\mathcal{A}\mathcal{A}^* + \frac{1}{6}\mathcal{A}^2\right),\\
a_3 &= \frac{1}{1152} i \mathcal{A}^7 - \frac{17}{1536} i \mathcal{A}^6 \mathcal{A}^* + \frac{17}{2304} i \mathcal{A}^5 - \frac{37}{1536} i \mathcal{A}^5 (\mathcal{A}^*)^2 + \frac{7}{256} i \mathcal{A}^4 \mathcal{A}^* + \frac{1}{128} i \mathcal{A}^3,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
y_0(t) &= \mathcal{A} e^{i(t-t_0)} + c.c.,\\
y_1(t) &= \left(\frac{1}{2}\mathcal{A}\left(1 - \mathcal{A}\mathcal{A}^*\right)(t-t_0)\, e^{i(t-t_0)} - \frac{1}{24} i \mathcal{A}^3 e^{3i(t-t_0)}\right) + c.c.,\\
y_2(t) &= \left(\frac{1}{16} i \mathcal{A}^3 (\mathcal{A}^*)^2 - \frac{1}{8} i \mathcal{A}\right)(t-t_0)\, e^{i(t-t_0)}\\
&\quad+\frac{1}{8}\mathcal{A}\left(\mathcal{A}\mathcal{A}^* - 1\right)\left(3\mathcal{A}\mathcal{A}^* - 1\right)\ (t-t_0)^2\ e^{i(t-t_0)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{3}{64}\mathcal{A}^4\mathcal{A}^*-\frac{1}{32}\mathcal{A}^3\right)e^{3i(t-t_0)}+\frac{1}{16}i\mathcal{A}^3\left(\mathcal{A}\mathcal{A}^*-1\right)\ (t-t_0)\ e^{3i(t-t_0)}\\
&-\frac{1}{192}\mathcal{A}^5e^{5i(t-t_0)}+c.c.,\\
y_3\left(t\right)=&\left(-\frac{13}{128}\mathcal{A}^4(\mathcal{A}^*)^3+\frac{11}{64}\mathcal{A}^3(\mathcal{A}^*)^2\right)(t-t_0)\,e^{i(t-t_0)}\\
&+\left(-\frac{3}{32}i\mathcal{A}^4(\mathcal{A}^*)^3+\frac{3}{32}i\mathcal{A}^3(\mathcal{A}^*)^2+\frac{1}{16}i\mathcal{A}^2\mathcal{A}^*-\frac{1}{16}i\mathcal{A}\right)(t-t_0)^2\,e^{i(t-t_0)}\\
&+\left(-\frac{5}{16}\mathcal{A}^4(\mathcal{A}^*)^3+\frac{9}{16}\mathcal{A}^3(\mathcal{A}^*)^2-\frac{13}{48}\mathcal{A}^2\mathcal{A}^*+\frac{1}{48}\mathcal{A}\right)(t-t_0)^3\,e^{i(t-t_0)}\\
&+\left(-\frac{37}{1536}i\mathcal{A}^5(\mathcal{A}^*)^2+\frac{1}{128}i\mathcal{A}^3+\frac{7}{256}i\mathcal{A}^4\mathcal{A}^*\right)e^{3i(t-t_0)}\\
&+\left(-\frac{7}{64}\mathcal{A}^5(\mathcal{A}^*)^2+\frac{21}{128}\mathcal{A}^4\mathcal{A}^*-\frac{1}{16}\mathcal{A}^3\right)(t-t_0)\,e^{3i(t-t_0)}\\
&+\left(-\frac{5}{64}i\mathcal{A}^5(\mathcal{A}^*)^2+\frac{1}{8}i\mathcal{A}^4\mathcal{A}^*-\frac{3}{64}i\mathcal{A}^3\right)(t-t_0)^2\,e^{3i(t-t_0)}\\
&+\left(\frac{17}{2304}i\mathcal{A}^5-\frac{17}{1536}i\mathcal{A}^6\mathcal{A}^*\right)e^{5i(t-t_0)}+\left(\frac{5}{384}\mathcal{A}^6\mathcal{A}^*-\frac{5}{384}\mathcal{A}^5\right)(t-t_0)\,e^{5i(t-t_0)}\\
&+\frac{1}{1152}i\mathcal{A}^7e^{7i(t-t_0)}+c.c.
\end{aligned}
$$

Remark 1 DeVille [7] have obtained same result correct upto $O\left(\varepsilon^3\right)$. However, their computed expression of a_2 is not correct. We have made the correction in the expression of a_2.

We observe that each of $y_1\left(t\right)$, $y_2\left(t\right)$, $y_3\left(t\right)$ contains secular terms. As a consequence the solution

$$y\left(t\right)=y_0\left(t\right)+y_1\left(t\right)\varepsilon+y_2\left(t\right)\varepsilon^2+y_3\left(t\right)\varepsilon^3+O\left(\varepsilon^4\right)$$

becomes divergent as $t\to\infty$. To regularize the perturbation series using RGM an arbitrary time τ is introduced and $t-t_0$ is split as $(t-\tau)+(\tau-t_0)$. The terms containing $\tau-t_0$ is observed in the renormalized counterpart $\mathcal{A}$ of the constant of integration A. Since the final solution should not depend upon the choice of the arbitrary time τ, so

$$\left.\frac{\partial y}{\partial\tau}\right|_{\tau=t}=0 \tag{28}$$

for any t. However, DeVille et. al. [7] have simplified this condition and proposed an equivalent condition as

$$\left.\frac{\partial y}{\partial t_0}\right|_{t_0=t}=0. \tag{29}$$

We note that renormalized counterpart $\mathcal{A}$ is no longer a constant of motion in RGM. The RG condition (29) is developed in such a manner that one need to differentiate the terms containing $e^{i(t-t_0)}$, $e^{-i(t-t_0)}$, $(t-t_0)\,e^{i(t-t_0)}$ and $(t-t_0)\,e^{-i(t-t_0)}$ and thereafter substituting $t_0=t$ the resultant expression is equated to zero. The other terms related to higher harmonics are not involved in RG condition. Simplifying RG condition (29) we get

$$\frac{\partial\mathcal{A}}{\partial t_0}=\mathcal{A}i-\frac{1}{2}\mathcal{A}\left(\mathcal{A}\mathcal{A}^*-1\right)\varepsilon-\frac{1}{8}i\mathcal{A}\left(1-\frac{1}{2}\mathcal{A}^2(\mathcal{A}^*)^2\right)\varepsilon^2-\frac{1}{64}\mathcal{A}^3(\mathcal{A}^*)^2\left(\frac{13}{2}\mathcal{A}\mathcal{A}^*-11\right)\varepsilon^3$$

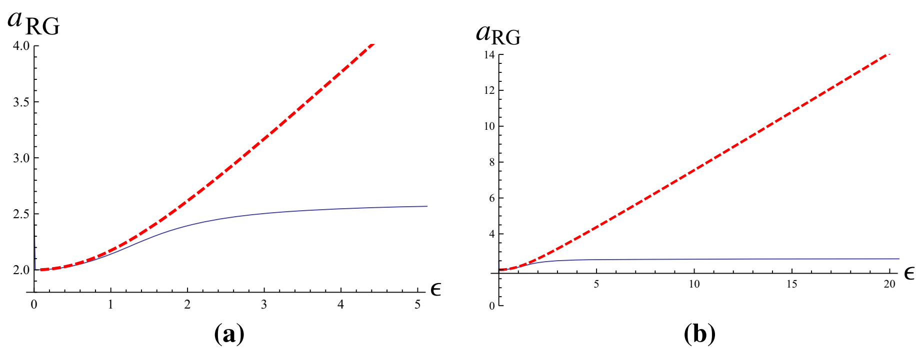

Fig. 3 Graph of $a_{RG}(\varepsilon)$ (by *solid lines*) correct upto $O\left(\varepsilon^4\right)$ and compared with exact graph of $a(\varepsilon)$ (by *dotted lines*) for $0 < \varepsilon \leq 5$ in (**a**) and for $0 < \varepsilon \leq 20$ in (**b**)

to the order $O\left(\varepsilon^4\right)$. Taking $\mathcal{A} = \frac{R}{2} e^{i(t+\theta)}$ we obtain corresponding amplitude and phase flow equations to the order $O\left(\varepsilon^4\right)$ as

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{1}{2} R \left(1 - \frac{R^2}{4}\right) \varepsilon + \frac{1}{1024} R^5 \left(11 - \frac{13}{8} R^2\right) \varepsilon^3 + O\left(\varepsilon^4\right), \tag{30}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{8} \left(1 - \frac{R^4}{32}\right) \varepsilon^2 + O\left(\varepsilon^4\right). \tag{31}$$

To the authors' knowledge these higher order flow equations are reported for the first time in the literature. We remark that above flow equations match exactly with $O(\varepsilon^3)$ flow equations of the Van der Pol equation [8]. Although not done explicitly, we expect that the $O\left(\varepsilon^4\right)$ VdP flow equations would also have the equivalent forms. For latter reference, we also write down the order $O(\varepsilon^2)$ solution of the Rayleigh equation [6]

$$y(t) = R(t)\cos(t+\theta) + \frac{\varepsilon}{96} R(t)^3 (\sin 3(t+\theta) - \sin(t+\theta)). \tag{32}$$

Solving the amplitude equation (30) by numerical method and taking the limit as $t \to \infty$ so that for a fixed value of ε we have $R \to a_{RG}(\varepsilon)$, the approximation of the amplitude of limit cycle of Rayleigh equation (1) by RGM, we obtain Fig. 3 representing ε dependence of the amplitude a_{RG} by solid lines.

Thus we observe that the RG flow equation to the order $O\left(\varepsilon^4\right)$ for the amplitude does not give good approximation to the exact solution for moderate and large values of ε.

Improved RG Method: Nonlinear Time

In RGM an arbitrary time τ is introduced in between current time t and the initial time t_0 so that $t - t_0 = (t - \tau) + (\tau - t_0)$ in order to remove the divergent terms in the naive perturbation expansion for the solution of the given differential equation. The solution is renormalized by suitable choice of the constants of integration to remove the terms containing $(\tau - t_0)$ and keeping the terms having $(t - \tau)$. Since the solution should be independent of the arbitrary time τ, the RG condition

$$\left.\frac{\partial y}{\partial \tau}\right|_{\tau=t} = 0$$

is applied to the renormalized solution. However, in the previous section we have seen that the method fails to produce good approximations to the exact solution for $\varepsilon \sim O(1)$. Our target is not only to remove the divergent terms in the solution but also to introduce some control parameter $h(\varepsilon)$ which can control the RG solution in such a manner that this solution ultimately converges to the exact solution. Moreover, our another goal is to achieve this accuracy by merely solving the differential equation to a minimal order of the expansion parameter, viz., upto $O\left(\varepsilon^2\right)$ or less.

Since the basic idea is to split the time difference $t - t_0$ by introduction of an arbitrary time, so we can write

$$t - t_0 = \left(t - \frac{\tau}{\varepsilon}\right) + \left(\frac{\tau}{\varepsilon} - t_0\right).$$

From now on let us assume that $0 << \varepsilon <\approx 1$. The case $\varepsilon >\approx 1$ will be commented upon later. The constants of integration can be renormalized in order to remove the terms containing $\left(\frac{\tau}{\varepsilon} - t_0\right)$ from the solution keeping the terms containing $\left(t - \frac{\tau}{\varepsilon}\right)$. Finally analogous to the classical RG method we put $t = \frac{\tau}{\varepsilon}$, i.e. $\tau = \varepsilon t$, in

$$\frac{\partial y}{\partial \tau} = 0 \tag{33}$$

giving rise to an improved form of the RG flow equation to remove secular terms involving $\left(t - \frac{\tau}{\varepsilon}\right)$. So far the improved method does not produce any qualitative new result compared to the RGM and so we must get the same phase and amplitude equation as deduced in "Computation of Amplitude by RG Method" section.

We next proceed one step further. As stated already in the Introduction, we now exploit the possibility of extending the original linear t dependence of τ viz., $\tau = t$ of RGM in removing the explicit divergences by a *nonlinear dependence* $\tau = \varepsilon t$ along with the *additional condition* that $\tau \to \varepsilon^{-n}\phi(\tilde{\tau})$, where ϕ a slowly varying scaling function of the O(1) rescaled variable $\tilde{\tau} = \varepsilon\tilde{t} \sim O(1)$, as the original linear time $t \to \infty$ following the *scales* $t \sim \varepsilon^{-n}\tilde{t},\ n = 1, 2, \ldots$. (Note that linear time flows with uniform rate 1 and τ is nonlinear since rate $\dot{\phi}(\tilde{\tau}) < 1$). It follows that for a given nonlinear differential system, such a nonlinear time dependence always exists and nontrivial, provided one invokes a *duality principle* transferring nonlinear influences from the far asymptotic region into the finite observable sector in a cooperative manner [14–16]. In Appendix, we give a brief overview of the *novel* analytic framework extending the standard classical analysis to one that supports naturally the above stated *duality structure* and the emergent nonlinear scaling patterns typical for a given nonlinear system.

In fact, as the linear time $t \to \infty$ following the above hierarchy of scales, there exists $\tilde{t}_n$ such that $1 << (\varepsilon t)^n < \varepsilon^{-n} < \tilde{t}_n$ and satisfying *the inversion law* $\tilde{t}_n/\varepsilon^{-n} \propto \varepsilon^{-n}/(\varepsilon t)^n$. This inversion law makes a room for transfer of effective influences, typical for the nonlinear system concerned, from nonobservable sector $t > \varepsilon^{-n}$ to the observable sector $t < \varepsilon^{-n}$ bypassing the dynamically generated singular points denoted by the scales ε^{-n}. Notice the nonlinear connection between scales of the form $\varepsilon^n\tilde{t}_n$ with the scale εt via duality structure (c.f. Appendix). Let $\tilde{t}(t) = \lim n \to \infty \left(\tilde{t}_n\right)^{1/n}$ so that $\varepsilon t < \varepsilon^{-1} < \tilde{t}(t)$ and $\tilde{t}/\varepsilon^{-1} \propto \varepsilon^{-1}/(\varepsilon t)$. Define

$$h_0(\tilde{\tau}) = \lim_{n\to\infty} \log_{\varepsilon^{-n}} \tilde{t}_n/\varepsilon^{-n}. \tag{34}$$

Here, the scaling exponent h_0 corresponds to the visibility norm (Appendix), that can access (encode) the non-perturbative region (information) of the nonlinear system and $\tilde{\tau}$ is an O(1) rescaled variable. The exponent h_0 is *scaling invariant* in the sense that it appears uniformly for every n as $t \to \infty$ through the scales $\tilde{t}_n = \varepsilon^{-n}\varepsilon^{-nh_0(\tilde{\tau})}$. As a consequence, a significant amount of asymptotic scaling information in the limit $t \to \infty$ could be simply retrieved by considering the scaling limit instead at $t = \varepsilon^{-1}$.

Exploiting the above insight, one now writes the nonperturbative scaling limit in the form

$$\tau = \lim \varepsilon t = \varepsilon^{-h_{RG}(\tilde{\tau})} > 1, \quad \varepsilon < 1 \tag{35}$$

as $t \to \varepsilon^{-1}$. Moreover, $h_{RG}(\tilde{\tau}) = 1 - h_0(\tilde{\tau})$. As noted already, the scaling exponent $h_0(\tilde{\tau})$ here encodes the *effective* cooperative influence of far asymptotic sector $t > \varepsilon^{-n}$ into the observable sector $1 < t < \varepsilon^{-n}$ by the inversion mediated duality principle. As pointed out in Appendix, the duality principle *does* allow asymptotic limiting (non-perturbative) behaviour of the nonlinear system to be encoded into the scaling exponents of the nonlinear time τ that, in turn, offers an efficient handle in uncovering key dynamical information of the said system. Notice that, in the absence of the said duality the linear time t can in principle attain the scale ε^{-1} (say), and as a consequence $h_0 = 0$, retrieving the ordinary scaling of $\tau = \varepsilon t \sim \varepsilon^{-1}$ as $t \sim \varepsilon^{-2}$. This also establishes, in retrospect, that the scaling exponent $h_0(\varepsilon)$ is well defined and can exist nontrivially i.e. $h_0 \sim O(1)$ in a nonlinear problem (c.f. Appendix). As a consequence, *the RG control parameter h_{RG} can be of both the signs, with relatively small numerical value in fully developed nonlinear systems $\varepsilon \gg 1$, but with a possible $O(1)$ variations for $\varepsilon \sim O(1)$ or less.*

The above construction actually tells somewhat more. Corresponding to the first generation scales ε^{-n}, one can, in fact, have the *second generation* nonlinear scales

$$\tau_m = \lim \varepsilon^m t = \varepsilon^{-mh^m_{RG}(\tilde{\tau})} > 1, \quad \varepsilon < 1, m > 1 \tag{36}$$

as $t \to \varepsilon^{-m}$ with $h^1_{RG} = h_{RG}$. The *nonlinear time* τ now stands for these hierarchy of scales $\{\tau_m\}$. Consequently, as the linear time t approaches ∞ through the first generation linear scales, the slowly varying nonlinear time τ approaches either to ∞ or 0 at slower and slower rates as represented by the numerically small RG scaling exponents $h^m_{RG}(\varepsilon)$, each of which remains almost constant over longer and longer intervals of ε^{-1} (as $\varepsilon^{-1} \to \infty$). In the present paper we show how the first two scaling exponents $h_i(\tilde{\tau}),\ i = 1, 2$ relate to the nonperturbative properties of the limit cycle. We expect higher order scaling exponents h_m would have vital role in bifurcation of nonautonomous systems. This problem will be investigated elsewhere.

Let us remark that for $\varepsilon > 1$, we consider instead the first generation scales as ε^n, and the duality is invoked for variables satisfying $t/\varepsilon < \varepsilon < \tilde{t}(t)$ so that the asymptotic scaling variables are derived as $\tau_m = \varepsilon^{mh^m_{RG}(\tilde{\tau})},\ \varepsilon > 1$ where $h^m_{RG} = 1 - h^m_0$. Moreover, said proliferation of nonlinear scales (36) actually continues ad infinitum. In fact, interpreting each second generation scale τ_m, m fixed, as first generation scale, and iterating above steps one associates third generation scales $\tau_{m_k},\ k = 1, 2, \ldots$, and so on.

It now follows, from the above general remarks on the behaviour of h_{RG}, that the nonlinear time τ actually approaches 0 or ∞ as $\tau \sim (\log \varepsilon)^{-\alpha}$ or $\tau \sim (\log \varepsilon)^{\alpha},\ \alpha > 0$ respectively as $\varepsilon \to \infty$. However, one must have $\tau = \varepsilon^{-h_{RG}(\varepsilon)} \to \infty$ as $\varepsilon \to 0$. An example of the asymptotic behaviours of h_{RG} is given by

$$\tau_m = \varepsilon^{\pm\alpha_m \frac{\log\log\varepsilon}{\log\varepsilon}} \quad \text{for } \varepsilon \to \infty,$$

which one expects to verify explicitly in evaluation of asymptotic quantities, such as amplitude of a periodic cycle, in a nonlinear system.

In the IRGM, we exploit this duality induced nontrivial scaling information to rewrite the lowest order perturbative flow Eqs. (30) and (31) as the asymptotic RG flow equations in the limit $t \to \infty$

$$\frac{\mathrm{d}a}{\mathrm{d}\tau_1} = \frac{1}{2}a\left(1 - \frac{a^2}{4}\right), \tag{37}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau_2} = -\frac{1}{8}\left(1 - \frac{a^4}{32}\right) \tag{38}$$

for the amplitude $a = a(\tilde{\tau})$ and the phase $\psi = \psi(\tilde{\tau})$ of the limit cycle of both the Rayleigh and Van der Pol equations, involving slowly varying nonlinear time scales $\tau_i,\ i = 1, 2$. The asymptotic scaling functions $\tau_1 = \phi_1(\varepsilon t) = \varepsilon^{h^1_{RG}}$ and $\tau_2 = \phi(\varepsilon^2 t) = \varepsilon^{2h^2_{RG}}$ are activated invoking nonlinear limits as in (36) as $t \to \varepsilon^{-1}$ and $t \to \varepsilon^{-2}$ successively in the above equations. The slowly varying almost constant scaling functions ϕ_1 and ϕ_2, satisfying $|\ddot{\phi}_i| << |\dot{\phi}_i^2| << 1$, are assumed to have a rhythmic pattern over the cycle: when ϕ_1 varies slowly, ϕ_2 remains almost constant i.e. $\dot{\phi}_1 > 0,\ \dot{\phi}_2 \approx 0$ and vice versa successively on the cycle (c.f. Appendix 2). Nontrivial ultrametric neighbourhood structure induced asymptotically by duality principle (c.f. Appendix 1) can indeed support such *locally constant* nonlinear rhythmic behaviour. The above flow equations may therefore be considered *exact* and encode non-perturbative information of the limit cycle variables a and ψ respectively. The conventional perturbative RG flow equations in the linear time t is now extended into the non-perturbative flow equations in the nontrivial scaling variable $\tau_i = \varepsilon^{ih^i_{RG}(\tilde{\tau})},\ i = 1, 2$ involving the nonlinearity parameter $\varepsilon > 1$. The perturbative fixed point for the amplitude equation at $a = 2$ for $t \to \infty$ corresponding to the periodic oscillation with $\varepsilon << 1$ is superimposed by *small scale periodic* flow of amplitude $a(\tau_1)$ over the entire cycle. The associated phase $\psi(\tau_2)$ then flow at a slower rate *linearly with the higher order scale* τ_2 when $a(\tau_1)$ remains almost constant over a relatively small period of time.

The RG estimated approximate formulae for the amplitude $a(\varepsilon)$ for the Rayleigh and Van der Pol limit cycles are obtained from the Eq. (37) in the "Approximate Formula for Amplitude" section, when appropriate boundary condition, derived either from exact computation or from perturbative analysis, is used for a suitable finite value of ε. In the next "Approximate Limit Cycle" section, we present the efficient graphs of the Rayleigh and VdP limit cycle parametrized by the nonlinear scales $\tau_i = \phi_i(\tilde{\tau}),\ \tilde{\tau} \sim O(1)$ for fixed values of the nonlinearity parameter ε.

As it turns out, entire onus in the improved RG analysis essentially rests in proper estimation/identification of the scaling functions h^i_{RG} (i.e. $\phi_i(\tilde{\tau})$) which should yield correct dynamical properties of a nonlinear system. We hope to undertake more detailed and systematic analysis for determining h^i_{RG} elsewhere. In this work we limit ourselves only to show that IRGM can indeed yield correct amplitude and solution for the Rayleigh and VdP systems provided one makes appropriate choice of h^i_{RG} based on clues from exact computations and previously known approximate results (for instance the perturbative RGM). We remark finally that the perturbative RG method is known to extend the conventional multiple scale method [6]. Nonlinear time formalism introduces new set of nonlinear scales $\phi_n(\tilde{\tau})$ associated with ordinary scales ε^n. We study here the nontrivial applications of such nonlinear scaling functions.

Approximate Formula for Amplitude

We shall now use the above asymptotic amplitude flow Eq. (37) to find analytic approximations of the amplitudes of the limit cycle for both the Rayleigh and Van der Pol equations.

By a direct integration, one obtains from (37)

$$\ln\left(a^2-4\right)-2\ln a=-\varepsilon^{h_{RG}}-0.87953 \tag{39}$$

as the Rayleigh limit cycle amplitude where we use the boundary condition the value $a = 2.17271$ for $\varepsilon = 1$ (this choice simplifies calculation). It follows immediately that for suitable choices of the control parameter h_{RG} one can achieve efficient matching for the estimated amplitude $a_E(\varepsilon)$. For example, using the HAM generated approximate formula (27) for $a_E(\varepsilon)$, we can determine the control parameter $h_{RG}(\varepsilon)$ by the formula

$$h_{RG}=\frac{1}{\ln\varepsilon}\ln\left\{\left|\ln\left(\frac{a^2}{a^2-4}\right)-0.87953\right|\right\}. \tag{40}$$

In Fig. 4, we display the typical piece-wise smooth form of $h_{RG}(\varepsilon)$ given by (40) for the Rayleigh limit cycle amplitude that would reproduce the HAM generated amplitude with relative error less that 1 %. Clearly, the graph reveals variability of h_{RG} for moderate values of ε, but the variability dies out fast for larger values ε, as expected.

We recall that the corresponding graph of the exact computed values of VdP amplitude $a(\varepsilon)$, on the other hand, has a hump like shape with a maximum roughly at $\varepsilon \approx 2.0235$ and having the asymptotic limits 2 as $\varepsilon \to 0$ and ∞. Lopez et al. [4] obtained HAM generated approximate formula for the VdP amplitude with relative error less than 0.05 % at the order $O\left(\varepsilon^4\right)$. It is interesting to note that the RG generated formula (39) can reproduce the exact computed values of the VdP amplitude with error less than 0.05 % directly from only the first order RG flow equation. To achieve this goal we first intuitively guess a piecewise smooth formula for the estimated amplitude a_E by

$$a_E(\varepsilon)=\begin{cases}1.998+\dfrac{0.015}{8.121\,e^{-2.139\,\varepsilon}+0.512\,e^{0.043\,\varepsilon}} & 0<\varepsilon<3,\\[2ex] 2.0025+\dfrac{0.031}{0.5\,e^{-2.033(\varepsilon-2.183)}+1.869\,e^{0.087(\varepsilon-6.376)}} & 3\le\varepsilon\le 50,\end{cases} \tag{41}$$

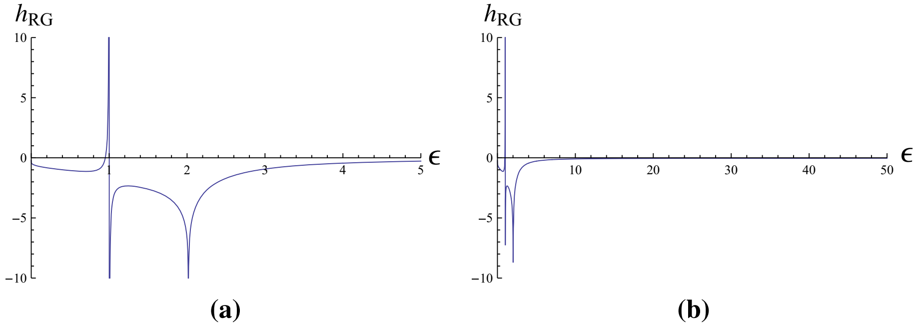

Fig. 4 The graph of $h_{RG}(\varepsilon)$ used for approximation of the amplitude of the Rayleigh equation (1) by HAM given by (40) for $0 < \varepsilon \le 5$ in (**a**) and for $0 < \varepsilon \le 50$ in (**b**)

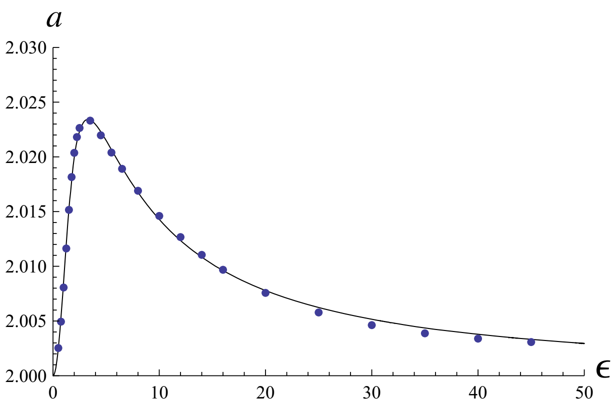

Fig. 5 The exact amplitude of Van der Pol Equation (2) (by *solid line*) and its approximation $a_E(\varepsilon)$ given by (41) (by *bold points*) for $0 < \varepsilon \leq 50$

keeping the maximum relative percentage error

$$\left|\frac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100\right|$$

less than 0.05 %. This shows that the approximation is quite accurate. The graph of $a_E(\varepsilon)$ is compared with the exact values in Fig. 5. One may as well use a least square fit of the exact data instead of the above fit. We do not pursue this approach here.

Using this efficient formula for the VdP amplitude, we then obtain the RG flow equation in the form

$$\ln\left(a^2 - 4\right) - 2\ln a = -\varepsilon^{h_{RG}} - 4.08785, \tag{42}$$

where we use the boundary condition $a = 2.0086$ for $\varepsilon = 1$ (for simplicity of calculation) for the VdP amplitude. Inverting this equation, we finally obtain the corresponding RG control parameter

$$h_{RG} = \frac{1}{\ln\varepsilon}\ln\left\{\left|\ln\left(\frac{a_E^2}{a_E^2 - 4}\right) - 4.08785\right|\right\}. \tag{43}$$

Figure 6 displays the piecewise smooth variation of h_{RG} with ε. The rapid $O(1)$ variation for moderate values of ε is evident in Fig. 6(a). As expected, h_{RG} dies out fast for larger values of ε. However, a change in sign is noticed here already for $\varepsilon > 20$ (Fig. 6(b)). One expects many more such small scale sign variations as $\varepsilon \to \infty$. This particular form of the control parameter h_{RG}, in turn, would reproduce the VdP amplitude with relative error less than 0.05 %. As this level of accuracy is achieved only at the order $O(\varepsilon)$, the improved RGM may be considered to be more efficient and advantageous compared to the HAM.

Alternatively, the amplitude equation (37) can be inverted as

$$a(\tilde{\tau}) = \frac{a_0}{\sqrt{e^{-\tilde{\tau}} + \frac{a_0^2}{4}(1 - e^{-\tilde{\tau}})}}, \tag{44}$$

where a_0 is estimated from the exact value of amplitude $a(\varepsilon_0)$ for a suitably chosen value of ε, for instance $\varepsilon = 1$. Recall that for the VdP equation $\tau \sim (\log\varepsilon)^\alpha$ and for the Rayleigh

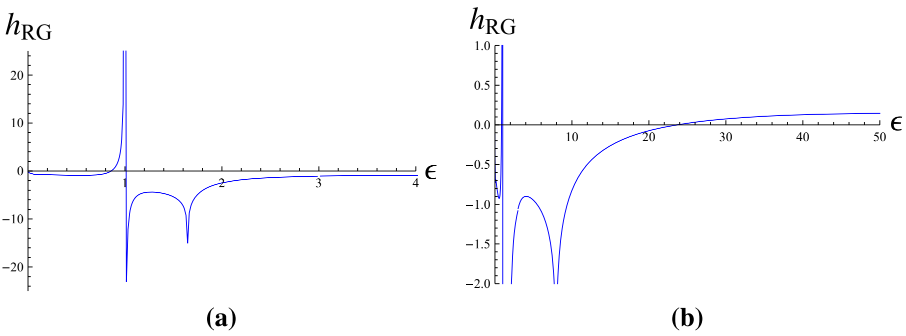

Fig. 6 The graph of h_{RG} (ε) used for approximation (43) of the amplitude of the Van der Pol equation (2) for $0 < \varepsilon \leq 4$ in (*a*) and for $0 < \varepsilon \leq 50$ in (*b*)

equation $\tau \sim (\log \varepsilon)^{-\alpha}$ for $\varepsilon \to \infty$ and $\alpha > 0$. By adjusting suitably the values of α over appropriate intervals on ε one should be able to obtain efficient matchings with the exact values of $a(\varepsilon)$.

To summarize, the recipe for deriving approximate formula for limit cycle amplitude of a nonlinear system can be stated as follows: Determine the first order (perturbative) RG flow equation for amplitude in the nonlinear time τ. This will yield an explicit formula for amplitude a as a function of the nonlinearity parameter ε and the control parameter h_{RG}. Efficient match with the exact amplitude can be achieved by right choice of the control parameter h_{RG} or α. Alternatively, determine an efficient formula for $a(\varepsilon)$ by inspection (expert guess) or by appropriate curve fitting methods. Then determine the control parameter h_{RG} by an inversion of the estimated amplitude $a_E(\varepsilon)$ as in Eq. (43) (and Fig. 6). Since the equations concerned form a closed system, this already gives a (numerical) proof of the unique existence of h_{RG} for a given nonlinear oscillation.

Approximate Limit Cycle

Here we calculate the approximate limit cycle orbit for the Rayleigh and VdP equations for a sufficiently large $\varepsilon > 1$. Perturbative RGM fails to give correct relaxation oscillation solution. The first order solution given in "Computation of Amplitude by HAM" section by HAM is also found insufficient. In this work we do not under take the problem of computing approximate limit cycle by HAM, which has been addressed by Lopez et al. [4] for the VdP equation. Our aim here is to highlight the strength of IRGM over perturbative RGM.

For a sufficiently large time $t \to \varepsilon^n$, n large, but fixed, the slowly varying nonlinear scales τ_1 and τ_2 are activated in a successive rhythmic manner, as explained in "Improved RG Method: Nonlinear Time" section (see also Appendix 2), so that the perturbative solution given in (32) is extended to the asymptotic limit cycle (relaxation oscillation) solution

$$y(\tau_1, \tau_2) = a(\tau_1)\cos(\varepsilon^n + \psi(\tau_2)) + Y, \tag{45}$$

where the amplitude a and phase ψ *flow* along the cycle following the nonperturbative flow Eqs. (37) and (38) in the asymptotic scaling variables τ_1 and τ_2 respectively. Here, Y encodes all the renormalized perturbative terms depending on higher order, slowly varying nonlinear scales τ_i, $i > 2$. The corresponding velocity component $\dot{y} = \frac{\partial y}{\partial t}$ at $t = \varepsilon^n$ has the form

$$\dot{y}(\tau_1, \tau_2) = -a(\tau_1)\sin(\varepsilon^n + \psi(\tau_2)) + \sum_i \dot{\tau}_i \frac{\partial \tilde{Y}}{\partial \tau_i}. \tag{46}$$

where $\tilde{Y}$ represents possible slow variations in all the nonlinear scales $\tau_i, i \geq 1$. Equations (45) and (46) are the parametric equations of the limit cycle, parametrized by multiple nonlinear scales, when slowly varying amplitude a and phase ψ are computed from (37) and (38) respectively. An alternative derivation of (45) and (46) based purely on duality induced nonlinear scales is given in Appendix 2. We remark that (45) and (46) actually represent the general form of the limit cycle for a much larger class of Lienard systems having unique limit cycle. *Typical geometric shape of the periodic cycle of a given nonlinear system is controlled entirely by the rhythmic cooperative, almost constant variations of the nonlinear scales τ_i*. As explained in "Improved RG Method: Nonlinear Time" section (See also Appendix 1), scale invariance of the scaling functions $\tau_1 = \phi_1(\tilde{t}_1/\varepsilon)$ and $\tau_2 = \phi_1(\tilde{t}_2/\varepsilon^2)$ tells that as the linear time $t \to \varepsilon^n$, $\tau_1 \to \phi_1(1) = 1$ and $\tau_2 \to \phi_2(1) = 1$ with $\tilde{t}_1 \to \varepsilon$ and $\tilde{t}_2 \to \varepsilon^2$. We now set the initial conditions $a(1) = a_{\text{amp}}(\varepsilon)$, $a'(1) = \frac{a(1)}{2}(1 - \frac{a(1)^2}{4})$ and $\psi(1) = 0$, where $a_{\text{amp}}(\varepsilon)$ is the exact (experimental) value of the amplitude of the limit cycle. Setting further $\tau_1 = 1 + \eta_1$ and $\tau_2 = 1 + \eta_2$ as $t \to \varepsilon^n$, both amplitude and phase flow Eqs. (37) and (38) now yield linear flow of amplitude and phase relative to the respective small scale slow, almost constant variables η_1 and η_2, satisfying $|\eta_i^2| << 1$ and $|\ddot{\eta}_i| << |\dot{\eta}_i^2| << 1$, those vary in a rhythmic manner (Appendix 2). The exact (experimental) limit cycle could now be approximated with any desired accuracy by smooth matching of elementary straight line segments of the form (i) $z - Z_0 = k(y - Y_0)$ and circular arcs of the form (ii) $(y - Y_0)^2 + (z - Z_0)^2 = a^2(\tau_1)$ over judiciously chosen intervals in y in the (y, z) plane, where $z = \dot{y}$, $k = -\tan(\varepsilon^n + \psi(\tau_2))$, $Y_0 = Y$ and $Z_0 = \sum_i \dot{\tau}_i \frac{\partial \tilde{Y}}{\partial \tau_i}$. Because of the availability of cooperatively evolving resource of nonlinear scales, such a matching is always possible theoretically. In the alternative derivation of limit cycle equations in Appendix 2, we have outlined an approach to gain more analytic understanding of the rhythmic, cooperative variations of the nonlinear scaling functions. We hope to address the question of determining the precise analytic properties of the scaling functions $\phi_i(\tilde{\tau})$ corresponding to a given nonlinear system in future communications.

In Fig. 7(a), (b), we display the (y, z) phase plane relaxation oscillation for the Rayleigh and VdP equations with $\varepsilon = 5$. In Appendix 3, the piece-wise smooth matching curves approximating these cycles are presented in tabular forms. However, the smoothness at the joining points are achieved at the level of one decimal only. More accurate approximation may be achieved with smarter efforts. Judicious choice of slowly varying centres (Y_0, Z_0) and radii $a(\tau_1)$ of circular arcs of right sizes (a straight line segment being an arc with sufficiently large radius) should give better approximations with a given exact (experimental) cycle that can be obtained on a symbolic computation platform, Mathematica for instance. The phase plane dynamics of these slowly varying centres and radii is expected to reveal interesting new insights into asymptotic properties of the nonlinear oscillation . It transpires from Appendix 3 that radii, for instance, vary much faster in Rayleigh than that in VdP oscillator, in which case radii fluctuate between small and large values through intermediate steps. This might be compared with fast and slow energy build ups in Rayleigh and VdP relaxation oscillations respectively. One would like to interpret this phase plane dynamics as cooperative evolution of multiple nonlinear scales driving amplitude and phase of the nonlinear oscillation to flow in such a fashion as to generate little (elementary) circular arcs and linear segments which join smoothly together to form the complete orbit. Making an accurate plot then boils down to finding right

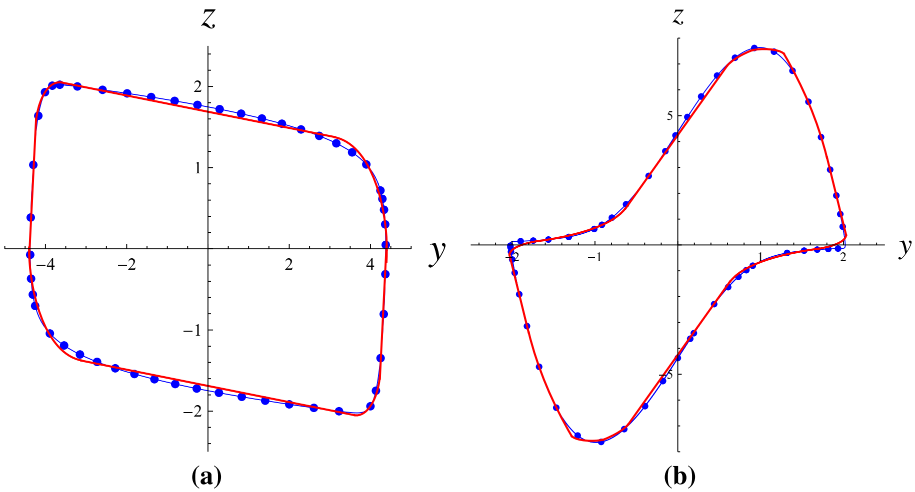

Fig. 7 Approximate limit cycle for (**a**) Rayleigh equation and (**b**) Van der Pol equation, *solid* (*red*) line for approximate curve, *dotted* (*blue*) line for exact curve ($\varepsilon = 5$). (Color figure online)

kind of such arcs and line segments. Intricate dependence of the trajectory itself into the definition of the nonlinear scales (Appendix Remark 3) tells in retrospect that one needs to look for a novel iteration scheme that would allow one to extract the trajectory systematically as a limit process. This problem will be considered in detail elsewhere.

Concluding Remarks

In this paper we have presented a comparative study of the homotopy analysis method and the RGM. The approximate formulae for the amplitudes of the limit cycles of the Rayleigh and the Van der Pol systems are derived using both the methods and are compared with the exact results. It turns out that the higher order perturbative calculations based on the conventional RGM would fail to give efficient formula for the limit cycle amplitudes for these nonlinear oscillators. However, an improved version of the Renormalization group analysis exploiting a novel concept of nonlinear time is shown to yield efficient amplitude formulae for all values of ε. Exploiting multiple nonlinear scales of the associated nonlinear time the improved RG method is also found to yield good plots for relaxation oscillation orbits for the Rayleigh and VdP systems.

In Appendix we have presented brief review of the nonlinear time formalism and also given an alternative approach in deriving non-perturbative flow equations of amplitude and phase of a limit cycle problem. Non-perturbative information of asymptotic quantities get naturally encoded into nonlinear scales, that can be exploited judiciously to extract desired asymptotic properties of a relevant dynamical quantity. More detailed analysis of the nonlinear time formalism in several other nonlinear systems will be considered in future.

Acknowledgments Authors thank the anonymous reviewers for constructive criticisms improving the quality of the paper.

Appendix 1: Formal Structure

The idea of nonlinear time can be given a rigorous meaning in a nonclassical extension of the ordinary analysis [16,17]. Recall that the real number system $\mathbf{R}$ is generally constructed as the metric completion of the rational field $\mathbf{Q}$ under the Euclidean metric $|x-y|,\ x,y \in \mathbf{Q}$. More specifically, let S be the set of all Cauchy sequences $\{x_n\}$ of rational numbers $x_n \in \mathbf{Q}$. Then S is a ring under standard component-wise addition and multiplication of two rational sequences. Then the real number field $\mathbf{R}$ is the quotient space S/S_0, where the set S_0 is the set of all Cauchy sequences converging to $0 \in Q$ and is a maximal ideal in the ring S. Alternatively, $\mathbf{R}$ can be considered as the set $[S]$ of equivalence classes, when two sequences in S are said to be equivalent if their difference belongs to S_0.

The nonclassical extension $\mathbf{R}^*$ of $\mathbf{R}$ is based on a *finer* equivalence relation that is defined in S_0 as follows: let $\{a_n\} \in S_0$. Consider an associated family of Cauchy sequences of the form $S_{0a} := \{A^{\pm}|A^{\pm} = \{a_n \times a_n^{\pm a^{\pm}_{m_n}}\}\}$ where $a^{\pm}_{m_n} \neq 0$ is Cauchy for $m_n > N$ and N sufficiently large. Clearly, $S_{0a} \subset S_0$, and sequences of S_{0a} also converges to 0 in the metric $|.|$. As a parametrizes sequences in S, it follows that $\bigcup_{\{a\}} S_{0a} = S$. Assume further that $a^{\pm}_{m_n}$ respect the *duality structure* defined by $(a^{-}_{m_n})^{-1} \propto a^{+}_{m_n}$ for $m_n > N$. The duality structure extends also over the limit elements: viz., $\mathbf{R} \ni\ (a^{-})^{-1} \propto a^{+}$ where $a^{\pm}_{m_n} \to a^{\pm}$ as $m_n \to \infty$ such that $a^{\pm}$ are close to 1 in $\mathbf{R}$.

Next define an equivalence relation in S_{0a} declaring two sequences A_1, A_2 in the set S_{0a} equivalent if the associated exponentiated sequences $a^1_{m_n}$ and $a^2_{m_n}$ differ by an element of S_0 for $m_n > N$. In particular, one may impose the condition that $A_1 \equiv A_2$ if and only if $\exists M$ such that $a^1_{m_n} = a^2_{m_n}\ \forall\, m_n > M$. Clearly, the usual metric $|.|$ fails to distinguish elements belonging to two distinct such finer equivalent classes. However, the metric defined as the natural logarithmic extension of the Euclidean norm, generically called the *asymptotically visibility metric* is introduced by $h(A_1, A_2) = \lim n \to \infty| \log_{|A_0|^{-1}} |A_1 - A_2|/|A_0||$ where $A_0 = \{a_n\} \in S_0$. The sequence A_0 is said to define a natural scale relative to which elements in S_0 gets nontrivial values and hence become distinguishable. The limit exists because of concerned sequences $a^{\pm}_{m_n}$ being Cauchy. Note that the mapping $h : S_0 \to \mathbf{R}^+$ defined by $h(A) = \lim n \to \infty| \log_{|A_0|^{-1}} |A|/|A_0||$ is actually a nontrivial norm [16,17] (for simplicity of notation, we use same symbol to denote both the norm and metric).

The extended real number system $\mathbf{R}^*$ admitting duality induced *fine structure* is given, by definition, as the equivalence class under this finer equivalence relation viz., $\mathbf{R}^* := S/S_0$ when convergence is induced naturally by the asymptotically visibility metric $h(x, y)$. Clearly, under the usual norm $|.|$, $\mathbf{R}^*$ reduces to $\mathbf{R}$ as the exponentiated elements $a^{\pm}$ are essentially invisible. The natural application of the visibility norm on $\mathbf{R}^*$ is activated in the following steps. For any two distinct elements $x,\ y \in \mathbf{R} \subset \mathbf{R}^*$, *set*, by definition, $h(x, y) = 0,\ x \neq y$; $h(x, y)$ being nontrivial only for $y \in x + S_0$. This choice is natural as for any element $x \in \mathbf{R}$, the corresponding limiting h norm viz, $h(x) = \lim n \to \infty \log_{|A_0|^{-1}} |x/A_0|{=}1$ and so the definition $h(x, y) = 0,\ \forall x, y \in \mathbf{R}$ makes sense. For nontrivial values of $h(x, y),\ x,\ y \in \mathbf{R}^*$, the definition of the visibility metric extends over to $h(x, y) = \lim \varepsilon \to 0| \log_{\varepsilon^{-1}} |\frac{x-y}{\varepsilon}||$, which exists by construction.

Next, consider the metric $d : \mathbf{R}^* \to \mathbf{R}^+$ by $d(x, y) = |x - y| + h(x, y)$. Clearly, $d(x, y) = |x - y|$ for any $x,\ y \in \mathbf{R}$ and $d(x, y) = h(x, y)$ for $x,\ y \in \mathbf{R}^* - \mathbf{R}$ and hence $(\mathbf{R}^*, d)$ is a complete metric space. The metric $h(x, y)$ acting nontrivially on S_0 is essentially an ultrametric: $h(x, y) \leq \max\{h(x, z), h(z, y)\}$. This follows immediately from the observation that h maps $\mathbf{R}$ to the singleton set $\{1\}$. Further, the ultrametric h must be discretely valued [17] and hence the nontrivial value set of h viz., $h(S_0)$ is countable. As a consequence, the set S_0 is totally disconnected and perfect in the induced topology.

More detailed analytic aspects (including the idea of smooth jump differentiability and jump derivative) of the extended system $\mathbf{R}^*$ equipped with the metric d will be reported elsewhere [16]. Here, we make a few relevant remarks.

1. Even as the size of a $\delta-$ neighbourhood of a point $x \in \mathbf{R}$ vanishes linearly, the same for $x^* \in \mathbf{R}^*$ need not vanish at the same rate and may only vanish at a slower rate $\delta h(\delta)$. The real number model $\mathbf{R}$ is called the hard or string model when the space $\mathbf{R}^*$ is called the soft or fluid model of real numbers [15]. The ordinary differential measure dx gets extended in $\mathbf{R}^*$ as $d(h(x)x)$.
2. Consider the open interval $(\delta, \delta^{-1}) \subset \mathbf{R}^*$. In the asymptotic limit $\delta \to 0^+$, the duality structure identifies the right neighbourhood of δ with the left neighbourhood of δ^{-1} in a nontrivial manner. As a consequence, the linear (translation) group action on $\mathbf{R}$ is extended to a nonlinear SL(2, R) group on $\mathbf{R}^*$. Infact, the translation subgroup acts on $\mathbf{R}$, when the inversion acts nontrivially only on $\mathbf{R}^*$ in the sense that the visibility norm h is invariant under inversion $\hat{i}$: $h(\hat{i}A) = \hat{i}(h(A))$ where $\hat{i}(A) = \{a_n^{-1} \times a_n^{(a_{m_n}^-)^{-1}}\}$, $A = \{a_n \times a_n^{-a_{m_n}^-}\}$. For a translation T_r by a shift r, on the other hand, $h(T_r(A)) = h(A)$ and hence $T_r(A) = A \;\Rightarrow r = 0$ (i.e. T acts trivially on $\mathbf{R}^* - \mathbf{R}$). Above two *salient* properties of the duality structure are expected to have significant application in nonlinear problems.
3. To give an example of the intricate nonlinear structure that can get encoded into a well behaved (smooth) function in $\mathbf{R}$, let us consider the simplest case of a real variable x. In $\mathbf{R}^*$ the variable x gets extended to, say, $X = xe^{\phi(\log X)}$. The function ϕ exposing the nonlinear dependence is also assumed to be differentiable. Differentiating X with respect to x one gets $xX'(1-\phi') = X$, where $'$ denotes derivation with the argument. We now assume that $\phi(\log X)$ is vanishingly small (i.e. less than accuracy level δ in any given application) for $0 < x < \infty$ and O(1) when $\log X >> 1$ i.e. $x \to 0$ or ∞. As a consequence, existence of ϕ is felt only in the asymptotic neighbourhoods (Remark 2) of 0 or ∞. We now make a further assumption that $\phi' = 0$ almost everywhere in an asymptotic neighbourhood, but everywhere in $0 < x < \infty$. Then X satisfies $xX' = X$ a.e. in $\mathbf{R}^*$. Thus ordinary variable $x \in \mathbf{R}$ gets extended in $\mathbf{R}^*$ as X which has the intermittent property of a Cantor devil's Stairecase function in an asymptotic neighbourhood. Since, under duality structure, such a neighbourhood has ultrametric topology, X in fact satisfies the above scale invariant equation everywhere in $\mathbf{R}^*$, because ordinary non-differentiability at the points of the associated Cantor set is removed by inversion mediated jump increments [16,17]. This example tells that an ordinary function can have nonlinear and nonlocal functional dependence with itself, along with rhythmic (intermittent) variability that can have significant amplification in an asymptotic sector.
4. The asymptotic scaling variables $h_0(\varepsilon)$ and $H_{RG}(\tilde{\tau})$ introduced in "Improved RG Method: Nonlinear Time" section correspond to the associated visibility norm $h(A)$ defined above. A real variable $t \in \mathbf{R}$ approaching asymptotically either to 0 or ∞ has natural images in $\mathbf{R}^*$ in the form $\tau_0 = t \times t^{-h^-(\varepsilon t)}$, $h^-(\varepsilon t) < 1$ and $\tau_\infty = t \times t^{h^+(\varepsilon t)}$, $h^+(\varepsilon t) > 1$ respectively. The scaling exponents $h^{\pm}$ encode asymptotic scaling information of a given nonlinear system. Further, $(h^-(\varepsilon t))^{-1} \propto h^+(\varepsilon t)$ by duality. In "Improved RG Method: Nonlinear Time" section, we discuss how such information

can be systematically extracted in the case of a limit cycle for a nonlinear oscillator (see also below).

5. The fine structures in $\mathbf{R}^*$ remain inactive (passive/hidden) in absence of any stimulus, either intrinsic or external. In presence of an external input, say, the actions of the nontrivial component of the metric d and the associated duality structure become manifest. The RG analysis makes a room for direct implementation of the intrinsically realized duality structure in the context of a nonlinear system in the soft model $\mathbf{R}^*$.

Appendix 2: Application: Limit Cycle

Consider a general nonlinear oscillator given by

$$\ddot{x} + x = \varepsilon f(x, \dot{x}). \tag{47}$$

We assume f such that the system admits a unique isolated cycle for $\varepsilon > 0$ and other relevant parameter values. For a finite nonlinearity $\varepsilon > 1$ (say), the usual linear time t is extended to one enjoying *right* asymptotic correction $t \to T_i = t\phi_i(\tilde{\tau}(t))$ as $t \to \infty$ through linear scales ε^i, where $\phi_i(\tilde{\tau})$ stands succinctly for the nontrivial intrinsically generated *slowly varying* scaling components arising from the associated visibility norm. Here, $\tilde{\tau}$, as usual denotes an O(1) rescaled variable in the neighbourhood of linear scales ε^i. In the case of a nonlinear planar autonomous system the relevant dynamical quantities are only amplitude and phase of the nonlinear oscillation and so we have only two asymptotic scaling functions $\phi_i(\tilde{\tau}),\ i = 1, 2$ which get *selected* naturally so as to facilitate direct non-perturbative calculation of the asymptotic properties i.e. the amplitude and phase of the limit cycle of the system. An implementation of this non-pertubative scheme in the perturbative RG formalism is presented in "Approximate Formula for Amplitude" section for computation of the amplitude of the concerned oscillators. In "Approximate Limit Cycle" section, we have demonstrated that the computed plot of the limit cycle for the Rayleigh and VdP oscillators could be matched arbitrarily closely for appropriate choices of the slowly varying nonlinear time when the amplitude and phase of the unperturbed periodic solution *flow linearly* in the appropriately chosen nonlinear scaling time variables.

Here, we give an alternative derivation of the nonperturbative *relaxation oscillation* flow equations ab-initio from the slowly varying nonlinear time in the context of the Rayleigh equation (1) with $\varepsilon >> 1$. It will transpire that the new approach is free of any divergence problem because of its inbuilt RG cancellations via duality principle. Since we are interested in the planar limit cycle properties, we assume that all the relevant quantities e.g. the solution y, amplitude a and phase ψ are functions of asymptotic time variable $t \sim \varepsilon^n,\ n >> 1$ and the associated nontrivial scaling variables $\tau_1 = \phi_1(\tilde{\tau})$ and $\tau_2 = \phi_2(\tilde{\tau})$ for a rescaled $\tilde{\tau} \sim O(1)$. Higher order scaling variables $\tau_n,\ n > 2$ of the nonlinear structure of time variable may become relevant for a non-planar system. Accordingly, we write the ansatz $y(t, \tau_1, \tau_2) = y_0(t) + Y_1(\tau_1) + Y_2(\tau_2)$ for the limit cycle solution involving multiple time scales (only three for the planar system). Assuming slow variations of nonlinear scales $\tau_i = \phi_i,\ i = 1, 2$, viz. $|\phi_i''| << |\phi_i'^2| << 1,\ \phi_i' = \frac{\mathrm{d}\phi_i}{\mathrm{d}\tilde{\tau}}$, as $t \to \varepsilon^n,\ n \to \infty$ and noting that $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y_0}{\partial t} + \sum_i \dot{\tilde{\tau}}\phi_i' \frac{\partial Y_i}{\partial \tau_i}$ etc., the Rayleigh equation (1) simplifies to

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 + \sum_i Y_i = \varepsilon \left(\frac{\partial y_0}{\partial t} + \sum_i \dot{\tilde{\tau}}\phi_i' \frac{\partial Y_i}{\partial \tau_i} \right) - \frac{\varepsilon}{3} \left(\left(\frac{\partial y_0}{\partial t} \right)^3 + 3 \left(\frac{\partial y_0}{\partial t} \right)^2 \sum_i \dot{\tilde{\tau}}\phi_i' \frac{\partial Y_i}{\partial \tau_i} \right), \tag{48}$$

where we drop all higher order terms involving ϕ_i'' and $\phi_i'^2$. Assuming $y_0(t) = a(\tau_1, \tau_2)\cos(t + \psi(\tau_1, \tau_2))$ with *flowing* amplitude and phase in scaling times τ_1 and τ_2 so that

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0, \tag{49}$$

we next get a simplified linearized evolution for the nonlinear components of the asymptotic limit cycle solution in the form

$$\left(1 - \left(\frac{\partial y_0}{\partial t}\right)^2\right)\sum \dot{\phi}_i \frac{\partial Y_i}{\partial \tau_i} = \left\{\frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3 - \frac{\partial y_0}{\partial t}\right\} + \varepsilon^{-1}\sum Y_i, \tag{50}$$

where $\dot{\phi}_i = \frac{d\phi_i}{dt}$. As a consequence, under the assumption of slowly varying nonlinear time scales, a second order nonlinear planar system (1) would decompose into a linear second order partial differential equation (49) for the zero level solution y_0 and an associated first order partial differential equation (50) for the nonlinear scale dependent components Y_i. Clearly, analogous decomposition holds actually for a larger class of planar autonomous systems (47) having a unique limit cycle solution. Extension of this result to multiple limit cycles would be considered separately.

We note here that since the system (49) and (50) is *under determined*, there is room for further restrictions to solve the system self-consistently. To re-derive the RG flow Eqs. (37) and (38) from (50), we now make following assumptions: we write (i) $a(\tau_1, \tau_2) = a(\tau_1)$, $\psi(\tau_1, \tau_2) = \psi(\tau_2)$ so that amplitude varies slowly with first order scale τ_1 when the second order scale τ_2 and phase ψ remain almost constant. On the other hand as a stabilizes to an almost constant value, the phase begins to flow, though slowly with the second order scale τ_2. Such slow, almost constant, rhythmic cooperative variations of τ_1 and τ_2 are modeled, depending on the specific problem under consideration (see below and c.f. "Approximate Limit Cycle" section), to retrieve the RG flow equations correctly. As shown in the example (Remark 3, Appendix 1) such a rhythmic nonlinear variation does exist in an ultrametric neighbourhood in $\mathbf{R}^*$. To further quantify the slow variation of dynamical variables, we next impose the condition that (ii) *the total variation of the exact solution* $y(t, \tau_1, \tau_2)$ *with respect to each slow variable* τ_i *along the full periodic cycle* C *must vanish viz,* $\int_C \frac{\partial y}{\partial \tau_i}\mathrm{d}t = 0$ *for each* i. To avoid trivialities i.e. $\int_C \cos(t + \psi)\mathrm{d}t = 0$ etc., we, however, evaluate the concerned integrals only on the quarter cycle, with the understanding that phase shifts of $\pi/2$ are absorbed in the definition of ψ.

In the sufficiently large $\varepsilon > 1$ relaxation oscillation, one can further simplify (50) by dropping the ε^{-1} term to obtain

$$\sum \dot{\phi}_i \frac{\partial Y_i}{\partial \tau_i} = \frac{\frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3 - \frac{\partial y_0}{\partial t}}{1 - \left(\frac{\partial y_0}{\partial t}\right)^2} \equiv \Phi(y_{0t}), \quad y_{0t} = \frac{\partial y_0}{\partial t}. \tag{51}$$

To make contact with RG flow Eqs. (37) and (38) one now exploits *the freedom of right choice* in the functional forms of nonlinear scales. For the Rayleigh equation, we now set for slow, cooperatively active functional dependence (a) $\dot{\phi}_1 = \Phi(y_{0t})S_1^{-1}(a, \psi, t)$, $\dot{\phi}_2 = 0$ and (b) $\dot{\phi}_1 = 0$, $\dot{\phi}_2 = \Phi(y_{0t})S_2^{-1}(a, \psi, t)$ for successive slow variations, as described in (i), of the scales τ_1 and τ_2 respectively, where, $S_1 = \frac{1}{2}a(\frac{a^2}{4} - 1)\cos(t + \psi)$ and $S_2 = \frac{1}{8}(1 - \frac{a^4}{32})\sin(t + \psi)$ (recall the example in Remark 3 of Appendix 1 above highlighting wide possible choices and intricate functional dependence). These choices for S_1 and S_2 would yield the RG flow equations when condition (ii) is invoked.

Note that the relations in both (a) and (b) are truly nonlinear; the dynamical variables a and ψ in S_1 and S_2 depend implicitly in ϕ_1 and ϕ_2 respectively, which, in turn are *slowly varying* as the linear parameter t is assumed to vary in a neighbourhood of ε^n for a large but fixed n. Invoking the global slow variation condition (ii) for each i, in conjunction with the ansatz (a) and (b), one finally deduce the amplitude and phase flow Eqs. (37) and (38) in slow variables τ_1 and τ_2 respectively.

In the present format, the flow equations, however, have got *new* interpretations: Amplitude and phase must flow in successive rhythmic manner; phase remains almost constant (i.e. $\frac{\partial\psi}{\partial\tau_i} = 0$ for each i) when amplitude varies slowly with τ_1 towards an almost constant value. Subsequently, the flowing of a is halted temporarily (i.e. $\frac{\partial a}{\partial\tau_i} = 0$), initiating flowing of ψ in next level variable τ_2. This rhythmic oscillation would obviously continue indefinitely over a cycle. The RG flow equations could be treated as non-perturbative because of *implicit* connections of nonlinear scaling time functions with amplitude and phase via intrinsically defined duality principle (c.f. Remark 3 above).

To summarize, based on perturbative RG formalism we have presented an alternative approach in deriving non-perturbative flow equations of relevant asymptotic dynamical quantities of a planar autonomous limit cycle problem, from nonlinear scale invariant time scales, which become available in an extended analytic framework incorporating duality structure. Non-perturbative information of asymptotic quantities get naturally encoded into nonlinear scales, that can be exploited judiciously to extract desired asymptotic analytic properties of a relevant dynamical quantity. An algorithmic procedure of extracting such information is explained in estimating both the limit cycle amplitude and trajectory for Rayleigh and VdP equations.

Appendix 3: Matching Arcs and Segments

(a) Piecewise smooth matching curves for upper half of the approximate Rayleigh limit cycle for $\varepsilon = 5$:

$$z(y) = \begin{cases} 0.02 - \sqrt{1.96 - (y+3)^2} & -4.96 < y \le -4.393, \\ 10y + 43.9 & -4.393 < y \le -4.23 \\ 1.46 + \sqrt{0.35 - (y+3.65)^2} & -4.23 < y \le -3.6, \\ -0.1y + 1.689 & -3.6 < y \le 3.12, \\ -0.02 + \sqrt{1.96 - (y-3)^2} & 3.12 < y \le 4.96. \end{cases}$$

(b) Piecewise smooth matching curves for upper half of the approximate VdP limit cycle for $\varepsilon = 5$:

$$z(y) = \begin{cases} -0.388 + \sqrt{0.352 - (y+1.438)^2} & -2.05 < y \le -1.7, \\ 1.8 - \sqrt{3.2 - (y+2.38)^2} & -1.7 < y \le -0.633, \\ 4.5y + 4.25 & -0.633 < y \le 0.6, \\ 6.5 + \sqrt{2.76 - (y-2.2)^2} & 0.6 < y \le 0.9, \\ 7.325 + \sqrt{0.063 - (y-1.04)^2} & 0.90 < y \le 1.28, \\ 0.38 + \sqrt{1530 - (y+37.2)^2} & 1.28 < y \le 1.8, \\ -13y + 26.8 & 1.8 < y \le 2.033, \\ 0.388 + \sqrt{0.352 - (y-1.438)^2} & 2.033 < y \le 2.05. \end{cases}$$

References

1. Jordan, D.W., Smith, P.: Nonlinear Ordinary Differential Equations: An Introduction to Dynamical Systems, 3rd edn. Oxford University Press, Oxford (2003)
2. Bender, C.M., Orszag, S.A.: Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory. Springer, Berlin (1999)
3. Liao, S.: Beyond Perturbation: Introduction to Homotopy Analysis Method. Chapman & Hall/CRC, London (2003)
4. Lopez, J.L., Abbasbandy, S., Lopez-Ruiz, R.: Formulas for the amplitude of the van der Pol limit cycle through the homotopy analysis method. Sch. Res. Exch. **2009**, 854060 (2009)
5. Chen, L., Goldenfeld, N., Oono, Y.: Renormalization group theory for global asymptotic analysis. Phys. Rev. Lett. **73**, 1311–15 (1994)
6. Chen, L., Goldenfeld, N., Oono, Y.: The renormalization group and singular perturbations: multiple-scales, boundary layers and reductive perturbation theory. Phys. Rev. E **54**, 376-394 (1996)
7. DeVille, R.E.L., Harkin, A., Holzer, M., Josic, K., Kaper, T.J.: Analysis of a renormalization group method and normal form theory for perturbed ordinary differential equations. Phys. D **237**(8), 1029–1052 (2008)
8. Sarkar, A., Bhattacharjee, J.K.: Renormalization group as a probe for dynamical systems. J. Phys. Conf. Ser. **319**, 012017 (2011)
9. Keller, J.B., Miksis, M.: Bubble oscillations of large amplitude. J. Acoust. Soc. Am. **68**, 628–633 (1980)
10. Veskos, P., Demiris, Y.: Experimental comparison of the Van der Pol and Rayleigh nonlinear oscillators for a robotic swinging task. In: Proceedings of the AISB 2006 Conference, Adaptation in Artificial and Biological Systems, Bristol, pp. 197–202 (2006)
11. Liao, S.: An analytic approximation for free oscillations of self-excited systems. Int. J. Non-linear Mech. **39**, 271–280 (2004)
12. Meijer, H.G.E., Kalmar-Nagy, T.: The Hopf-van der Pol system: failure of a homotopy method. Differ. Equ. Dyn. Syst. **20**, 323–328 (2012)
13. Datta, D.P., Raut, S.: The arrow of time. Complexity and the scale free analysis. Chaos Solitons Fractals **28**, 581–589 (2006)
14. Datta, D.P.: Novel late time asymptotics: applications to anomalous transport in turbulent flows. Radiat. Eff. Defects Solids **167**, 789–801 (2013)
15. Datta, D.P., Sen, S.: Excitation of flow instabilities due to nonlinear scale invariance. Phys. Plasma **21**, 052311 (2014)
16. Datta, D. P.: On jump differentiability of fractal curves, In preperation
17. Datta, D.P., Raut, S., Raychoudhuri, A.: Ultrametric cantor sets and growth of measure. p-Adic Numbers Ultametr. Anal. Appl. **3**, 7–22 (2011)