# Chapter 7

# Improved Renormalization Group Method

## 7.1 Introduction

We have discussed Homotopy Analysis Method (HAM) and Renormalization Group Method (RGM) in the previous two chapters. Both of these two methods have the objective to derive global solutions to nonlinear differential equations. HAM involves a convergence control parameter $h$ by which one can control the convergence of an approximate solution to any level of precision. It helps us to evaluate analytic approximation to different quantities such as amplitude, frequency etc. related to a limit cycle. However, we found in Chapter 6 that classical RG solution fails to give good approximation to the amplitude of the limit cycle for Rayleigh equation

$$\ddot{y} + \varepsilon\left(\frac{1}{3}\dot{y}^3 - \dot{y}\right) + y = 0 \tag{7.1}$$

with moderately large values of $\varepsilon >\approx O(1)$, where the dots are used to designate the derivatives with respect to time $t$ (c.f. Figure 6.1 in Chapter 6). Let us recall that there are some limitations in the conventional form of RGM (c.f. Chapter 6). The actual convergence of the RG expansions is not well addressed and should require further investigations. We observe that in RGM we do not have any control parameter $h$ as we have in case of HAM. In this Chapter we shall improve the classical RGM by introducing similar control parameters which will help us to control the convergence of the approximate solution to the exact one for a given nonlinear ordinary differential equation (ODE) to any desired degree of precision. In this chapter, along with the Rayleigh equation we shall also study the Van der Pol equation

$$\ddot{x} + \varepsilon\, \dot{x}\left(x^2 - 1\right) + x = 0. \tag{7.2}$$

In IRGM, we advocate the concept of *nonlinear time* [40,76,77] that extends the original RG idea of eliminating the divergent secular term of the form $(t - t_0)\sin t$, where $t_0$ is the initial time, in the naive perturbation series for the solution of the nonlinear problem, by exploiting the arbitrariness in fixing the initial moment $t_0$. In the framework of nonlinear time, we suppose the arbitrary initial time $\tau$ to depend explicitly on the nonlinearity parameter (coupling strength) $\varepsilon$ of the nonlinear equation, so that one can write $\tau/\varepsilon = \varepsilon^h$ where $h = h\left(\varepsilon t\right),\ \varepsilon t > 1$ is a slowly varying (almost constant), free (asymptotic) control parameter for $t \rightarrow \infty$ and $\varepsilon\ \rightarrow$ either to 0 or $\infty$, to be utilized judiciously to improve the convergence and non-perturbative global asymptotic behaviour of the original RG proposal ($h < 0$ for $0 < \varepsilon < 1$). We note that there is a slight abuse of the notation $\tau$ as discussed in Chapter 5. In Section 7.2, we give an overview, in brief, of an extended analytic framework that naturally supports nontrivial existence of such an asymptotic scaling parameter $h(\tilde{\tau})$ as a function of the rescaled $O(1)$ variable $\tilde{\tau} = \varepsilon t \sim O(1)$, satisfying what we call the *principle of duality structure*. The secular terms in the naive perturbation series would now be altered instead as $(t - \tau/\varepsilon + \tau/\varepsilon - t_0)\sin t$ and we obtain the new RG flow equations in the form

$$\frac{dR}{d\tau} = f_0\left(R\right)\left(1 + O\left(\varepsilon^2\right)\right),\ \frac{d\theta}{d\tau} = \varepsilon g_1\left(R\right)\left(1 + O\left(\varepsilon^3\right)\right), \tag{7.3}$$

where $f_0(R)$ and $g_1(R)$ are nonzero, minimal order $R$ dependent terms in the respective perturbation series. Following the analogy of RG prescription in annulling secular divergence through corresponding 'flowing' of the renormalized perturbative amplitude and phase, we next make *the key assumption that there exists, for a given nonlinear oscillation, a set of right control parameters $h^i$ that would absorb any possible secular or other kind of divergence in the higher order pertur-*

*bation series*, so that in the asymptotic limit $t \to \infty$, one obtains the finite, non-perturbative flow equations directly for the periodic orbit of the nonlinear system

$$\frac{da}{d\tau_1} = f_0(a), \ \frac{d\theta}{d\tau_2} = g_1(a), \tag{7.4}$$

where

$$\tau_i = \varepsilon^{i \times h^i_{RG}(\tilde{\tau})},$$

and $h^i_{RG}(\tilde{\tau})$ is a finite *scale* independent control parameter in the rescaled variable $\tilde{\tau} \sim O(1)$ and $a(\varepsilon) = \lim_{t\to\infty} R(\varepsilon, t)$ is the $\varepsilon$- dependent amplitude of the limit cycle. A simple quadrature formula should then relate the control parameter $h_{RG} := h^1_{RG}$ with the amplitude $a(\varepsilon)$. As a consequence, adjusting the control parameter $h_{RG}$ suitably, one can generate an efficient algorithm to estimate the amplitude $a(\varepsilon)$ that would compare well with the exact values, upto any desired accuracy. It will transpire that the control parameter $h_{RG}(\varepsilon)$ must respect some asymptotic conditions depending on the characteristic features of a particular relaxation oscillation (c.f. Section 7.3).

Exploiting the rescaling symmetry, one may as well rewrite the above non-perturbative flow equations (7.4) in the equivalent $\tilde{\tau} \sim O(1)$ dependent scaling variable $\tau = \tilde{\tau}^{H_{RG}(\tilde{\tau})}$, $\left(H_{RG}(\tilde{\tau}) = h_{RG}(\tilde{\tau})\frac{\log \tilde{\tau}}{\log \varepsilon}\right)$, for each fixed value of the nonlinearity parameter $\varepsilon$ that should expose small scale $\tilde{\tau} \sim O(1)$ dependent variation of the amplitude. As a by-product that would allow one to retrieve an efficient approximation of the limit cycle orbit for the nonlinear oscillator. It turns out that the general framework of IRGM is quite successful in obtaining excellent fits for the limit cycle orbit even for relaxation oscillation corresponding to nonlinearity parameters $\varepsilon \geq 1$.

It follows that the application of the idea of nonlinear time in RG formalism offers one with a robust formalism for global asymptotic analysis for a general nonlinear system that might even be advantageous in many respects compared to HAM. The application of nonlinear time in HAM will be considered separately.

The chapter is organized as follows. In Section 7.2 we give a brief overview of the *novel* analytic framework extending the standard classical analysis to one that supports naturally the above stated *duality structure* and the emergent nonlinear scaling patterns typical for a given nonlinear system. The IRGM is proposed using the idea of nonlinear time in Section 7.3. Approximate formulae for the amplitude of limit cycle solution have been deduced there in the context of Rayleigh and VdP equations. The application of nonlinear time on limit cycles of the Rayleigh and Van der Pol equations is included in Section 7.4. Finally, some concluding remarks are included in Section 7.5.

## 7.2 Nonlinear Time: Formal Structure

The idea of nonlinear time can be given a rigorous meaning in a nonclassical extension of the ordinary analysis [42, 78]. Recall that the real number system $\mathbb{R}$ is generally constructed as the metric completion of the rational field $\mathbb{Q}$ under the Euclidean metric $|x-y|$, $x, y \in \mathbb{Q}$. More specifically, let $S$ be the set of all Cauchy sequences $\{x_n\}$ of rational numbers $x_n \in \mathbb{Q}$. Then $S$ is a ring under standard component-wise addition and multiplication of two rational sequences. Then the real number field $\mathbb{R}$ is the quotient space $S/S_0$, where the set $S_0$ is the set of all Cauchy sequences converging to $0 \in \mathbb{Q}$ and is a maximal ideal in the ring $S$. Alternatively, $\mathbb{R}$ can be considered as the set $[S]$ of equivalence classes, when two sequences in $S$ are said to be equivalent if their difference belongs to $S_0$.

The nonclassical extension $\mathbb{R}^*$ of $\mathbb{R}$ is based on a *finer* equivalence relation that is defined in $S_0$ as follows: let $\{a_n\} \in S_0$. Consider an associated family of Cauchy sequences of the form

$$S_{0a} := \left\{ A^{\pm} \mid A^{\pm} = \{a_n \times a_n^{\pm a^{\pm}_{m_n}}\} \right\},$$

where $a^{\pm}_{m_n} \neq 0$ is Cauchy for $m_n > N$ and $N$ sufficiently large. Clearly, $S_{0a} \subset S_0$, and sequences of $S_{0a}$ also converges to 0 in the metric $|\cdot|$. As $a$ parametrizes sequences in $S$, it follows that $\bigcup\limits_{\{a\}} S_{0a} = S$. Assume further that $a^{\pm}_{m_n}$ respect the *duality structure* defined by $\left(a^{-}_{m_n}\right)^{-1} \propto a^{+}_{m_n}$ for $m_n > N$. The duality structure extends also over the limit elements: viz., $\mathbb{R} \ni \ (a^{-})^{-1} \propto a^{+}$ where $a^{\pm}_{m_n} \to a^{\pm}$ as $m_n \to \infty$ such that $a^{\pm}$ are close to 1 in $\mathbb{R}$.

Next define an equivalence relation in $S_{0a}$ declaring two sequences $A_1,\ A_2$ in the set $S_{0a}$ equivalent if the associated exponentiated sequences $a^1_{m_n}$ and $a^2_{m_n}$ differ by an element of $S_0$ for $m_n > N$. In particular, one may impose the condition that $A_1 \equiv A_2$ if and only if $\exists\ M$ such that $a^1_{m_n} = a^2_{m_n}\ \forall\ m_n > M$. Clearly, the usual metric $|\cdot|$ fails to distinguish elements belonging to two distinct such finer equivalent classes. However, the metric defined as the natural logarithmic extension of the Euclidean norm, generically called the *asymptotically visibility metric* is introduced by

$$h\left(A_1, A_2\right) = \lim_{n\to\infty} \left|\log_{|A_0|^{-1}} \left|\frac{A_1}{A_2}\right|\right|,$$

where $A_0 = \{a_n\} \in S_0$. The sequence $A_0$ is said to define a natural scale relative to which elements in $S_0$ gets nontrivial values and hence become distinguishable. The limit exists because of concerned sequences $a^{\pm}_{m_n}$ being Cauchy. Note that the mapping $h : S_0 \longrightarrow \mathbb{R}^+$ defined by

$$h(A) = \lim_{n\to\infty} \left|\log_{|A_0|^{-1}} \left|\frac{A}{A_0}\right|\right|$$

is actually a nontrivial norm [42, 78] (for simplicity of notation, we use same symbol to denote both the norm and metric).

The extended real number system $\mathbb{R}^*$ admitting duality induced *fine structure* is given, by definition, as the equivalence class under this finer equivalence relation viz., $\mathbb{R}^* := S/S_0$ when convergence is induced naturally by the asymptotically visibility metric $h\left(x, y\right)$. Clearly, under the usual norm $|\cdot|$, $\mathbb{R}^*$ reduces to $\mathbb{R}$ as the exponentiated elements $a^{\pm}$ are essentially invisible. The natural application of the visibility norm on $\mathbb{R}^*$ is activated in the following steps. For any two distinct elements $x,\ y \in \mathbb{R} \subset \mathbb{R}^*$, *set*, by definition, $h\left(x, y\right) = 0,\ x \neq y$; $h\left(x, y\right)$ being nontrivial only for $y \in x + S_0$. This choice is natural as for any element $x \in \mathbb{R}$, the corresponding limiting $h$ norm viz.,

$$h\left(x\right) = \lim_{n\to\infty} \log_{|A_0|^{-1}} \left|\frac{x}{A_0}\right| = 1 \text{ and } h\left(x, y\right) = 0,\ \forall\ x, y \in \mathbb{R}.$$

For nontrivial values of $h\left(x, y\right),\ x,\ y \in \mathbb{R}^*$, the definition of the visibility metric extends over to $h(x,y) = \lim_{n\to\infty} \left| \log_{\varepsilon^{-n}} \left| \frac{x-y}{\varepsilon^n} \right| \right|$, which exists by construction, where $A_0 = \{\varepsilon^n\},\ 0 < \varepsilon < 1$.

Next, consider the metric $d : \mathbb{R}^* \to R^+$ by $d(x,y) = |x-y| + h(x,y)$. Clearly, $d(x,y) = |x-y|$ for any $x,\ y \in \mathbb{R}$ and $d(x,y) = h(x,y)$ for $x,\ y \in \mathbb{R}^* - \mathbb{R}$ and hence $(\mathbb{R}^*, d)$ is a complete metric space. The metric $h(x,y)$ acting nontrivially on $S_0$ is essentially an ultrametric: $h(x,y) \le \max\{h(x,y), h(x,y)\}$. This follows immediately from the observation that $h$ maps $\mathbb{R}$ to the singleton set $\{1\}$. Further, the ultrametric $h$ must be discretely valued [78] and hence the nontrivial value set of $h$ viz., $h(S_0)$ is countable. As a consequence, the set $S_0$ is totally disconnected and perfect in the induced topology.

More detailed analytic aspects (including the idea of smooth jump differentiability and jump derivative) of the extended system $\mathbb{R}^*$ equipped with the metric $d$ will be reported elsewhere [42]. Here, we make a few relevant remarks.

**Remark 7.2.1** *Even as the size of a $\delta-$ neighbourhood of a point $x \in \mathbb{R}$ vanishes linearly, the same for $x^* \in \mathbb{R}^*$ need not vanish at the same rate and may only vanish at a slower rate $\delta h(\delta)$. The real number model $\mathbb{R}$ is called the hard or string model when the space $\mathbb{R}^*$ is called the soft or fluid model of real numbers [40]. The ordinary differential measure $dx$ gets extended in $\mathbb{R}^*$ as $d(h(x)x)$.*

**Remark 7.2.2** *Consider the open interval $(\delta, \delta^{-1}) \subset \mathbb{R}^*$. In the asymptotic limit $\delta \to 0^+$, the duality structure identifies the right neighbourhood of $\delta$ with the left neighbourhood of $\delta^{-1}$ in a nontrivial manner. As a consequence, the linear (translation) group action on $\mathbb{R}$ is extended to a nonlinear $SL(2,R)$ group on $\mathbb{R}^*$. In fact, the translation subgroup acts on $\mathbb{R}$, when the inversion acts nontrivially only on $\mathbb{R}^*$ in the sense that the visibility norm $h$ is invariant under inversion $\hat{\imath}$ : $h(\hat{\imath}A) = \hat{\imath}(h(A))$ where $\hat{\imath}(A) = \left\{a_n^{-1} \times a_n^{(a_{m_n}^{-})^{-1}}\right\}$, $A = \left\{a_n \times a_n^{-a_{m_n}^{-}}\right\}$. For a translation $T_r$ by a shift $r$, on the other hand, $h(T_r(A)) = h(A)$ and hence $T_r(A) = A \ \Rightarrow r = 0$ (i.e. $T$ acts trivially). Above two* salient *properties of the duality structure are expected to have significant application in nonlinear problems.*

**Remark 7.2.3** *To give an example of the intricate nonlinear structure that can get encoded into a well behaved (smooth) function in* $\mathbb{R}$*, let us consider the simplest case of a real variable* $x$*. In* $\mathbb{R}^*$ *the variable* $x$ *gets extended to, say,* $X = xe^{\phi(\log X)}$*. The function* $\phi$ *exposing the nonlinear dependence is also assumed to be differentiable. Differentiating* $X$ *with respect to* $x$ *one gets* $xX'(1-\phi') = X$*, where "′" denotes derivation with the argument. We now assume that* $\phi(\log X)$ *is vanishingly small (i.e. less than accuracy level* $\delta$ *in any given application) for* $0 < x < \infty$ *and* $O(1)$ *when* $|\log X| \gg 1$ *i.e.* $x \to 0$ *or* $\infty$*. As a consequence, existence of* $\phi$ *is felt only in the asymptotic neighbourhoods (Remark 7.2.2) of* $0$ *or* $\infty$*. We now make a further assumption that* $\phi' = 0$ *almost everywhere in an asymptotic neighbourhood, but every where in* $0 < x < \infty$*. Then* $X$ *satisfies* $xX' = X$ *a.e. in* $\mathbb{R}^*$*. Thus ordinary variable* $x \in \mathbb{R}$ *gets extended in* $\mathbb{R}^*$ *as* $X$ *which has the intermittent property of a Cantor devil's Staircase function in an asymptotic neighbourhood. Since, under duality structure, such a neighbourhood has ultrametric topology,* $X$ *in fact satisfies the above scale invariant equation everywhere in* $\mathbb{R}^*$*, because ordinary non-differentiability at the points of the associated Cantor set is removed by inversion mediated jump increments* [42,78]*. This example tells that an ordinary function can have nonlinear and nonlocal functional dependence with itself, along with rhythmic (intermittent) variability that can have significant amplification in an asymptotic sector.*

**Remark 7.2.4** *The asymptotic scaling variables* $h_0(\varepsilon)$ *and* $H_{RG}(\tilde{\tau})$ *introduced in Subsection 7.3.1 correspond to the associated visibility norm* $h(A)$ *defined above. A real variable* $t \in \mathbb{R}$ *approaching asymptotically either to* $0$ *or* $\infty$ *has natural images in* $\mathbb{R}^*$ *in the form* $\tau_0 = t \times t^{-h^-(\varepsilon t)}$*,* $h^-(\varepsilon t) < 1$ *and* $\tau_\infty = t \times t^{h^+(\varepsilon t)}$*,* $h^+(\varepsilon t) > 1$ *respectively. The scaling exponents* $h^\pm$ *encode asymptotic scaling information of a given nonlinear system. Further,* $(h^-(\varepsilon t))^{-1} \propto h^+(\varepsilon t)$ *by duality. In Section 7.3.1, we discuss how such information can be systematically extracted in the case of a limit cycle for a nonlinear oscillator.*

**Remark 7.2.5** *The fine structures in* $\mathbb{R}^*$ *remain inactive (passive/hidden) in absence of any stimulus, either intrinsic or exter-*

*nal. In presence of an external input, say, the actions of the nontrivial component of the metric d and the associated duality structure become manifest. The RG analysis makes room for direct implementation of the intrinsically realized duality structure in the context of a nonlinear system in the soft model* $\mathbb{R}^*$.

## 7.3 Improved RG Method: use of Nonlinear Time

In RGM an arbitrary time $\tau$ is introduced in between current time $t$ and the initial time $t_0$ so that $t-t_0 = (t-\tau)+(\tau-t_0)$ in order to remove the divergent terms in the naive perturbation expansion for the solution of the given differential equation. The solution is renormalized by suitable choice of the constants of integration to remove the terms containing $(\tau - t_0)$ and keeping the terms having $(t-\tau)$. Since the solution should be independent of the arbitrary time $\tau$, the RG condition

$$\left.\frac{\partial y}{\partial \tau}\right|_{\tau=t} = 0$$

is applied to the renormalized solution. However, in the previous section we have seen that the method fails to produce good approximations to the exact solution for $\varepsilon \sim O(1)$. Our target is not only to remove the divergent terms in the solution but also to introduce some control parameter $h(\varepsilon)$ which can control the RG solution in such a manner that this solution ultimately converges to the exact solution. Moreover, our another goal is to achieve this accuracy by merely solving the differential equation to a minimal order of the expansion parameter, viz., upto $O\left(\varepsilon^2\right)$ or less.

Since the basic idea is to split the time difference $t-t_0$ by introduction of an arbitrary time, so we can write $t - t_0 = \left(t - \frac{\tau}{\varepsilon}\right) + \left(\frac{\tau}{\varepsilon} - t_0\right)$. From now on let us assume that $0 \ll \varepsilon <\approx 1$. The case $\varepsilon >\approx 1$ will be commented upon later. The constants of integration can be renormalized in order to remove the terms containing $\left(\frac{\tau}{\varepsilon} - t_0\right)$ from the solution keeping the terms containing $\left(t - \frac{\tau}{\varepsilon}\right)$. Finally analogous to the classical RG method we put $t = \dfrac{\tau}{\varepsilon}$, i.e. $\tau = \varepsilon t$, in

$$\frac{\partial y}{\partial \tau} = 0 \tag{7.5}$$

giving rise to an improved form of the RG flow equation to remove secular terms involving $\left(t - \dfrac{\tau}{\varepsilon}\right)$. So far the improved method does not produce any qualitative new result compared to the RGM and so we must get the same phase and amplitude equation as deduced in Section 6.3 in Chapter 6.

We next proceed one step further. As stated already in Section 7.1, we now exploit the possibility of extending the original linear $t$ dependence of $\tau$ viz., $\tau = t$ of RGM in removing the explicit divergences by a *nonlinear dependence* $\tau = \varepsilon t$ along with the *additional condition* that $\tau \to \varepsilon^{-n}\phi(\tilde{\tau})$, where $\phi$ a slowly varying scaling function of the $O\,(1)$ rescaled variable $\tilde{\tau} = \varepsilon\tilde{t} \sim O\,(1)$, as the original linear time $t \to \infty$ following the *scales* $t \sim \varepsilon^{-n}\ \tilde{t},\ n = 1, 2, \ldots$. (Note that linear time flows with uniform rate 1 and $\tau$ is nonlinear since the rate $\dot{\phi}\,(\tilde{\tau}) < 1$). It follows that for a given nonlinear differential system, such a nonlinear time dependence always exists and nontrivial, provided one invokes a *duality principle* transferring nonlinear influences from the far asymptotic region into the finite observable sector in a cooperative manner [40, 42, 77].

In fact, as the linear time $t \to \infty$, following the above hierarchy of scales, there exists $\tilde{t}_n$ such that $1 \ll (\varepsilon t)^n < \varepsilon^{-n} < \tilde{t}_n$ and satisfying *the inversion law*

$$\frac{\tilde{t}_n}{\varepsilon^{-n}} \propto \frac{\varepsilon^{-n}}{(\varepsilon t)^n}. \tag{7.6}$$

This inversion law makes a room for transfer of effective influences, typical for the nonlinear system concerned, from nonobservable sector $t > \varepsilon^{-n}$ to the observable sector $t < \varepsilon^{-n}$ bypassing the dynamically generated singular points denoted by the scales $\varepsilon^{-n}$. Notice the nonlinear connection between scales of the form $\varepsilon^n\ \tilde{t}_n$ with the scale $\varepsilon t$ via duality structure (c.f. Section 7.2). Let $\tilde{t}\,(t) = \lim\limits_{n\to\infty} \left(\tilde{t}_n\right)^{1/n}$ so that

$\varepsilon t < \varepsilon^{-1} < \tilde{t}\,(t)$ and $\dfrac{\tilde{t}_n}{\varepsilon^{-1}} \propto \dfrac{\varepsilon^{-1}}{(\varepsilon t)}$. Define

$$h_0\,(\tilde{\tau}) = \lim_{n\to\infty} \log_{\varepsilon^{-n}} \left(\frac{\tilde{t}_n}{\varepsilon^{-n}}\right). \tag{7.7}$$

Here, the scaling exponent $h_0$ corresponds to the visibility norm (c.f. Section 7.2), that can access (encode) the non-perturbative region (information) of the nonlinear system and $\tilde{\tau}$ is an $O\,(1)$ rescaled variable. The exponent $h_0$ is *scaling invariant* in the sense that it appears uniformly for every $n$ as $t \to \infty$ through the scales $\tilde{t}_n = \varepsilon^{-n}\varepsilon^{-n\ h_0(\tilde{\tau})}$. As a consequence, a significant amount of asymptotic scaling information in the limit $t \to \infty$ could be simply retrieved by considering the scaling limit instead at $t = \varepsilon^{-1}$.

Exploiting the above insight, one now writes the nonperturbative scaling limit in the form

$$\tau = \lim \varepsilon t = \varepsilon^{-h_{RG}(\tilde{\tau})} > 1,\ \varepsilon < 1 \tag{7.8}$$

as $t \to \varepsilon^{-1}$. Moreover, $h_{RG}\,(\tilde{\tau}) = 1 - h_0\,(\tilde{\tau})$. As noted already, the scaling exponent $h_0\,(\tilde{\tau})$ here encodes the *effective* cooperative influence of far asymptotic sector $t > \varepsilon^{-n}$ into the observable sector $1 < t < \varepsilon^{-n}$ by the inversion mediated duality principle. As pointed out in Section 7.2, the duality principle *does* allow asymptotic limiting (non-perturbative) behaviour of the nonlinear system to be encoded into the scaling exponents of the nonlinear time $\tau$ that, in turn, offers an efficient handle in uncovering key dynamical information of the said system. Notice that, in the absence of the said duality the linear time $t$ can in principle attain the scale $\varepsilon^{-1}$ (say), and as a consequence $h_0 = 0$, retrieving the ordinary scaling of $\tau = \varepsilon t \sim \varepsilon^{-1}$ as $t \sim \varepsilon^{-2}$. This also establishes, in retrospect, that the scaling exponent $h_0\,(\varepsilon)$ is well defined and can exist nontrivially i.e. $h_0 \sim O\,(1)$ in a nonlinear problem. Further, the scaling variable $\tau$ is also positively directed with $t$. As a consequence, *the RG control parameter $h_{RG}$ can be of both the signs, with relatively small numerical value in fully developed nonlinear systems $\varepsilon \gg 1$, but with a possible $O\,(1)$ variations for $\varepsilon \sim O\,(1)$ or less.*

The above construction actually tells somewhat more. Corresponding to the first generation scales $\varepsilon^{-n}$, one can, in fact, have the *second generation* nonlinear scales

$$\tau_m = \lim \varepsilon^m t = \varepsilon^{-m\ h^m_{RG}(\tilde{\tau})} > 1,\ \varepsilon < 1 \tag{7.9}$$

as $t \to \varepsilon^{-m}$ with $h^1_{RG} = h_{RG}$. The *nonlinear time* $\tau$ now stands for these hierarchy of directed scales $\{\tau_m\}$. Consequently, as the linear time $t$ approaches $\infty$ through the first generation linear scales, the slowly varying nonlinear time $\tau$ $\left(\text{or } \tau^{-1}\right)$ approaches to $\infty$ at slower and slower rates as represented by the numerically small RG scaling exponents $h^m_{RG}(\varepsilon)$, each of which remains almost constant over longer and longer intervals of $\varepsilon^{-1}$ (as $\varepsilon^{-1} \to \infty$ ). In the present chapter we show how the first two scaling exponents $h_i(\tilde{\tau}),\ i = 1, 2$ relate to the nonperturbative properties of the limit cycle. We expect higher order scaling exponents $h_m$ would have vital role in bifurcation of nonautonomous systems. This problem will be investigated elsewhere.

Let us remark that for $\varepsilon > 1$, we consider instead the first generation scales as $\varepsilon^n$, and the duality is invoked for variables satisfying $\frac{t}{\varepsilon} < \varepsilon < \tilde{t}(t)$ so that the asymptotic scaling variables are derived as $\tau_m = \varepsilon^{m\ h^m_{RG}(\tilde{\tau})}$, $\varepsilon > 1$ where $h^m_{RG} = 1 - h^m_0$. Moreover, said proliferation of nonlinear scales (7.9) actually continues ad infinitum. In fact, interpreting each second generation scale $\tau_m$, $m$ fixed, as first generation scale, and iterating above steps one associates third generation scales $\tau_{m_k},\ k = 1, 2, \ldots$, and so on.

It now follows, from the above general remarks on the behaviour of $h_{RG}$, that the nonlinear time $\tau$ actually approaches 0 or $\infty$ as $\tau \sim (\log \varepsilon)^{-\alpha}$ or $\tau \sim (\log \varepsilon)^{\alpha}$ , $\alpha > 0$ respectively as $\varepsilon \to \infty$. However, one must have $\tau = \varepsilon^{-h_{RG}(\varepsilon)} \to \infty$ as $\varepsilon \to 0$. An example of the asymptotic behaviours of $h_{RG}$ is given by $\tau_m = \varepsilon^{\pm \alpha_m \frac{\log \log \varepsilon}{\log \varepsilon}}$ for $\varepsilon \to \infty$, which one expects to verify explicitly in evaluation of asymptotic quantities, such as amplitude of a periodic cycle, in a nonlinear system.

In the IRGM, we exploit this duality induced nontrivial scaling information to rewrite the lowest order perturbative flow equations (6.6) and (6.7) in Chapter 6 as the asymptotic RG flow equations in the limit

$t \to \infty$

$$\frac{da}{d\tau_1} = \frac{1}{2}a\left(1 - \frac{a^2}{4}\right) \tag{7.10}$$
$$\frac{d\psi}{d\tau_2} = -\frac{1}{8}\left(1 - \frac{a^4}{32}\right) \tag{7.11}$$

for the amplitude $a = a(\tilde{\tau})$ and the phase $\psi = \psi(\tilde{\tau})$ of the limit cycle of both the Rayleigh and Van der Pol equations, involving slowly varying nonlinear time scales $\tau_i,\ i = 1, 2$. The asymptotic scaling functions $\tau_1 = \phi_1(\varepsilon t) = \varepsilon^{h^1_{RG}}$ and $\tau_2 = \phi\left(\varepsilon^2 t\right) = \varepsilon^{2h^2_{RG}}$ are activated invoking nonlinear limits in (7.9) as $t \to \varepsilon^{-1}$ and $t \to \varepsilon^{-2}$ successively in the above equations. The slowly varying almost constant scaling functions $\phi_1$ and $\phi_2$, satisfying $|\ddot{\phi}_i| \ll |\dot{\phi}^2_i| \ll 1$, are assumed to have a rhythmic pattern over the cycle: when $\phi_1$ varies slowly, $\phi_2$ remains almost constant i.e. $\dot{\phi}_1 > 0,\ \dot{\phi}_2 \approx 0$ and vice versa successively on the cycle (c.f. Section 7.4). Nontrivial ultrametric neighbourhood structure induced asymptotically by duality principle (c.f. Section 7.2) can indeed support such *locally constant* nonlinear rhythmic behaviour. The above flow equations may therefore be considered *exact* and encode non-perturbative information of the limit cycle variables $a$ and $\psi$ respectively. The conventional perturbative RG flow equations in the linear time $t$ is now extended into the non-perturbative flow equations in the nontrivial scaling variable $\tau_i = \varepsilon^{i\ h^i_{RG}(\tilde{\tau})},\ i = 1, 2$ involving the nonlinearity parameter $\varepsilon > 1$. The perturbative fixed point for the amplitude equation at $a = 2$ for $t \to \infty$ corresponding to the periodic oscillation with $\varepsilon \ll 1$ is extended to the by *small scale periodic* flow of amplitude $a(\tau_1)$ over the entire cycle. The associated phase $\psi(\tau_2)$ then flow at a slower rate *linearly with the higher order scale* $\tau_2$ when $a(\tau_1)$ remains almost constant over a relatively small period of time.

The RG estimated approximate formulae for the amplitude $a(\varepsilon)$ for the Rayleigh and Van der Pol limit cycles are obtained from the equation (7.10) in the Subsection 7.3.1, when appropriate boundary condition, derived either from exact computation or from perturbative analysis, is used for a suitable finite value of $\varepsilon$. In Subsection 7.4.1, we present the efficient graph of the Rayleigh and VdP limit cycle parametrized by the nonlinear scales $\tau_i = \phi_i(\tilde{\tau})$, $\tilde{\tau} \sim O(1)$ for fixed values of the nonlinearity parameter $\varepsilon$.

As it turns out, entire onus in the improved RG analysis essentially rests in proper estimation/identification of the scaling functions $h^i_{RG}$ (i.e. $\phi_i(\tilde{\tau})$) which should yield correct dynamical properties of a nonlinear system. We hope to undertake more detailed and systematic analysis for determining $h^i_{RG}$ elsewhere. In this work we limit ourselves only to show that IRGM can indeed yield correct amplitude and solution for the Rayleigh and VdP systems provided one makes appropriate choice of $h^i_{RG}$ based on clues from exact computations and previously known approximate results (for instance the perturbative RGM). We remark finally that the perturbative RG method is known to extend the conventional multiple scale method [31]. Nonlinear time formalism introduces new set of nonlinear scales $\phi_n(\tilde{\tau})$ associated with ordinary scales $\varepsilon^n$. We study here the nontrivial applications of such nonlinear scaling functions.

### 7.3.1 Approximate Formula for Amplitude

We shall now use the above asymptotic amplitude flow equation (7.10) to find analytic approximations of the amplitudes of the limit cycle for both the Rayleigh and Van der Pol equations.

By a direct integration, one obtains from (7.10)

$$\ln\left(a^2 - 4\right) - 2\ln a = -\varepsilon^{h_{RG}} - 0.87953 \tag{7.12}$$

as the Rayleigh limit cycle amplitude where we use the boundary condition the value $a = 2.17271$ for $\varepsilon = 1$ (this choice simplifies calculation). It follows immediately that for suitable choices of the control parameter $h_{RG}$ one can achieve efficient matching for the estimated amplitude $a_E(\varepsilon)$. For example, using the HAM generated approximate formula (5.24) in Chapter 5 for $a_E(\varepsilon)$, we can determine the control parameter $h_{RG}(\varepsilon)$ by the formula

$$h_{RG} = \frac{1}{\ln\varepsilon}\ln\left\{\left|\ln\left(\frac{a^2}{a^2-4}\right) - 0.87953\right|\right\} \tag{7.13}$$

In Figure 7.1, we display the typical piece-wise smooth form of $h_{RG}(\varepsilon)$

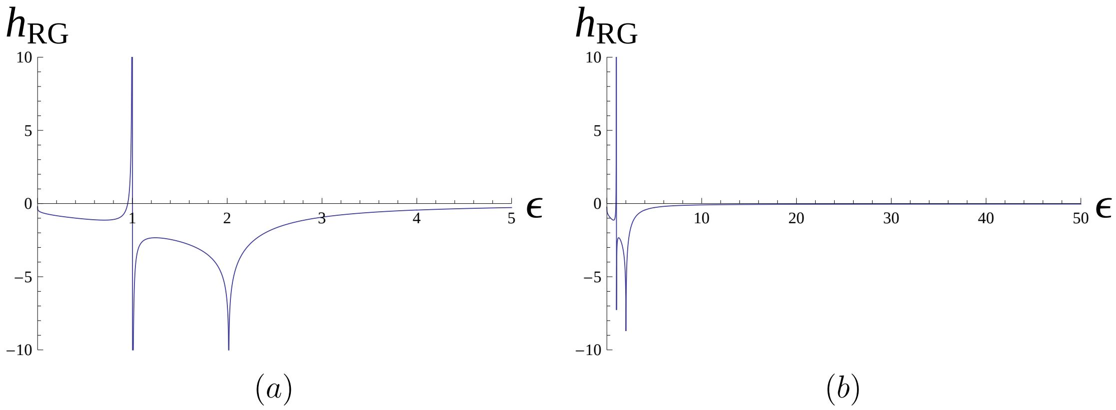



Figure 7.1: The graph of $h_{RG}(\varepsilon)$ used for approximation of the amplitude of the Rayleigh equation (7.1) by HAM given by (7.13) for $0 < \varepsilon \leq 5$ in $(a)$ and for $0 < \varepsilon \leq 50$ in $(b)$.

given by (7.13) for the Rayleigh limit cycle amplitude that would reproduce the HAM generated amplitude with relative error less that 1%. Clearly, the graph reveals variability of $h_{RG}$ for moderate values of $\varepsilon$, but the variability dies out fast for larger values $\varepsilon$, as expected.

We recall that the corresponding graph of the exact computed values of VdP amplitude $a(\varepsilon)$, on the other hand, has a hump like shape with a maximum roughly at $\varepsilon \approx 2.0235$ and having the asymptotic limits 2 as $\varepsilon \to 0$ and $\infty$. Lopez et al. [18] obtained HAM generated approximate formula for the VdP amplitude with relative error less than 0.05% at the order $O\left(\varepsilon^4\right)$. It is interesting to note that the RG generated formula (7.12) can reproduce the exact computed values of the VdP amplitude with error less than 0.05% directly from only the first order RG flow equation. To achieve this goal we first intuitively guess an estimated piecewise smooth formula for the estimated amplitude $a_E$ by

$$a_E(\varepsilon) = \begin{cases} 1.998 + \dfrac{0.015}{8.121\ e^{-2.139\ \varepsilon} + 0.512\ e^{0.043\ \varepsilon}}, & 0 < \varepsilon < 3, \\ 2.0025 + \dfrac{0.031}{0.5\ e^{-2.033(\varepsilon-2.183)} + 1.869\ e^{0.087(\varepsilon-6.376)}}, & 3 \leq \varepsilon \leq 50, \end{cases} \tag{7.14}$$

keeping the maximum relative percentage error $\left|\dfrac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100\right|$ less than 0.05%. This shows that the approximation is quite accurate. The graph of $a_E(\varepsilon)$ is compared with the exact values in Figure 7.2. One may as well use a least square fit of the exact data instead of the above fit. We do not pursue this approach here.

Using this efficient formula for the VdP amplitude, we then obtain

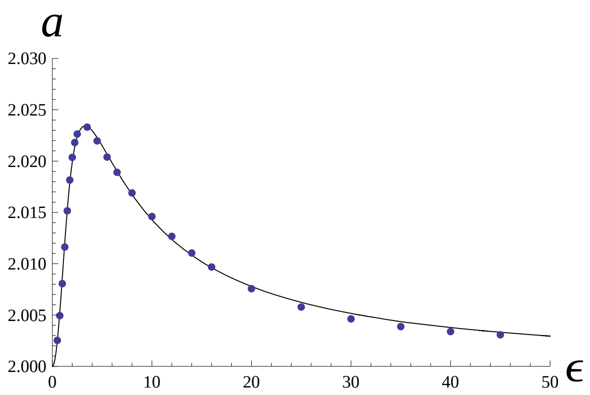



Figure 7.2: The exact amplitude of Van der Pol Equation (7.2) (by solid line) and its approximation $a_E(\varepsilon)$ given by (7.14) (by bold points) for $0 < \varepsilon \leq 50$.

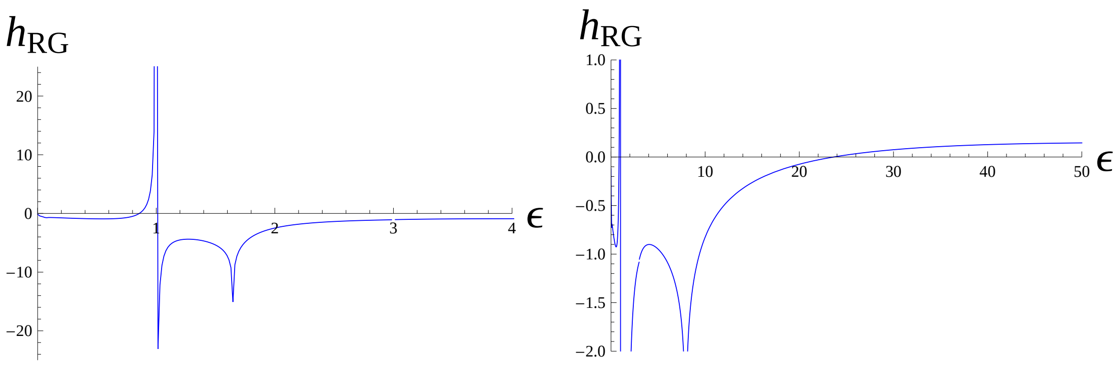
the RG flow equation in the form

$$\ln\left(a^2 - 4\right) - 2\ln a = -\varepsilon^{h_{RG}} - 4.08785 \tag{7.15}$$

where we use the boundary condition $a = 2.0086$ for $\varepsilon = 1$ (for simplicity of calculation) for the VdP amplitude. Inverting this equation, we finally obtain the corresponding RG control parameter

$$h_{RG} = \frac{1}{\ln\varepsilon}\ln\left\{\left|\ln\left(\frac{a_E^2}{a_E^2 - 4}\right) - 4.08785\right|\right\} \tag{7.16}$$

Figure 7.3 displays the piecewise smooth variation of $h_{RG}$ with $\varepsilon$. The




$(a)$ $(b)$

Figure 7.3: The graph of $h_{RG}(\varepsilon)$ used for approximation (7.16) of the amplitude of the Van der Pol equation (7.2) for $0 < \varepsilon \leq 4$ in $(a)$ and for $0 < \varepsilon \leq 50$ in $(b)$.

rapid $O(1)$ variation for moderate values of $\varepsilon$ is evident in Figure 7.3$(a)$. As expected, $h_{RG}$ dies out fast for larger values of $\varepsilon$. However, a change in sign is noticed here already for $\varepsilon > 20$ (Figure 7.3 $(b)$). One expects many more such small scale sign variations as $\varepsilon \to \infty$. This particular form of the control parameter $h_{RG}$, in turn, would reproduce the VdP amplitude with relative error less than 0.05%. As this level of accuracy is achieved only at the order $O(\varepsilon)$, the improved RGM may be considered to be more efficient and advantageous compared to the HAM.

Alternatively, the amplitude equation (7.10) can be inverted as

$$a(\tilde{\tau}) = \frac{a_0}{\sqrt{e^{-\tilde{\tau}} + \frac{a_0^2}{4}(1 - e^{-\tilde{\tau}})}}, \tag{7.17}$$

where $a_0$ is estimated from the exact value of amplitude $a(\varepsilon_0)$ for a suitably chosen value of $\varepsilon$, for instance $\varepsilon = 1$. Recall that for the VdP equation $\tau \sim (\log \varepsilon)^{\alpha}$ and for the Rayleigh equation $\tau \sim (\log \varepsilon)^{-\alpha}$ for $\varepsilon \to \infty$ and $\alpha > 0$. By adjusting suitably the values of $\alpha$ over appropriate intervals on $\varepsilon$ one should be able to obtain efficient matching with the exact values of $a(\varepsilon)$.

To summarize, the recipe for deriving approximate formula for limit cycle amplitude of a nonlinear system can be stated as follows: Determine the first order (perturbative) RG flow equation for amplitude in the nonlinear time $\tau$. This will yield an explicit formula for amplitude $a$ as a function of the nonlinearity parameter $\varepsilon$ and the control parameter $h_{RG}$. Efficient match with the exact amplitude can be achieved by right choice of the control parameter $h_{RG}$ or $\alpha$. Alternatively, determine an efficient formula for $a(\varepsilon)$ by inspection (expert guess) or by appropriate curve fitting method. Then determine the control parameter $h_{RG}$ by an inversion of the estimated amplitude $a_E(\varepsilon)$ as in equation (7.16) (and Figure 7.3). Since the equations concerned form a closed system, this already gives a proof of the unique existence of $h_{RG}$ for a given nonlinear oscillation.

## 7.4 Application to Nonlinear Time: Limit Cycle

Consider a general nonlinear oscillator given by

$$\ddot{x} + x = \varepsilon f\left(x, \dot{x}\right) \tag{7.18}$$

We assume $f$ such that the system admits a unique isolated cycle for $\varepsilon > 0$ and other relevant parameter values. For a finite nonlinearity $\varepsilon > 1$ (say), the usual linear time $t$ is extended to one enjoying *right* asymptotic correction $t \rightarrow \ T_i = t\ \phi_i\left(\tilde{\tau}\left(t\right)\right)$ as $t \rightarrow \infty$ through linear scales $\varepsilon^i$, where $\phi_i\left(\tilde{\tau}\right)$ stands succinctly for the nontrivial intrinsically generated *slowly varying* scaling components arising from the associated visibility norm. Here, $\tilde{\tau}$, as usual denotes an $O\left(1\right)$ rescaled variable in the neighbourhood of linear scales $\varepsilon^i$. In the case of a nonlinear planar autonomous system the relevant dynamical quantities are only amplitude and phase of the nonlinear oscillation and so we have only two asymptotic scaling functions $\phi_i\left(\tilde{\tau}\right),\ i = 1, 2$ which get *selected* naturally so as to facilitate direct non-perturbative calculation of the asymptotic properties i.e. the amplitude and phase of the limit cycle of the system. An implementation of this non-pertubative scheme in the perturbative RG formalism is presented in Subsection 7.3.1 for computation of the amplitude of the concerned oscillators. In Subsection 7.4.1, we shall show that the computed plot (c.f. Figure 7.4) of the limit cycle for the Rayleigh and VdP oscillators could be matched arbitrarily closely for appropriate choices of the slowly varying nonlinear time when the amplitude and phase of the unperturbed periodic solution *flow linearly* in the appropriately chosen nonlinear scaling time variables.

Here, we give an alternative derivation of the nonperturbative *relaxation oscillation* flow equations ab-initio from the slowly varying nonlinear time in the context of the Rayleigh equation (7.1) with $\varepsilon \gg 1$. It will transpire that the new approach is free of any divergence problem because of its inbuilt RG cancellations via duality principle. Since we are interested in the planar limit cycle properties, we assume that all the relevant quantities e.g. the solution $y$, amplitude $a$ and phase $\psi$ are functions of asymptotic time variable $t \sim \varepsilon^n,\ n \gg 1$ and the associated nontrivial scaling variables $\tau_1 = \phi_1\left(\tilde{\tau}\right)$ and $\tau_2 = \phi_2\left(\tilde{\tau}\right)$ for a rescaled

$\tilde{\tau} \sim O(1)$. Higher order scaling variables $\tau_n,\ n > 2$ of the nonlinear structure of time variable may become relevant for a non-planar system. Accordingly, we write the ansatz $y(t, \tau_1, \tau_2) = y_0(t) + Y_1(\tau_1) + Y_2(\tau_2)$ for the limit cycle solution involving multiple time scales (only three for the planar system). Assuming slow variations of nonlinear scales $\tau_i = \phi_i,\ i = 1, 2$, viz. $|\phi_i''| \ll |\phi_i'^2| \ll 1,\ \phi_i' = \dfrac{d\phi_i}{d\tilde{\tau}}$, as $t \to \varepsilon^n,\ n \to \infty$ and noting that $\dfrac{dy}{dt} = \dfrac{\partial y_0}{\partial t} + \sum_i \dot{\tilde{\tau}}\phi_i' \dfrac{\partial Y_i}{\partial \tau_i}$ etc., the Rayleigh equation (7.1) simplifies to

$$\begin{aligned}\frac{\partial^2 y_0}{\partial t^2} + y_0 + \sum_i Y_i = \varepsilon \left( \frac{\partial y_0}{\partial t} + \sum_i \dot{\tilde{\tau}}\phi_i' \frac{\partial Y_i}{\partial \tau_i} \right) \\ - \frac{\varepsilon}{3} \left( \left( \frac{\partial y_0}{\partial t} \right)^3 + 3 \left( \frac{\partial y_0}{\partial t} \right)^2 \sum_i \dot{\tilde{\tau}}\phi_i' \frac{\partial Y_i}{\partial \tau_i} \right) \qquad (7.19)\end{aligned}$$

where we drop all higher order terms involving $\phi_i''$ and $\phi_i'^2$. Assuming $y_0(t) = a(\tau_1, \tau_2)\cos(t + \psi(\tau_1, \tau_2))$ with *flowing* amplitude and phase in scaling times $\tau_1$ and $\tau_2$ so that

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0 \qquad (7.20)$$

we next get a simplified linearized evolution for the nonlinear components of the asymptotic limit cycle solution in the form

$$\left(1 - \left(\frac{\partial y_0}{\partial t}\right)^2\right) \sum \dot{\phi}_i \frac{\partial Y_i}{\partial \tau_i} = \left\{ \frac{1}{3} \left( \frac{\partial y_0}{\partial t} \right)^3 - \frac{\partial y_0}{\partial t} \right\} + \varepsilon^{-1} \sum Y_i \quad (7.21)$$

where $\dot{\phi}_i = \dfrac{d\phi_i}{dt}$. As a consequence, under the assumption of slow varying nonlinear time scales, a second order nonlinear planar system (7.1) would decompose into a linear second order partial differential equation (7.20) for the zero level solution $y_0$ and an associated first order partial differential equation (7.21) for the nonlinear scale dependent components $Y_i$. Clearly, analogous decomposition holds actually for a larger class of planar autonomous systems (7.18) having a unique limit cycle solution. Extension of this result to multiple limit cycles would be considered separately.

We note here that since the system (7.20) and (7.21) is *under determined*, there is room for further restrictions to solve the system self-consistently. To re-derive the RG flow equations (7.10) and (7.11) from (7.21), we now make following assumptions: we write $(i)$ $a(\tau_1, \tau_2) = a(\tau_1)$, $\psi(\tau_1, \tau_2) = \psi(\tau_2)$ so that amplitude varies slowly with first order scale $\tau_1$ when the second order scale $\tau_2$ and phase $\psi$ remain almost constant. On the other hand as $a$ stabilizes to an almost constant value, the phase begins to flow, though slowly with the second order scale $\tau_2$. Such slow, almost constant, rhythmic cooperative variations of $\tau_1$ and $\tau_2$ are modeled, depending on the specific problem under consideration (see below and c.f. Subsection 7.4.1), to retrieve the RG flow equations correctly. As shown in the example (Remark 7.2.3) such a rhythmic nonlinear variation does exist in an ultrametric neighbourhood in $\mathbb{R}^*$. To further quantify the slow variation of dynamical variables, we next impose the condition that $(ii)$ *the total variation of the exact solution* $y(t, \tau_1, \tau_2)$ *with respect to each slow variable* $\tau_i$ *along the full periodic cycle* $C$ *must vanish viz.,* $\int\limits_C \frac{\partial y}{\partial \tau_i} dt = 0$ *for each* $i$. To avoid trivialities i.e. $\int\limits_C \cos(t+\psi)dt = 0$ etc., we, however, evaluate the concerned integrals only on the quarter cycle, with the understanding that phase shifts of $\frac{\pi}{2}$ are absorbed in the definition of $\psi$.

In the sufficiently large $\varepsilon > 1$ relaxation oscillation, one can further simplify (7.21) by dropping the $\varepsilon^{-1}$ term to obtain

$$\sum \dot{\phi}_i \frac{\partial Y_i}{\partial \tau_i} = \frac{\frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3 - \frac{\partial y_0}{\partial t}}{1 - \left(\frac{\partial y_0}{\partial t}\right)^2} \equiv \Phi(y_{0t}), \; y_{0t} = \frac{\partial y_0}{\partial t}. \tag{7.22}$$

To make contact with RG flow equations (7.10) and (7.11) one now exploits *the freedom of right choice* in the functional forms of nonlinear scales. For the Rayleigh equation, we now set for slow, cooperatively active functional dependence $(a)$ $\dot{\phi}_1 = \Phi(y_{0t}) S_1^{-1}(a, \psi, t)$, $\dot{\phi}_2 = 0$ and $(b)$ $\dot{\phi}_1 = 0$, $\dot{\phi}_2 = \Phi(y_{0t}) S_2^{-1}(a, \psi, t)$ for successive slow variations, as described in $(i)$, of the scales $\tau_1$ and $\tau_2$ respectively, where,

$S_1 = \frac{1}{2}a\left(\dfrac{a^2}{4} - 1\right)\cos(t+\psi)$ and $S_2 = \frac{1}{8}\left(1 - \dfrac{a^4}{32}\right)\sin(t+\psi)$ (recall the example in Remark 7.2.3 above highlighting wide possible choices and intricate functional dependence). These choices for $S_1$ and $S_2$ would yield the RG flow equations when condition $(ii)$ is invoked.

Note that the relations in both $(a)$ and $(b)$ are truly nonlinear; the dynamical variables $a$ and $\psi$ in $S_1$ and $S_2$ depend implicitly in $\phi_1$ and $\phi_2$ respectively, which, in turn are *slowly varying* as the linear parameter $t$ is assumed to vary in a neighbourhood of $\varepsilon^n$ for a large but fixed $n$. Invoking the global slow variation condition $(ii)$ for each $i$, in conjunction with the ansatz $(a)$ and $(b)$, one finally deduce the amplitude and phase flow equations (7.10) and (7.11) in slow variables $\tau_1$ and $\tau_2$ respectively.

In the present format, the flow equations, however, have got *new* interpretations: Amplitude and phase must flow in successive rhythmic manner; phase remains almost constant (i.e. $\dfrac{\partial\psi}{\partial\tau_i} = 0$ for each $i$) when amplitude varies slowly with $\tau_1$ towards an almost constant value. Subsequently, the flowing of $a$ is halted temporarily (i.e. $\dfrac{\partial a}{\partial\tau_i} = 0$), initiating flowing of $\psi$ in next level variable $\tau_2$. This rhythmic oscillation would obviously continue indefinitely over a cycle. The RG flow equations could be treated as non-perturbative because of *implicit* connections of nonlinear scaling time functions with amplitude and phase via intrinsically defined duality principle (c.f. Remark 7.2.3).

### 7.4.1 Approximating Limit Cycle

Here we calculate the approximate limit cycle orbit for the Rayleigh and VdP equations for a sufficiently large $\varepsilon > 1$. Perturbative RGM fails to give correct relaxation oscillation solution. The first order solution given in Section 5.2 in Chapter 5 by HAM is also found insufficient. In this work we do not undertake the problem of computing approximate limit cycle by HAM, which has been addressed by Lopez et al. [18] for the VdP equation. Our aim here is to highlight the strength of IRGM over perturbative RGM.

For a sufficiently large time $t \to \varepsilon^n$, $n$ large, but fixed, the slowly varying nonlinear scales $\tau_1$ and $\tau_2$ are activated in a successive rhythmic manner, as explained in Section 7.3, so that the perturbative solution given in (6.8) of Chapter 6 is extended to the asymptotic limit cycle (relaxation oscillation) solution

$$y\left(\tau_1, \tau_2\right) = a\left(\tau_1\right)\cos\left(\varepsilon^n + \psi\left(\tau_2\right)\right) + Y \tag{7.23}$$

where the amplitude $a$ and phase $\psi$ *flow* along the cycle following the nonperturbative flow equations (7.10) and (7.11) in the asymptotic scaling variables $\tau_1$ and $\tau_2$ respectively. Here, $Y$ encodes all the renormalized perturbative terms depending on higher order, slowly varying nonlinear scales $\tau_i,\ i > 2$. The corresponding velocity component $\dot{y} = \dfrac{\partial y}{\partial t}$ at $t = \varepsilon^n$ has the form

$$\dot{y}\left(\tau_1, \tau_2\right) = -a\left(\tau_1\right)\sin\left(\varepsilon^n + \psi\left(\tau_2\right)\right) + \sum_i \dot{\tau}_i \frac{\partial Y}{\partial \tau_i}. \tag{7.24}$$

Equations (7.23) and (7.24) are the parametric equations of the limit cycle, parametrized by multiple nonlinear scales, when slowly varying amplitude $a$ and phase $\psi$ are computed from (7.10) and (7.11) respectively. An alternative derivation of (7.23) and (7.24) based purely on duality induced nonlinear scales is given in Section 7.4. We remark that (7.23) and (7.24) actually represent the general form of the limit cycle for a much large class of Lienard system having unique limit cycle. *Typical geometric shape of the periodic cycle of a given nonlinear system is controlled entirely by the rhythmic cooperative, almost constant variations of the nonlinear scales $\tau_i$.* As explained in Section 7.3, scale invariance of the scaling functions $\tau_1 = \phi_1\left(\tilde{t}_1/\varepsilon\right)$ and $\tau_2 = \phi_1\left(\tilde{t}_2/\varepsilon^2\right)$ tells that as the linear time $t \to \varepsilon^n$, $\tau_1 \to \phi_1(1) = 1$ and $\tau_2 \to \phi_2(1) = 1$ with $\tilde{t}_1 \to \varepsilon$ and $\tilde{t}_2 \to \varepsilon^2$. We now set the initial conditions $a(1) = a_{\mathrm{amp}}(\varepsilon)\,,\ a'(1) = \dfrac{a(1)}{2}\left(1 - \dfrac{a(1)^2}{4}\right)$ and $\psi(1) = 0$, where $a_{\mathrm{amp}}(\varepsilon)$ is the exact (experimental) value of the amplitude of the limit cycle. Setting further $\tau_1 = 1 + \eta_1$ and $\tau_2 = 1 + \eta_2$ as $t \to \varepsilon^n$, both amplitude and phase flow equations (7.10) and (7.11) now yield linear flow of amplitude and phase relative to the respective small scale slow, almost constant variables $\eta_1$ and $\eta_2$, satisfying

$|\eta_i^2| \ll 1$ and $|\ddot{\eta}_i| \ll |\dot{\eta}_i^2| \ll 1$, those vary in a rhythmic manner. The exact (experimental) limit cycle could now be approximated with any desired accuracy by smooth matching of the straight line segments of the form $(i)$ $z - Z_0 = k\,(y - Y_0)$ and circular arcs of the form $(ii)$ $(y - Y_0)^2 + (z - Z_0)^2 = a^2\,(\tau_1)$ over judiciously chosen intervals in $y$ in the $(y, z)$ plane, where $z = \dot{y}$, $k = -\tan\left(\varepsilon^n + \psi\,(\tau_2)\right)$, $Y_0 = Y$ and $Z_0 = \sum_i \dot{\tau}_i \dfrac{\partial Y}{\partial \tau_i}$. Because of the availability of cooperatively evolving resource of nonlinear scales, such a matching is always possible theoretically. In the alternative derivation of limit cycle equations in Section 7.4, we have outlined an approach to gain more analytic understanding of the rhythmic, cooperative variations of the nonlinear scaling functions. We hope to address the question of determining the precise analytic properties of the scaling functions $\phi_i\,(\tilde{\tau})$ corresponding to a given nonlinear system in future communications.

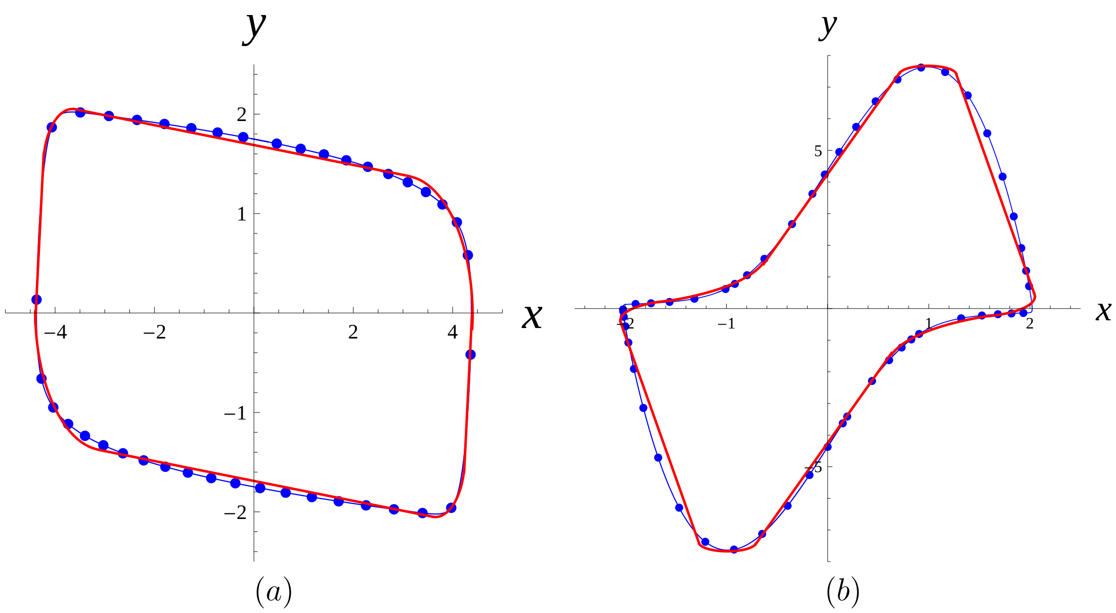



Figure 7.4: Approximate limit cycle for $(a)$ Rayleigh equation and $(b)$ Van der Pol equation; solid (red) line for approximate curve, dotted (blue) line for exact curve $(\varepsilon = 5)$.

In Figure 7.4$(a)$ and Figure 7.4$(b)$, we display the $(y, z)$ phase plane relaxation oscillation for the Rayleigh and VdP equations with $\varepsilon = 5$. A piecewise smooth matching curve for upper half of the approximate

Rayleigh limit cycle for $\varepsilon = 5$ is

$$z(y) = \begin{cases} 0.02 - \sqrt{1.96 - (y+3)^2}, & -4.96 < y \le -4.393, \\ 10y + 43.9, & -4.393 < y \le -4.23, \\ 1.46 + \sqrt{0.35 - (y+3.65)^2}, & -4.23 < y \le -3.6, \\ -0.1y + 1.689, & -3.6 < y \le 3.12, \\ -0.02 + \sqrt{1.96 - (y-3)^2}, & 3.12 < y \le 4.96; \end{cases}$$

given by solid (red) lines in Figure 7.4($a$) and another such curve for upper half of the approximate VdP limit cycle for $\varepsilon = 5$ is

$$z(y) = \begin{cases} -0.388 + \sqrt{0.352 - (y+1.438)^2}, & -2.05 < y \le -1.7, \\ 1.8 - \sqrt{3.2 - (y+2.38)^2}, & -1.7 < y \le -0.633, \\ 4.5y + 4.25, & -0.633 < y \le 0.6, \\ 6.5 + \sqrt{2.76 - (y-2.2)^2}, & 0.6 < y \le 0.9, \\ 7.325 + \sqrt{0.063 - (y-1.04)^2}, & 0.90 < y \le 1.28, \\ 0.38 + \sqrt{1530 - (y+37.2)^2}, & 1.28 < y \le 1.8, \\ -13y + 26.8, & 1.8 < y \le 2.033, \\ 0.388 + \sqrt{0.352 - (y-1.438)^2}, & 2.033 < y \le 2.05; \end{cases}$$

shown as solid (red) lines in Figure 7.4($b$).

Here, the piece-wise smooth matching curves approximating these cycles are presented in tabular forms. However, the smoothness at the joining points are achieved at the level of one decimal only. More accurate approximation may be achieved with smarter efforts. Judicious choice of slowly varying centres $(Y_0, Z_0)$ and radii $a(\tau_1)$ of circular arcs of right sizes (a straight line segment being an arc with sufficiently large radius) should give better approximations with a given exact (experimental) cycle that can be obtained on a symbolic computation platform, Mathematica for instance. The phase plane dynamics of these slowly varying centres and radii is expected to reveal interesting new insights into asymptotic properties of the nonlinear oscillation.

It transpires from above discussion that radii, for instance, vary much faster in Rayleigh than that in VdP oscillator, in which case radii fluctuate between small and large values through intermediate steps. This might be compared with fast and slow energy build ups in Rayleigh and VdP relaxation oscillations respectively. One would like to interpret this phase plane dynamics as cooperative evolution of multiple nonlinear scales driving amplitude and phase of the nonlinear oscillation to flow in such a fashion as to generate little circular arcs and linear segments which join smoothly together to form the complete orbit. Making an accurate plot then boils down to finding right kind of such arcs and line segments. Intricate dependence of the trajectory itself into the definition of the nonlinear scales (c.f. Remark 7.2.3) tells in retrospect that one needs to look for a novel iteration scheme that would allow one to extract the trajectory systematically as a limit process. This problem will be considered in detail elsewhere.

## 7.5 Concluding Remarks

In this chapter we have presented a comparative study of the homotopy analysis method and the Renormalization Group method. It turns out that the higher order perturbative calculations based on the conventional Renormalization group method would fail to give efficient formula for the limit cycle amplitudes for these nonlinear oscillators. However, an improved version of the Renormalization group analysis exploiting a novel concept of nonlinear time is shown to yield efficient amplitude formulae for all values of $\varepsilon$. Exploiting multiple nonlinear scales of the associated nonlinear time the improved RG method is also found to yield good plots for relaxation oscillation orbits for the Rayleigh and VdP systems.

We have presented brief review of the nonlinear time formalism and also given an alternative approach in deriving non-perturbative flow equations of amplitude and phase of a limit cycle problem. Non-perturbative information of asymptotic quantities get naturally encoded into nonlinear scales, that can be exploited judiciously to extract desired asymptotic properties of a relevant dynamical quantity. An algorithmic procedure of extracting such information is explained in estimating both the limit cycle amplitude and trajectory for Rayleigh and VdP equations. More detailed analysis of the nonlinear time formalism in several other nonlinear systems will be considered in future.