# Part II

# Approximate Limit Cycle: Amplitude and Shape

# Chapter 5

# Analytic Approximation of Amplitude of Limit Cycles by Homotopy Analysis Method

## 5.1 Introduction

We have extended the results available in literature in search of sufficient condition for existence of exactly one or multiple limit cycles in case of symmetric Lienard systems and for existence of at least $N$ limit cycles in case of nonsymmetric Lienard systems in the previous chapters. In the current and subsequent chapter we shall study some results on the characteristics such as size, amplitude etc. of limit cycles in the case of a special class of Lienard systems. Indeed, we shall restrict our study in case of Rayleigh equation

$$\ddot{y} + \varepsilon \left( \frac{1}{3} \dot{y}^3 - \dot{y} \right) + y = 0 \tag{5.1}$$

and Van der Pol (VdP) equation

$$\ddot{x} + \varepsilon \, \dot{x} \left( x^2 - 1 \right) + x = 0, \tag{5.2}$$

which arise very frequently in modeling the dynamics of different nonlinear systems in the fields of Physics, Biology, Acoustics, Robotics, Engineering etc. [10,11]. Here the dots are used to designate the derivative with respect to time $t$. Many nonlinear systems can be converted to the above systems by suitable transformations. Although a lot of research have been done on these systems using analytical and numerical techniques, determination of analytic formula for shape, size etc.

of limit cycles have been quite a challenging task for a long time, specially in nonperturbative regime. Mostly, perturbation techniques are used based on the existence of small or large parameters. These techniques use perturbation parameters to convert a nonlinear problem into infinite number of linear subproblems. The solution of the original nonlinear problem is then approximated by the solution of first few finite number of these linear subproblems. However, the presence of these parameters in these techniques impose some significant restrictions on the approximate solution in long time scale. As a result the asymptotic phenomena cannot be properly described by perturbative techniques. We know that limit cycle is an isolated closed curve that arise only in nonlinear system. This is an isolated closed curve $\Gamma$ (say) in the phase plane so that any path in its suitable small neighbourhood starting from a point, specified by some given initial condition, ultimately converges to (or diverge from) $\Gamma$ in long asymptotic time. As a result perturbation techniques do not give good approximations of different characteristics of limit cycles such as its size, shape etc.

In recent past some new techniques have been introduced such as multiple scale analysis, method of boundary layers, WKB method and so on [1,3] in order to find good, uniformly valid approximate solutions to a nonlinear problem, specially in nonperturbative regime. The recently developed homotopy analysis method (HAM) [18,68] helps us to find good analytic approximation to the exact solution. The aim of this new improved method is to derive in an unified manner uniformly valid asymptotic quantities of interest for a given nonlinear dynamical problem, i.e., the approximate analytic formulae for different asymptotic quantities remains valid uniformly for all values of the perturbative (or nonlinearity) parameters involved in the problem [41].

The Rayleigh and the Van der Pol (VdP) equations represent two closely related nonlinear systems and it is easy to observe that differentiating (5.1) with respect to time $t$ and putting $\dot{y}(t) = x(t)$ we obtain (5.2). Both these systems have unique isolated periodic orbit (limit cycle). The amplitude of a periodic oscillation $y(t)$ (or $x(t)$) is generally defined by $\max|y(t)|$ (or $\max|x(t)|$) over the entire cycle. It is well known that the naive perturbative solutions of these equations are useful when $0 < \varepsilon \ll 1$ and yields the asymptotic value $a(\varepsilon) \approx 2$ of the amplitude for the limit cycle correctly. For $\varepsilon \gg 1$, simple analysis based on singular perturbation theory also yields the asymptotic amplitude for the relaxation oscillation as $a(\varepsilon) \approx 2$ for the VdP equation. However, the conventional perturbative approaches fail when $\varepsilon$ is finite. One of the aim of this chapter is to determine efficient approximate formulae for the amplitude of the limit cycle for the Rayleigh systems by HAM. Lopez et al. [18] have reported an efficient formula for the amplitude of the VdP limit cycle by HAM. We note here that a key difference in Rayleigh and VdP oscillators is the fact that with increase in input energy (voltage), the amplitude of the Rayleigh periodic oscillation increases, when that of the VdP oscillator remains almost constant at the value 2, with possible increase in the corresponding frequency only. For large $\varepsilon$ ($\geq 1$) relaxation oscillations, on the other hand, the Rayleigh system shows up a rather fast building up and slow subsequent release of internal energy, when the VdP models the reverse behaviour, with slow rise and fast drop in the accumulated energy.

As remarked above, HAM is formulated to determine the uniformly valid global asymptotic behaviours of relevant dynamical quantities like amplitude, period, frequency etc. related to periodic solutions of these equations for finite values of $\varepsilon$, by devising efficient methods in eliminating divergent secular terms of the naive perturbation theory. HAM seems to have the advantage of yielding uniformly convergent solutions of very high order in the nonlinearity parameter $\varepsilon$ utilizing a freedom in the choice of a free parameter $h$. The computation of higher order term could be facilitated by symbolic computational algorithms. This method is used to obtain good approximate solutions for the VdP equation by a number of authors [18, 69]. Lopez et al. [18] derived efficient formulae for estimating the amplitude of the limit cycle of the VdP equation for all values of $\varepsilon > 0$. Although, HAM is now considered to be an efficient method in the study of non-perturbative asymptotic analysis, it is recently pointed out [19] that this method might fail even in some innocent looking nonlinear problems.

Here we compute an analytic expressions of the amplitude of the periodic solutions of the Rayleigh equation (5.1) as functions of $\varepsilon$. Same has been already reported by Lopez et al. [18] for the VdP equation (5.2). The HAM contains a control parameter $h = h(\varepsilon)$ which controls the convergence of the approximation to the numerically computed exact value of the amplitude for all values of $\varepsilon$. Suitable choice of $h$ can control the relative percentage error. In the Section 5.2 we have deduced the solution to the Rayleigh equation (5.1) by HAM.

## 5.2 Computation of Amplitude by HAM

The Homotopy Analysis method proposed by Liao [68, 69] is used to obtain the solution of non-linear equation even if the problem does not contain a small or large parameter. HAM always gives a family of functions at any given order of approximation. An auxiliary parameter $h$ is introduced in HAM to control the convergence region of approximating series involved in this method to the exact solution. HAM is based on the idea of homotopy in topology. In simple language, it involves continuous deformation of the solution of a linear ordinary differential equation (ODE) to that of desired nonlinear ODE. The solution of linear ODE gives a set of functions called *base functions*. One advantage of HAM is that it can be used to approximate a nonlinear problem by efficient choice of different sets of base functions. A suitable choice of the set of base functions and the convergence control parameter can speed up the convergence process.

In this paper we consider the self-excited system (5.1), which can be written as the ODE

$$\ddot{U}(t) + \varepsilon\left(\frac{1}{3}\dot{U}^3(t) - \dot{U}(t)\right) + U(t) = 0, \quad t \geq 0, \tag{5.3}$$

where the dot denotes the derivative with respect to the time $t$. We know that a limit cycle represents an isolated periodic motion of a self-excited system. We have mentioned earlier that it is an isolated closed curve $\Gamma$ (say) in the phase plane so that any path in its suitable small neighbourhood starting from a point, specified by some given initial condition, ultimately converges to (or diverge from) $\Gamma$. Consequently, this periodic motion represented by limit cycle is independent of initial conditions. It, however, involves the frequency $\omega$ and the amplitude $a$

of the oscillation. Therefore, without loss of generality, we consider an initial condition

$$U(0) = a, \qquad \dot{U}(0) = 0. \tag{5.4}$$

In [31], an alternative initial condition i.e. $U(0) = 0,\ \dot{U}(0) = a$ was considered. Let,

$$\tau = \omega t \text{ and } U(t) = a\ u(\tau)$$

so that (5.3) and (5.4) respectively become

$$\omega^2 u''(\tau) + \varepsilon\left(\frac{1}{3}a^2\omega^2 u'^2(\tau) - 1\right)\omega u'(\tau) + u(\tau) = 0 \tag{5.5}$$

and

$$u(0) = 1, \quad u'(0) = 0. \tag{5.6}$$

Since the limit cycle represents a periodic motion, so we suppose that the initial approximation to the solution $u(\tau)$ to (5.5) can be taken as

$$u_0(\tau) = \cos\tau.$$

Let, $\omega_0$ and $a_0$ respectively denote the initial approximations of the frequency $\omega$ and the amplitude $a$.

We consider a linear operator

$$\mathcal{L}[\phi(\tau,p)] = \omega_0^2\left[\frac{\partial^2\phi(\tau,p)}{\partial\tau^2} + \phi(\tau,p)\right] \tag{5.7}$$

so that for the coefficients $C_1$ and $C_2$

$$\mathcal{L}(C_1\sin\tau + C_2\cos\tau) = 0. \tag{5.8}$$

We further consider a nonlinear operator

$$\begin{aligned} &\mathcal{N}[\phi(\tau,p), \Omega(p), A(p)] \\ &= \Omega^2(p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2} \\ &\quad + \varepsilon\left[\frac{1}{3}A^2(p)\,\Omega^3(p)\left(\frac{\partial\phi(\tau,p)}{\partial\tau}\right)^3 - \Omega(p)\left(\frac{\partial\phi(\tau,p)}{\partial\tau}\right)\right] + \phi(\tau,p). \end{aligned} \tag{5.9}$$

Next, we construct a homotopy as

$$\mathcal{H}\left[\phi\left(\tau,p\right),h,p\right]=\left(1-p\right)\mathcal{L}\left[\phi\left(\tau,p\right)-u_0\left(\tau\right)\right] \\ -h\ p\,\mathcal{N}\left[\phi\left(\tau,p\right),\Omega\left(p\right),A\left(p\right)\right], \tag{5.10}$$

where $p\in[0,1]$ is the embedding parameter and $h$ a non-zero auxiliary (control) parameter used to improve the convergence of series expansions. Setting $\mathcal{H}\left[\phi\left(\tau,p\right),h,p\right]=0$ we obtain zero-th order deformation equation

$$\left(1-p\right)\mathcal{L}\left[\phi\left(\tau,p\right)-u_0\left(\tau\right)\right]-h\ p\,\mathcal{N}\left[\phi\left(\tau,p\right),\Omega\left(p\right),A\left(p\right)\right]=0 \tag{5.11}$$

subject to the initial conditions

$$\phi\left(0,p\right)=1,\qquad \left.\frac{\partial\phi\left(\tau,p\right)}{\partial\tau}\right|_{\tau=0}=0. \tag{5.12}$$

Clearly, as $p$ increases from $p=0$ to $p=1$, (5.11) changes from $\mathcal{L}\left[\phi\left(\tau,p\right)-u_0\left(\tau\right)\right]=0$ to $\mathcal{N}\left[\phi\left(\tau,p\right),\Omega\left(p\right),A\left(p\right)\right]=0$ and as a consequence $\phi\left(\tau,p\right)$ varies from the initial guess $\phi\left(\tau,0\right)=u_0\left(\tau\right)=\cos\tau$ to the exact solution $\phi\left(\tau,1\right)=u\left(\tau\right)$, so does $\Omega\left(p\right)$ from $\omega_0$ to exact frequency $\omega$ and $A\left(p\right)$ from $a_0$ to the exact amplitude $a$. It can be shown that assuming $\phi\left(\tau,p\right)$, $\Omega\left(p\right)$, $A\left(p\right)$ analytic in $p\in[0,1]$ so that

$$u_k\left(\tau\right)=\frac{1}{k!}\left.\frac{\partial^k}{\partial p^k}\phi\left(\tau,p\right)\right|_{p=0},\ \omega_k=\frac{1}{k!}\left.\frac{\partial^k}{\partial p^k}\Omega\left(p\right)\right|_{p=0},\ a_k=\frac{1}{k!}\left.\frac{\partial^k}{\partial p^k}A\left(p\right)\right|_{p=0} \tag{5.13}$$

we have,

$$u\left(\tau\right)=\sum_{k=0}^{\infty}u_k\left(\tau\right), \tag{5.14}$$

$$\omega=\sum_{k=0}^{\infty}\omega_k, \tag{5.15}$$

$$a=\sum_{k=0}^{\infty}a_k, \tag{5.16}$$

where $u_k\left(\tau\right)$ are solutions of the $k$-th order deformation equation

$$\mathcal{L}\left[u_k\left(\tau\right)-\chi_k u_{k-1}\left(\tau\right)\right]=h\ R_k\left(\tau\right) \tag{5.17}$$

subject to the initial conditions

$$u_k(0) = 0, \qquad u'_k(0) = 0 \tag{5.18}$$

in which

$$\begin{aligned}
R_k(\tau) &= \frac{1}{(k-1)!}\,\frac{\partial^{k-1}}{\partial p^{k-1}}\mathcal{N}\left[\phi(\tau,p),\Omega(p),A(p)\right]\Big|_{p=0} \\
&= \sum_{n=0}^{k-1} u''_{k-1-n}(\tau)\sum_{j=0}^{n}\omega_j\omega_{n-j} + u_{k-1}(\tau) \\
&+ \frac{\varepsilon}{3}\sum_{n=0}^{k-1}\sum_{i=0}^{n}\left(\sum_{r=0}^{i}a_r a_{i-r}\right)\times\left(\sum_{s=0}^{n-i}\omega_s\sum_{h=0}^{n-i-s}\omega_h\omega_{n-i-s-h}\right) \\
&\times\left(\sum_{j=0}^{k-1-n}u'_j(\tau)\sum_{m=0}^{k-1-n-j}u'_m(\tau)\,u'_{k-1-n-j-m}(\tau)\right) - \varepsilon\sum_{n=0}^{k-1}\omega_n u'_{k-1-n}(\tau)
\end{aligned} \tag{5.19}$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases} . \tag{5.20}$$

To ensure that the solution to the $k$-th order deformation equation (5.17) do not contain the secular terms $\tau\sin\tau$ and $\tau\cos\tau$ the coefficients of $\sin\tau$ and $\cos\tau$ in the expressions of $R_k$ in (5.19) must vanish giving successive values of $\omega_k$ and $a_k$.

The linear equation $\mathcal{L}(\phi(\tau,p)) = 0$ represents a simple harmonic motion with frequency 1. So, we choose the initial guess of $\omega$ as $\omega_0 = 1$. Again, by perturbation method [3] we find $a \to 2$ as $\varepsilon \to 0$. So, we choose the initial guess of $a$ as $a_0 = 2$. Solving the differential equations given by (5.11), (5.12), (5.17), (5.18) and avoiding the generation of secular terms in each iteration we obtain

$$u_1(\tau) = -\frac{1}{24}h\varepsilon\sin 3\tau + \frac{1}{8}h\varepsilon\sin\tau,\ \omega_1 = -\frac{1}{16}h\varepsilon^2,\ a_1 = \frac{1}{8}h\varepsilon^2,$$

$$\begin{aligned}
u_2(\tau) = &\left(\frac{1}{384}h^2\varepsilon^3 - \frac{1}{24}h^2\varepsilon - \frac{1}{24}h\varepsilon\right)\sin 3\tau - \frac{1}{64}h^2\varepsilon^2\cos 3\tau \\
&+ \frac{1}{64}h^2\varepsilon^2\cos\tau + \left(\frac{1}{8}h^2\varepsilon - \frac{1}{128}h^2\varepsilon^3 + \frac{1}{8}h\varepsilon\right)\sin\tau
\end{aligned}$$

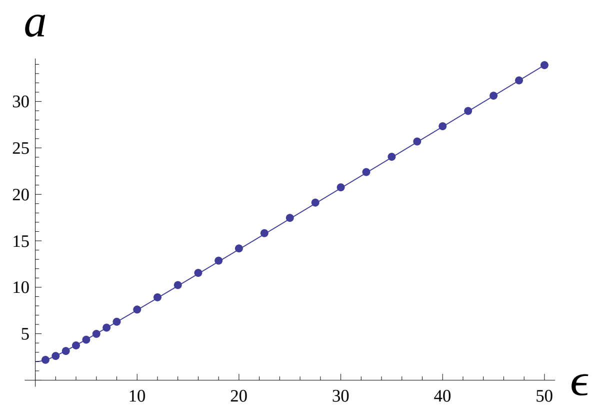


Figure 5.1: The exact amplitude of Rayleigh Equation (by solid line) and its approximation $a_E(\varepsilon)$ given by (5.24) (by bold points) for $0 < \varepsilon \leq 50$.

so that

$$R_1 = \frac{1}{3}\varepsilon \sin 3\tau,$$
$$R_2 = \frac{1}{24}\left[3h\varepsilon^2 \cos 3\tau + \left(8h\varepsilon - \frac{1}{2}h\varepsilon^3\right)\sin 3\tau\right].$$

Computing $R_k$ successively, we can find the successive expressions of $u_k(\tau)$, $\omega_k$ and $a_k$. The first order approximation to the amplitude in (5.16) is

$$a \approx a_0 + a_1 = 2 + \frac{1}{8}h\varepsilon^2 = a_E(\varepsilon) \quad (\text{say})\,. \tag{5.21}$$

The above first order expression for the amplitude involves as yet arbitrary control parameter $h$. Lopez et. al. [18] proposed specific $\varepsilon$-dependent expressions for $h$ to obtain an efficient formula for the VdP limit cycle amplitude. They made the proposal that $h$, besides being continuous, must also vanish in the limits of $\varepsilon \to 0$ and $\varepsilon \to \infty$ to reproduce the zeroth order perturbative solutions. In our application of HAM for the Rayleigh limit cycle amplitude, we have chosen a different set of base functions and so can weaken the condition considerably, both on the continuity and the asymptotic limit $\varepsilon \to \infty$. From careful inspections of the graph of the exact amplitude (Figure 5.1), it turns out that an appropriate ansatz for the control parameter $h$ is given by

$$h = \frac{1}{0.5 + \varepsilon\, b(\varepsilon)}, \tag{5.22}$$

where, $b(\varepsilon)$ is taken as the step function in the domain $0 < \varepsilon \le 50$ as follows:

| $\varepsilon$ : | $0<\varepsilon\le 4$ | $4<\varepsilon\le 5$ | $5<\varepsilon\le 7$ | $7<\varepsilon\le 8$ | $8<\varepsilon\le 9$ |
|---|---|---|---|---|---|
| $b(\varepsilon)$ : | 0.162 | 0.165 | 0.168 | 0.171 | 0.174 |
| $\varepsilon$ : | $9<\varepsilon\le 11$ | $11<\varepsilon\le 15$ | $15<\varepsilon\le 20$ | $20<\varepsilon\le 30$ | $30<\varepsilon\le 50$ |
| $b(\varepsilon)$ : | 0.176 | 0.179 | 0.181 | 0.183 | 0.185 |

With this particular form of $h$, we are able to find an analytic approximation $a_E(\varepsilon)$ to the numerically computed exact value $a = a(\varepsilon)$ in the domain $0 < \varepsilon \le 50$ with maximum relative percentage error $\left|\dfrac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100\right|$ less than 1%. Obviously, better accuracy fit can be obtained by considering finer subdivisions in the definition of $b(\varepsilon)$. We remark that a piece-wise continuous $\varepsilon$ dependence of $h$ as above is admissible in the framework of HAM.

Since the exact graph of $a(\varepsilon)$ is almost a straight line for sufficiently large $\varepsilon$ $(7 < \varepsilon \le 50)$, we can reduce the number of steps to 4 only. Let us choose

$$h = \frac{8m}{\varepsilon} - \frac{56m}{\varepsilon^2} + \frac{8c}{\varepsilon^2} - \frac{16}{\varepsilon^2}, \qquad 7 < \varepsilon \le 50 \tag{5.23}$$

so that (5.21) becomes

$$a_E(\varepsilon) = \begin{cases} 2 + \frac{1}{8}\left(\frac{1}{0.5+0.162\ \varepsilon}\right)\varepsilon^2, & 0 < \varepsilon \le 4; \\ 2 + \frac{1}{8}\left(\frac{1}{0.5+0.165\ \varepsilon}\right)\varepsilon^2, & 4 < \varepsilon \le 5; \\ 2 + \frac{1}{8}\left(\frac{1}{0.5+0.168\ \varepsilon}\right)\varepsilon^2, & 5 < \varepsilon \le 7; \\ m(\varepsilon - 7) + c, & 7 < \varepsilon \le 50; \end{cases}, \tag{5.24}$$

where $m$ and $c$ are computed from the exact solution as

$$m = \frac{a(50) - a(7)}{50 - 7} = 0.657692 \text{ and } c = a(7) = 5.63108$$

keeping the maximum relative percentage error $\left|\dfrac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100\right|$ less than 1%. The plot of $a_E(\varepsilon)$ given by (5.24) is shown by bold points in Figure 5.1 (explicit discontinuities of $h$ at $\varepsilon = 4, 5$ and 7 are not visible at the resolution of the plotted figure) . As remarked above,

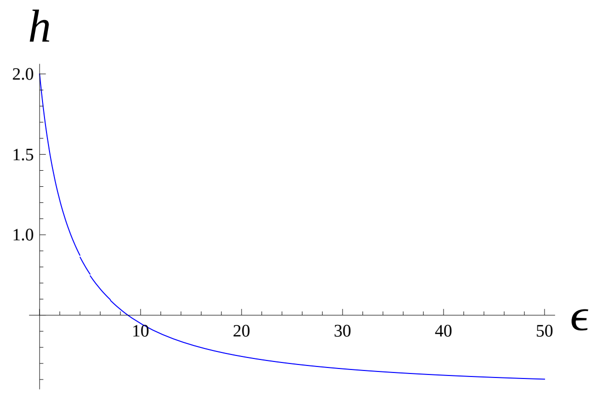


Figure 5.2: The graph of $h(\varepsilon)$ used for approximation of the amplitude by HAM given by (5.24) for $0 < \varepsilon \leq 50$.

Lopez et. al. [18] proposed that a reasonable property for $h$ would be to vanish in the limits as $\varepsilon \to 0$ and $\varepsilon \to \infty$. However, from (5.22) and (5.23) we observe that a suitable approximation to the amplitude of Rayleigh equation can be obtained even if $h$ do not follow this property. The graph of $h(\varepsilon)$ is given in Figure 5.2 for $0 < \varepsilon \leq 50$ (discontinuity in $h$ is not visible at the level of resolution in the figure).

## 5.3 Concluding Remarks

In Section 5.2 we find that one can obtain more accurate approximate formula by suitable choices of the control parameter $h(\varepsilon)$ upto any desired level of accuracy. We also note that a piecewise continuous control parameter $h$ enables us to obtain good approximation by solving only the first order deformation equation. However, 5.21 shows that the first order HAM estimated amplitude $a(\varepsilon)$ is $O\left(\varepsilon^2\right)$. We do not undertake the computation of the VdP amplitude by HAM separately, as that was already reported by Lopez et al. [18]. We shall use these results in the subsequent chapters and compare them with our newly developed *Improved Renormalization Group Method* for estimation of different characteristics of limit cycles in the context of Rayleigh and Van der Pol equations [41].